Original Problems Proposed by Stanley Rabinowitz 1963–2005

Problem 193. *Mathematics Student Journal*
10(Mar 1963)6

In triangle ABC, angle C is $30°$. Equilateral triangle ABD is erected outwardly on side AB. Prove that CA, CB, CD can be the sides of a right triangle.

Problem 242. *Mathematics Student Journal*
13(Jan 1966)7

D is the midpoint of side BC in $\triangle ABC$. A perpendicular to AC erected at C meets AD extended at point E. If $\angle BAD = 2\angle DAC$, prove that $AE = 2AB$.

Problem 252. *Mathematics Student Journal*
13(May 1966)6

Corrected version of problem 242.

Problem 637. *Mathematics Magazine*
39(Nov 1966)306

Prove that a triangle is isosceles if and only if it has two equal symmedians.

Problem 262. *Mathematics Student Journal*
14(Jan 1967)6

$ACJD$, $CBGH$, and $BAEF$ are squares constructed outwardly on the sides of $\triangle ABC$. DE, FE, and HJ are drawn. If the sum of the areas of squares $BAEF$ and $CBGH$ is equal to the area of the rest of the figure, find the measure of $\angle ABC$.

Problem 191. *Pi Mu Epsilon Journal*
4(Spring 1967)258

Let P and P' denote points inside rectangles $ABCD$ and $A'B'C'D'$, respectively. If $PA = a + b$, $PB = a + c$, $PC = c + d$, $PD = b + d$, $P'A' = ab$, $P'B' = ac$, $P'C' = cd$, prove that $P'D' = bd$.

Problem 661. *Mathematics Magazine*
40(May 1967)163

Find all differentiable functions satisfying the functional equation $f(xy) = yf(x) + xf(y)$.

Problem 198. *Pi Mu Epsilon Journal*
4(Fall 1967)296

A semi-regular solid is obtained by slicing off sections from the corners of a cube. It is a solid with 36 congruent edges, 24 vertices and 14 faces, 6 of which are regular octagons and 8 are equilateral triangles. If the length of an edge of this polytope is e, what is its volume?

Problem E2017. *American Mathematical Monthly*
74(Oct 1967)1005

Let h be the length of an altitude of an isosceles tetrahedron and suppose the orthocenter of a face divides an altitude of that face into segments of lengths h_1 and h_2. Prove that $h^2 = 4h_1h_2$.

Problem E2035*. *American Mathematical Monthly*
74(Dec 1967)1261

Can the Euler line of a nonisosceles triangle pass through the Fermat point of the triangle? (Lines to the vertices from the Fermat point make angles of $120°$ with each other.)

Problem H-125*. *Fibonacci Quarterly*
5(Dec 1967)436

Define a sequence of integers to be *left-normal* if given any string of digits, there exists a member of the given sequence beginning with this string of digits, and define the sequence to be *right-normal* if given any string of digits, there exists a member of the given sequence ending with this string of digits.

Show that the sequence whose nth terms are given by the following are left-normal but not right-normal.

a) $P(n)$, where $P(x)$ is a polynomial function with integral coefficents

b) P_n, where P_n is the nth prime

c) $n!$

d) F_n, where F_n is the nth Fibonacci number.

Problem H-129. *Fibonacci Quarterly*
6(Feb 1968)51

Define the Fibonacci polynomials by $f_1(x) = 1$, $f_2(x) = x$,

$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x), n \geq 0.$$

Solve the equation

$$(x^2 + 4)f_n^2(x) = 4k(-1)^{n-1}$$

in terms of radicals, where k is a constant.

Problem E2064. *American Mathematical Monthly*
75(Feb 1968)190

Let A_n be an $n \times n$ determinant in which the entries, 1 to n^2, are put in order along the diagonals. For example,

$$A_4 = \begin{vmatrix} 1 & 2 & 4 & 7 \\ 3 & 5 & 8 & 11 \\ 6 & 9 & 12 & 14 \\ 10 & 13 & 15 & 16 \end{vmatrix}.$$

Show that if $n = 2k$ then $A_n = \pm k(k + 1)$, and if $n = 2k + 1$, $A_n = \pm(2k^2 + 2k + 1)$.

Problem 693. *Mathematics Magazine*
41(May 1968)158

A square sheet of one cycle by one cycle log log paper is ruled with n vertical lines and n horizontal lines. Find the number of perfect squares on this sheet of logarithmic graph paper.

Problem 203. *Pi Mu Epsilon Journal*
4(Spring 1968)354

Let P denote any point on the median AD of $\triangle ABC$. If BP meets AC at E and CP meets AB at F, prove that $AB = AC$, if and only if, $BE = CF$.

Problem E2102*. *American Mathematical Monthly*
75(Jun 1968)671

Given an equilateral triangle of side one. Show how, by a straight cut, to get two pieces which can be rearranged so as to form a figure with maximal diameter (a) if the figure must be convex; (b) otherwise.

Problem 68–11.** *Siam Review*
10(Jul 1968)376

Two people, A and B, start initially at given points on a spherical planet of unit radius. A is searching for B, i.e., A will travel along a search path at constant speed until he comes within detection distance of B (say λ units). One kind of optimal search path for A is the one which maximizes his probability of detecting B within a given time t. Describe A's optimal search path under the following conditions:

1. B remains still.
2. B is trying to be found, i.e. B is moving in such a way as to maximize A's probability of detecting him in the given time t.
3. B is trying not to be detected.
4. B is moving in some given random way.

Problem E2122. *American Mathematical Monthly*
75(Oct 1968)898

Let D, E, and F be points in the plane of a nonequilateral triangle ABC so that triangles BDC, CEA, and AFB are directly similar. Prove that triangle DEF is equilateral if and only if the three triangles are isosceles (with a side of triangle ABC as base) with base angles of 30°.

Problem E2139. *American Mathematical Monthly*
75(Dec 1968)1114

Consider the following four points of the triangle: the circumcenter, the incenter, the orthocenter, and the nine point center. Show that no three of these points can be the vertices of a nondegenerate equilateral triangle.

Problem 5641*. *American Mathematical Monthly*
75(Dec 1968)1125

From the set $\{1, 2, 3, ..., n^2\}$ how many arrangements of the n^2 elements are there such that there is no subsequence of $n + 1$ elements either monotone increasing or monotone decreasing?

Problem E2150. *American Mathematical Monthly*
76(Feb 1969)187

Let $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ be any three equilateral triangles in the plane (vertices labelled clockwise). Let the midpoints of segments C_2B_3, C_3B_1, C_1B_2 be M_1, M_2, M_3 respectively. Let the points of trisection of segments A_1M_1, A_2M_2, A_3M_3 nearer M_1, M_2, M_3, be T_1, T_2, T_3 respectively. Prove that triangle $T_1T_2T_3$ is equilateral.

Problem 723. *Mathematics Magazine*
42(Mar 1969)96

Find the ratio of the major axis to the minor axis of an ellipse which has the same area as its evolute.

Problem 219. *Pi Mu Epsilon Journal*
4(Spring 1969)440

Consider the following method of solving $x^3 - 11x^2+36x-36 = 0$. Since $(x^3-11x^2+36x)/36 = 1$, we may substitute this value for 1 back in the first equation to obtain $x^3-11x^2+36x(x^3-11x^2+36x)/36-36 = 0$, or $x^4-10x^3+25x^2-36 = 0$, with roots -1, 2, 3 and 6. We find that $x = -1$ is an extraneous root. Generalize the method and determine what extraneous roots are generated.

Problem 759*. *Mathematics Magazine*
43(Mar 1970)103

Circles A, C, and B with radii of lengths a, c, and b, respectively, are in a row, each tangent to a straight line DE. Circle C is tangent to circles A and B. A fourth circle is tangent to each of these three circles. Find the radius of the fourth circle.

Problem 765. *Mathematics Magazine*
43(May 1970)166

Let ABC be an isosceles triangle with right angle at C. Let $P_0 = A$, $P_1 =$ the midpoint of BC, $P_{2k} =$ the midpoint of AP_{2k-1}, and $P_{2k+1} =$ the midpoint of BP_{2k} for $k = 1, 2, 3, ...$. Show that the cluster points of the sequence $\{P_n\}$ trisect the hypotenuse.

Problem 790. *Mathematics Magazine*
44(Mar 1971)106

(1) Find all triangles ABC such that the median to side a, the bisector of angle B, and the altitude to side c are concurrent.

(2) Find all such triangles with integral sides.

Problem 279. *Pi Mu Epsilon Journal*
5(Spring 1972)297

Let $F_0,\ F_1,\ F_2,\ \ldots$ be a sequence such that for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n}.$$

Problem 838. *Mathematics Magazine*
45(Sep 1972)228

Mr. Jones makes n trips a day to his bank to remove money from his account. On the first trip he withdrew $1/n^2$ percent of the account. On the next trip he withdrew $2/n^2$ percent of the balance. On the kth trip he withdrew k/n^2 percent of the balance left at that time. This continued until he had no money left in the bank. Show that the time he removed the largest amount of money was on his last trip of the tenth day.

Problem 941. *Mathematics Magazine*
48(May 1975)181

Show that each of the following expressions is equal to the nth Legendre polynomial.

(i)
$$\frac{1}{n!}\begin{vmatrix} x & 1 & 0 & 0 & \ldots & 0 \\ 1 & 3x & 2 & 0 & \ldots & 0 \\ 0 & 2 & 5x & 3 & \ldots & 0 \\ 0 & 0 & 3 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & n-1 \\ 0 & 0 & 0 & \ldots & n-1 & (2n-1)x \end{vmatrix}$$

(ii)
$$\frac{1}{n!}\begin{vmatrix} x & 1 & 0 & 0 & \ldots & 0 \\ 1 & 3x & 1 & 0 & \ldots & 0 \\ 0 & 4 & 5x & 1 & \ldots & 0 \\ 0 & 0 & 9 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & 0 & \ldots & (n-1)^2 & (2n-1)x \end{vmatrix}$$

Problem 703. *Crux Mathematicorum*
8(Jan 1982)14

A right triangle ABC has legs $AB = 3$ and $AC = 4$. A circle γ with center G is drawn tangent to the two legs and tangent internally to the circumcircle of the triangle, touching the circumcircle in H. Find the radius of γ and prove that GH is parallel to AB.

Problem 720*. *Crux Mathematicorum*
8(Feb 1982)49

On the sides AB and AC of a triangle ABC as bases, similar isosceles triangles ABE and ACD are drawn outwardly. If $BD = CE$, prove or disprove that $AB = AC$.

Problem 728. *Crux Mathematicorum*
8(Mar 1982)78

Let $E(P,Q,R)$ denote the ellipse with foci P and Q which passes through R. If A, B, C are distinct points in the plane, prove that no two of $E(B,C,A)$, $E(C,A,B)$, and $E(A,B,C)$ can be tangent.

Problem 738. *Crux Mathematicorum*
8(Apr 1982)107

Find in terms of p, q, r, a formula for the area of a triangle whose vertices are the roots of

$$x^3 - px^2 + qx - r = 0$$

in the complex plane.

Problem 744. *Crux Mathematicorum*
8(May 1982)136

(a) Prove that, for all nonnegative integers n,

$$\begin{aligned} 5 &| 2^{2n+1} + 3^{2n+1}, \\ 7 &| 2^{n+2} + 3^{2n+1}, \\ 11 &| 2^{8n+3} + 3^{n+1}, \\ 13 &| 2^{4n+2} + 3^{n+2}, \\ 17 &| 2^{6n+3} + 3^{4n+2}, \\ 19 &| 2^{3n+4} + 3^{3n+1}, \\ 29 &| 2^{5n+1} + 3^{n+3}, \\ 31 &| 2^{4n+1} + 3^{6n+9}. \end{aligned}$$

(b) Of the first eleven primes, only 23 has not figured in part (a). Prove that there do not exist polynomials f and g such that

$$23 \mid 2^{f(n)} + 3^{g(n)}$$

for all positive integers n.

Problem A-3. *AMATYC Review*
3(Spring 1982)52

Solve the system of equations:

$$\begin{aligned} x + xy + xyz &= 12 \\ y + yz + yzx &= 21 \\ z + zx + zxy &= 30 \end{aligned}$$

Problem 222.** *Two Year College Math. Journal*
13(Jun 1982)207

Four n-sided dice are rolled. What is the probability that the sum of the highest 3 numbers rolled is $2n$?

Problem 758. *Crux Mathematicorum*
8(Jun 1982)174

Find a necessary and sufficient condition on p, q, r so that the roots of the equation

$$x^3 + px^2 + qx + r = 0$$

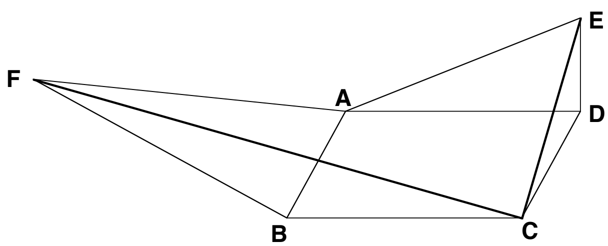
are the vertices of an equilateral triangle in the complex plane.

Problem 766. *Crux Mathematicorum*
8(Aug 1982)210

Let ABC be an equilateral triangle with center O. Prove that, if P is a variable point on a fixed circle with center O, then the triangle whose sides have lengths PA, PB, PC has a constant area.

Problem 523. *Pi Mu Epsilon Journal*
7(Fall 1982)479

Let $ABCD$ be a parallelogram. Erect directly similar right triangles ADE and FBA outwardly on sides AB and DA (so that angles ADE and FBA are right angles). Prove that CE and CF are perpendicular.

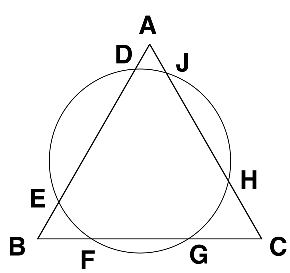


Problem 529. *Pi Mu Epsilon Journal*
7(Fall 1982)480

Show that there is no "universal field" that contains an isomorphic image of every finite field.

Problem 184. *Mathematics and Computer Education*
16(Fall 1982)222

A circle intersects an equilateral triangle ABC in six points, D, E, F, G, H, J. In traversing the perimeter of the triangle, these points occur in the order A, D, E, B, F, G, C, H, J. Prove that $AD + BF + CH =$



$AJ + BE + CG$.

