Conic Construction of a Triangle from the Feet of Its Angle Bisectors

Paul Yiu

Abstract. We study an extension of the problem of construction of a triangle from the feet of its internal angle bisectors. Given a triangle ABC, we give a conic construction of points which are the incenter or excenters of their own anticevian triangles with respect to ABC. If the given triangle contains a right angle, a very simple ruler-and-compass construction is possible. We also examine the case when the feet of the three external angle bisectors are three given points on a line.

1. The angle bisectors problem

In this note we address the problem of construction of a triangle from the endpoints of its angle bisectors. This is Problem 138 in Wernick's list [3]. The corresponding problem of determining a triangle from the lengths of its angle bisectors have been settled by Mironescu and Panaitopol [2].

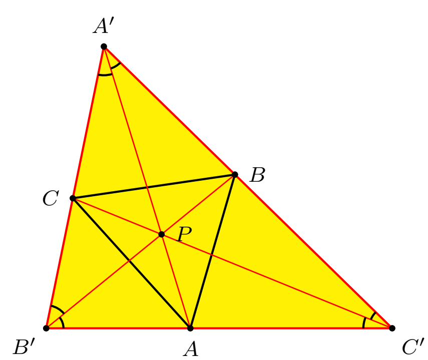

Figure 1. The angle bisectors problem

Given a triangle ABC, we seek, more generally, a triangle $A'B'C'$ such that the lines $A'A$, $B'B$, $C'C$ bisect the angles $B'A'C'$, $C'A'B'$, $A'C'B'$, internally or externally. In this note, we refer to this as the *angle bisectors problem*. With reference to triangle ABC, $A'B'C'$ is the anticevian triangle of a point P, which is the incenter or an excenter of triangle $A'B'C'$. It is an excenter if two of the lines $A'P$, $B'P$, $C'P$ are external angle bisectors and the remaining one an internal angle bisector. For a nondegenerate triangle ABC, we show in §3 that the angle bisectors problem always have real solutions, as intersections of three cubics. We proceed to provide a conic solution in §§4, 5, 6. The particular case of right triangles has an

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elegant ruler-and-compass solution which we provide in §7. Finally, the construction of a triangle from the feet of its external angle bisectors will be considered in §8. In this case, the three feet are collinear. We make free use of standard notations of triangle geometry (see [4]) and work in homogeneous barycentric coordinates with respect to ABC.

2. The cubic $\mathscr{K}_a$

We begin with the solution of a locus problem: to find the locus of points at which two of the sides of a given triangle subtend equal angles.

Proposition 1. *Given a triangle ABC with $b \neq c$, the locus of a point Q for which QA is a bisector of the angles between QB and QC is the isogonal conjugate of the A-Apollonian circle.*

Proof. The point A lies on a bisector of angle BQC if and only if $\cos AQB = \pm\cos AQC$, *i.e.*, $\cos^2 AQB = \cos^2 AQC$. In terms of the distances, this is equivalent to

$$\begin{aligned}&(QA^4 - QB^2\cdot QC^2)(QB^2 - QC^2) - 2QA^2(b^2\cdot QB^2 - c^2\cdot QC^2)\\ &-2(b^2-c^2)QB^2\cdot QC^2 + b^4\cdot QB^2 - c^4\cdot QC^2 = 0.\end{aligned} \tag{1}$$

Let Q have homogeneous barycentric coordinates $(x : y : z)$ with respect to triangle ABC. We make use of the distance formula in barycentric coordinates in [4, §7.1, Exercise 1]:

$$QA^2 = \frac{c^2y^2 + (b^2+c^2-a^2)yz + b^2z^2}{(x+y+z)^2}$$

and analogous expressions for QB^2 and QC^2. Substitution into (1) leads to the cubic

$$\mathscr{K}_a: \qquad x(c^2y^2 - b^2z^2) + yz((c^2+a^2-b^2)y - (a^2+b^2-c^2)z) = 0$$

after canceling a factor $\frac{-(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{(x+y+z)^4}\cdot x$. Note that the factor x can be suppressed because points on BC do not lie on the locus.

We obtain the isogonal conjugate of the cubic $\mathscr{K}_a$ by replacing, in its equation, x, y, z respectively by a^2yz, b^2zx, c^2xy. After clearing a factor $b^2c^2x^2yz$, we obtain

$$(b^2-c^2)(a^2yz + b^2zx + c^2xy) + a^2(x+y+z)(c^2y - b^2z) = 0.$$

This is the circle through $A = (1:0:0)$ and $(0:b:\pm c)$, the feet of the bisectors of angle A on the sideline BC. It is the A-Apollonian circle of triangle ABC, and is the circle orthogonal to the circumcircle at A and with center on the line BC. See Figure 2. □

Remark. If $b = c$, this locus is the circumcircle.

