SKOLIAD No. 101

Robert Bilinski

Please send your solutions to the problems in this edition by ***1 October, 2007***. A copy of **MATHEMATICAL MAYHEM Vol. 3** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

This month we present a selection of problems from the 6th Annual CNU Regional High School Mathematics Contest. Thanks go to R. Persky, Christopher Newport University, Newport News, VA.

6th Annual CNU Regional High School Mathematics Contest (2005)

1. There are 8 girls and 6 boys at the Math Club at Central High School. The Club needs to send a delegation to a conference, and the delegation must contain exactly two girls and two boys. The number of possible delegations that can be formed from the membership of the club is

(A) 480 (B) 420 (C) 576 (D) 1680

4. The remainder of 7^{100} divided by 9 is

(A) 3 (B) 4 (C) 7 (D) 5

7. When $(x^{\frac{1}{2}} - x^{\frac{2}{3}})^7$ is multiplied out and simplified, one of the terms has the form Kx^4 where K is a constant. Find K.

(A) 7 (B) -7 (C) 35 (D) -35

8. Two points are picked at random on the unit circle $x^2 + y^2 = 1$. What is the probability the the chord joining the two points has length at least 1?

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$

11. Let m be a constant. The graphs of the lines $y = x - 2$ and $y = mx + 3$ intersect at a point whose x–coordinate and y–coordinate are both positive if and only if

(A) $m = 1$ (B) $m < 1$ (C) $m > -\frac{3}{2}$ (D) $-\frac{3}{2} < m < 1$

13. Let $f(x)$ be a function such that $f(x) + 2f(-x) = \sin x$ for every real number x. What is the value of $f(\frac{\pi}{2})$?

(A) -1 (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) 1

15. $\sqrt{7+4\sqrt{3}} - \sqrt{7-4\sqrt{3}} =$

(A) 4 (B) $2\sqrt{3}$ (C) $\sqrt{6}$ (D) 2

29. One root of $mx^2 - 10x + 3 = 0$ is two thirds of the other root. What is the sum of the roots?

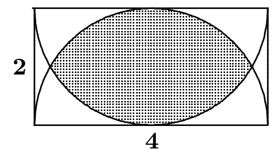
(A) $\frac{3}{2}$ (B) $\frac{5}{2}$ (C) $\frac{7}{2}$ (D) $\frac{5}{4}$

33. Calculate the expression $1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n!$.

(A) $(n^2+n+1)n!$ (B) $(n+1)! - 1$

(C) $(n+2)! - n!$ (D) $(n!)^2 - 1$

36. A rectangle has length 4 and height 2. What is the area of the shaded region, which is the intersection of the two semicircles pictured?



(A) $\frac{4\pi}{3} + 2\sqrt{3}$ (B) $\frac{4\pi}{3} - 2\sqrt{3}$ (C) $\frac{8\pi}{3} - 2\sqrt{3}$ (D) $\frac{8\pi}{3} + 2\sqrt{3}$

6ième Concours CNU Régional de Mathématiques (2005)

1. Il y a 8 filles et 6 garçons au Club de Maths de l'école. Le Club doit former une délégation à envoyer à un congrès, et la délégation doit se composer exactement de deux filles et de deux garçons. Le nombre possible de délégations qui peuvent être formées à partir des membres du Club est

(A) 480 (B) 420 (C) 576 (D) 1680

4. Le reste de 7^{100} divisé par 9 est

(A) 3 (B) 4 (C) 7 (D) 5

7. Quand $(x^{\frac{1}{2}} - x^{\frac{2}{3}})^7$ est développé et simplifié, un des termes a la forme Kx^4 où K est une constante. Trouver K.

(A) 7 (B) -7 (C) 35 (D) -35

8. Deux points sont choisis au hasard sur le cercle unitaire $x^2 + y^2 = 1$. Quelle est la probabilité que la corde joignant les deux points ait une longeur d'au moins 1?

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$

11. Soit m une constante. Les dessins des lignes $y = x - 2$ et $y = mx + 3$ s'intersectent au point d'abscisse et d'ordonnée toutes deux positives si et seulement si

(A) $m = 1$ (B) $m < 1$ (C) $m > -\frac{3}{2}$ (D) $-\frac{3}{2} < m < 1$

13. Soit $f(x)$ une fonction telle que $f(x) + 2f(-x) = \sin x$ pour tout nombre réel x. Quelle est la valeur de $f(\frac{\pi}{2})$?

(A) -1 (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) 1

15. $\sqrt{7 + 4\sqrt{3}} - \sqrt{7 - 4\sqrt{3}} =$

(A) 4 (B) $2\sqrt{3}$ (C) $\sqrt{6}$ (D) 2

29. Une racine de $mx^2 - 10x + 3 = 0$ est les deux tiers de l'autre racine. Quelle est la somme des racines ?

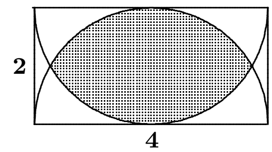
(A) $\frac{3}{2}$ (B) $\frac{5}{2}$ (C) $\frac{7}{2}$ (D) $\frac{5}{4}$

33. Que vaut l'expression $1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n!$.

(A) $(n^2 + n + 1)n!$ (B) $(n + 1)! - 1$

(C) $(n + 2)! - n!$ (D) $(n!)^2 - 1$

36. Un rectangle a une longueur de 4 et une hauteur de 2. Quelle est l'aire de la région hachurée, qui est l'intersection des deux demi-cercles dessinés ?



(A) $\frac{4\pi}{3} + 2\sqrt{3}$ (B) $\frac{4\pi}{3} - 2\sqrt{3}$ (C) $\frac{8\pi}{3} - 2\sqrt{3}$ (D) $\frac{8\pi}{3} + 2\sqrt{3}$

Next we give the official solutions to the 22^{nd} W.J. Blundon contest [2006 : 354–356].

