Stanford Mathematics PhD Qualifying Exam

Algebra – Spring 2005

Morning Session

1. Suppose p and q are odd primes and $p < q$. Let G be a finite group of order $p^3 q$.

(a) Prove that G has a normal Sylow subgroup.

(b) Let n_p and n_q denote the number of p-Sylow and q-Sylow subgroups of G. Determine, with proof, all ordered pairs (n_p, n_q) that are possible for groups of order $p^3 q$.

2. Let $f(X) \in \mathbb{Q}[X]$ be a monic irreducible polynomial of degree 4 with roots $\alpha, \beta, \gamma, \delta$.

The *discriminant* of a polynomial with roots $r_1, \cdots, r_n$ is $\prod_{i<j} (r_i - r_j)^2$.

(a) Prove that $\lambda = \alpha\beta + \gamma\delta$ is the root of a monic cubic polynomial $g(X) \in \mathbb{Q}[X]$ whose discriminant is the same as the discriminant of f.

(b) If $f \in \mathbb{Z}[X]$ prove that $g \in \mathbb{Z}[X]$.

3. Let M be a finitely-generated module over the Noetherian commutative ring R. Prove that if $f: M \longrightarrow M$ is an R-module homomorphism, and if f is surjective, then f is also injective. *Hint*: consider the submodules $\ker(f^n)$.

4. Let G be the nonabelian group of order 16 with generators x and y subject to the relations

$$x^8 = y^2 = 1, \quad yxy^{-1} = x^3.$$

Determine the conjugacy classes of G and compute its character table.

5. If B is a positive-definite symmetric real matrix, show that there exists a unique positive-definite symmetric real matrix C such that $C^2 = B$.

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Afternoon Session

1. Suppose that $A \subset B$ is an integral extension of commutative rings with unit.

(a) If $\mathfrak{q}$ is a maximal ideal of B, prove that $\mathfrak{p} = \mathfrak{q} \cap A$ is a maximal ideal of A.

(b) Outline the proof that for any prime ideal $\mathfrak{p} \subset A$ there exists a prime ideal $\mathfrak{q}$ of B with $\mathfrak{p} = \mathfrak{q} \cap A$.

2. Let $F = \mathbb{Z}/2\mathbb{Z}$, and let $F[X, Y]$ be the polynomial ring in two variables. Let I be the ideal generated by $X^5 + X^3 + X$ and $Y^3 + (X^3 + 1)Y + 1$, and let R be the quotient ring $F[X, Y]/I$. Determine the number of maximal ideals in the ring R.

Hint: if $a \in \mathbb{F}_4$, what is a^3?

3. If G is a permutation group acting on a set S we say G is *n-transitive* if $|S| \geqslant n$ and whenever $x_1, \cdots, x_n$ are distinct elements of S and $y_1, \cdots, y_n$ are distinct elements of S there exists $g \in G$ such that $g(x_i) = y_i$. We will denote by $\chi(g)$ the number of fixed points of g. Prove that a necessary and sufficient condition for G to be 3-transitive is that

$$\frac{1}{|G|} \sum \chi(g)^3 = 5.$$

4. Suppose that A is an $n \times n$ matrix over $\mathbb{C}$ with minimal polynomial $(X - \lambda)^n$ where $\lambda \neq 0$. Find the Jordan form of A^2. What if $\lambda = 0$?

5. Find the Galois group of the polynomial $X^5 + 99X - 1$ over the fields $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/5\mathbb{Z}$, $\mathbb{Z}/11\mathbb{Z}$ and $\mathbb{Q}$.