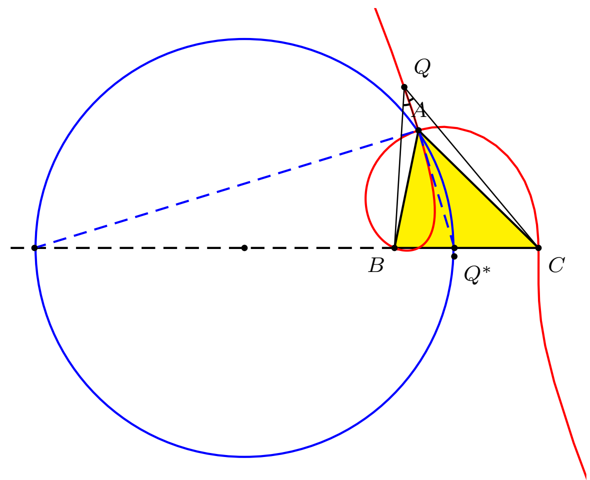

Figure 2. The cubic $\mathscr{K}_a$ and the A-Apollonian circle

3. Existence of solutions to the angle bisectors problem

Let $P = (x : y : z)$ be a point whose anticevian triangle $A'B'C'$ is such that the line $A'A$ is a bisector, internal or external, of angle $B'A'C'$, which is the same as angle $CA'B$. By Proposition 1 with $Q = A' = (-x : y : z)$, we have the equation $F_a = 0$ below. Similarly, if $B'B$ and $C'C$ are angle bisectors of $C'B'A'$ and $A'C'B'$, then by cyclic permutations of a, b, c and x, y, z, we obtain $F_b = 0$ and $F_c = 0$. Here,

$$\begin{aligned}
F_a &:= -x(c^2y^2 - b^2z^2) + yz((c^2 + a^2 - b^2)y - (a^2 + b^2 - c^2)z), \\
F_b &:= -y(a^2z^2 - c^2x^2) + zx((a^2 + b^2 - c^2)z - (b^2 + c^2 - a^2)x), \\
F_c &:= -z(b^2x^2 - a^2y^2) + xy((b^2 + c^2 - a^2)x - (c^2 + a^2 - b^2)y).
\end{aligned}$$

Theorem 2. *The angle bisectors problem for a nondegenerate triangle ABC always has real solutions*, i.e., *the system of equations $F_a = F_b = F_c = 0$ has at least one nonzero real solution.*

Proof. This is clear for equilateral triangles. We shall assume triangle ABC non-equilateral, and $B > \frac{\pi}{3} > C$. From $F_a = 0$, we write x in terms of y and z. Substitutions into the other two equations lead to the same homogeneous equation in y and z of the form

$$c^2((c^2 + a^2 - b^2)^2 - c^2a^2)y^4 + \cdots + b^2((a^2 + b^2 - c^2)^2 - a^2b^2)z^4 = 0. \quad (2)$$

Note that

$$\begin{aligned}
c^2((c^2 + a^2 - b^2)^2 - c^2a^2) &= c^4a^2(2\cos 2B + 1) < 0, \\
b^2((a^2 + b^2 - c^2)^2 - a^2b^2) &= a^2b^4(2\cos 2C + 1) > 0.
\end{aligned}$$

It follows that a nonzero real solution (y, z) of (2) exists, leading to a nonzero real solution (x, y, z) of the system $F_a = F_b = F_c = 0$. $\square$

Figure 3 illustrates a case of two real intersections. For one with four real intersections, see 6.