1. An automobile went up a hill at an average speed of 30 km/hr and down the same distance at an average speed of 60 km/hr. What was the average speed for the trip?

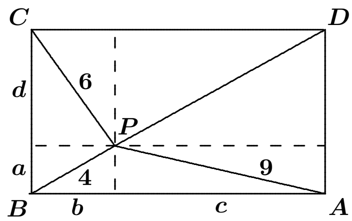
Official solution.

Let d be the distance one way, t_1 the time going up the hill, and t_2 the time going down. Since $30t_1 = d = 60t_2$, then $t_1 = 2t_2$. The required speed is $\dfrac{2d}{t_1 + t_2} = \dfrac{120t_2}{2t_2 + t_2} = 40$ km/hr.

2. Let P be a point in the interior of rectangle $ABCD$. If $PA = 9$, $PB = 4$, and $PC = 6$, find PD.

Official solution.

Since $PD^2 = c^2 + d^2$, $c^2 = 9^2 - a^2$, and $d^2 = 6^2 - b^2$, we have



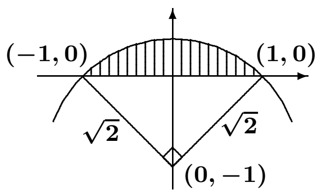
$$\begin{aligned} PD^2 &= 9^2 - a^2 + 6^2 - b^2 \\ &= 117 - (a^2 + b^2) \\ &= 117 - 16 = 101 . \end{aligned}$$

Hence, $PD = \sqrt{101}$.

3. Find the area of the region above the x–axis and below the graph of $x^2 + (y+1)^2 = 2$.

Official solution.

The graph of the equation $x^2 + (y+1)^2 = 2$ is a circle of radius $\sqrt{2}$ with centre at $(0, -1)$. The circle intersects the x–axis at $(\pm 1, 0)$. The area of the required region is clearly a quarter of the circle of radius $\sqrt{2}$ minus the area of the triangle with base length $\sqrt{2}$ and height $\sqrt{2}$. That is,

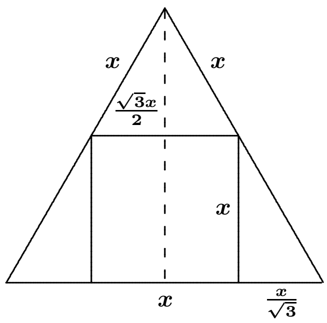


$$\text{area of the region} = \frac{1}{4}\pi\left(\sqrt{2}\right)^2 - \frac{1}{2}\left(\sqrt{2}\right)^2 = \frac{\pi}{2} - 1 .$$

4. A square is inscribed in an equilateral triangle. Find the ratio of the area of the square to the area of the triangle.

Official solution.

Let x be the length of each side of the square. Note that the top triangle is equilateral and all the right triangles are $30°$–$60°$–$90°$ triangles. Using the values of $\tan 60°$ and $\sin 60°$, the sides of the right triangles are calculated as shown. The base of the equilateral triangle is $x + \frac{2x}{\sqrt{3}}$ and the height is $x + \frac{\sqrt{3}x}{2}$.



The required ratio is $\dfrac{x^2}{\frac{1}{2}\left(x + \frac{2x}{\sqrt{3}}\right)\left(x + \frac{\sqrt{3}x}{2}\right)} = 28\sqrt{3} - 48.$

5. Find the number of solutions to the equation $2x + 5y = 2005$ for which both x and y are positive integers.

Official solution, expanded by the editor.

The given equation can be rewritten as $2x = 5(401 - y)$. If x and y are integers satisfying the equation, then x must be divisible by 5; that is, $x = 5t$ for some integer t. Then $10t = 5(401 - y)$, which simplifies to $2t = 401 - y$. If $y > 0$, then $2t < 401$ and hence $t \le 200$ (since t is an integer). We also want $t > 0$ to get $x > 0$. Thus, $1 \le t \le 200$.

Each integer t such that $1 \le t \le 200$ gives positive integers $x = 5t$ and $y = 401 - 2t$ which are solutions of the original equation. Hence, there are exactly 200 solutions which are positive integers.

6. For what values of a does the equation $4x^2 + 4ax + a + 6 = 0$ have real solutions?

Official solution, modified by the editor.

A quadratic equation has real solutions if and only if its discriminant is non-negative. The discriminant of the given equation is

$$\Delta = (4a)^2 - 4(4)(a+6) = 16(a^2 - a - 6) = 16(a-3)(a+2)\,.$$

We see that $\Delta \ge 0$ if and only if $a \ge 3$ or $a \le -2$.

7. Ace runs with constant speed and Flash runs x times as fast, $x > 1$. Flash gives Ace a head start of y metres, and, at a given signal, they start off in the same direction. Find the distance Flash must run to catch Ace.

Official solution.

Let d be the distance Flash must travel to catch Ace, let v be Ace's speed, and let t be the time needed to catch up. Then we have $d = vxt$ and also $d - y = vt$. Eliminating v, we have $d - y = \frac{d}{x}$. Hence, $d = \frac{xy}{x-1}$.

8. Show that $3^n - 2n - 1$ is divisible by 4 for any positive integer n.

Official solution.

We consider two cases.

For n even, we write $n = 2m$. Then

$$\begin{aligned} 3^n - 2n - 1 &= 3^{2m} - 2(2m) - 1 = 3^{2m} - 4m - 1 \\ &= (3^m - 1)(3^m + 1) - 4m\,. \end{aligned}$$

Clearly, $3^m - 1$ and $3^m + 1$ are even; whence, 4 divides $(3^m - 1)(3^m + 1)$. Thus, 4 divides $3^n - 2n - 1$.