Problem 780. *Crux Mathematicorum*
8(Oct 1982)247

Prove that one can take a walk on Pascal's triangle, stepping from one element only to one of its nearest neighbors, in such a way that each element $\binom{m}{n}$ gets stepped on exactly $\binom{m}{n}$ times.

Problem 3917. *School Science and Mathematics*
82(Oct 1982)532

If A, B, and C are the angles of an acute or obtuse triangle, prove that

$$\begin{vmatrix} \tan A & \tan B & \tan C \\ 1 & 1 & 1 \\ \sin 2A & \sin 2B & \sin 2C \end{vmatrix} = 0.$$

Problem 1155. *Mathematics Magazine*
55(Nov 1982)299

A plane intersects a sphere forming two spherical segments. Let S be one of these segments and let A be the point furthest from the segment S. Prove that the length of the tangent from A to a variable sphere inscribed in the segment S is a constant.

Problem 784. *Crux Mathematicorum*
8(Nov 1982)277

Let $F_n = a_i/b_i, i = 1, 2, \ldots, m$, be the Farey sequence of order n, that is, the ascending sequence of irreducible fractions between 0 and 1 whose denominators do not exceed n. (For example,

$$F_5 = (\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}),$$

with $m = 11$.) Prove that, if $P_0 = (0, 0)$ and $P_i = (a_i, b_i), i = 1, 2, \ldots, m$ are lattice points in a Cartesian coordinate plane, then $P_0 P_1 \ldots P_m$ is a simple polygon of area $(m-1)/2$.

Problem C-3. *AMATYC Review*
4(Fall 1982)54

Prove or disprove that if n is a non-negative integer, then $2^{7n+1} + 3^{2n+1} + 5^{10n+1} + 7^{6n+1}$ is divisible by 17.

Problem 798*. *Crux Mathematicorum*
8(Dec 1982)304

For a nonnegative integer n, evaluate

$$I_n \equiv \int_0^1 \binom{x}{n} dx.$$

Problem 808*. *Crux Mathematicorum*
9(Jan 1983)22

Find the length of the largest circular arc contained within the right triangle with sides $a \leq b < c$.

Problem 3930. *School Science and Mathematics*
83(Jan 1983)83

If two distinct squares of the same area can be inscribed in a triangle, prove that the triangle is isosceles.

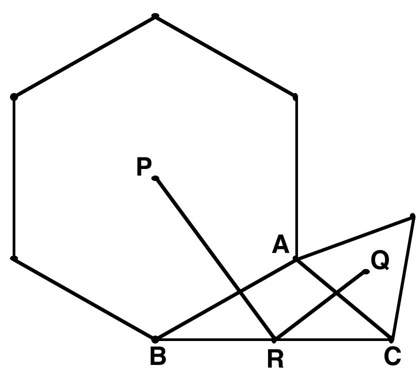
Problem 817. *Crux Mathematicorum*
9(Feb 1983)46

(a) Suppose that to each point on the circumference of a circle we arbitrarily assign the color red or green. Three distinct points of the same color will be said to form a *monochromatic triangle*. Prove that there are monochromatic isosceles triangles.

(b) Prove or disprove that there are monochromatic isosceles triangles if to every point on the circumference of a circle we arbitrarily assign one of k colors, where $k \geq 2$.

Problem 3936. *School Science and Mathematics*
83(Feb 1983)172

Let P be the center of a regular hexagon erected outwardly on side AB of triangle ABC. Also, let Q be the center of an equilateral triangle erected outwardly on side AC. If R is the midpoint of side BC, prove that angle PRQ is a right angle.



Problem 1168. *Mathematics Magazine*
56(Mar 1983)111

Let P be a variable point on side BC of triangle ABC. Segment AP meets the incircle of triangle ABC in two points, Q and R, with Q being closer to A. Prove that the ratio AQ/AP is a minimum when P is the point of contact of the excircle opposite A with side BC.

Problem 1248. *Journal of Recreational Mathematics*
15(1982-1983)302

A *semi-anti-magic square* of order n is an $n \times n$ array of distinct integers that has the property that the row sums and the column sums form a set of $2n$ consecutive integers. For example, in the 4×4 array

4	10	13	14
7	12	9	16
8	1	21	15
23	20	3	2

the row and column sums comprise the set {41, 42, 43, 44, 45, 46, 47, 48}.

a. Prove that there is no semi-anti-magic square of order 5.

b. Find a semi-anti-magic square of order 6.

c. For which n do semi-anti-magic squares of order n exist?

d. Can a semi-anti-magic square contain only the positive integers less than or equal to n^2?

Problem 821. *Crux Mathematicorum*
9(Mar 1983)78

Solve the alphametic

CRUX=[MATHEMAT/CORUM],

where the brackets indicate that the remainder of the division, which is less than 500, is to be discarded.

Problem 191. *Mathematics and Computer Education*
17(Spring 1983)142

Find all positive solutions of

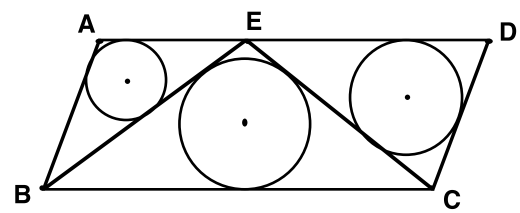
a. $x^x = \frac{1}{2}\sqrt{2}$

b. $x^x = \frac{3}{4}\sqrt{6}$

c. Let $x^x = c$. For what values of c will there be no positive real solution, one positive solution, etc.

Problem D–2. *AMATYC Review*
4(Spring 1983)62-63

Let $ABCD$ be a parallelogram and let E be any point on side AD. Let r_1, r_2, and r_3 represent the inradii of triangles ABE, BEC, and CED, respectively. Prove that $r_1 + r_3 \geq r_2$.



Problem 535. *Pi Mu Epsilon Journal*
7(Spring 1983)542

In the small hamlet of Abacinia, two base systems are in common use. Also, everyone speaks the truth. One resident said, "26 people use my base, base 10, and only 22 people speak base 14." Another said, "Of the 25 residents, 13 are bilingual and 1 is illiterate." How many residents are there?

Problem 541. *Pi Mu Epsilon Journal* 7(Spring 1983)543-544

A line meets the boundary of an annulus A_1 (the ring between two concentric circles) in four points P, Q, R, S with R and S between P and Q. A second annulus A_2 is constructed by drawing circles on PQ and RS as diameters. Find the relationship between the areas of A_1 and A_2.

Problem 545. *Pi Mu Epsilon Journal* 7(Spring 1983)544

Let F_n denote the nth Fibonacci number ($F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_n + F_{n+1}$ for n a positive integer). Find a formula for F_{m+n} in terms of F_m and F_n (only).

Problem 847. *Crux Mathematicorum* 9(May 1983)143-144

Prove that

$$\sum_{j=0}^{n} \frac{\binom{n}{2j-n-1}}{5^j} = \frac{1}{2}(0.4)^n F_n,$$

where F_n is the nth Fibonacci number. (Here we make the usual assumption that $\binom{a}{b} = 0$ if $b < 0$ or $b > a$.)

Problem B-496. *Fibonacci Quarterly* 21(May 1983)147

Show that the centroid of the triangle whose vertices have coordinates

(F_n, L_n), (F_{n+1}, L_{n+1}), and (F_{n+6}, L_{n+6})

is (F_{n+4}, L_{n+4}).

Problem B-497. *Fibonacci Quarterly* 21(May 1983)147

For d an odd positive integer, find the area of the triangle with vertices (F_n, L_n), (F_{n+d}, L_{n+d}), and (F_{n+2d}, L_{n+2d}).

Problem 868. *Crux Mathematicorum* 9(Aug 1983)209

The graph of $x^3 + y^3 = 3axy$ is known as the *folium of Descartes*. Prove that the area of the loop of the folium is equal to the area of the region bounded by the folium and its asymptote $x + y + a = 0$.

Problem E3013. *American Mathematical Monthly* 90(Oct 1983)566

Let ABC be a fixed triangle in the plane. Let T be the transformation of the plane that maps a point P into its isotomic conjugate (relative to ABC). Let G be the transformation that maps P into its isogonal conjugate. Prove that the mappings TG and GT are affine collineations (linear transformations).

Problem 875*. *Crux Mathematicorum* 9(Oct 1983)241

Can a square be dissected into three congruent nonrectangular pieces?

Problem 1292. *Journal of Recreational Mathematics* 16(1983-1984)137

a. For some n, partition the first n perfect squares into two sets of the same size and same sum.

b. For some n, partition the first n triangular numbers into two sets of the same size and same sum. (Triangular numbers are of the form $T_n = n(n+1)/2$.)

c. For some n, partition the first n perfect cubes into two sets of the same size and same sum.

d. For some n, partition the first n perfect fourth powers into two sets of the same size and same sum.

Problem 1299. *Journal of Recreational Mathematics* 16(1983-1984)139

Show how to dissect a 3-4-5 right triangle into four pieces that may be rearranged to form a square.

Problem H-362*. *Fibonacci Quarterly* 21(Nov 1983)312-313

Let Z_n be the ring of integers modulo n. A *Lucas number* in this ring is a member of the sequence $\{L_k\}, k = 0, 1, 2, ...,$ where $L_0 = 2$, $L_1 = 1$, and $L_{k+2} \equiv L_{k+1} + L_k$ for $k \geq 0$. Prove that, for $n > 14$, all members of Z_n are Lucas numbers if and only if n is a power of 3.

Problem 881. *Crux Mathematicorum* 9(Nov 1983)275

Find the unique solution to the following "areametic", where A, B, C, D, E, N, and R represent distinct decimal digits:

$$\int_{\mathrm{B}}^{\mathrm{D}} \mathrm{C}x^{\mathrm{N}} dx = \mathrm{AREA}.$$

Problem 894*. *Crux Mathematicorum* 9(Dec 1983)313

(a) Find necessary and sufficient conditions on the complex numbers a, b, ω so that the roots of $z^2 + 2az + b = 0$ and $z - \omega = 0$ shall be collinear in the complex plane.

(b) Find necessary and sufficient conditions on the complex numbers a, b, c, d so that the roots of $z^2 + 2az + b = 0$ and $z^2 + 2cz + d = 0$ shall all be collinear in the complex plane.

Problem E-3. *AMATYC Review* 5(Fall 1983)56

Let E be a fixed non-circular ellipse. Find the locus of a point P in the plane of E with the property that the two tangents from P to E have the same length.

Problem 903. *Crux Mathematicorum*
10(Jan 1984)19

Let ABC be an acute-angled triangle with circumcenter O and orthocenter H.

(a) Prove that an ellipse with foci O and H can be inscribed in the triangle.

(b) Show how to construct, with straightedge and compass, the points L, M, N where this ellipse is tangent to the sides BC, CA, AB, respectively, of the triangle.

Problem 3988. *School Science and Mathematics*
84(Feb 1984)175

Let b be an arbitrary complex number. Find all 2 by 2 matrices X with complex entries that satisfy the equation

$$(X - bI)^2 = 0,$$

where I is the 2 by 2 identity matrix.

Problem 913*. *Crux Mathematicorum*
10(Feb 1984)53

Let

$$f_n(x) = x^n + 2x^{n-1} + 3x^{n-2} + 4x^{n-3} + \cdots + nx + (n+1).$$

Prove or disprove that the discriminant of $f_n(x)$ is

$$(-1)^{n(n-1)/2} \cdot 2(n+2)^{n-1}(n+1)^{n-2}.$$

Problem H-366*. *Fibonacci Quarterly*
22(Feb 1984)90

The *Fibonacci polynomials* are defined by the recursion

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$$

with the initial conditions $f_1(x) = 1$ and $f_2(x) = x$. Prove that the discriminant of $f_n(x)$ is

$$(-1)^{(n-1)(n-2)/2} 2^{n-1} n^{n-3}$$

for $n > 1$.

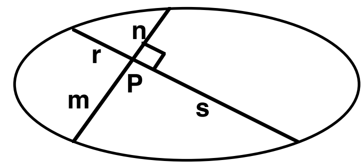
Problem 1186. *Mathematics Magazine*
57(Mar 1984)109

(a) Show how to arrange the 24 permutations of the set $\{1, 2, 3, 4\}$ in a sequence with the property that adjacent members of the sequence differ in each coordinate. (Two permutations (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) differ in each coordinate if $a_i \neq b_i$ for $i = 1, 2, 3, 4$.)

(b) For which n can the $n!$ permutations of the integers from 1 through n be arranged in a similar manner?

Problem 926. *Crux Mathematicorum*
10(Mar 1984)89

Let P be a fixed point inside an ellipse, L a variable chord through P, and L' the chord through P that is perpendicular to L. If P divides L into two segments of lengths m and n, and if P divides L' into two segments of lengths r and s, prove that $1/mn + 1/rs$ is a constant.



Problem 936. *Crux Mathematicorum*
10(Apr 1984)115

Find all eight-digit palindromes in base 10 that are also palindromes in at least one of the bases two, three, ..., nine.

Problem 1322. *Journal of Recreational Mathematics*
16(1983-1984)222

How should one select n integral weights that may be used to weigh the maximal number of consecutive integral weights (beginning with 1)?

The weighing process involves a pan balance and the unknown integral weight may be placed on either pan. The selected weights may be placed on either pan, also. Furthermore, one may reason that if an unknown weight weighs less than $k + 1$ but greater than $k - 1$, then it must weigh exactly k.

Problem 946. *Crux Mathematicorum*
10(May 1984)156

The nth differences of a function f at r are defined as usual by $\triangle^0 f(r) = f(r)$ and

$$\triangle^1 f(r) = \triangle f(r) = f(r+1) - f(r),$$

$$\triangle^n f(r) = \triangle(\triangle^{n-1} f(r)), n = 1, 2, 3, \ldots.$$

Prove or disprove that, if $\triangle^n f(1) = n$ for $n = 0, 1, 2, \ldots$, then

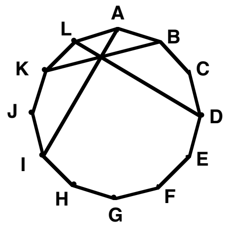
$$f(n) = (n-1) \cdot 2^{n-2}.$$

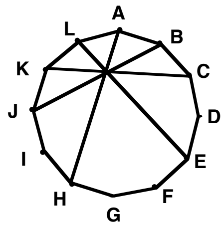
Problem F-4. *AMATYC Review*
5(Spring 1984)55

Let $ABCDEFGHIJKL$ be a regular dodecagon.

(a) Prove that the diagonals AI, BK, DL concur.

(b) Do the same for the diagonals AH, BJ, CK, EL.