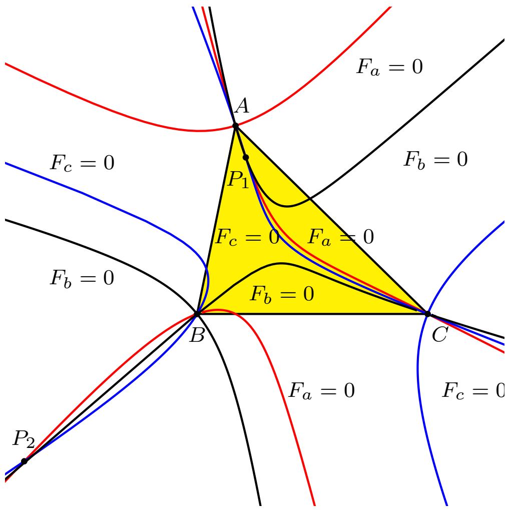

Figure 3. The cubics $F_a = 0$, $F_b = 0$ and $F_c = 0$

4. The hyperbola $\mathscr{C}_a$

The isogonal conjugate of the cubic curve $F_a = 0$ is the conic

$$\mathscr{C}_a : \ f_a(x, y, z) := a^2(c^2y^2 - b^2z^2) + b^2(c^2 + a^2 - b^2)zx - c^2(a^2 + b^2 - c^2)xy = 0.$$

See Figure 4.

Proposition 3. *The conic $\mathscr{C}_a$ is the hyperbola through the following points: the vertex A, the endpoints of the two bisectors of angle A, the point X which divides the A-altitude in the ratio $2 : 1$, and its traces on sidelines CA and AB.*

Proof. Rewriting the equation of $\mathscr{C}_a$ in the form

$$a^2(b^2 - c^2)yz + b^2(2a^2 - b^2 + c^2)zx - c^2(2a^2 + b^2 - c^2)xy + a^2(x + y + z)(c^2y - b^2z) = 0,$$

we see that it is homothetic to the circumconic which is the isogonal conjugate of the line

$$(b^2 - c^2)x + (2a^2 - b^2 + c^2)y - (2a^2 + b^2 - c^2)z = 0.$$

This is the perpendicular through the centroid to BC. Hence, the circumconic and $\mathscr{C}_a$ are hyperbolas. The hyperbola $\mathscr{C}_a$ clearly contains the vertex A and the endpoints of the A-bisectors, namely, $(0 : b : \pm c)$. It intersects the sidelines CA and AB at

$$Y = (a^2 : 0 : c^2 + a^2 - b^2) \quad \text{and} \quad Z = (a^2 : a^2 + b^2 - c^2 : 0)$$

respectively. These are the traces of $X = (a^2 : a^2 + b^2 - c^2 : c^2 + a^2 - b^2)$, which divides the A-altitude AH_a in the ratio $AX : XH_a = 2 : 1$. See Figure 5. $\square$

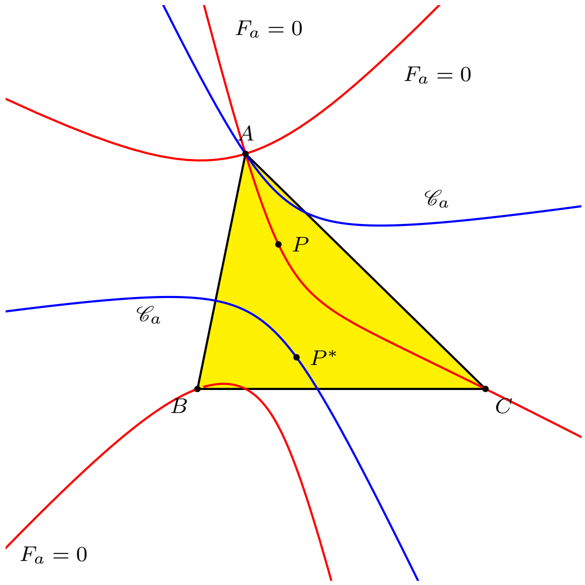

Figure 4. The cubic $F_a = 0$ and its isogonal conjugate conic $\mathscr{C}_a$

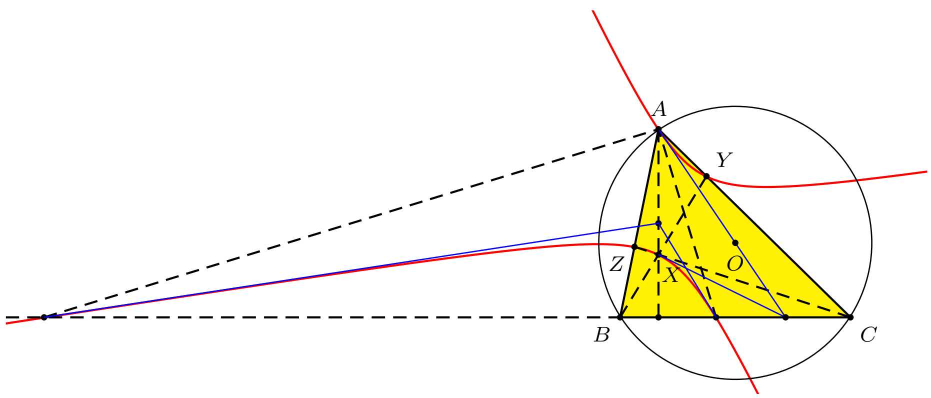

Figure 5. The hyperbola $\mathscr{C}_a$

Remark. The tangents of the hyperbola $\mathscr{C}_a$
(i) at $(0 : b : \pm c)$ pass through the midpoint of the A-altitude,
(ii) at A and X intersect at the trace of the circumcenter O on the sideline BC.