For n odd, we write $n = 2m + 1$. Then

$$\begin{aligned} 3^n - 2n - 1 &= 3^{2m+1} - 2(2m+1) - 1 = 3^{2m+1} - 3 - 4m \\ &= 3(3^m - 1)(3^m + 1) - 4m\,. \end{aligned}$$

As above, 4 divides $(3^m - 1)(3^m + 1)$. Thus, 4 divides $3^n - 2n - 1$.

9. If the polynomial $P(x) = x^3 - x^2 + x - 2$ has the three zeroes a, b, and c, find $a^3 + b^3 + c^3$.

Official solution.

We have

$$\begin{aligned} P(a) &= a^3 - a^2 + a - 2 = 0\,, \\ P(b) &= b^3 - b^2 + b - 2 = 0\,, \\ P(c) &= c^3 - c^2 + c - 2 = 0\,. \end{aligned}$$

Summing these three equations, we get

$$a^3 + b^3 + c^3 - (a^2 + b^2 + c^2) + (a + b + c) - 6 = 0\,.$$

Since $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$, we get

$$a^3 + b^3 + c^3 = (a + b + c)^2 - 2(ab + bc + ca) - (a + b + c) + 6\,.$$

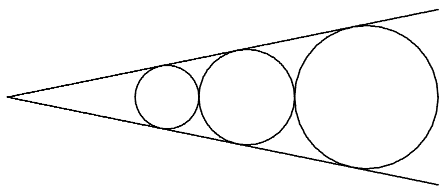
But we also have

$$\begin{aligned} x^3 - x^2 + x - 2 &= (x - a)(x - b)(x - c) \\ &= x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc\,, \end{aligned}$$

which implies that $a + b + c = 1$ and $ab + bc + ca = 1$. Therefore,

$$a^3 + b^3 + c^3 = 1^2 - 2(1) - 1 + 6 = 4\,.$$

10. A circle of radius 2 is tangent to both sides of an angle. A circle of radius 3 is tangent to the first circle and both sides of the angle. A third circle is tangent to the second circle and both sides of the angle. Find the radius of the third circle.



Official solution.

Let x be the radius of the third circle, and let a be the shortest distance from the vertex of the angle to the first circle. By similar triangles, we have $\frac{a+2}{2} = \frac{a+7}{3}$, and hence $a = 8$. By similar triangles again, we have $\frac{a+10+x}{x} = \frac{a+2}{2}$, implying that $\frac{18+x}{x} = 5$. Hence, $x = \frac{9}{2}$.

That brings us to the end of another issue. Please send in solutions! We had no readers' solutions to feature this month.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

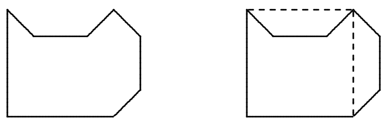
Please send your solutions to the problems in this edition by **1 August 2007**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M288. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

The following figure can be cut into two pieces and reassembled into a square, by simply cutting off the 'tab' and placing it in the cutaway at the top, as shown in the second image.



Determine a method to cut the given figure into three pieces which can be reassembled to form a square. (Find a method which is essentially different from cutting it into two pieces; for example, cutting the tab into two pieces would not be considered different from the two-piece dissection.)

M289. *Proposed by K.R.S. Sastry, Bangalore, India.*

Solve the following equation for real x:

$$\log\left(x+\sqrt{5x-\tfrac{13}{4}}\right) \;=\; -\log\left(x-\sqrt{5x-\tfrac{13}{4}}\right)\,.$$

M290. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Give a purely geometric proof that $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$.

M291. *Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.*

The right triangle having sides 3, $\sqrt{7}$, and 4, has the strange property that the two integer lengths sum to the value under the square root sign for the length of the third side.

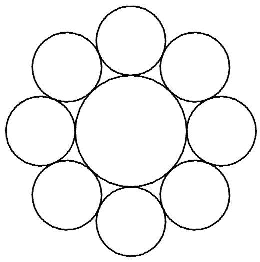
1. Find another such triangle.
2. Prove that there are infinitely many such triangles, and show how to construct them.
3. Does the formula work only for integers?

M292. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let x be a positive number. Prove that $\sqrt{\dfrac{[x]}{x+\{x\}}} + \sqrt{\dfrac{\{x\}}{x+[x]}} > 1$, where $[x]$ and $\{x\}$ represent the integer part and the fractional part of x, respectively.

M293. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Eight equal circles are mutually tangent in pairs and tangent externally to a unit circle. Determine the common radii of the eight smaller circles.



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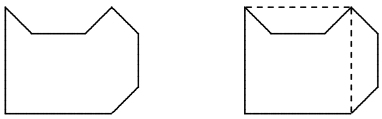
M288. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

La figure ci-dessous peut être coupée en deux morceaux qu'on peut réarranger pour former un carré, comme le montre le second dessin.



Trouver une méthode pour couper la figure donnée en trois morceaux pouvant former un carré par réarrangement. (Cette méthode doit être essentiellement différente de la première ; simplement couper en deux le morceau ajouté pour former le premier carré ne compte pas.)

M289. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Trouver les solutions réelles de l'équation :

$$\log\left(x+\sqrt{5x-\tfrac{13}{4}}\right) \;=\; -\log\left(x-\sqrt{5x-\tfrac{13}{4}}\right)\,.$$

M290. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Trouver une démonstration purement géométrique de l'égalité $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{4}$.