Problem 960. *Crux Mathematicorum*
10(Jun 1984)196

If the altitude of a triangle is also a symmedian, prove that the triangle is either an isosceles triangle or a right triangle.

Problem 210. *Mathematics and Computer Education*
18(Fall 1984)229

Prove that the determinant of a magic square with integer entries is divisible by the magic constant.

Problem 963. *Crux Mathematicorum*
10(Aug 1984)216

Find consecutive squares that can be split into two sets with equal sums.

Problem 4010. *School Science and Mathematics*
84(Oct 1984)534

Prove that there is no triangle whose side lengths are prime numbers and which has an integral area.

Problem 4011. *School Science and Mathematics*
84(Oct 1984)534

The lengths of the sides of a triangle are a, b, and c with $c > b$.

(a) Find the condition on a, b, and c so that the altitude to side a is tangent to the circumcircle of the triangle.

(b) Find such a triangle with sides of integral length.

Problem 972*. *Crux Mathematicorum*
10(Oct 1984)262

(a) Prove that two equilateral triangles of unit side cannot be placed inside a unit square without overlapping.

(b) What is the maximum number of regular tetrahedra of unit side that can be packed without overlapping inside a unit cube?

(c) Generalize to higher dimensions.

Problem 581. *Pi Mu Epsilon Journal*
8(Fall 1984)43

If a triangle similar to a 3-4-5 right triangle has its vertices at lattice points (points with integral coordinates) in the plane, must its legs be parallel to the coordinate axes?

Problem 986. *Crux Mathematicorum*
10(Nov 1984)292

Let
$$x = \sqrt[3]{p+\sqrt{r}} + \sqrt[3]{q-\sqrt{r}},$$
where p, q, r are integers and $r \geq 0$ is not a perfect square. If x is rational ($x \neq 0$), prove that $p = q$ and x is integral.

Problem 1026. *Crux Mathematicorum*
11(Mar 1985)83

D, E, and F are points on sides BC, CA, and AB, respectively, of triangle ABC and AD, BE, and CF concur at point H. If H is the incenter of triangle DEF, prove that H is the orthocenter of triangle ABC. (This is the converse of a well-known property of the orthocenter.)

Problem 1050. *Crux Mathematicorum*
11(May 1985)148

In the plane, you are given the curve known as the folium of Descartes. Show how to construct the asymptote to this curve using straightedge and compasses only.

Problem 1216*. *Mathematics Magazine*
58(May 1985)177

Find all differentiable functions f that satisfy
$$f(x) = xf'\left(\frac{x}{\sqrt{3}}\right)$$
for all real x.

Problem 1053. *Crux Mathematicorum*
11(Jun 1985)187

Exhibit a bijection between the points in the plane and the lines in the plane.

Problem H-2. *AMATYC Review*
6(Spring 1985)59

Under what condition(s) are at least two roots of $x^3 - px^2 + qx - r = 0$ equal? Here, the numbers, p, q, r, are (at worst) complex, and we seek necessary and sufficient conditions.

Problem 212. *Mathematics and Computer Education* 19(Winter 1985)67

Let ABC be an arbitrary triangle with sides a, b, and c. Let P, Q, and R be points on sides BC, CA, and AB respectively. Let $AQ = x$ and $AR = y$. If triangles ARQ, BPR, and CQP all have the same area, then prove that either $xy = bc$ or $x/b + y/c = 1$.

Problem 1070. *Crux Mathematicorum* 11(Sep 1985)221

Let O be the center of an n-dimensional sphere. An $(n-1)$-dimensional hyperplane, H, intersects the sphere (O) forming two segments.

Another n-dimensional sphere, with center C, is inscribed in one of these segments, touching sphere (O) at point B and touching hyperplane H at point Q. Let AD be the diameter of sphere (O) that is perpendicular to hyperplane H, the points A and B being on opposite sides of H. Prove that A, Q, and B colline.

Problem 592. *Pi Mu Epsilon Journal* 8(Spring 1985)122

Find all 2 by 2 matrices A whose entries are distinct non-zero integers such that for all positive integers n, the absolute value of the entries of A^n are all less than some finite bound M.

Problem 596. *Pi Mu Epsilon Journal* 8(Spring 1985)123

Two circles are externally tangent and tangent to a line L at points A and B. A third circle is inscribed in the curvilinear triangle bounded by these two circles and L and it touches L at point C. A fourth circle is inscribed in the curvilinear triangle bounded by line L and the circles at A and C and it touches the line at D. Find the relationship between the lengths AD, DC, and CB.

Problem 597. *Pi Mu Epsilon Journal* 8(Spring 1985)123

Find the smallest n such that there exists a polyhedron of non-zero volume and with n edges of lengths $1, 2, 3, \ldots, n$.

Problem 1078. *Crux Mathematicorum* 11(Oct 1985)250

Prove that

$$\sum_{k=1}^{n} \binom{n}{k} \cdot \frac{1}{k} = \sum_{k=1}^{n} \frac{2^k - 1}{k}.$$

Problem 1101. *Crux Mathematicorum* 12(Jan 1986)11

Independently solve each of the following alphametics in base 10:

$$6 \cdot \text{FLOCK} = \text{GEESE},$$

$$7 \cdot \text{FLOCK} = \text{GEESE},$$

$$8 \cdot \text{FLOCK} = \text{GEESE}.$$

Problem 1119. *Crux Mathematicorum* 12(Feb 1986)27

The following problem, for which I have been unable to locate the source, has been circulating. A rectangle is partitioned into smaller rectangles. If each of the smaller rectangles has the property that one of its sides has integral length, prove that the original rectangle also has this property.

Problem 1124. *Crux Mathematicorum* 12(Mar 1986)51 with Peter Gilbert

If $1 < a < 2$ and k is an integer, prove that

$$[a[k/(2-a)] + a/2] = [ak/(2-a)]$$

where $[x]$ denotes the greatest integer not larger than x.

Problem 1133. *Crux Mathematicorum* 12(Apr 1986)78

The incircle of triangle ABC touches sides BC and AC at points D and E respectively. If $AD = BE$, prove that the triangle is isosceles.

Problem 4102. *School Science and Mathematics* 86(May 1986)446

A piece of wood is made as follows. Take four unit cubes and glue a face of each to a face of a fifth (central) cube in such a manner that the two exposed faces of the central cube are not opposite each other. Prove that twenty-five of these pieces cannot be assembled to form a 5 by 5 by 5 cube.

Problem 1148. *Crux Mathematicorum* 12(May 1986)108

Find the triangle of smallest area that has integral sides and integral altitudes.

Problem 1157*. *Crux Mathematicorum* 12(Jun 1986)140

Find all triples of positive integers (r, s, t), $r \le s, t$, for which $(rs + r + 1)(st + s + 1)(tr + t + 1)$ is divisible by $(rst - 1)^2$.

Problem 235. *Mathematics and Computer Education* 20(Fall 1986)211

Prove or disprove: If every side of triangle A is larger than the corresponding side of triangle B, then triangle A has a larger area than triangle B.

Problem K-1. *AMATYC Review*
8(Sep 1986)67

Everyone is familiar with the linear recurrence, $x_n = x_{n-1} + x_{n-2}$, $n \geq 2$, which generates the familiar Fibonacci sequence with the initial conditions $x_0 = x_1 = 1$. Can you find a linear recurrence, with initial conditions, that will generate precisely the sequence of perfect squares?

Problem 1187. *Crux Mathematicorum*
12(Nov 1986)242

Find a polynomial with integer coefficients that has $2^{1/5} + 2^{-1/5}$ as a root.

Problem 1193*. *Crux Mathematicorum*
12(Dec 1986)282

Is there a Heronian triangle (having sides and area rational) with one side twice another?

Problem 1206. *Crux Mathematicorum*
13(Jan 1987)15

Let X be a point inside triangle ABC, let Y be the isogonal conjugate of X and let I be the incenter of $\triangle ABC$. Prove that X, Y, and I colline if and only if X lies on one of the angle bisectors of $\triangle ABC$.

Problem 236. *Mathematics and Computer Education*
21(Winter 1987)69

Let A and B be two distinct points in the plane. For which point, P, on the perpendicular bisector of AB does the circle determined by A, B, and P have the smallest radius?

Problem 1261. *Mathematics Magazine*
60(Feb 1987)40

(a) What is the area of the smallest triangle with integral sides and integral area?

(b) What is the volume of the smallest tetrahedron with integral sides and integral volume?

Problem 1227. *Crux Mathematicorum*
13(Mar 1987)86

Find all angles θ in $[0, 2\pi)$ for which

$$\sin\theta + \cos\theta + \tan\theta + \cot\theta + \sec\theta + \csc\theta = 6.4.$$

Problem 1240. *Crux Mathematicorum*
13(Apr 1987)120

Find distinct positive integers a, b, c such that

$$a + b + c, \quad ab + bc + ca, \quad abc$$

forms an arithmetic progression.

Problem 1251. *Crux Mathematicorum*
13(May 1987)179
(Dedicated to Léo Sauvé)

(a) Find all integral n for which there exists a regular n-simplex with integer edge and integer volume.

(b)* Which such n-simplex has the smallest volume?

Problem 1262. *Crux Mathematicorum*
13(Sep 1987)215
(Dedicated to Léo Sauvé)

Pick a random n-digit decimal integer, leading 0's allowed, with each integer being equally likely. What is the expected number of distinct digits in the chosen integer?

Problem 1574. *Journal of Recreational Mathematics*
19.3(1987)232

(a) Prove that there is no 4×4 magic square, consisting of distinct positive integers, whose top row consists of the entries 1, 9, 8, 7 in that order.

(b) Find a 4×4 magic square consisting of distinct integers each larger than -5, whose top row consists of the entries 1, 9, 8, 7 in that order.

Problem 660. *Pi Mu Epsilon Journal*
8(Fall 1987)470

Recently the elderly numerologist E. P. B. Umbugio read the life of Leonardo Fibonacci and became interested in the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, ..., where each number after the second one is the sum of the two preceding numbers. He is trying to find a 3×3 magic square of distinct Fibonacci numbers (but $F_1 = 1$ and $F_2 = 1$ can both be used), but has not yet been successful. Help the professor by finding such a magic square or by proving that none exists.

Problem 245. *Mathematics and Computer Education*
21(Fall 1987)?

Equilateral triangle ABC is inscribed in a unit circle. Chord BD intersects AC in point E, with D closer to A than C. If the length of chord BD is $\sqrt{14}/2$, find the length of DC.

Problem 1278. *Crux Mathematicorum*
13(Oct 1987)257

(a) Find a non-constant function $f(x, y)$ such that $f(ab + a + b, c)$ is symmetric in a, b, and c.

(b)* Find a non-constant function $g(x, y)$ such that $g(ab(a + b), c)$ is symmetric in a, b, and c.

Problem 1596. *Journal of Recreational Mathematics*
19.4(1987)307

What is the shortest possible chess game in which White moves his King only (except on the first move) and mates Black?

Problem 1281*. *Crux Mathematicorum*
13(Nov 1987)289

Find the area of the largest triangle whose vertices lie in or on a unit n-dimensional cube.

Problem 1293. *Crux Mathematicorum*
13(Dec 1987)320
with Steve Maurer

Solve the following "twin" problems. In both problems, O is the center of a circle, A is a point inside the circle, $OA \perp AB$ with point B lying on the circle. C is a point on chord BD.

(a) If $AB = BC$ and $\angle ABC = 60°$, prove $CD = OA\sqrt{3}$.

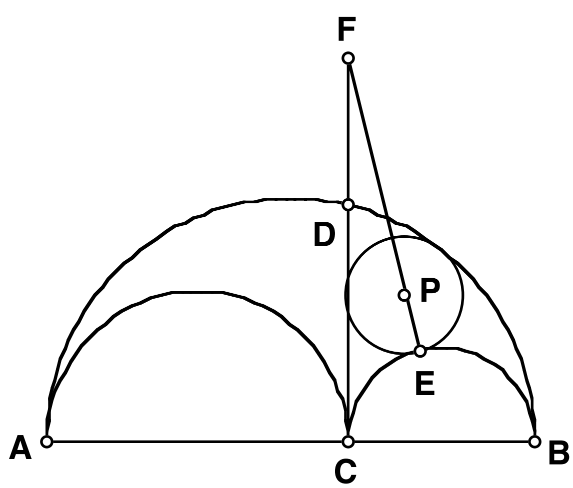
(b) If $OA = BC$ and $\angle ABC = 30°$, prove $CD = AB\sqrt{3}$.

Problem 20.8. *Mathematical Spectrum*
20(1987/1988)59

Let P be a point on side BC of triangle ABC. If $(AP)^2 = (AB)(BC) - (PB)(PC)$, prove that either $AB = AC$ or that AP bisects angle BAC.

Problem Cover1988. *Arbelos*
6(1988)cover
(Dedicated to Sam Greitzer)

An arbelos is formed by erecting semicircles on segments AC, CB, and AB. $CD \perp AB$. A circle, center P, is drawn touching the semicircle on BC at E and touching the semicircle on AB and tangent to CD. EP meets CD at F. Prove that $EF = AC$.



Problem 1613. *Journal of Recreational Mathematics*
20.1(1988)78

A lattice point is a point in the plane with integer coordinates. A lattice triangle is a triangle whose vertices are lattice points. Find a lattice triangle with the property that its centroid, circumcenter, incenter, and orthocenter are also lattice points.

Problem 4180. *School Science and Mathematics*
88(Feb 1988)178

Let A be a variable point on a fixed ellipse with foci B and C. Prove that the area of triangle ABC is proportional to $\tan(A/2)$.

Problem 20.9. *Mathematical Spectrum*
20(1987/1988)95

Prove that $15^n - 2^{3n+1} + 1$ is divisible by 98 for all positive integers n.

Problem N-3. *AMATYC Review*
9(Spring 1988)71

Over C, the field of complex numbers, the polynomial $x^2 + y^2$ factors as $(x + iy)(x - iy)$. Does the polynomial $x^2 + y^2 + z^2$ factor over C?

Problem 1312. *Crux Mathematicorum*
14(Feb 1988)44

Find all 27 solutions of the system of equations

$$\begin{aligned} y &= 4x^3 - 3x \\ z &= 4y^3 - 3y\ . \\ x &= 4z^3 - 3z \end{aligned}$$

Problem 1325. *Crux Mathematicorum*
14(Mar 1988)77

Let P be any point inside a unit circle. Perpendicular chords AB and CD pass through P. Two other chords passing through P form four angles of θ radians each with these two chords (measured counterclockwise). Prove that the area of the portion of these four angles contained within the circle is 2θ.