5. Conic solution of the angle bisectors problem

Suppose now P is a point which is the incenter (or an excenter) of its own anticevian triangle with respect to ABC. From the analysis of the preceding section, its isogonal conjugate lies on the hyperbola $\mathscr{C}_a$ as well as the two analogous hyperbolas

$$\mathscr{C}_b : \ f_b(x,y,z) := b^2(a^2z^2 - c^2x^2) + c^2(a^2+b^2-c^2)xy - a^2(b^2+c^2-a^2)yz = 0,$$

and

$$\mathscr{C}_c:\ f_c(x,y,z) := c^2(b^2x^2-a^2y^2)+a^2(b^2+c^2-a^2)yz-b^2(c^2+a^2-b^2)zx = 0.$$

Since $f_a + f_b + f_c = 0$, the three hyperbolas generate a pencil. The isogonal conjugates of the common points of the pencil are the points that solve the angle bisectors problem. Theorem 2 guarantees the existence of common points. To distinguish between the incenter and the excenter cases, we note that a nondegenerate triangle ABC divides the planes into seven regions (see Figure 6), which we label in accordance with the signs of the homogeneous barycentric coordinates of points in the regions:

$$+++,\ -++,\ -+-,\ ++-,\ +--,\ +-+,\ --+$$

In each case, the sum of the homogeneous barycentric coordinates of a point is adjusted to be positive.

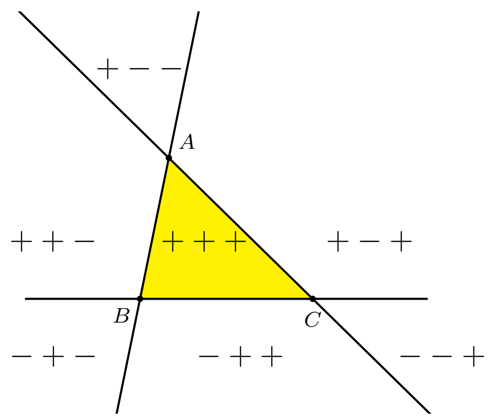

Figure 6. Partition of the plane by the sidelines of a triangle

In the remainder of this section, we shall denote by ε_a, ε_b, ε_c a triple of plus and minus signs, not all minuses.

Lemma 4. *A point is in the $\varepsilon_a\varepsilon_b\varepsilon_c$ region of its own anticevian triangle* (*with respect to ABC*) *if and only if it is in the $\varepsilon_a\varepsilon_b\varepsilon_c$ region of the medial triangle of ABC.*

The isogonal conjugates (with respect to ABC) of the sidelines of the medial triangle divide the plane into seven regions, which we also label $\varepsilon_a\varepsilon_b\varepsilon_c$, so that the isogonal conjugates of points in the $\varepsilon_a\varepsilon_b\varepsilon_c$ region are in the corresponding region partitioned by the lines of the medial triangle. See Figure 7.

Proposition 5. *Let Q be a common point of the conics $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$ in the $\varepsilon_a\varepsilon_b\varepsilon_c$ region of the partitioned by the hyperbolas. The isogonal conjugate of Q is a point whose anticevian triangle $A'B'C'$ has P as incenter or excenter according as all or not of ε_a, ε_b, ε_c are plus signs.*