M291. *Proposé par Robert Bilinski, Collège Montmorency, Laval, QC.*

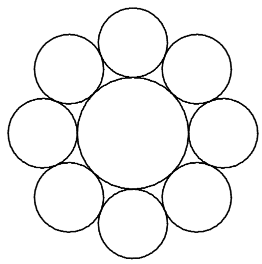
Le triangle rectangle de côtés 3, $\sqrt{7}$ et 4 possède la curieuse propriété qu'un de ses côtés est la racine carrée de la somme des côtés mesurés par des entiers.

1. Trouver un autre tel triangle.
2. Montrer qu'il existe une infinité de tels triangles et décrire leur construction.
3. La formule n'est-elle valable que pour des entiers ?

M292. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit x un nombre positif. Montrer que $\sqrt{\dfrac{[x]}{x+\{x\}}}+\sqrt{\dfrac{\{x\}}{x+[x]}}>1$, où $[x]$ et $\{x\}$ désignent respectivement les parties entière et fractionnaire de x.

M293. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*



On couronne le cercle unité avec huit petits cercles égaux tangents et tangents deux à deux. Trouver leur rayon commun.

Mayhem Solutions

M238. *Proposé par John Grant McLoughlin, Université du Nouveau-Brunswick, Fredericton, NB.*

Soit PQ une corde d'une parabole et soit R le point milieu de PQ. Soit S un point sur la parabole tel que la tangente en S est parallèle à PQ. Si T désigne le point d'intersection des tangentes en P et Q, montrer que R, S et T sont colinéaires.

Solution par Jean-David Houle, Cégep de Drummondville, Drummondville, QC.

Plaçons la parabole et les points sur un plan cartésien. On choisit nos axes tels que le sommet de la parabole se situe à l'origine. Soit la parabole d'équation $y = ax^2$ et les points $P(p, ap^2)$ et $Q(q, aq^2)$. On va prouver que, sous ces conditions, les points R, S, et T ont la même abscisse, et sont par le fait même colinéaires.

Soit r l'abscisse du point R. Puisque R est le point milieu de la corde joignant P et Q, son abscisse est donc la moyenne de celles des points P et Q ; c'est-à-dire $r = \frac{1}{2}(p+q)$.

Soit s l'abscisse du point S. Puisque la tangente en S et la corde PQ sont parallèles, elles ont la même pente. Nous avons donc l'équation suivante à résoudre pour s :

$$\left.\frac{d(ax^2)}{dx}\right|_{x=s} = 2as = \frac{aq^2 - ap^2}{q-p} = \frac{a(q^2-p^2)}{q-p}.$$

Puisque $p \neq q$, on a $2as = a(p+q)$ et on obtient $s = \frac{1}{2}(p+q)$.

Maintenant, on va trouver l'équation des tangentes aux points P et Q. Les dérivées de la parabole aux points P et Q nous donneront les pentes de ces tangentes. Ainsi, les équations des tangentes sont :

$$y = 2apx - ap^2 \quad \text{et} \quad y = 2aqx - aq^2 .$$

Pour trouver l'abscisse t du point T, on résout l'équation suivante pour t :

$$\begin{aligned} 2apt - ap^2 &= 2aqt - aq^2 , \\ 2at(p-q) &= a(p^2 - q^2) . \end{aligned}$$

Ainsi, $t = \frac{1}{2}(p+q)$, parce que $p \neq q$.

Donc, les points R, S et T ont la même abscisse $\frac{1}{2}(p+q)$ et sont colinéaires.

En outre résolu par HASAN DENKER, Istanbul, Turquie ; et TITU ZVONARU, Comăneşti, Roumanie. Une solution incorrecte a aussi été soumise.

M239. *Proposed by Yakub N. Aliyev, Baku State University, Baku, Azerbaijan.*

If a, b, $c > 0$, prove that

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \le \frac{(a+b+c)^2}{6abc}.$$

Solution by Vedula N. Murty, Dover, PA, USA.

We have $(a+b)^2 - 4ab = (a-b)^2 \ge 0$, and hence, $4ab \le (a+b)^2$. Similarly, $4bc \le (b+c)^2$ and $4ca \le (c+a)^2$. Therefore,

$$\begin{aligned} 4abc\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) &= \frac{4ab}{a+b}c + \frac{4bc}{b+c}a + \frac{4ca}{c+a}b \\ &\le (a+b)c + (b+c)a + (c+a)b \\ &= 2(ab+bc+ca). \qquad (1) \end{aligned}$$

Using the well-known inequality $ab + bc + ca \le a^2 + b^2 + c^2$, we obtain

$$\begin{aligned} 3(ab+bc+ca) &\le a^2+b^2+c^2+2(ab+bc+ca) \\ &= (a+b+c)^2. \qquad (2) \end{aligned}$$

Combining (1) and (2), we have

$$4abc\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \le \frac{2}{3}(a+b+c)^2.$$

Dividing by $4abc$ gives the desired result.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ARKADY ALT, San Jose, CA, USA; MIHÁLY BENCZE, Brasov, Romania; HASAN DENKER, Istanbul, Turkey; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JEAN-DAVID HOULE, Cégep de Drummondville, Drummondville, QC; BABIS STERGIOU, Chalkida, Greece; and TITU ZVONARU, Comăneşti, Romania.

M240. *Proposé par l'Équipe de Mayhem.*

En utilisant une seule fois chacun des chiffres de 0 à 9, trouver quatre carrés parfaits (positifs) tels qu'il y en ait un de quatre chiffres, un de trois, un de deux et un dernier de un chiffre. (Note : Il y a plus d'une solution. Combien pouvez-vous en trouver ?)

Solution par Jean-David Houle, Cégep de Drummondville, Drummondville, QC.