Problem 1334. *Crux Mathematicorum*
14(Apr 1988)109

(a) Suppose Fibonacci had wanted to set up an annuity that would pay F_n lira on the nth year after the plan was established, for $n = 1, 2, 3, \ldots$ ($F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n > 2$). To fund such an annuity, Fibonacci had planned to invest a sum of money with the Bank of Pisa [they'd held a lien on a tower he once owned], which paid 70% interest per year, and instruct them to administer the trust. How much money did he have to invest so that the annuity could last in perpetuity?

(b) When he got to the bank, Fibonacci found that their interest rate was only 7% (he had misread their ads), not enough for his purposes. Despondently, he went looking for another bank with a higher interest rate. What rate must he seek in order to allow for a perpetual annuity?

Problem E3264. *American Mathematical Monthly* 95(Apr 1988)352

Let P_n/Q_n be the nth convergent for the continued fraction

$$\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{4+}\cdots,$$

i.e. let $P_1 = 1$, $Q_1 = 1$, $P_2 = 2$, $Q_2 = 3$, and

$$P_n = nP_{n-1} + P_{n-2}, \quad Q_n = nQ_{n-1} + Q_{n-2} \quad (n \geq 3).$$

Give asymptotic estimates for P_n and Q_n.

Problem B-616. *Fibonacci Quarterly* 26(May 1988)181

(a) Find the smallest positive integer a such that $L_n \equiv F_{n+a} \pmod 6$ for $n = 0, 1, \ldots$.

(b) Find the smallest positive integer b such that $L_n \equiv F_{5n+b} \pmod 5$ for $n = 0, 1, \ldots$.

Problem B-617. *Fibonacci Quarterly* 26(May 1988)181

Let R be a rectangle each of whose vertices has Fibonacci numbers as its coordinates x and y. Prove that the sides of R must be parallel to the coordinate axes.

Problem 1341. *Crux Mathematicorum* 14(May 1988)140

An ellipse has center O and the ratio of the lengths of the axes is $2+\sqrt{3}$. If P is a point on the ellipse, prove that the (acute) angle between the tangent to the ellipse at P and the radius vector PO is at least $30°$.

Problem 1353. *Crux Mathematicorum* 14(Jun 1988)174

(a) Find a linear recurrence with constant coefficients whose range is the set of all integers.

(b)* Is there a linear recurrence with constant coefficients whose range is the set of all Gaussian integers (complex numbers $a + bi$ where a and b are integers)?

Problem 255. *Mathematics and Computer Education* 22(Spring 1988)137

Let P be a point on the parabola $y = ax^2$, but not the vertex. Let R be the projection of P on the axis of the parabola, and let Q be the point on the axis such that PQ is perpendicular to the parabola at P. Prove that as P moves around the parabola, the length of QR remains constant.

Problem 991. *Elemente der Mathematik* 43(Jul 1988)125

Prove that if the area of face S of a tetrahedron is the average of the areas of the other three faces, then the line joining the incenter to the centroid of the tetrahedron is parallel to face S.

Problem E3279. *American Mathematical Monthly* 95(Aug 1988)655

For some $n > 1$, find a simplex in E^n with integer edges and volume 1.

Problem H-423. *Fibonacci Quarterly* 26.3(Aug 1988)283

Prove that each root of the equation

$$F_n x^n + F_{n+1}x^{n-1} + F_{n+2}x^{n-2} + \cdots + F_{2n-1}x + F_{2n} = 0$$

has absolute value near ϕ, the golden ratio.

Problem 1364. *Crux Mathematicorum* 14(Sep 1988)202

Let a and b be integers. Find a polynomial with integer coefficients that has $\sqrt[3]{a} + \sqrt[3]{b}$ as a root.

Problem 673. *Pi Mu Epsilon Journal* 8(Spring 1988)533

Let AB be an edge of a regular tesseract (a four-dimensional cube) and let C be the tesseract's vertex that is furthest from A. Find the measure of angle ACB.

Problem 4205. *School Science and Mathematics* 88(Oct 1988)535 (editorial revision)

Find a single explicit formula (no "cases" allowed) for the n-th term of the sequence

$$2, 4, 6_1, 8, 12_3, 16, 24_7, 32, 48_{15}, 64, \ldots$$

where m_n denotes the number m repeated n times.

Problem 4210. *School Science and Mathematics* 88(Nov 1988)626

A student was given as an assignment to write down the first twenty rows of Pascal's triangle. He made one mistake, however. Except for one number, every number in the triangle was the sum of the two numbers above it. The teacher noticed that the last row began 1, 20, 190, 1090, ... whereas it should have begun 1, 20, 190, 1140, ... Also, the sum of the numbers of the last row was 1046976 whereas it should have been 1048576. From this information, the teacher was able to pinpoint the student's mistake. At which point in Pascal's triangle was the mistake made? What was the erroneous entry? What should have been the correct entry?

Problem 1384. *Crux Mathematicorum*
14(Nov 1988)269

If the center of curvature of every point on an ellipse lies inside the ellipse, prove that the eccentricity of the ellipse is at most $1/\sqrt{2}$.

Problem H-425. *Fibonacci Quarterly*
26(Nov 1988)377

Let $F_n(x)$ be the n^{th} Fibonacci polynomial defined by $F_1(x) = 1$, $F_2(x) = x$, $F_{n+2}(x) = xF_{n+1}(x)+F_n(x)$. Evaluate:

(a) $\int_0^1 F_n(x)\,dx$;

(b) $\int_0^1 F_n^2(x)\,dx$.

Problem 4213. *School Science and Mathematics*
88(Dec 1988)715

B is a point on line segment AC. Semicircles are erected on the same side of AC with AB, BC, and AC as diameters. From C, a line is drawn tangent to the semicircle on AB touching it at point D. This line meets the semicircles on AC and BC at E and F, respectively. Prove that $DE = DF$.

Problem 4229. *School Science and Mathematics*
89(Feb 1989)176

The common tangent, L, to two externally tangent circles touches the circle with radius R at point A and the circle with radius r ($r < R$) at point P_0. Circle C_1 is inscribed in the region bounded by the two circles and L and touches L at point P_1. A sequence of circles is constructed as follows: circle C_n touches line L at P_n, circle C_{n-1}, and the circle with radius R. Let x_n denote the distance from A to P_n. Find a formula for x_n in terms of R and r.

Problem 1426. *Crux Mathematicorum*
15(Mar 1989)74

Prove that if n is a positive integer, then

$$512 \mid 3^{2n} - 32n^2 + 24n - 1.$$

Problem 4235. *School Science and Mathematics*
89(Mar 1989)264

Let F_n denote the n-th Fibonacci number ($F_1 = 1$, $F_2 = 1$, $F_n = F_{n-2} + F_{n-1}$ for $n > 2$) and let L_k denote the k-th Lucas number ($L_1 = 1$, $L_2 = 3$, $L_k = L_{k-2} + L_{k-1}$ for $k > 2$). Determine k as a function of n, $k = k(n)$, such that

$$F_n + L_{k(n)} \equiv 0 \pmod 4$$

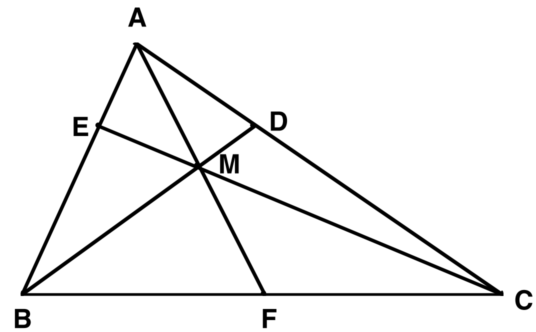
for all positive integers n.

Problem 4236. *School Science and Mathematics*
89(Apr 1989)354

Find all polynomials in two variables, $f(x, y)$, with complex coefficients, whose value is real for all complex values of its arguments.

Problem 264. *Mathematics and Computer Education*
23(Spring 1989)140

Let ABC be a triangle with D and E being points on sides AC and AB respectively. Let M be the intersection of BD and CE. If the areas of $\triangle BEM$ and $\triangle CDM$ are equal, prove that M lies on the median to side BC.



Problem 1441*. *Crux Mathematicorum*
15(May 1989)147

Let

$$S_n = (x - y)^n + (y - z)^n + (z - x)^n.$$

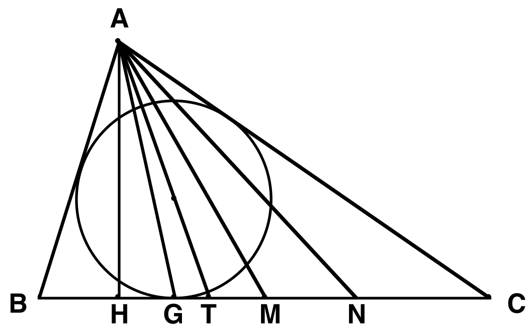
It is easy to see that if p is a prime, S_p/p is a polynomial with integer coefficients. Prove that

$$\frac{S_2}{2} \mid S_{2+6k},$$
$$\frac{S_3}{3} \mid S_{3+6k},$$
$$\frac{S_5}{5} \mid S_{5+6k},$$
$$\frac{S_7}{7} \mid S_{7+6k},$$

for all $k = 1, 2, 3, \ldots$, where $\mid$ denotes polynomial divisibility.

Problem 4245. *School Science and Mathematics*
89(May 1989)445

Points H, G, T, M, and N lie on side BC of triangle ABC (with $AB < AC$). AH is an altitude, AM a median, AT bisects angle A, G is the point at which the incircle touches BC and N is the point at which the excircle (opposite A) touches BC. Prove that $AH < AG < AT < AM < AN$.



Problem 11. *Missouri Journal of Math. Sciences*
1.2(Spring 1989)29

A *strip* is the closed region bounded between two parallel lines in the plane. Prove that a finite number of strips cannot cover the entire plane.

Problem E3327. *American Mathematical Monthly*
96(May 1989)445–446

Suppose n is a positive integer greater than 3. Put

$$F_n(x) = \sum_{k=0}^{n} \left| \binom{n}{k} - x^k \right|, \quad S_n = \min_{x>0} F_n(x).$$

(a) Prove that

$$c2^n < S_n < F_n(1) = 2^n - n - 1$$

for a suitable positive c.

(b) Is it true that $\lim_{n\to\infty} 2^{-n} S_n = 1$?

Problem E3334. *American Mathematical Monthly*
96(Jun 1989)524–525

Consider the cubic curve $y = x^3 + ax^2 + bx + c$, where a, b, c are real numbers with $a^2 - 3b > 3\sqrt{3}$. Prove that there are exactly two lines that are perpendicular (normal) to the cubic at two points of intersection and that these two lines intersect at the point of inflection of the curve.

Problem 1014. *Elemente der Mathematik*
44(Jul 1989)110

A circle intersects each side of a regular n-gon, $A_1A_2A_3 \ldots A_n$ in two points. The circle cuts side A_iA_{i+1} (with $A_{n+1} = A_1$) in points B_i and C_i with B_i lying between A_i and A_{i+1} and C_i lying between B_i and A_{i+1}. Prove that

$$\sum_{i=1}^{n} |A_iB_i| = \sum_{i=1}^{n} |C_iA_{i+1}|.$$

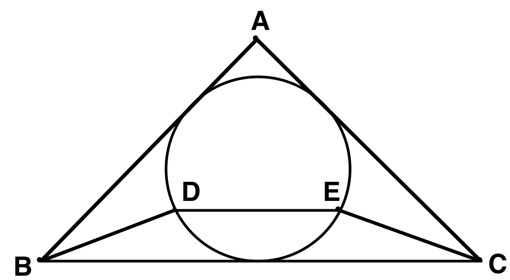
Problem 1463. *Crux Mathematicorum*
15(Sep 1989)207

Prove that if n and r are integers with $n > r$, then

$$\sum_{k=1}^{n} \cos^{2r}\left(\frac{k\pi}{n}\right) = \frac{n}{4^r}\binom{2r}{r}.$$

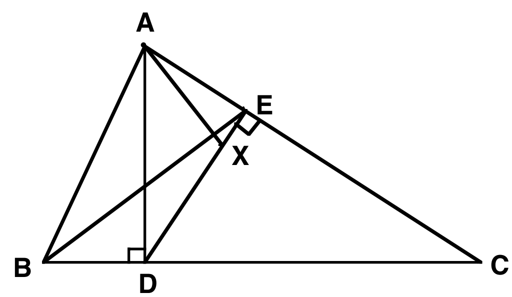
Problem 4251. *School Science and Mathematics*
89(Oct 1989)536

Let DE be a chord of the incircle of triangle ABC that is parallel to side BC. If $BD = CE$, prove that $AB = AC$.



Problem 14. *Missouri Journal of Math. Sciences*
1.3(Fall 1989)40

In triangle ABC, AD is an altitude (with D lying on segment BC). $DE \perp AC$ with E lying on AC. X is a point on segment DE such that $\frac{EX}{XD} = \frac{BD}{DC}$. Prove that $AX \perp BE$.



Problem 1535. *Crux Mathematicorum*
16.?(? 1990)109

Let P be a variable point inside an ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Through P draw two chords with slopes b/a and $-b/a$ respectively. The point P divides these two chors into four peices of lengths d_1, d_2, d_3, d_4. Prove that $d_1^2 + d_2^2 + d_3^2 + d_4^2$ is independent of the location of P and in fact has the value $2(a^2 + b^2)$.

Problem 1544. *Crux Mathematicorum*
16.?(? 1990)143

One root of $x^3+ax+b=0$ is λ times the difference of the other two roots ($|\lambda| \neq 1$, $a \neq 0$). Find this root as a simple rational function of a, b, and λ.

Problem 421. *College Mathematics Journal*
21.2(Mar 1990)150

Let P be any point on the median to side BC of triangle ABC. Extend the line segment $BP(CP)$ to meet $AC(AB)$ at $D(E)$. If the circles inscribed in triangles BPE and CPD have the same radius, prove that $AB = AC$.

