Ph. D Qualifying Examination
Complex analysis–Spring 2002

Work all six problems. All problems have equal weight. Write the solution to each problem in a separate bluebook.

Convention: We denote by Δ the open unit disk and by $\bar{A}$ the closure of A in $\mathbb{C}$.

1. It is known that for any smooth complex valued function g on $\overline{\Delta}$, there is a smooth function in Δ so that $\frac{\partial u}{\partial \bar{z}} = g$. Now let $\Omega \subset \mathbb{C}$ be Δ with the origin and the line segment $[\frac{1}{2}, 1]$ deleted:

$$\Omega = \Delta \setminus \{z \in \mathbb{R} : z = 0 \text{ or } \tfrac{1}{2} \le z \le 1\}.$$

Prove that given any function $f \in C^\infty(\Omega)$ there is a smooth function $u \in C^\infty(\Omega)$ so that $\frac{\partial u}{\partial \bar{z}} = f$.

Hint: Use the Runge approximation theorem.

2. (1). Write down an entire function $f : \mathbb{C} \to \mathbb{C}$ in the form of an infinite product so that its set of zeros equals $\{\log n : n = 2, 3, 4, \dots\}$.
(2) State the definition of an entire function being finite order.
(3) Is there an entire function of finite order whose zero set is $\{\log n : n = 2, 3, 4, \dots\}$? Prove the existence or the non-existence of such functions.

3. Prove that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \frac{z}{2}\, e^{i(\frac{1}{2} - t)z}}{z}\, dz = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{if } t < 0 \text{ or } t > 1. \end{cases}$$

4. Let Δ^* be the punctured disc. We let $\mathcal{P}$ be the set of all subharmonic functions v on Δ^* such that $\limsup_{z \to z_0} v(z) \le 0$ for all $z_0 \in \partial\Delta$ and $\limsup_{z \to 0} v(z) \le 2002$. Prove that for any $v \in \mathcal{P}$, $v(z) \le 0$ in Δ^*.

5. Show that the function

$$w = \log\left(\frac{1+z}{1-z}\right) + \frac{2z}{1+z^2},$$

maps Δ one-one and onto the full w-plane with four half-lines deleted. Find the locations of the four end points of the four half-lines.

6. (1). We let A be the interior of an equilateral triangle. Prove that any one-one, onto holomorphic map $f : A \to \Delta$ can be extended to a holomorphic function $\tilde{f} : U \to \mathbb{C}$ defined on an open neighborhood $U \supset \bar{A}$.
(2). We let B be the interior of a triangle whose three interior angles are $\frac{\pi}{5}$, $\frac{2\pi}{5}$ and $\frac{2\pi}{5}$. Prove that no one-one, onto holomorphic map $f : B \to \Delta$ can be extended to a holomorphic function $\tilde{f} : U \to \mathbb{C}$ defined on an open neighborhood $U \supset \bar{B}$.