Évidemment, nous ne devons pas considérer les carrés qui comportent au moins 2 nombres identiques. Sous cette condition, on peut démontrer qu'aucun carré de 1, 2, ou 3 chiffres ne contienne de 0. En rédigeant une table comprenant tous les nombres carrés à considérer (de 1, 2, 3, ou 4 chiffres, sans répétition, et comprenant le chiffre 0 dans le cas des nombres à 4 chiffres), on obtient 37 nombres.

Pour chaque carré à 4 chiffres, on inscrit les carrés à 2 chiffres possibles (ceux dont les chiffres n'apparaissent pas dans le carré à 4 chiffres). Pour chaque paire, on inscrit ensuite les carrés à 1 chiffre qui n'apparaissent pas dans le carré à 4 chiffres ou dans celui à 2 chiffres. En vérifiant si il est possible de trouver un carré à 3 chiffres comprenant les 3 chiffres non-utilisés, on obtient les 4 solutions suivantes : $(9, 81, 576, 2304)$, $(9, 16, 784, 3025)$, $(9, 81, 324, 7056)$, et $(1, 36, 784, 9025)$.

Autres solutions soumises par HASAN DENKER, Istanbul, Turquie ; et TITU ZVONARU, Comăneşti, Roumanie. Une solution incomplète a aussi été soumise.

M241. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Three gunfighters, called Quick, Fast, and Slow, stand one at each vertex of an equilateral triangle. Quick is faster on the draw than Fast, and Fast is faster than Slow. If x intends to fire at y, we will say that x *targets* y. We will assume that if x fires at y, then y will be hit, and that if x and y both target each other, the one who is slower on the draw will be hit before he can fire. A combatant cannot fire once he has been hit.

In the first phase of the confrontation, each combatant targets one of the other two and fires a maximum of one round. No man knows how fast the other two are, and the targeting choices are made randomly and cannot be changed during the first phase.

If two combatants survive the first phase, they face each other in a second phase and the fastest draw wins. If only one combatant survives the first phase, he is the winner (and there is no second phase).

Find the probability that:

(a) Quick survives; (b) Fast survives; (c) Slow survives.

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

There are 8 targeting possibilities in the first round as shown by the table of outcomes below, where Quick, Fast, and Slow are denoted by Q, F, and S, respectively.

Targets for Q, F, and S	[*Ed.:* First Round Survivor]	Final Survivor
F, Q, Q	Slow	Slow
F, Q, F	Quick and Slow	Quick
F, S, Q	None	None
F, S, F	Quick	Quick
S, Q, Q	Fast	Fast
S, Q, F	None	None
S, S, Q	Quick and Fast	Quick
S, S, F	Quick and Fast	Quick

Thus, the probability of survival for Quick is $\frac{1}{2}$, for Fast is $\frac{1}{8}$, and for Slow is $\frac{1}{8}$.

Also solved by Jean-David Houle, Cégep de Drummondville, Drummondville, QC. A solution submitted by Hasan Denker, Istanbul, Turkey used the assumption that the one who is slower on the draw will always be hit before he can fire. In that case, the probability that Quick survives is $\frac{1}{2}$, the probability that Fast survives is $\frac{1}{4}$, and the probability that Slow survives is $\frac{1}{4}$.

M242. *Proposé par Houda Anoun, Bordeaux, France.*

Pour quels nombres naturels x le nombre $x^4 + x^3 + x^2 + x + 1$ est-il un carré parfait ?

Solution par Jean-David Houle, Cégep de Drummondville, Drummondville, QC.

Disons que $f(x) = x^4 + x^3 + x^2 + x + 1$.

Cas 1. x est un nombre pair.

Notons que

$$\begin{aligned} \left(x^2 + \tfrac{1}{2}x\right)^2 &= x^4 + x^3 + \tfrac{1}{4}x^2 \;<\; f(x) \\ \text{et} \quad \left(x^2 + \tfrac{1}{2}x + 1\right)^2 &= x^4 + x^3 + \tfrac{9}{4}x^2 + x + 1 \;\geq\; f(x)\,, \end{aligned}$$

alors $\left(x^2 + \frac{1}{2}x\right)^2 < f(x) \leq \left(x^2 + \frac{1}{2}x^2 + 1\right)^2$.

L'égalité survient si $x = 0$. Dans tous les autres cas, le polynôme $f(x)$ est compris entre deux carrés parfaits consécutifs et ne peut donc pas être, lui aussi, un carré parfait.

Cas 2. x est un nombre impair.

Pour $x \geq 5$, on a

$$\begin{aligned} \left(x^2 + \tfrac{1}{2}x - \tfrac{1}{2}\right)^2 &= x^4 + x^3 - \tfrac{3}{4}x^2 - \tfrac{1}{2}x + \tfrac{1}{4} \;<\; f(x) \\ \text{et} \quad \left(x^2 + \tfrac{1}{2}x + \tfrac{1}{2}\right)^2 &= x^4 + x^3 + \tfrac{5}{4}x^2 + \tfrac{1}{2}x + \tfrac{1}{4} \;>\; f(x)\,, \end{aligned}$$

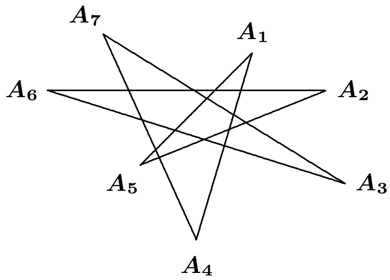
donc $\left(x^2 + \frac{1}{2}x - \frac{1}{2}\right)^2 < f(x) < \left(x^2 + \frac{1}{2}x + \frac{1}{2}\right)^2$.

Le polynôme $f(x)$ est compris entre deux carrés parfaits consécutifs et ne peut donc pas être, lui aussi, un carré parfait.