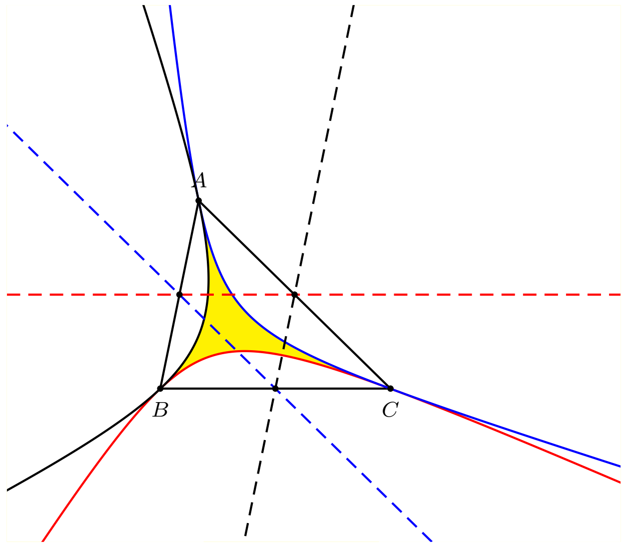

Figure 7. Partition of the plane by three branches of hyperbolas

6. Examples

Figure 8 shows an example in which the hyperbolas $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$ have four common points Q_0, Q_a, Q_b, Q_c, one in each of the regions $+++$, $-++$, $+-+$, $++-$. The isogonal conjugate P_0 of Q_0 is the incenter of its own anticevian triangle with respect to ABC. See Figure 9.

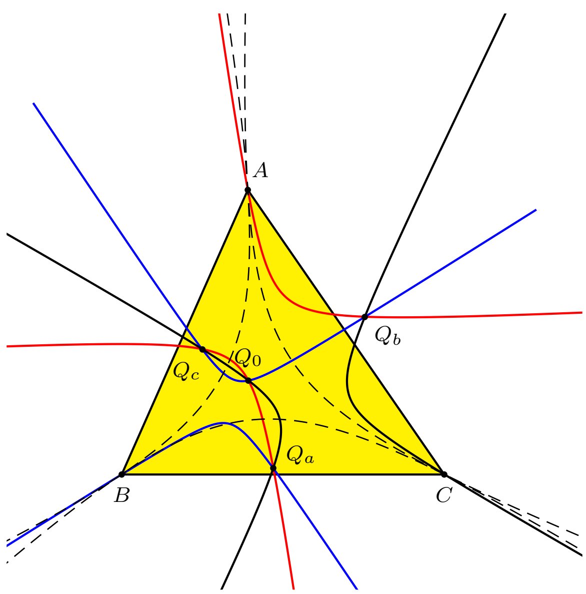

Figure 8. Pencil of hyperbolas with four real intersections

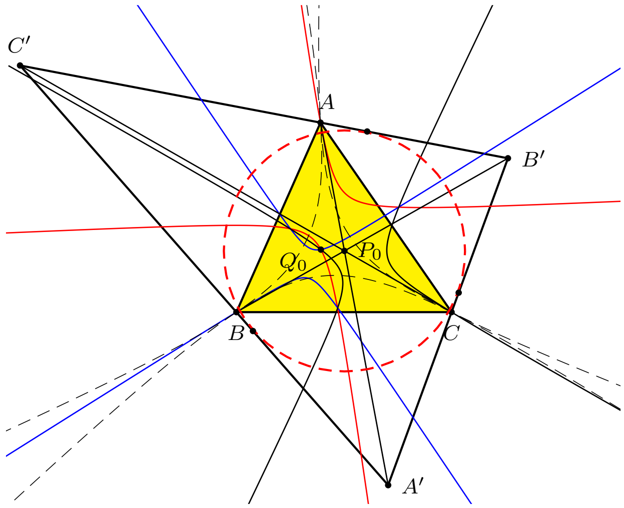

Figure 9. P_0 as incenter of its own anticevian triangle

Figure 10 shows the hyperbolas $\mathscr{C}_a$, $\mathscr{C}_b$, $\mathscr{C}_c$ corresponding to the cubics in Figure 3. They have only two real intersections Q_1 and Q_2, none of which is in the region $+++$. This means that there is no triangle $A'B'C'$ for which A, B, C are the feet of the internal angle bisectors. The isogonal conjugate P_1 of Q_1 has anticevian triangle $A_1B_1C_1$ and is its A_1-excenter. Likewise, P_2 is the isogonal conjugate of Q_2, with anticevian triangle $A_2B_2C_2$, and is its B_2-excenter.