Problem 17. *Missouri Journal of Math. Sciences*
2.1(Winter 1990)35

Let $ABCD$ be an isosceles tetrahedron. Denote the dihedral angle at edge AB by $\angle AB$. Prove that

$$\frac{AB}{\sin \angle AB} = \frac{AC}{\sin \angle AC} = \frac{AD}{\sin \angle AD}$$

Problem 21. *Missouri Journal of Math. Sciences*
2.2(Spring 1990)80

Find distinct postivie integers a, b, c, d, such that

$$a+b+c+d+abcd = ab+bc+ca+ad+bd+cd$$
$$+abc+abd+acd+bcd.$$

Problem 26. *Missouri Journal of Math. Sciences*
2.3(Fall 1990)140

Prove that

$$\sum_{k=1}^{38} \sin \frac{k^8 \pi}{38} = \sqrt{19}.$$

Problem 1585. *Crux Mathematicorum*
16.9(Nov 1990)267

We are given a triangle $A_1A_2A_3$ and a real number $r > 0$. Inside the triangle, inscribe a rectangle R_1 whose height is r times its base, with its base lying on side A_2A_3. Let B_1 be the midpoint of the base of R_1 and let C_1 be the center of R_1. In a similar manner, locate points B_2, C_2 and B_3, C_3, using rectangle R_2 and R_3.

(a) Prove that lines A_iB_i, i =1, 2, 3, concur.

(b) Prove that lines A_iC_i, i =1, 2, 3, concur.

Problem 1364. *Mathematics Magazine*
64.1(Feb 1991)60

The incircle of triangle ABC touches sides BC, CA, and AB at points D, E, and F, repsectively. Let P be any point inside triangle ABC. Line PA meets the incircle at two points; of these let X be the point that is closer to A. In a similar manner, let Y and Z be the points where PB and PC meet the incircle respectively. Prove that DX, EY, and FZ are concurrent.

(((needs figure)))

Problem B-685. *Fibonacci Quarterly*
29.1(Feb 1991)84
with Gareth Griffith

For integers $n \geq 2$, find k as a function of n such that

$$F_{k-1} \leq n < F_k.$$

Problem 1617. *Crux Mathematicorum*
17.2(Feb 1991)44

If p is a prime and a and k are positive integers such that $p^k \mid (a-1)$, then prove that

$$p^{n+k} \mid (a^{p^n} - 1)$$

for all positive integers n.

Problem 1623. *Crux Mathematicorum*
17.3(Mar 1991)78

Let l be any line through vertex A of triangle ABC that is external to the triangle. Two circles with radii r_1 and r_2 are each external to the triangle and each tangent to l and to line BC, and are respectively tangent to AB and AC.

(a) If $AB = AC$, prove that as l varies, $r_1 + r_2$ remains constant and equal to the height of A above BC.

(b) If $\triangle ABC$ is arbitrary, find constants k_1 and k_2, depending only on the triangle, so that $k_1r_1 + k_2r_2$ remains constant as l varies.

Problem 1632. *Crux Mathematicorum*
17.4(Apr 1991)113

Find all x and y which are rational multiples of π (with $0 < x < y < \pi/2$) such that $\tan x + \tan y = 2$.

Problem 755. *Pi Mu Epsilon Journal*
9.4(Spring 1991)255–256

In triangle ABC, a circle of radius p is inscribed in the wedge bounded by sides AB and BC and the incircle of the triangle. A circle of radius q is inscribed in the wedge bounded by sides AC and BC and the incircle. If $p = q$, prove that $AB = AC$.

((needs figure)))

Problem 1659*. *Crux Mathematicorum*
17.6(Jun 1991)172

For any integer $n > 1$, prove or disprove that the largest coefficient in the expansion of

$$(1 + 2x + 3x^2 + 4x^3)^n$$

is the coefficient of x^{2n}.

Problem 1668. *Crux Mathematicorum*
17.7(Sep 1991)208

What is the envelope of the ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

as a and b vary so that $a^2 + b^2 = 1$?

Problem 38. *Missouri Journal of Math. Sciences*
3.3(Fall 1991)149

Consider the equation: $\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} = 0$. Bring the $\sqrt{x_3}$ term to the right-hand side and then square both sides. Then isolate the $\sqrt{x_2}$ term on one side and square again. The result is a polynomial and we say that we have rationalized the original equation.

Can the equation

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} = 0$$

be rationalized in a similar manner, by successive transpositions and squarings?

Problem Q781. *Mathematics Magazine*
64.4(Oct 1991)275

Let P be a point inside triangle ABC. Let X, Y, and Z be the centroids of triangles BPC, CPA, and APB respectively. Prove that segments AX, BY and CZ are concurrent.

Problem 1684. *Crux Mathematicorum*
17.9(Nov 1991)270

Let

$$f(x, y, z) = x^4 + x^3z + ax^2z^2 + bx^2y + cxyz + y^2.$$

Prove that for any real numbers b, c with $|b| > 2$, there is a real number a such that f can be written as the product of two polynomials of degree 2 with real coefficients; furthermore, if b an c are rational, a will also be rational.

Problem 3468. *American Mathematical Monthly*
98.9(Nov 1991)853
with Curtis Cooper and Robert E. Kennedy

Suppose m and n are positive integers such that all prime factors of n are larger than m.

(a) Prove that

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} \sin^{2m} \frac{k\pi}{n} = \frac{\phi(n) - \mu(n)}{4^m} \binom{2m}{m}.$$

(Here ϕ and μ denote the arithmetic functions of Euler and Möbius, respectively.)

(b) Find a similar formula for

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} \cos^{2m} \frac{k\pi}{n}.$$

Problem H–459. *Fibonacci Quarterly*
29.4(Nov 1991)377

Prove that for all $n > 3$,

$$\frac{13\sqrt{5} - 19}{10} L_{2n+1} + 4.4(-1)^n$$

is very close to the square of an integer.

Problem 1704. *Crux Mathematicorum*
18.1(Jan 1992)13

Two chords of a circle (neither a diameter) intersect at right angles inside the circle, forming four regions. A circle is inscribed in each region. The radii of the four circles are r, s, t, u in cyclic order. Show that

$$(r - s + t - u) \left(\frac{1}{r} - \frac{1}{s} + \frac{1}{t} - \frac{1}{u} \right) = \frac{(rt - su)^2}{rstu}.$$

Problem 467. *College Mathematics Journal*
23.1(Jan 1992)69

The cosines of the angles of a triangle are in the ratio 2:9:12. Find the ratio of the sides of the triangle.

Problem 1724. *Crux Mathematicorum*
18.3(Mar 1992)75

A fixed plane intersects a fixed sphere forming two spherical segments. Each segment is a region bounded by the plane and one of the spherical caps it cuts from the sphere. Let S be one of these segments and let A be the point on the sphere furthest from S. A variable chord of the sphere through A meets the boundary of S in two points P and Q. Let λ be a variable sphere whose only constraint is that it passes through P and Q. Prove that the length of the tangent from A to λ is a constant.

Problem B–716. *Fibonacci Quarterly*
30.2(May 1992)183
(Dedicated to Dr. A. P. Hillman)

If a and b have the same parity, prove that $L_a + L_b$ cannot be a prime larger than 5.

Problem 55. *Missouri Journal of Math. Sciences*
5.1(Winter 1993)40

Let F_n and L_n denote the n-th Fibonacci and Lucas numbers, respectively. Find a polynomial $f(x, y)$ with constant coefficients such that $f(F_n, L_n)$ is identically zero for all positive integers n or prove that no such polynomial exists.

Problem 64. *Missouri Journal of Math. Sciences*
5.3(Fall 1993)132

For n a positive integer, let M_n denote the $n \times n$ matrix (a_{ij}) where $a_{ij} = i+j$. Is there a simple formula for $\text{perm}(M_n)$?

Problem 822. *Pi Mu Epsilon Journal*
9.9(Fall 1993)822

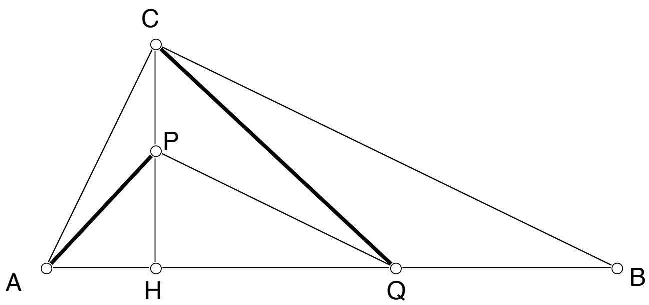
If α is a root of the equation

$$x^5 + x - 1 = 0,$$

then find an equation that has $\alpha^4 + 1$ as a root.

Problem H133. *Mathematical Mayhem*
6.2(Nov 1993)18

Let P be any point on altitude CH of right triangle ABC (with right angle at C). Let Q be a point on side AB such that $PQ \| CB$. Prove that $AP \perp CQ$.



Problem 93–17*. *Siam Review*
35.4(Dec 1993)642
with Peter J. Costa

In the construction of the Kalman filter, one needs to choose a matrix that optimizes a cost function that is the trace of a symmetric matrix. To find the desired optimum, use is made of the following theorem, which can be proved directly (expanding the elements of the trace in summations and differentiating).

Theorem. If $B \in M_{n\times n}(R)$ is a symmetric matrix and $A \in M_{m\times n}(R)$, then

$$\frac{\partial}{\partial A} tr(ABA^T) = 2AB.$$

Matrix differentiation is defined as follows: if $A = [a_{ij}] \in M_{m\times n}$ and c is a scalar, then $\partial c/\partial A \in M_{m\times n}$ is given by

$$\frac{\partial c}{\partial A} = \left[\frac{\partial c}{\partial a_{ij}}\right].$$

Can the above theorem be proved via matrix methods or identities? The result appears, without proof in [1, p. 109].

Similarly, computer algebra studies suggest that the symmetric matrix $C = ABA^T$ (for B symmetric) also satisfies the following identity:

$$\frac{\partial[\det(C)]}{\partial A} A^T = 2\det(C) I_{m\times m}.$$

Can this identity be shown without expanding $\det(C)$?

REFERENCE

[1] A. Gelb, ed., *Applied Optimal Estimation*, MIT Press, Cambridge, MA, 1974.

Problem B–756. *Fibonacci Quarterly*
32(Feb 1994)

Find a formula expressing P_n in terms of Fibonacci and/or Lucas numbers.

Problem 65*. *Missouri Journal of Math. Sciences*
6.1(Winter 1994)47

Evaluate

$$\sum_{k=0}^{n} \left| \binom{n}{k} - 2^k \right|.$$

Problem H–487. *Fibonacci Quarterly*
32.2(May 1994)187

Suppose H_n satisfies a second-order linear recurrence with constant coefficients. Let $\{a_i\}$ and $\{b_i\}$, $i = 1, 2, \ldots, r$ be integer constants and let $f(x_0, x_1, x_2, \ldots, x_r)$ be a polynomial with integer coefficients. If the expression

$$f\big((-1)^n, H_{a_1 n+b_1}, H_{a_2 n+b_2}, \ldots, H_{a_r n+b_r}\big)$$

vanishes for all integers $n > N$, prove that the expression vanishes for all integral n.

[As a special case, if an identity involving Fiboancci and Lucas numbers is true for all positive subscripts, then it must also be true for all negative subscripts as well.]

Problem 10387*. *American Mathematical Monthly*
101.5(May 1994)474
with Peter J. Costa

Let $T_n = (t_{i,j})$ be the $n \times n$ matrix with $t_{i,j} = \tan(i + j - 1)x$, i.e.,

$$T_n = \begin{pmatrix} \tan x & \tan 2x & \tan 3x & \cdots & \tan nx \\ \tan 2x & \tan 3x & \tan 4x & \cdots & \tan(n+1)x \\ \tan 3x & \tan 4x & \tan 5x & \cdots & \tan(n+2)x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tan nx & \tan(n+1)x & \tan(n+2)x & \cdots & \tan(2n-1)x \end{pmatrix}.$$

Computer experiments suggest that

$$\det(T_n) = (-1)^{\lfloor n/2 \rfloor} \sec^n nx$$

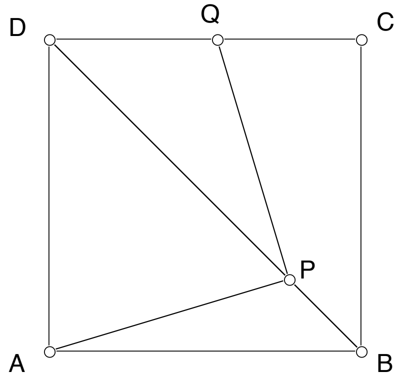
$$\times \prod_{r=1}^{n-1} (\sin^2(n-r)x \sec rx \sec(2n-r)x)^r$$

$$\times \begin{cases} \sin n^2 x, & \text{if } n \text{ is odd,} \\ \cos n^2 x, & \text{if } n \text{ is even.} \end{cases}$$

Prove or disprove this conjecture.

Problem 827. *Pi Mu Epsilon Journal* 9.10(Spring 1994)690

Let P be a point on diagonal BD of square $ABCD$ and let Q be a point on side CD such that $AP \perp PQ$. Prove that $AP = PQ$.



Problem I–1. *Mathematics & Informatics Quarterly* 4.4(Nov 1994)188

(a) Change one digit of

51553025557084342053471885588936120253166702774336

to obtain a perfect square.

(b) Change one digit of

37394298437314895265088274739213418872712112606929

to obtain a perfect square.

Problem I–2. *Mathematics & Informatics Quarterly* 4.4(Nov 1994)189

Given two circles, C_1 and C_2, in the plane, with centers O_1 and O_2 respectively, find an algorithm that constructs a common external tangent to the two circles using straightedge and compasses. The algorithm may not contain any conditional steps (i.e. no "if/then/else" constructs).

For example, the following is an algorithm that is frequently taught in high school. We assume that C_1 is the larger circle.

`ALGORITHM HS1:`

Given circles C_1, C_2 (with centers O_1, O_2), construct T, a common external tangent.

1. Let $A = \texttt{Intersection}(\overrightarrow{O_1O_2}, C_1)$
2. Let $B = \texttt{Intersection}(\overrightarrow{O_2O_1}, C_2)$
3. Let $C_3 = \texttt{Circle}(A, BO_2)$
4. Let $D = \texttt{Intersection}(\overrightarrow{AO_1}, C_3)$
5. Let $C_4 = \texttt{Circle}(O_1, O_1D)$
   [note: C_4 has radius $R_1 - R_2$]
6. Let $M = \texttt{Midpoint}(O_1O_2)$
7. Let $C_5 = \texttt{Circle}(M, MO_1)$
8. Let $P = \texttt{Intersection}(C_5, C_4)$
   [note: O_2P will be tangent to C_4]
9. Let $E = \texttt{Intersection}(\overrightarrow{O_1P}, C_1)$
10. Then $T = \texttt{Perpendicular}(O_1E, E)$

The notation should be obvious. For example, $\texttt{Circle}(A, BO_2)$ represents the circle with center A and radius equal to the length of line segment BO_2. If two elements X and Y intersect, then $\texttt{Intersection}(X, Y)$ returns one of the points of intersection (at random). The notation $\overrightarrow{AB}$ denotes the ray from A toward B. We assume all the basic constructions can be performed unconditionally.