Ceci étant dit, il ne reste qu'à vérifier les valeurs de x qui sont impaires et inférieures à 5. Si $x = 1$, alors $x^4 + x^3 + x^2 + x + 1 = 5$, qui n'est pas un carré parfait. Si $x = 3$, alors $x^4 + x^3 + x^2 + x + 1 = 121 = 11^2$, qui donne une solution.

Le seul nombre naturel x satisfaisant l'énoncé est $x = 3$.

Autres solutions soumises par ALINA ALT et ARKADY ALT, San José, CA, É-U; RICHARD I. HESS, Rancho Palos Verdes, CA, É-U; EDWARD T.H. WANG, Université Wilfrid Laurier, Waterloo, ON; et TITU ZVONARU, Comăneşti, Roumanie. Une solution incomplète a aussi été soumise.

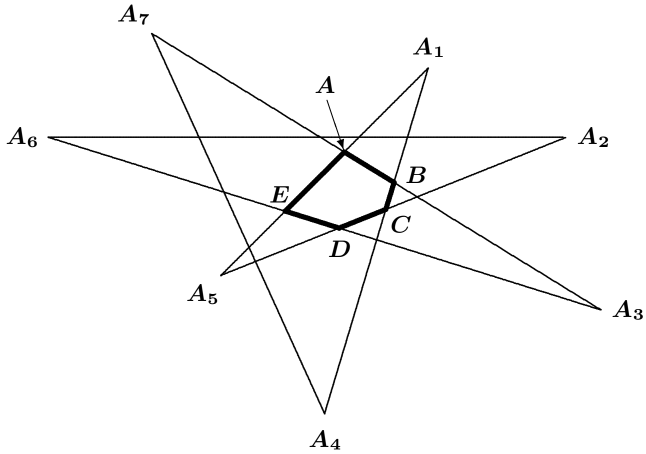


M243. *Proposed by K.R.S. Sastry, Bangalore, India.*

In the 7-point star shown, no three lines are concurrent. Find the sum $A_1 + A_2 + \ldots + A_7$.

Solution by Hasan Denker, Istanbul, Turkey.

The given 7-point star, with no three lines concurrent, generates pentagon $ABCDE$, as shown.



Considering triangles AEA_3, A_7A_4B, A_1A_5C, and A_2A_6D, the following relationships are obtained:

$$\begin{aligned} A + E &= 180^\circ - A_3\,, \\ B &= 180^\circ - A_4 - A_7\,, \\ C &= 180^\circ - A_1 - A_5\,, \\ D &= 180^\circ - A_2 - A_6\,. \end{aligned}$$

Summing these equations, we can then conclude that

$$\begin{aligned} & A + B + C + D + E \\ &\quad = (180^\circ - A_3) + (180^\circ - A_4 - A7) \\ &\qquad + (180^\circ - A_1 - A_5) + (180^\circ - A_2 - A_6) \\ &\quad = 720^\circ - (A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7)\,. \end{aligned}$$

However, $A + B + C + D + E = 540^\circ$. Hence,

$$540^\circ = 720^\circ - (A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7)\,.$$

Therefore, $A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 = 180^\circ$.

There was one incomplete solution submitted.

Problem of the Month

Ian VanderBurgh

This month, we give some thought about repeating decimals (decimals, decimals, decimals, decimals...)

Problem #1 (2006 Canadian Open Mathematics Challenge)
Suppose n and D are integers with n positive and $0 \le D \le 9$. Determine n if $\frac{n}{810} = 0.\overline{9D5} = 0.9D59D59D5\ldots$.

I knew that I should have paid more attention in elementary school! If you're like me, you probably remember that $\frac{1}{3} = 0.33333\ldots$ which can also be written as $0.\overline{3}$. Maybe you remember that $\frac{7}{9} = 0.\overline{7} = 0.7777\ldots$. How about $\frac{1}{7}$? Do you know that this equals $0.\overline{142857}$?

To solve Problem #1, it would be helpful to convert the repeating decimal to a fraction. But how do we do this? Let's look at two different ways.

In the first approach, we set $x = 0.\overline{9D5} = 0.9D59D59D5\ldots$. Then $1000x = 9D5.9D59D5\ldots = 9D5.\overline{9D5}$. Thus,

$$\begin{aligned} 999x &= 1000x - x = 9D5.\overline{9D5} - 0.\overline{9D5} = 9D5 \qquad \text{(an integer!)} \\ x &= \frac{9D5}{999}\,. \end{aligned}$$

Does this method look familiar? It may, if you have ever tried to prove that $0.\overline{9}$ actually equals 1.

For a second approach, we rewrite $0.9D59D59D5\ldots$ as

$$\frac{9D5}{10^3} + \frac{9D5}{10^6} + \frac{9D5}{10^9} + \cdots$$

This is an infinite geometric series with first term $a = \frac{9D5}{10^3}$ and common ratio $r = \frac{1}{10^3}$; thus, its sum is

$$\frac{a}{1-r} = \frac{\frac{9D5}{10^3}}{1 - \frac{1}{10^3}} = \frac{9D5}{1000-1} = \frac{9D5}{999}\,.$$

It is reassuring to get the same answer in two different ways. Try using one or both of these methods to show that $0.\overline{1234} = \frac{1234}{9999}$ and $0.\overline{abc} = \frac{abc}{999}$. Can you come up with some general rules for converting repeating decimals to fractions?

Now we are ready to solve Problem #1.