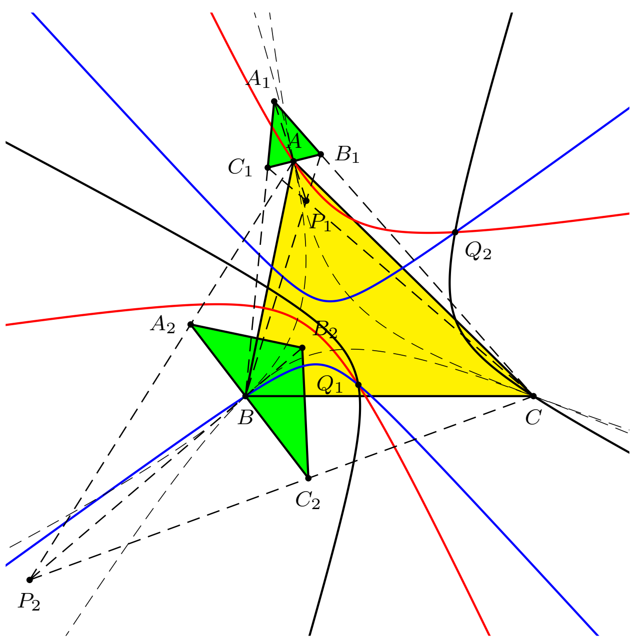

Figure 10. Pencil of hyperbolas with two real intersections

7. The angle bisectors problem for a right triangle

If the given triangle ABC contains a right angle, say, at vertex C, then the point P can be constructed by ruler and compass. Here is an easy construction. In fact, if $c^2 = a^2 + b^2$, the cubics $F_a = 0$, $F_b = 0$, $F_c = 0$ are the curves

$$x((a^2+b^2)y^2 - b^2z^2) - 2a^2y^2z = 0,$$
$$y((a^2+b^2)x^2 - a^2z^2) - 2b^2x^2z = 0,$$
$$z(b^2x^2 - a^2y^2) - 2xy(b^2x - a^2y) = 0.$$

A simple calculation shows that there are two real intersections

$$P_1 = (a(\sqrt{3}a-b) : b(\sqrt{3}b-a) : (\sqrt{3}a-b)(\sqrt{3}b-a)),$$
$$P_2 = (a(\sqrt{3}a+b) : b(\sqrt{3}b+a) : -(\sqrt{3}a+b)(\sqrt{3}b+a)).$$

These two points can be easily constructed as follows. Let ABC_1 and ABC_2 be equilateral triangles on the hypotenuse AB of the given triangle (with C_1 and C on opposite sides of AB). Then P_1 and P_2 are the reflections of C_1 and C_2 in C. See Figure 11. Each of these points is an excenter of its own anticevian triangle with respect to ABC, except that in the case of P_1, it is the incenter when the acute angles A and B are in the range $\arctan\frac{\sqrt{3}}{2} < A, B < \arctan\frac{2}{\sqrt{3}}$.

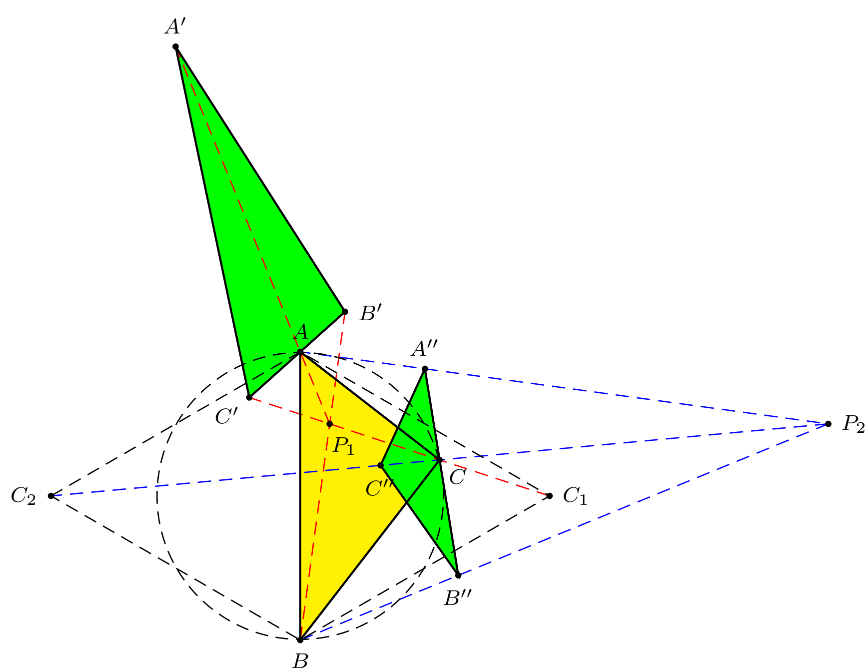

Figure 11. The angle bisectors problems for a right triangle

Remark. The cevian triangle of the incenter contains a right angle if and only if the triangle contains an interior angle of 120° angle (see [1]).

