The Forty-Eighth Annual William Lowell Putnam Competition
Saturday, December 5, 1987

A–1 Curves A, B, C and D are defined in the plane as follows:

$$\begin{aligned} A &= \left\{(x,y) : x^2 - y^2 = \frac{x}{x^2+y^2}\right\}, \\ B &= \left\{(x,y) : 2xy + \frac{y}{x^2+y^2} = 3\right\}, \\ C &= \left\{(x,y) : x^3 - 3xy^2 + 3y = 1\right\}, \\ D &= \left\{(x,y) : 3x^2y - 3x - y^3 = 0\right\}. \end{aligned}$$

Prove that $A \cap B = C \cap D$.

A–2 The sequence of digits

$$123456789101112131415161718192021\ldots$$

is obtained by writing the positive integers in order. If the 10^n-th digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

A–3 For all real x, the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x.$$

(a) If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x? Explain.

(b) If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x? Explain.

A–4 Let P be a polynomial, with real coefficients, in three variables and F be a function of two variables such that

$$P(ux, uy, uz) = u^2 F(y-x, z-x) \quad \text{for all real } x, y, z, u,$$

and such that $P(1,0,0) = 4$, $P(0,1,0) = 5$, and $P(0,0,1) = 6$. Also let A, B, C be complex numbers with $P(A,B,C) = 0$ and $|B - A| = 10$. Find $|C - A|$.

A–5 Let

$$\vec{G}(x,y) = \left(\frac{-y}{x^2+4y^2}, \frac{x}{x^2+4y^2}, 0\right).$$

Prove or disprove that there is a vector-valued function

$$\vec{F}(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$$

with the following properties:

(i) M, N, P have continuous partial derivatives for all $(x,y,z) \neq (0,0,0)$;

(ii) $\operatorname{Curl} \vec{F} = \vec{0}$ for all $(x,y,z) \neq (0,0,0)$;

(iii) $\vec{F}(x,y,0) = \vec{G}(x,y)$.

A–6 For each positive integer n, let $a(n)$ be the number of zeroes in the base 3 representation of n. For which positive real numbers x does the series

$$\sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3}$$

converge?

B–1 Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$

B–2 Let r, s and t be integers with $0 \leq r$, $0 \leq s$ and $r + s \leq t$. Prove that

$$\frac{\binom{s}{0}}{\binom{t}{r}} + \frac{\binom{s}{1}}{\binom{t}{r+1}} + \cdots + \frac{\binom{s}{s}}{\binom{t}{r+s}} = \frac{t+1}{(t+1-s)\binom{t-s}{r}}.$$

B–3 Let F be a field in which $1 + 1 \neq 0$. Show that the set of solutions to the equation $x^2 + y^2 = 1$ with x and y in F is given by $(x,y) = (1,0)$ and

$$(x,y) = \left(\frac{r^2-1}{r^2+1}, \frac{2r}{r^2+1}\right)$$

where r runs through the elements of F such that $r^2 \neq -1$.

B–4 Let $(x_1, y_1) = (0.8, 0.6)$ and let $x_{n+1} = x_n \cos y_n - y_n \sin y_n$ and $y_{n+1} = x_n \sin y_n + y_n \cos y_n$ for $n = 1, 2, 3, \ldots$. For each of $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$, prove that the limit exists and find it or prove that the limit does not exist.

B–5 Let O_n be the n-dimensional vector $(0, 0, \cdots, 0)$. Let M be a $2n \times n$ matrix of complex numbers such that whenever $(z_1, z_2, \ldots, z_{2n})M = O_n$, with complex z_i, not all zero, then at least one of the z_i is not real. Prove that for arbitrary real numbers $r_1, r_2, \ldots, r_{2n}$, there are complex numbers $w_1, w_2, \ldots, w_n$ such that

$$\operatorname{re}\left[M \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}\right] = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}.$$

(Note: if C is a matrix of complex numbers, $\operatorname{re}(C)$ is the matrix whose entries are the real parts of the entries of C.)

B–6 Let F be the field of p^2 elements, where p is an odd prime. Suppose S is a set of $(p^2-1)/2$ distinct nonzero elements of F with the property that for each $a \neq 0$ in F, exactly one of a and $-a$ is in S. Let N be the number of elements in the intersection $S \cap \{2a : a \in S\}$. Prove that N is even.