The above algorithm is not a valid solution to our problem, because it assumes that C_1 is the larger circle. It fails if C_1 is the smaller circle. Your task is to find an algorithm that works in all cases, regardless of the size or location of the given circles. You may assume that the circles have nonzero radius.

Problem I–4. *Mathematics & Informatics Quarterly* 5.1(Mar 1995)31

Find the exact solution (in terms of radicals) for the following polynomial equation:

$$x^7 + (3-\sqrt{3})x^6 + (6-3\sqrt{3})x^5 + (7-6\sqrt{3})x^4$$
$$+(3-7\sqrt{3})x^3 - 3\sqrt{3}x^2 + 2x - 2\sqrt{3} = 0.$$

You may assume that we know how to solve equations of degree less than 5, so you may express your answer in terms of the zeros of quadratic, cubic, or quartic polynomials.

Problem I–5. *Mathematics & Informatics Quarterly* 5.1(Mar 1995)31

A *Pythagorean triangle* is a right triangle with integer sides. A *Heronian triangle* is a triangle with integer sides and integer area. It is well-known that the lengths of the sides of all Pythagorean triangles are generated by the formulas $k(m^2 - n^2)$, $k(2mn)$, and $k(m^2 + n^2)$ where k, m, and n are positive integers.

Find a formula or an algorithm that generates all Heronian triangles.

Problem I–6. *Mathematics & Informatics Quarterly*
5.1(Mar 1995)31
with Mark Saul

You have an infinite chessboard with a gold coin on each square. Some coins are genuine and some are counterfeit. (There is at least one of each.) Counterfeit coins are heavier than genuine coins. To find the counterfeit coins, you will perform some weighings using an infinite supply of balance scales that you had previously built in case the need arose. At each stage of the weighing process, you can pair the coins up in any way you want and, in each pair, weigh one coin against the other using a balance scale.

Devise an algorithm (using the smallest k) such that you can locate all the countefeit coins with k stages of weighings.

Problem 2046. *Crux Mathematicorum*
21.5(May 1995)158

Find integers a and b so that

$$x^3 + xy^2 + y^3 + 3x^2 + 2xy + 4y^2 + ax + by + 3$$

factors over the complex numbers.

Problem 2056. *Crux Mathematicorum*
21.6(Jun 1995)203

Find a polynomial of degree 5 whose roots are the tenth powers of the roots of the equation $x^5 - x - 1 = 0$.

Problem 2065. *Crux Mathematicorum*
21.7(Sep 1995)235

Find a monic polynomial $f(x)$ of lowest degree and with integer coefficients such that $f(n)$ is divisible by 1995 for all integers n.

Problem I–11. *Mathematics & Informatics Quarterly*
5.3(Sep 1995)134
with Larry Zimmerman

(a) Find a positive integer n such that $n + i$ is divisible by exactly i distinct primes, for $i = 1, 2, 3, 4, 5$.

(b) Find a positive integer n such that $n + i$ is divisible by exactly i distinct primes, for $i = 1, 2, 3, 4, 5, 6$.

Problem I–12. *Mathematics & Informatics Quarterly*
5.3(Sep 1995)134

Find the smallest positive integer k for which the following identity is *false*:

$$(x+3)^k + 21(x+28)^k + 209(x+5)^k + 1310(x+26)^k$$
$$+5796(x+7)^k + 19228(x+24)^k + 49588(x+9)^k$$
$$+101706(x+22)^k + 118864(x+15)^k + 168245(x+11)^k$$
$$+208012(x+18)^k + 225929(x+20)^k + 245157(x+13)^k$$
$$= (x+29)^k + 21(x+4)^k + 209(x+27)^k + 1310(x+6)^k$$
$$+5796(x+25)^k + 19228(x+8)^k + 49588(x+23)^k$$
$$+101706(x+10)^k + 118864(x+17)^k + 168245(x+21)^k$$
$$+208012(x+14)^k + 225929(x+12)^k + 245157(x+19)^k.$$

Problem 2074. *Crux Mathematicorum*
21.8(Oct 1995)277

The number 3774 is divisible by 37, 34, and 74 but not by 77. Find another 4-digit integer `abcd` that is divisible by the 2-digit numbers `ab`, `ac`, `ad`, `bd`, and `cd` but is not divisible by `bc`.

Problem 2083. *Crux Mathematicorum*
21(Nov? 1995)306

The numerical identity

$$\cos^2 14^\circ - \cos 7^\circ \cos 21^\circ = \sin^2 7^\circ$$

is a special case of the more general identity

$$\cos^2 2x - \cos x \cos 3x = \sin^2 x.$$

In a similar manner, find a generalization for each of the following numerical identities:
(a) $\tan 55^\circ - \tan 35^\circ = 2\tan 20^\circ$
(b) $\tan 70^\circ = \tan 20^\circ + 2\tan 40^\circ + 4\tan 10^\circ$
(c)* $\csc 10^\circ - 4\sin 70^\circ = 2$

Problem 10483. *American Mathematical Monthly*
102.9(Nov 1995)841

Given an odd positive integer n, let $A_1A_2\ldots A_n$ be a regular n-gon with circumcircle Γ. A circle O_i with radius r is drawn externally tangent to Γ at A_i, for $i = 1, 2, \ldots, n$. Let P be any point on Γ between A_n and A_1. A circle C (with any radius) is drawn externally tangent to Γ at P. Let t_i be the length of the common external tangent between the circles C and O_i.

Prove that

$$\sum_{i=1}^{n} (-1)^i t_i = 0.$$

Problem Q841. *Mathematics Magazine*
68.5(Dec 1995)400
with Murray S. Klamkin

Prove that the sequence $u_n = 1/n$, $n = 1, 2, \ldots$, cannot be the solution of a nonhomogeneous linear finite-order difference equation with constant coefficients.

Problem B–802. *Fibonacci Quarterly*
34.1(Feb 1996)81
(using the pseudonym Al Dorp)

For all $n > 0$, let $T_n = n(n+1)/2$ denote the nth triangular number. Find a formula for T_{2n} in terms of T_n.

Problem B–804. *Fibonacci Quarterly*
34.1(Feb 1996)81

Find integers a, b, c, and d (with $1 < a < b < c < d$) that make the following an identity:

$$F_n = F_{n-a} + 9342F_{n-b} + F_{n-c} + F_{n-d}.$$

Problem 2129*. *Crux Mathematicorum*
22.3(Apr 1996)123

For $n > 1$ and $i = \sqrt{-1}$, prove that

$$\frac{1}{4i} \sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{4n} i^k \tan \frac{k\pi}{4n}$$

is an integer.

Problem 4572. *School Science and Mathematics*
96(May 1996)?

The numerical identity $\cos^2 12° - \cos 6° \cos 18° = \sin^2 6°$ is a special case of the more general identity $\cos^2 2x - \cos x \cos 3x = \sin^2 x$.

Find and prove a general identity for each of the following numerical identities:
(a) $2 \sin 40° \sin 50° =° \sin 80°$
(b) $4 \cos 24° \cos 36° \cos 84° = \sin 18°$
(c) $\sin^2 10° + \cos^2 20° - \sin 10° \cos 20° = 3/4$

Problem 89. *Missouri Journal of Math. Sciences*
8.1(Winter 1996)36

Let ω be a primitive 49th root of unity. Prove that

$$\prod_{\substack{k=1 \\ \gcd(k,49)=1}}^{49} \left(1 - \omega^k\right) = 7.$$

Problem 95. *Missouri Journal of Math. Sciences*
8.2(Spring 1996)90
with Curtis Cooper and Robert E. Kennedy

Let n be a positive integer. It is known that

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} 1 = \phi(n)$$

where ϕ is Euler's phi function.

(a) Prove

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} k = \frac{1}{2} n \cdot \phi(n).$$

(b) Prove

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} k^2 = \frac{1}{6} n \left(2n \cdot \phi(n) + \phi^{-1}(n)\right)$$

where ϕ^{-1} is the Dirichlet inverse of Euler's phi function.

(c)* Let m be a positive integer, $m \geq 3$. Find a formula for

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} k^m.$$

Problem B–820. *Fibonacci Quarterly*
34.5(Nov 1996)468

Find a recurrence (other than the usual one) that generates the Fibonacci sequence.

Problem 10577. *American Mathematical Monthly*
104.2(Feb 1997)169
with Mark Bowron

It is well known that a maximum of 14 distinct sets are obtainable from one set in a topological space by repeatedly applying the operations of closure and complement in any order. Is there any bound on the number of sets that can be generated if we further allow arbitrary unions to be taken in addition to closures and complements?

Problem B–826. *Fibonacci Quarterly*
35.2(May 1997)181

Find a recurrence consisting of positive integers such that each positive integer n occurs exactly n times.

Problem B–830. *Fibonacci Quarterly*
35.2(May 1997)181
(using the pseudonym Al Dorp)

(a) Prove that if $n = 84$, then $(n + 3) \mid F_n$.

(b) Find a positive integer n such that $(n+19) \mid F_n$.

(c) Is there an integer a such that $n + a$ never divides F_n?

Problem B–831. *Fibonacci Quarterly*
35.3(Aug 1997)277

Find a polynomial $f(x, y)$ with integer coefficients such that $f(F_n, L_n) = 0$ for all integers n.

Problem B–833. *Fibonacci Quarterly*
35.3(Aug 1997)277
(using the pseudonym Al Dorp)

For n a positive integer, let $f(x)$ be the polynomial of degree $n-1$ such that $f(k) = L_k$, for $k = 1, 2, 3, \ldots, n$. Find $f(n+1)$.

Problem B–836. *Fibonacci Quarterly*
35.4(Nov 1997)371
(using the pseudonym Al Dorp)

Replace each of "W", "X", "Y", and "Z" by either "F" or "L" to make the following an identity:

$$W_n^2 - 6X_{n+1}^2 + 2Y_{n+2}^2 - 3Z_{n+3}^2 = 0.$$

Problem B–840. *Fibonacci Quarterly*
35.4(Nov 1997)372

Let

$$A_n = \begin{pmatrix} F_n & L_n \\ L_n & F_n \end{pmatrix}.$$

Find a formula for A_{2n} in terms of A_n and A_{n+1}.

Problem B–842. *Fibonacci Quarterly*
36.1(Feb 1998)85

Prove that no Lucas polynomial is exactly divisible by $x-1$.

Problem B–844. *Fibonacci Quarterly*
36.1(Feb 1998)85
(using the pseudonym Mario DeNobili)

If $a+b$ is even and $a > b$, show that

$$[F_a(x) + F_b(x)]\,[F_a(x) - F_b(x)] = F_{a+b}(x)F_{a-b}(x).$$

Problem H–537. *Fibonacci Quarterly*
36.1(Feb 1998)91
(corrected)

Let $\langle w_n \rangle$ be any sequence satisfying

$$w_{n+2} = Pw_{n+1} - Qw_n.$$

Let $e = w_0w_2 - w_1^2$ and assume $e \neq 0$ and $Q \neq 0$.

Computer experiments suggest the following formula, where k is an integer larger than 1:

$$w_{kn} = \frac{1}{e^{k-1}} \sum_{i=0}^{k} c_{k-i} \binom{k}{i} (-1)^i w_n^{k-i} w_{n+1}^i,$$

where

$$c_i = \sum_{j=0}^{k-2} \binom{k-2}{j} (-Qw_0)^j w_1^{k-2-j} w_{i-j}.$$

Prove or disprove this conjecture.

Problem 4658. *School Science and Mathematics*
98.3(March 1998)164

In right triangle ABC with hypotenuse AB, let D be a point on leg BC such that $\angle DAC = 2\angle BAD$. Prove that if AC/AD is rational, then CD/DB is rational.

Problem 4663. *School Science and Mathematics*
98.4(Apr 1998)?

The numerical identity $\cos^2 12^\circ - \cos 6^\circ \cos 18^\circ = \sin^2 6^\circ$ is a special case of the more general identity $\cos^2 2x - \cos x \cos 3x = \sin^2 x$.

Find and prove a general identity for the numerical identity

$$\sin 47^\circ + \sin 61^\circ - \sin 11^\circ - \sin 25^\circ = \cos 7^\circ.$$

Problem B–850. *Fibonacci Quarterly*
36.2(May 1998)181

Find distinct positive integers a, b, and c so that

$$F_n = 17F_{n-a} + cF_{n-b}$$

is an identity.

Problem B–852. *Fibonacci Quarterly*
36.2(May 1998)181

Evaluate

$$\begin{vmatrix} F_0 & F_1 & F_2 & F_3 & F_4 \\ F_9 & F_8 & F_7 & F_6 & F_5 \\ F_{10} & F_{11} & F_{12} & F_{13} & F_{14} \\ F_{19} & F_{18} & F_{17} & F_{16} & F_{15} \\ F_{20} & F_{21} & F_{22} & F_{23} & F_{24} \end{vmatrix}.$$

Problem H–541. *Fibonacci Quarterly*
36.2(May 1998)187–188

The simple continued fraction expansion for F_{13}^5/F_{12}^5 is (with $x = 375131$)

$$11 + \cfrac{1}{11 + \cfrac{1}{x + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{9 + \cfrac{1}{11}}}}}}}}}}}}}$$

which can be written more compactly using the notation $[11, 11, 375131, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 9, 11]$. To be even more concise, we can write this as $[11^2, 375131, 1^9, 2, 9, 11]$ where the superscript denotes the number of consecutive occurrences of the associated number in the list.

If $n > 0$, prove that the simple continued fraction expansion for $(F_{10n+3}/F_{10n+2})^5$ is

$$[11^{2n}, x, 1^{10n-1}, 2, 9, 11^{2n-1}]$$

where x is an integer and find x.

Problem B–855. *Fibonacci Quarterly*
36.4(Aug 1998)373

Let $r_n = F_{n+1}/F_n$ for $n > 0$. Find a recurrence for r_n.

Problem B–857. *Fibonacci Quarterly*
36.4(Aug 1998)373

Find a sequence of integers $\langle w_n \rangle$ satisfying a recurrence of the form $w_{n+2} = Pw_{n+1} - Qw_n$ for $n \geq 0$, such that for all $n > 0$, w_n has precisely n digits (in base 10).