Solution to Problem #1: From the given information and our comments above, we know that $\frac{n}{810} = 0.\overline{9D5} = \frac{9D5}{999}$. Clearing the fractions yields

$999n = 810(9D5)$. We can simplify this by dividing both sides by 27, giving $37n = 30(9D5)$. Since 37 is a factor of the left side, it must be a factor of the right side. Since 37 and 30 have no common factors, then 37 must divide exactly into $9D5$. How can we determine D? One way would be to get out a calculator and try to find a multiple of 37 that is between 900 and 1000 and ends with a 5. This wouldn't be too hard.

Here is another approach. We first note that

$$37 \times 20 \;=\; 740 \;<\; 9D5 \;<\; 1120 \;=\; 37 \times 30\,.$$

Hence, $9D5 = 37 \times 25$ because no other number in the permissible range when multiplied by 37 will end in 5. Therefore, $D = 2$.

But we want the value of n. Recall that $37n = 30(9D5) = 30(925)$. Hence, $n = 30(925)/37 = 750$.

As with many problems involving a repeating decimal, the decimal gets converted to a fraction. So the amount of knowledge of repeating decimals that we need is not enormous.

Here is another such problem to keep you busy over the next month:

Problem #2 (1992 AIME) Let S be the set of all rational numbers r with $0 < r < 1$, that have a repeating decimal expansion of the form $0.\overline{abc}$, where the digits a, b, and c are not necessarily distinct. To write the elements of S as fractions in lowest terms, how many different numerators are required?

Good luck! I've put a few hints at the end.

In February's Problem of the Month, we looked at a problem involving determining the average number of "change points" in sequences of 0s and 1s. This involved counting the total number of change points over all such sequences in a clever way.

Imagine my surprise on the first Saturday in December (about the time I was writing the February column), when I saw the following problem on the 2006 William Lowell Putnam Mathematical Competition. (I have modified this problem slightly to remove the special case of $n = 2$ and to remove some of the more technical notation.)

Problem #3. A permutation π of $\{1, 2, \ldots, n\}$ (with $n \geq 3$) has a local maximum at position k if the two neighbouring numbers (or, in case $k = 1$ or $k = n$, the one neighbouring number) are both smaller than the number in position k. (For example, if $n = 5$, then $2, 1, 4, 5, 3$ has local maxima in positions 1 and 4.) What is the average number of local maxima of a permutation of $\{1, 2, \ldots, n\}$, averaging over all such permutations?

We will try to solve this problem by the same technique that we used in February: fixing a position and counting the total number of permutations with a local maximum in that position.

Solution to Problem #3: First consider position 1. How many permutations have a local maximum in position 1? Whether or not there is a local maximum at position 1 depends on the numbers in positions 1 and 2. Any pair of numbers can give a local maximum at position 1 if they are arranged with the larger number first. (For example, the pair 3 and 5 gives a local maximum in position 1 if the 5 comes before the 3.)

There are $\binom{n}{2}$ possible pairs of numbers that can be placed in positions 1 and 2. There is only one way to arrange a given pair to get a local maximum in position 1. There are then $(n-2)!$ ways of filling out the rest of the permutation. Thus, there are

$$\binom{n}{2}(n-2)! \;=\; \frac{n!}{(n-2)!2!}(n-2)! \;=\; \frac{n!}{2}$$

permutations with a local maximum in position 1. In other words, among all such permutations, there are $\frac{1}{2}n!$ local maxima in position 1. By a similar argument, there are $\frac{1}{2}n!$ local maxima in position n.

Now consider a position k with $1 < k < n$. How many local maxima are there at position k? Whether there is a local maximum at position k depends on the numbers in positions $k-1$, k, and $k+1$. Any triple of numbers can be arranged to form a local maximum at position k in two ways. For example, if we choose 1, 3, 7, then a local maximum occurs in the middle if (and only if) they are arranged as 1, 7, 3 or 3, 7, 1. There are $\binom{n}{3}$ ways of choosing the three numbers that will go in positions $k-1$ through $k+1$, two ways of arranging these numbers to form a local maximum at position k, and $(n-3)!$ ways to arrange the remaining $n-3$ numbers in the permutation. Thus, there are

$$2\binom{n}{3}(n-3)! \;=\; \frac{2n!}{(n-3)!3!}(n-3)! \;=\; \frac{n!}{3}$$

permutations with a local maximum at position k. In other words, there are $\frac{1}{3}n!$ local maxima at position k among all such permutations. (Remember that there are $n-2$ values for k that we have to keep track of in this case.)

Hence, the total number of local maxima over all such permutations is

$$\tfrac{1}{2}n! + \tfrac{1}{2}n! + (n-2)\left(\tfrac{1}{3}n!\right) \;=\; \tfrac{1}{3}(n+1)n!\,.$$

Since the total number of permutations of $\{1, 2, \ldots, n\}$ is $n!$, the average number of local maxima is $\frac{1}{3}(n+1)$.

It's always neat to see an old technique come in handy. That's part of the reason why we practice solving problems—the more we practice, the more techniques we learn, and the more likely we are to think, "Hey, wait a second! I know what to do here."

Hints for Problem #2:

- Convert the repeating decimal to a fraction.
- When is this fraction irreducible? How many of these cases are there?
- If the given fraction is reducible, what happens? What are the possible denominators when reduced? What are the possible numerators?

Pólya's Paragon

The Pigeonhole Principle

Jeff Hooper

In problem-solving, we can sometimes get to the answer using the most direct approach (which is often the first one we think of). But there may be approaches to a problem that are *indirect* or *non-constructive*; they force a solution or situation to happen, but not explicitly. In fact, even in cases where a direct attack works, these alternative methods sometimes provide simpler, more elegant solutions. In this issue we will explore one of these ideas and look at a number of problems in which it can be applied.

