Ph. D Qualifying Examination
Complex analysis–Autumn 2003

Work all 6 problems. All problems have equal weight. Write each problem in a separate bluebook.

1. Let D be a bounded region in $\mathbf{C}$ whose boundary consists of n-smooth disjoint Jordan curves. Thus D is n-connected. We denote by $\overline{D}$ the closure of D.

Suppose $f(z)$ is a non-constant continuous function on $\overline{D}$ and is analytic in D. Suppose further that

$$|f(w)| = 1 \quad \text{for all } w \in \partial D.$$

Show that f has at least n zeros (counting multiplicities) in D.

2. Show that

$$\int_0^\infty \frac{(\log x)^2}{(1+x^2)}\, dx \;=\; \frac{\pi^3}{8}.$$

Hint: You may need to compute $\int_0^\infty \frac{1}{(1+x^2)}\, dx$ along the way.

Remark: You need to provide details to justify each step in your computation.

3. Let D be the open unit disk and let $f : D \to \mathbf{C}$ be an odd univalent (i.e. one-one) function. Show that there is a univalent analytic function $g : D \to \mathbf{C}$ such that

$$f(z) = \sqrt{g(z^2)}.$$

4. Prove that the function

$$w = \log(z) + \frac{z^2 - 1}{z^2 + 1}$$

is a 1-1 mapping from the half-plane defined by $Re(z) > 0$ onto a region Ω in the w plane. Describe the region Ω as explicitly as you can.

5. Define

$$F(z) = \int_0^\infty x^{z-1} e^{-x^2} \, dx.$$

(a) Prove that F is an analytic function on the region $Re(z) > 0$.

(b) Prove that F extends to a meromorphic function on the whole complex plane.

(c) Find all the poles of F and find the singular parts of F at these poles.

6. Let ω_1 and ω_2 be two non-zero complex numbers with non-real ratio ω_1/ω_2. Let Λ be the lattice $\Lambda = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ and let a and b be two complex numbers not congruent to each other. We form the linear space V of all elliptic functions of period Λ with *at most* simple poles at a and b.

(a) Prove that $\dim_{\mathbf{C}} V$ is at most 2.

(b) Using the method of infinite series, construct explicitly a two dimensional families of elliptic functions in V, thereby proving that $\dim V = 2$.

Remark: In (b) one needs to provide details to why the series converge, why they have period Λ, and why they provide a two dimensional family.