Problem B–861. *Fibonacci Quarterly*
36.5(Nov 1998)467

The sequence $w_0, w_1, w_2, w_3, w_4, \ldots$ satisfies the recurrence $w_n = Pw_{n-1} - Qw_{n-2}$ for $n > 1$. If every term of the sequence is an integer, must P and Q both be integers?

Problem B–863. *Fibonacci Quarterly*
36.5(Nov 1998)468

Let

$$A = \begin{pmatrix} -9 & 1 \\ -89 & 10 \end{pmatrix}, \quad B = \begin{pmatrix} -10 & 1 \\ -109 & 11 \end{pmatrix},$$

$$C = \begin{pmatrix} -7 & 5 \\ -11 & 8 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} -4 & 19 \\ -1 & 5 \end{pmatrix}$$

and let n be a positive integer.

Simplify $30A^n - 24B^n - 5C^n + D^n$.

Problem B–864. *Fibonacci Quarterly*
36.4(Nov 1998)468

The sequence $\langle Q_n \rangle$ is defined by $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n > 1$ with initial conditions $Q_0 = 2$ and $Q_1 = 2$.

(a) Show that $Q_{7n} \equiv L_n \pmod{159}$ for all n.

(b) Find an integer $m > 1$ such that $Q_{11n} \equiv L_n \pmod{m}$ for all n.

(c) Find an integer a such that $Q_{an} \equiv L_n \pmod{31}$ for all n.

(d) Show that there is no integer a such that $Q_{an} \equiv L_n \pmod{7}$ for all n.

(e) Extra credit: Find an integer $m > 1$ such that $Q_{19n} \equiv L_n \pmod{m}$ for all n.

Problem B–866. *Fibonacci Quarterly*
37.1(Feb 1999)85

For n an integer, show that $L_{8n+4} + L_{12n+6}$ is always divisible by 25.

Problem B–867. *Fibonacci Quarterly*
37.1(Feb 1999)85

Find small positive integers a and b so that 1999 is a member of the sequence $\langle u_n \rangle$, defined by $u_0 = 0$, $u_1 = 1$, $u_n = au_{n-1} + bu_{n-2}$ for $n > 1$.

Problem B–868. *Fibonacci Quarterly*
37.1(Feb 1999)85
(based on a proposal by Richard André-Jeannin)

Find an integer $a > 1$ such that, for all integers n,

$$F_{an} \equiv aF_n \pmod{25}.$$

Problem B–869. *Fibonacci Quarterly*
37.1(Feb 1999)85
(based on a proposal by Larry Taylor)

Find a polynomial $f(x)$ such that, for all integers n,

$$2^n F_n \equiv f(n) \pmod{5}.$$

Problem B–879. *Fibonacci Quarterly*
37(Aug 1999)
(using the pseudonym Mario DeNobili)

Let $\langle c_n \rangle$ be defined by the recurrence

$$c_{n+4} = 2c_{n+3} + c_{n+2} - 2c_{n+1} - c_n$$

with initial conditions $c_0 = 0$, $c_1 = 1$, $c_2 = 2$, and $c_3 = 6$. Express c_n in terms of Fibonacci and/or Lucas numbers.

Problem H–557. *Fibonacci Quarterly*
37.4(Nov 1999)377

Let $\langle w_n \rangle$ be any sequence satisfying the second-order linear recurrence $w_n = Pw_{n-1} - Qw_{n-2}$, and let $\langle v_n \rangle$ denote the specific sequence satisfying the same recurrence but with the initial conditions $v_0 = 2$, $v_1 = P$.

If k is an integer larger than 1, and $m = \lfloor k/2 \rfloor$, prove that for all integers n,

$$v_n \sum_{i=0}^{m-1} (-Q^n)^i w_{(k-1-2i)n}$$

$$= w_{kn} - (-Q^n)^m \times \begin{cases} w_0, & \text{if } k \text{ is even,} \\ w_n, & \text{if } k \text{ is odd.} \end{cases}$$

Problem B–889. *Fibonacci Quarterly*
38(Feb 2000)

Find 17 consecutive Fibonacci numbers whose average is a Lucas number.

Problem B–890. *Fibonacci Quarterly*
38(Feb 2000)

If $F_{-a}F_bF_{a-b} + F_{-b}F_cF_{b-c} + F_{-c}F_aF_{c-a} = 0$, show that either $a = b$, $b = c$, or $c = a$.

Problem B–891. *Fibonacci Quarterly*
38(Feb 2000)

Let $\langle P_n \rangle$ be the Pell numbers defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$. Find integers a, b, and m such that $L_n \equiv P_{an+b} \pmod{m}$ for all integers n.

Problem B–892. *Fibonacci Quarterly* 38(Feb 2000)

Show that, modulo 47, $F_n^2 - 1$ is a perfect square if n is not divisible by 16.

Problem B–893. *Fibonacci Quarterly* 38(Feb 2000)

Find integers a, b, c, and d so that

$$F_xF_yF_z + aF_{x+1}F_{y+1}F_{z+1} + bF_{x+2}F_{y+2}F_{z+2}$$

$$+cF_{x+3}F_{y+3}F_{z+3} + dF_{x+4}F_{y+4}F_{z+4} = 0$$

is true for all x, y, and z.

Problem B–894. *Fibonacci Quarterly* 38(Feb 2000)

Solve for x:

$$F_{110}^x + 442F_{115}^x + 13F_{119}^x = 221F_{114}^x + 255F_{117}^x.$$

Problem B-942. *Fibonacci Quarterly* 40.4(Aug 2002)372

(a) For $n > 3$, find the Fibonacci number closest to L_n.

(b) For $n > 3$, find the Fibonacci number closest to L_n^2.

Problem 4730. *School Science and Mathematics* 102.4(2002)191

Prove that $44^{12n+2} + 17^{6n+1} + 7^{12n+2}$ is divisible by 2002 for all positive integers n.

Problem 4731. *School Science and Mathematics* 102.5(2002)232

Let $ABCD$ be a quadrilateral, none of whose angles is a right angle. Prove that

$$\frac{\tan A+\tan B+\tan C+\tan D}{\cot A+\cot B+\cot C+\cot D} = \tan A \tan B \tan C \tan D.$$

Problem 4737. *School Science and Mathematics* 102.6(2002)4737

The lengths of the sides of a triangle are $\sin x$, $\sin y$, and $\sin z$ where $x + y + z = \pi$. Find the radius of the circumcircle of the triangle.

Problem B-947. *Fibonacci Quarterly* 40.5(Nov 2002)467

(a) Find a non-square polynomial $f(x, y, z)$ with integer coefficients such that $f(F_n, F_{n+1}, F_{n+2})$ is a perfect square for all n.

(b) Find a non-square polynomial $g(x, y)$ with integer coefficients such that $g(F_n, F_{n+1})$ is a perfect square for all n.

Problem B–951. *Fibonacci Quarterly* 41.1(Feb 2003)85

The sequence $\langle u_n \rangle$ is defined by the recurrence

$$u_{n+1} = \frac{3u_n + 1}{5u_n + 3}$$

with the initial condition $u_1 = 1$. Express u_n in terms of Fibonacci and/or Lucas numbers.

Problem B–964. *Fibonacci Quarterly* 41.4(Aug 2003)375

Find a recurrence for $r_n = F_n/L_n$.

Problem B–966. *Fibonacci Quarterly* 41.5(Nov 2003)466

Find a recurrence for $r_n = \frac{1}{1+F_n}$.

Problem 2900*. *Crux Mathematicorum* 29.8(Dec 2003)518

Let I be the incentre of $\triangle ABC$, r_1 the inradius of $\triangle IAB$ and r_2 the inradius of $\triangle IAC$. Computer experiments using Geometer's Sketchpad suggest that $r_2 < \frac{5}{4}r_1$.

(a) Prove or disprove this conjecture.

(b) Can 5/4 be replaced by a smaller constant?

Problem 2901. *Crux Mathematicorum* 30.1(Jan 2004)38

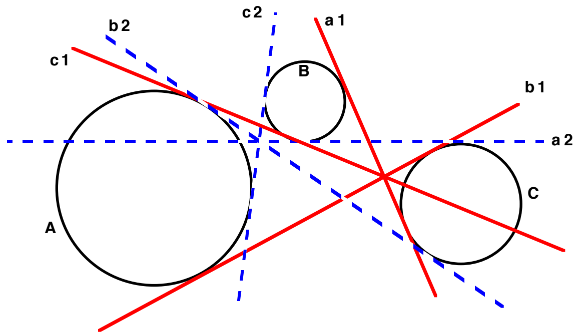
Let I be the incentre of $\triangle ABC$. The circles d, e, f inscribed in $\triangle IAB$, $\triangle IBC$, $\triangle ICA$ touch the sides AB, BC, CA at points D, E, F, respectively. The line IA is one of the two common internal tangents between the circles d and f. Let l be the other common internal tangent. Prove that l passes through the point E.

Problem 2902. *Crux Mathematicorum* 30.1(Jan 2004)38

Let P be a point in the interior of triangle ABC. Let D, E, and F be the feet of perpendiculars from P to BC, CA, AB, respectively. If the three quadrilaterals, $AEPF$, $BFPD$, $CDPE$ each have an incircle tangent to all four sides, prove that P is the incentre of $\triangle ABC$.

Problem 2903. *Crux Mathematicorum* 30.1(Jan 2004)39

Three disjoint circles A_1, A_2, and A_3 are given in the plane, none being interior to any other. The common internal tangents to A_j and A_k are α_{jk} and β_{jk}. If the α_{jk} are concurrent, prove that the β_{jk} are also concurrent.



Problems Submitted in 2005

Submitted to *Crux Mathematicorum*, 2005

Let a, b, and c be the lengths of the sides of a triangle ABC of area 1/2. Prove that

$$a^2 + \csc A \geq \sqrt{5}.$$

Submitted to *Crux Mathematicorum*, 2005

If A, B, and C are the angles of a triangle, prove that

$$\sin A + \sin B \sin C \leq \frac{1+\sqrt{5}}{2}.$$

When does equality occur?

Submitted to *Crux Mathematicorum*, 2005

Find a real number t, and polynomials $f(x)$, $g(x)$, and $h(x)$ with integer coefficients, such that

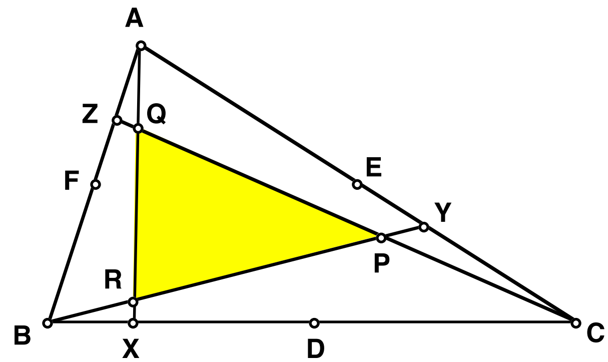
$$f(t) = \sqrt{2}, \qquad g(t) = \sqrt{3}, \qquad \text{and} \qquad h(t) = \sqrt{7}.$$

Submitted to *Crux Mathematicorum*, 2005

Let D, E, and F be the midpoints of sides BC, CA, and AB, respectively, in $\triangle ABC$. Let X, Y, and Z be points on segments BD, CE, and AF, respectively. The lines AX, BY, and CZ bound a central triangle PQR. Let X', Y', and Z' be the reflections of X, Y, and Z around D, E, and F, respectively. These give rise to a central triangle $P'Q'R'$.

Prove that

$$\frac{2+\sqrt{3}}{4} \leq \frac{[PQR]}{[P'Q'R']} \leq 8 - 4\sqrt{3}.$$



Submitted to *Crux Mathematicorum*, 2005

Prove that

$$\tan\frac{2\pi}{13} + 4\sin\frac{6\pi}{13} = \tan\frac{4\pi}{13} + 4\sin\frac{\pi}{13}$$

$$= \tan\frac{5\pi}{13} + 4\sin\frac{2\pi}{13} = \sqrt{13+2\sqrt{13}}.$$

Submitted to *Pentagon*, 2005

Let C be a unit circle centered at the point $(3,4)$. Let $O = (0,0)$ and let $A = (1,0)$. Let P be a variable point on C and let $PA = a$ and $PO = b$.

Find a non-constant polynomial $f(x,y)$ such that $f(a,b) = 0$ for all points P on C.

Submitted to *Pentagon*, 2005

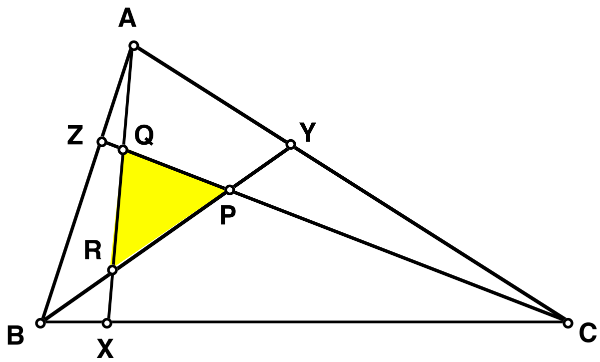
The points $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ are the vertices of a square S. Find an equation in x and y whose graph in the x-y plane is S.

Submitted to *Pentagon*, 2005

In $\triangle ABC$, let X, Y, and Z be points on sides BC, CA, and AB, respectively. Let $BX/XC = x$, $CY/YA = y$, and $AZ/ZB = z$. The lines AX, BY, and CZ bound a central triangle PQR.

Let X', Y', and Z' be the reflections of X, Y, and Z (respectively) around the midpoints of the sides of the triangle upon which they reside. These give rise to a central triangle $P'Q'R'$.

Prove that the area of $\triangle PQR$ is equal to the area of $\triangle P'Q'R'$ if and only if either $x = y$, $y = z$, or $z = x$.



Submitted to *Pentagon*, 2005

Express

$$\cos A\cos B\sin(A-B)+\cos B\cos C\sin(B-C)$$
$$+\cos C\cos A\sin(C-A)$$

as the product of three sines.

