Mathematics Department Stanford University
Real Analysis Qualifying Exam, Autumn 2003, Paper 1

1. Let f_n be a sequence of continuous functions on $\mathbb{R}$ satisfying $0 \leq f_n(x) \leq f_{n+1}(x) \leq 1$ for all $x \in \mathbb{R}$ and all $n \in \{1, 2, \ldots\}$. Let $f(x) = \lim f_n(x)$.

(a) Show that for all x we have $f(x) \leq \liminf_{y \to x} f(y)$.

(b) Assume that f is continuous at a point x. Show that for all $\varepsilon > 0$, there exist δ, N so that $|f_n(y) - f_n(x)| < \varepsilon$ whenever $|y - x| < \delta$ and $n > N$.

2. Suppose that $f_n : [0, 1] \to [0, \infty)$ are non-negative Lebesgue measurable functions with $f_n(x) \to 0$ a.e. $x \in [0, 1]$, and assume $\sup_n \int_0^1 \varphi(f_n(x))\, dx \leq 1$ for some continuous function $\varphi : [0, \infty) \to [0, \infty)$ such that $\lim_{t \to \infty} t^{-1}\varphi(t) = \infty$. Prove that $\int_0^1 f_n(t)\, dt \to 0$.

3. Suppose $f \in L^1([0, 2\pi])$ and $\hat{f}(n) = \int_0^{2\pi} f(x) e^{-inx}\, dx,\ n = 0, \pm 1, \ldots$. Prove the following:

(a) $\sum_{|n|=0}^{\infty} |\hat{f}(n)|^2 < \infty \Rightarrow f \in L^2([0, 2\pi])$.

(b) $\sum_{|n|=0}^{\infty} |n\hat{f}(n)| < \infty \Rightarrow$ the L^1 class of f has a representative f_0 which extends to all of $\mathbb{R}$ as a 2π-periodic C^1 function.

4. Suppose $f \in L^1(\mathbb{R})$ and $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx$. Prove:

(a) $\hat{f} \in C(\mathbb{R})$ with $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

(b) If f has compact support, $\hat{f}$ cannot have compact support unless $f = 0$.

5. Let f be an arbitrary real-valued function on $[0, 1]$, and for each $x \in (0, 1)$ define $\overline{D}f(x) = \limsup_{y \to x} \frac{f(y) - f(x)}{y - x}$.

(a) If $\beta \in \mathbb{R}$ and if $S_\beta = \{x \in (0, 1) : \overline{D}f(x) > \beta\}$, prove that for each $\varepsilon > 0$ $\exists$ pairwise disjoint subintervals $[a_1, b_1], \ldots, [a_N, b_N] \subset [0, 1]$ such that $m^*(S_\beta \setminus (\cup_{i=1}^N [a_i, b_i])) < \varepsilon$ and $\beta(b_i - a_i) < f(b_i) - f(a_i)$ for each $i = 1, \ldots, N$. (Here m^* denotes Lebesgue outer measure.)

(b) If f is increasing on $[0, 1]$ show that the result of (a) directly implies the fact that $\overline{D}f(x) < \infty$ a.e. $x \in (0, 1)$.

prove that

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1. Let B be a Banach space and S a linear map from B to $C([0,1])$ such that $\|v_n\| \to 0$ in $B \Rightarrow Sv_n \to 0$ pointwise a.e. on $[0,1]$. Prove that S is a bounded operator from B to $C([0,1])$, assuming that $C([0,1])$ is equipped with its usual sup norm.

2. Let $\mathcal{B}(X)$ denote the set of functions $X \to \mathbb{R}$, where X is a given non-empty set.

(a) Describe a topology $\mathcal{T}$ on $\mathcal{B}(X)$ such that pointwise convergence of f_n to f on X is equivalent to convergence of f_n to f with respect to the topology $\mathcal{T}$ whenever $f, f_1, f_2 \ldots$ are given functions in $\mathcal{B}(X)$.

(b) If X is uncountable, show that $\mathcal{T}$ (as in (a)) is not metrizable.

Hint: Assume a metric d for $\mathcal{B}(X)$ exists and consider the sets $\{x \in X : d(\delta_x, 0) > \varepsilon\}$ where $\varepsilon > 0$ and $\delta_x = 1$ at x and zero elsewhere.

3. Suppose $(X, \mathcal{A}, \mu)$ is an arbitrary measure space, $f_j \rightharpoonup f$ (weak convergence in L^2), and $g_j \to g$ pointwise with $g_j \geq 0$ and g_j μ-measurable for each j. Prove $\int_X gf^2\, d\mu \leq \liminf_{j\to\infty} \int_X g_j f_j^2\, d\mu$.

Hint: First prove $\sqrt{g_j} f_j$ converges weakly to $\sqrt{g} f$ in the case when $g_j \leq K$ for some fixed K.

4. Let X be a normed linear space. X is said to be uniformly convex if there is a strictly increasing continuous function η on $[0,\infty)$ with $\eta(0) = 0$ such that $\frac{1}{2}\|x+y\| \leq 1 - \eta(\|x-y\|)$ for every $x, y \in X$ with $\|x\| = \|y\| = 1$.

If X is a uniformly convex Banach space, show that for any decreasing sequence $\{K_n\}_{n=1,2,3,\ldots}$ of nonempty closed convex subsets of $\{x \in X : \|x\| \leq 1\}$, we have $\cap_{n=1}^{\infty} K_n \neq \emptyset$.

5. Suppose f is AC on $[0,1]$. Prove:

(a) $A \subset [0,1]$ Lebesgue measurable $\Rightarrow f(A)$ Lebesgue measurable.

Hint: Start by showing that $f(A)$ has measure zero if A has measure zero.

(b) If $f(0) = f(1)$ and $\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x}\, dx$, then $n\hat{f}(0) \to$ as $|n| \to \infty$.