8. Triangles from the feet of external angle bisectors

In this section we make a change of notations. Figure 12 shows the collinearity of the feet X, Y, Z of the external bisectors of triangle ABC. The line ℓ containing them is the trilinear polar of the incenter, namely, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$. If the internal bisectors of the angles intersect ℓ at X', Y', Z' respectively, then X, X' divide Y, Z harmonically, so do Y, Y' divide Z, X, and Z, Z' divide X, Y. Since the angles XAX', YBY' and ZCZ' are right angles, the vertices A, B, C lie on the circles with diameters XX', YY', ZZ' respectively. This leads to the simple solution of the external angle bisectors problem.

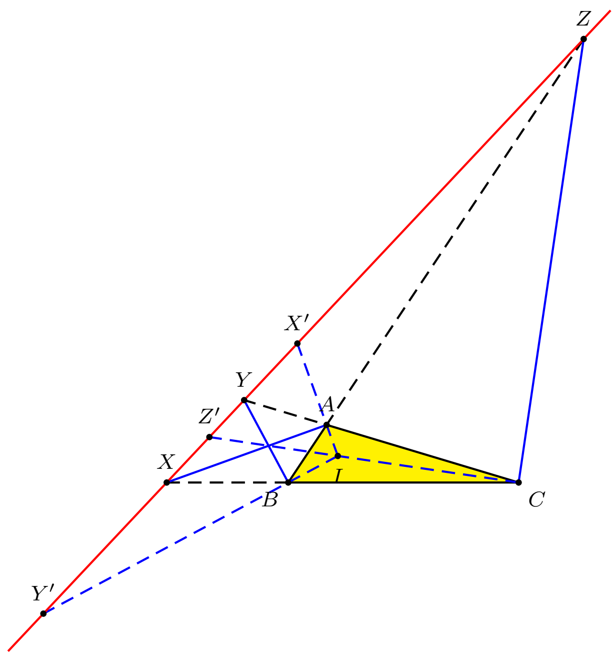

Figure 12. The external angle bisectors problem

We shall make use of the angle bisector theorem in the following form. Let $\varepsilon = \pm 1$. The ε-bisector of an angle is the internal or external bisector according as $\varepsilon = +1$ or -1.

Lemma 6 (Angle bisector theorem)**.** *Given triangle ABC with a point X on the line BC. The line AX is an ε-bisector of angle BAC if and only if*

$$\frac{BX}{XC} = \varepsilon \cdot \frac{AB}{AC}.$$

Here the left hand side is a signed ratio of directed segments, and the ratio $\frac{AB}{AC}$ on the right hand side is unsigned.

Given three distinct points X, Y, Z on a line ℓ (assuming, without loss of generality, Y in between, nearer to X than to Z, as shown in Figure 12), let X', Y', Z' be the harmonic conjugates of X, Y, Z in YZ, ZX, XY respectively. Here is a very simple construction of these harmonic conjugates and the circles with diameters XX', YY', ZZ'. These three circles are coaxial, with two common points F and F' which can be constructed as follows: if XYM and YZN are equilateral triangles erected on the same side of the line XYZ, then F and F' are the Fermat point of triangle YMN and its reflection in the line. See Figure 13.

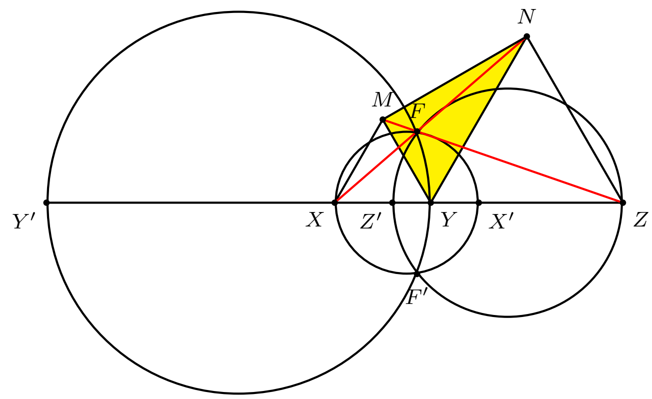

Figure 13. Coaxial circles with diameters XX', YY', ZZ'

Note that the circle (XX') is the locus of points A for which the bisectors of angle YAZ pass through X and X'. Since X' is between Y and Z, the internal bisector of angle YAZ passes through X' and the external bisector through X. Let the half-line YA intersect the circle (ZZ') at C. Then CZ is the external bisector of angle XCY. Let B be the intersection of the lines AZ and CX.