Submitted to *Math Horizons*, 2005

Prove that

$$\tan 20^\circ + 4\sin 20^\circ = \sqrt{3}.$$

Submitted to *College Mathematics Journal*, 2005

In problem 218 of this journal (1983, page 358), it was shown that

$$\tan\frac{3\pi}{11}+4\sin\frac{2\pi}{11}=\sqrt{11}.$$

Prove the related identity

$$\tan\frac{\pi}{11}+4\sin\frac{3\pi}{11}=\sqrt{11}.$$

Submitted to *School Science and Mathematics*, 2005

Find positive integers a, b, and c (each less than 12) such that

$$\sin\frac{a\pi}{24}+\sin\frac{b\pi}{24}=\sin\frac{c\pi}{24}.$$

Submitted to *Pi Mu Epsilon Journal*, 2005

Prove that

$$\sin 9^\circ+\cos 9^\circ=\frac{1}{2}\sqrt{3+\sqrt{5}}.$$

Submitted to *Pi Mu Epsilon Journal*, 2005

Find a rational function $f(x)$ with integer coefficients such that

$$\cos\theta = f(\sin\theta-\cos\theta)$$

is an identity or prove that no identity of this form exists.

Submitted to *Missouri Journal of Math. Sciences*, 2005

Find all positive integers a and n (with $n>1$ and $a<n$) such that

$$\sin\frac{\pi}{n}+\sin\frac{a\pi}{n}=\sin\frac{(a+2)\pi}{n}.$$

Submitted to *Mathematics Magazine*, 2005

Prove that

$$4\sin 3^\circ = \cos 15^\circ\csc 54^\circ - \sec 15^\circ\cos 18^\circ.$$

Submitted to *Fibonacci Quarterly*, 2005

Find positive integers a, b, and c such that

$$\sec F_a + \sec F_b = F_c$$

where all angles are measured in degrees.

Submitted to *Fibonacci Quarterly*, 2005

(a) Find a positive intger k and a polynomial $f(x)$ with rational coefficients such that

$$F_{kn}=f(L_n+F_n)$$

is an identity or prove that no identity of this form exists.

(b) Same question with $f(L_n-F_n)$ instead.

(c) Same question with $f(L_n\times F_n)$ instead.

Submitted to *Fibonacci Quarterly*, 2005

Find positive integers a, b, and m (with $m>1$) such that

$$F_n\equiv b^n-a^n \pmod{m}$$

is an identity (i.e. true for all n) or prove that no identity of this form exists.

Submitted to *American Mathematical Monthly*, 2005

Let n be an odd positive integer. Let $f(x)$ be the polynomial $U_n(x)/x$ where $U_n(x)$ is the n-th Chebyshev Polynomial of the 2nd kind. Let $S(n,k)$ be the sum of the k-th powers of the roots of $f(x)$. Prove that $S(n,k)$ is an integer for all negative integers k if and only if $n+1$ is a power of 2.

Submitted to *American Mathematical Monthly*, 2005

A triangle with sides a, b, and c has inradius r and circumradius R. If $a\le b\le c$, prove that

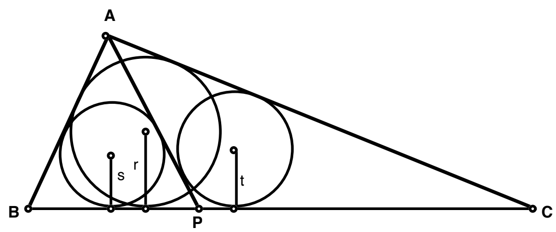
$$r\sqrt{22+10\sqrt{5}}\le a+b\le 2R\sqrt{3},$$
$$4r\sqrt{3}\le b+c\le 4R,$$
$$r\sqrt{\frac{47+13\sqrt{13}}{2}}\le c+a\le R\sqrt{\frac{207+33\sqrt{33}}{32}}.$$

Submitted to *American Mathematical Monthly*, 2005

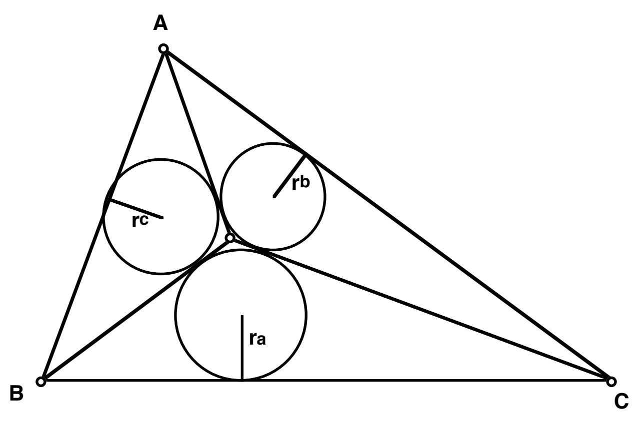
(a) Let P be a variable point on side $\overline{BC}$ of a fixed triangle ABC. Let r, s, and t be the inradii of triangles ABC, PAB, and PAC, respectively. Prove that

$$\frac{1}{s}+\frac{1}{t}-\frac{r}{st}$$

remains invariant as P varies along side $\overline{BC}$.



(b)* Let P be a variable point in the interior of a fixed triangle ABC. Let r, r_a, r_b, and r_c be the inradii of triangles ABC, PBC, PCA, and PAB, respectively. Find a non-constant function, f, such that $f(r, r_a, r_b, r_c)$ remains invariant as P varies inside $\triangle ABC$.

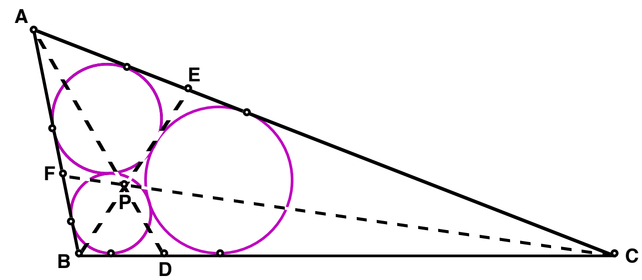


Submitted to *American Mathematical Monthly*, 2005

A sphere meets each face of a cube in a circle. The areas of the circles on three mutually adjacent faces of the cube are A_1, A_2, and A_3. The circles on the faces opposite these faces have areas B_1, B_2, and B_3, respectively. Find the relationship between A_1, A_2, A_3, B_1, B_2, and B_3.

Submitted to *American Mathematical Monthly*, 2005

Inside triangle ABC, the three Malfatti circles are drawn. That is, each circle is externally tangent to the other two and also tangent to two sides of the triangle. Along each side of the triangle lies a segment that is a common tangent to two of the Malfatti circles. Let D, E, and F be the midpoints of these segments, along sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Prove that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent.



Submitted to *Elemente der Mathematik*, 2005

(a) Express

$$\sin A \sin B \sin(A-B) + \sin B \sin C \sin(B-C)$$

$$+ \sin C \sin D \sin(C-D) + \sin D \sin A \sin(D-A)$$

as the product of three sines.

(b) Express

$$\cos A \cos B \sin(A-B) + \cos B \cos C \sin(B-C)$$

$$+ \cos C \cos D \sin(C-D) + \cos D \cos A \sin(D-A)$$

as the product of three sines.

Submitted to *Elemente der Mathematik*, 2005

Let $f(x) = x^{13} + 637x^3 + 1364x$. Find $\gcd_{n=1}^{\infty} f(n)$.

Submitted to *Elemente der Mathematik*, 2005

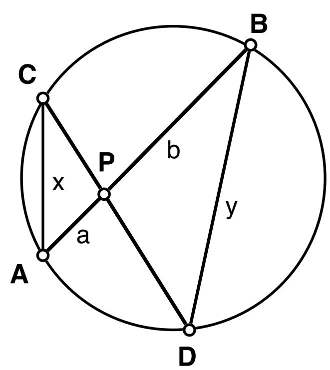
A triangle with sides a, b, and c has inradius r and circumradius R.
If $a \le b \le c$, prove that

$$\begin{aligned} a+b-c &\le 2r\sqrt{3}, \\ 2r\sqrt{3} \le b+c-a &\le 4R, \\ 2r \le c+a-b &\le 2R. \end{aligned}$$

Problems waiting to be submitted

Problem.

Let $\overline{AB}$ be a fixed chord of a fixed circle. Let P be a fixed point on the interior of this chord such that $PA = a$ and $PB = b$. Let C be a variable point on the circle and let $\overline{CD}$ be the chord of the circle that passes through P. Let $AC = x$ and $BD = y$. Find a non-constant function f (which can depend on the constants a and b) such that $f(x, y)$ remains invariant as C moves around on the circle.



Problem.

The point P lies in an obscure alcove on the floor of the Ceva Gallery in the Museum of Ancient Geometry. The point Q is tucked away in a corner on the floor of the Menelaus Room in the same museum. It is easy to walk between the two rooms because the floor is perfectly level; however the passageways are curved and twisty and the route is circuitous.

Show how to construct a ray emenating from P that points precisely at Q using only straightedge and compasses, using the museum's floor as a drawing board.

Problem.

with Gil Kessler

Is $x = 1$ the only positive rational solution to the Diophantine equation $x^4 + 2x + 1 = y^2$?

Problem.

Let A and B be two distinct points on a circle. Prove that the shortest path from A to B that does not go inside the circle lies along the circumference of the circle.

Problem.

Let $f_0(x) = x^2 + b_0x + c_0$ where b_0 and c_0 are real numbers with $b_0^2 > 4c_0$. For $n > 0$, let $f_n(x) = x^2 + b_nx + c_n$ where b_n and c_n are the roots of $f_{n-1}(x) = 0$ with $b_n > c_n$. Find $\lim_{n\to\infty} f_n(x)$.

Problem.

In each of the following problems, you are to insert parentheses and operators to the digits "1995", without changing the order of the digits, to form the number shown on the right. For example, to form the number 62, you could write $19 \times \sqrt{9} + 5 = 62$. The operators you are permitted to use are plus, minus, times, divide, juxtaposition, decimal point, square root, powers, and factorial. Thus, for example, the digits 1 and 9 could be combined to form $1 + 9$, $1 - 9$, 1×9, $1/9$, 19, 1.9, $1 + \sqrt{9}$, 1^9, $(-1 + 9)!$, etc.

```
(a) 1995=63
(b) 1995=78
(c) 1995=79
(d) 1995=142
(e) 1995=149
(f) 1995=152
(g) 1995=153
(h) 1995=155
(i) 1995=161
(j) 1995=166
(k) 1995=185
(l) 1995=194
(m) 1995=197
```

Problem.

Let M be the midpoint of side BC of triangle ABC. Lines through vertex C divide side AB into n equal parts for some positive integer n. These lines divide $\triangle ABM$ into n regions of which four consecutive ones have areas 2, 5, 7, and x, respectively. Find x.

Problem.

(a) Prove that there is no integer n such that x^5+x+n is irreducible in $\mathbf{Z}[x]$, yet $x^5 + x + n = 0$ is solvable by radicals.

(b) Find the unique integer n such that $x^5 + 11x + n$ is irreducible in $\mathbf{Z}[x]$, yet $x^5 + 11x + n = 0$ is solvable by radicals.

Problem.

The numerical identity $\sin 50^\circ \sin 30^\circ = \sin^2 40^\circ - \sin^2 10^\circ$ is a special case of the more general identity $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$ which is true for all x and y.

In a similar manner, find a 2-parameter generalization for each of the following numerical identities:

(a) $\cos 1^\circ - \cos 11^\circ = 2\sin 5^\circ \sin 6^\circ$

(b) $\cos 45^\circ + \cos 75^\circ = \cos 15^\circ$

(c) $\cos 60^\circ + 2\cos 70^\circ + \cos 80^\circ = 4\cos^2 5^\circ \cos 70^\circ$

(d) $\sin 10^\circ \cos 60^\circ + \sin 5^\circ \cos 45^\circ = \sin 15^\circ \cos 55^\circ$

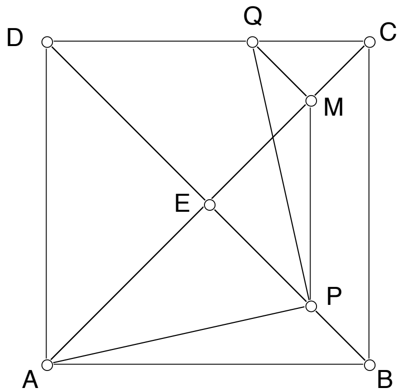
Problem.

Let $\mathbf{C}[x, y]$ denote the set of polynomials in x and y with complex coefficients.

Let $f(x, y) = ax^2 + bxy + cy^2 + dx + ey$ be a quadratic polynomial in $\mathbf{C}[x, y]$ with $b^2 - 4ac \neq 0$. Prove that there exists a unique complex number k such that $f(x, y) + k$ factors in $\mathbf{C}[x, y]$.

Problem.

Let E be the center of square $ABCD$ and let P be any point on BE. Let M be the point on CE such that $PM \| BC$ and let Q be the point on CD such that $MQ \| BD$. Prove that $AP = PQ$ and $AP \perp PQ$.



Problem.

A circle meets each side of a unit square in segments of lengths x, y, z, and w (going clockwise around the square). Find the relationship between x, y, z, and w.

Problem.

Find a polynomial, $f(x)$, with integer coefficients such that for all positive integers n,

$$19^n \equiv f(n) \pmod{72}.$$

Problem.

If r is a root of the equation

$$x^5 - x + 1 = 0,$$

find an equation that has $\frac{ar+b}{cr+d}$ as a root.

Problem.

A collection of spheres is said to surround a unit sphere if their interiors are disjoint and if each is tangent externally to the unit sphere. It is well-known that 12 unit spheres can surround a unit sphere, but that 13 cannot. Since the 12 unit spheres surrounding a unit sphere are not "tight", this suggests the following problems.

(a) What is the largest value of r such that 12 unit spheres and one sphere or radius r can surround a given unit sphere?

(b) What is the largest value of r such that 12 spheres of radius r can surround a given unit sphere?

(c) What is the largest value of r such that 13 spheres of radius r can surround a given unit sphere?

Problem.

A Heronian triangle is a triangle with integer sides and integer area.

(a) Find the acute Heronian triangle with smallest area.

(b) Find the obtuse Heronian triangle with smallest area.

(c) Find the acute scalene Heronian triangle with smallest area.

(d) Find the obtuse scalene Heronian triangle with smallest area.

Problem.

Let ω be a primitive n-th root of unity, where n is an odd positive integer. Prove that

$$\prod_{k=1}^{n-1} \left(1 - \omega^k - \omega^{2k}\right) = L_n$$

and that

$$\prod_{\substack{k=1 \\ \gcd(k,n)=1}}^{n-1} \left(1 - \omega^k - \omega^{2k}\right)$$

is always an integer that divides L_n.

Legend:

* The proposer did not supply a solution.

** No one supplied a solution.