Lemma 7. *The point B lies on the circle with diameter YY'.*

Proof. Applying Menelaus' theorem to triangle ABC and the transversal XYZ (with X on BC, Y on CA, Z on AB), we have

$$\frac{AY}{YC} \cdot \frac{CX}{XB} \cdot \frac{BZ}{ZA} = -1.$$

Here, each component ratio is negative. See Figure 12. We rearrange the numerators and denominators, keeping the signs of the ratios, but treating the lengths of the various segments without signs:

$$\left(-\frac{AY}{AZ}\right)\left(-\frac{CX}{CY}\right)\left(-\frac{BZ}{BX}\right) = -1.$$

Applying the angle bisector theorem to the first two ratios, we have

$$\frac{YX}{XZ} \cdot \frac{XZ}{ZY} \cdot \left(-\frac{BZ}{BX}\right) = -1.$$

Hence, $\frac{ZY}{YX} = \frac{BZ}{BX}$, and BY is the internal bisector of angle XBZ. This shows that B lies on the circle with diameter YY'. $\square$

The facts that X, Y, Z are on the lines BC, CA, AB, and that AX', BY, CZ' are bisectors show that AX, BY, CZ are the external bisectors of triangle ABC. This leads to a solution of a generalization of the external angle bisector problem.

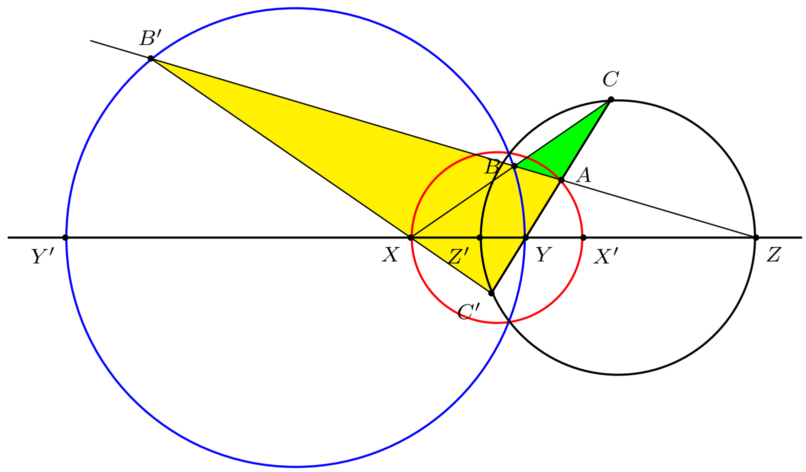

Figure 14. Solutions of the external angle bisectors problem

Let A be a point on the circle (XX'). Construct the line YA to intersect the circle (ZZ') at C and C' (so that A, C are on the same side of Y). The line AZ intersects CX and $C'X$ at points B and B' on the circle (YY'). The triangle ABC has AX, BY, CZ as external angle bisectors. At the same time, $AB'C'$ has internal bisectors AX, $B'Y$, and external bisector $C'Z$. See Figure 14.

We conclude with a characterization of the solutions to the external angle bisectors problem.

Proposition 8. *The triangles ABC with external bisectors AX, BY, CZ are characterized by*

$$a - b : b - c : a - c = XY : YZ : XZ.$$

Proof. Without loss of generality, we assume $a > b > c$. See Figure 12. The point Y is between X and Z. Since AX and CZ are the external bisector of angles BAC and ACB respectively, we have $\frac{BX}{XC} = \frac{-c}{b}$ and $\frac{AZ}{ZB} = \frac{-b}{a}$. From these, $\frac{CX}{BC} = \frac{b}{-(b-c)}$ and $\frac{AB}{ZA} = \frac{a-b}{b}$. Applying Menelaus' theorem to triangle XZB with transversal YAZ, we have

$$\frac{XY}{YZ} \cdot \frac{ZA}{AB} \cdot \frac{BC}{CX} = -1.$$

Hence, $\frac{XY}{YZ} = -\frac{CX}{BC} \cdot \frac{AB}{ZA} = \frac{a-b}{b-c}$. The other two ratios follow similarly. $\square$

References

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Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, 777 Glades Road, Boca Raton, Florida 33431-0991, USA
E-mail address: yiu@fau.edu