The simplest version of this idea is easy to explain. Suppose you have 10 balls and 9 boxes, and you must put all the balls into the boxes in some manner. There are of course lots of ways to do this. You could, for instance, put all of the balls in one box and leave the others empty, or you could try to distribute the balls evenly. But, no matter how you do it, *at least one of the boxes must get more than one ball!* This is because there are more balls than boxes.

Now I hope this is clear. Even if you tried to fill the boxes with one ball each, there would still be that one extra ball at the end, and it would need to go somewhere! Once you place it in a box, you must have (at least) two balls in one of the boxes.

It may surprise you that this idea is important enough to have a name. It is called the *pigeonhole principle*. Its name evokes an image of lots of pigeons fighting to get into a smaller number of holes to roost. It simply says that you cannot stuff lots of things into an insufficient number of boxes. A slightly more formal statement might be:

Pigeonhole Principle. If more than n objects (pigeons) are distributed into exactly n boxes (holes), then (at least) one of the boxes must contain more than one of the objects.

If the number of objects is a lot larger than the number of boxes, then we can make slightly stronger conclusions. Suppose we had 19 balls to place in 9 boxes. Can you see why it now must be the case that one (or possibly more) of the boxes must have at least 3 balls? So there is a more general version of the pigeonhole principle:

Pigeonhole Principle (General Version). Let $k \geq 1$. If more than kn objects (pigeons) are distributed into exactly n boxes (holes), then (at least) one of the boxes must contain more than k objects.

Even this generalization seems fairly obvious. What might surprise you is the number of situations in which this principle can be applied. Often there is some subtlety that makes the application not quite immediate. Let's look at some examples.

Example 1: At a conference there are 100 people participating. Show that there must be two of them who know the same number of other participants.

Solution: We will treat the 100 participants as 'pigeons'; we need to put them into 'holes'. But what sort of holes? It seems that we should assign to each participant the number of *other* participants he or she knows. This will be a number between 0 and 99. But wait! That's 100 holes! It seems possible that we might be able to assign all 100 numbers to the 100 different people. The pigeonhole principle does not seem to apply.

There's a subtlety though. Suppose person X receives the number 99. Then this person must know everybody else, and so nobody can be assigned the number 0! But now we must assign each participant a number from 1 to 99, and the pigeonhole principle applies. If no individual gets assigned the number 99, then the 100 people are each assigned one of the 99 numbers 0 through 98, and again we may apply the pigeonhole principle. In any case, two people must have the same number, which means that they know the same number of participants.

Example 2: Let $S = \{2, 3, 5, 7, 11, 13, 17, 19\}$ be the set of prime numbers less than 20. Show that there are four non-empty subsets of S with the same sum.

Solution: We will set up this problem by taking the possible sums to be the 'holes' and the various subsets to be the 'pigeons'. Since S has 8 elements, there are $2^8 - 1 = 255$ non-empty subsets of S. The sums which are possible for non-empty sets lie between 2 (corresponding to the subset $\{2\}$) and 77 (the sum of all the elements of S), for a total of 76 possible values. Since $255 = 3 \cdot 76 + 27$, the general version of the pigeonhole principle applies here with $k = 3$; namely, there must be a sum which corresponds to at least 4 subsets.

Here is an old favourite of mine that completely stumped me once when I was a student. (I wasn't thinking of the pigeonhole principle at the time.)

Example 3: Suppose that each square of a 3×7 chessboard is painted red or black at random. Show that the board must contain a rectangle whose four corner squares are all coloured the same.

Solution: At first glance, this does not look like the sort of problem where the pigeonhole principle would help. The squares look like boxes, but where are the pigeons? You may be tempted, as I was, to start working out the possibilities.

But wait a moment! Let's get a little more creative. Look at the columns of the board. There are 7 columns each containing 3 squares. No matter how the board is painted, each column must contain some pair of squares of the same colour, because there are 3 squares per column and only 2 colours. (We are applying the pigeon-hole principle with the 3 squares as the pigeons and the 2 colours as the holes.)

Now, in order for a rectangle to have its four corners coloured the same colour, there must be two different columns in which squares of the same colour are placed in the same two rows. For example, we might have black squares in rows 1 and 2 in two different columns i and j.

This leads us to consider the possible ways of placing pairs of squares of the same colour in a column of 3 squares. There are $\binom{3}{2} = 3$ ways to place a pair of black squares, and the same number of ways to place a pair of red squares. Thus, there are $2\binom{3}{2} = 6$ ways altogether. Since there are 7 columns in the board, there must be (at least) 2 different columns in which a pair of squares of the same colour are placed in the same way. Here we are applying the pigeonhole principle with the columns as the pigeons and the possible ways of placing a pair of like-coloured squares in a column as the holes. Now that's subtle! But it leads to the desired conclusion: the board must contain a rectangle whose four corner squares are painted with the same colour.

Problems for further study:

I now offer you a few problems to try out. Remember to keep in mind the idea of distributing things. Be on the lookout for the 'pigeons' you're trying to distribute and the 'holes' into which they are going. Identifying these may require a little creativity on your part. Good luck! Feel free to contact me for further discussion of your solutions (`jeff.hooper@acadiau.ca`).

1. Suppose we distribute 5 points in the interior of a square S of side length 2. Prove that some pair of these points must have distance less than $\sqrt{2}$.

2. Take any set A consisting of 10 natural numbers between 1 and 99. Show that there must be two disjoint subsets of the set A which have the same sum.

3. Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Show that there must be two distinct integers in A which sum to 104.

4. Suppose that 5 points are placed randomly on a sphere. Show that there must be a hemisphere which contains at least 4 of them.

5. Let x be any real number, and let $A = \{x, 2x, 3x, 4x, \ldots, (n-1)x\}$. Show that there must be at least one number in the set A which differs from an integer by at most $1/n$.

6. Suppose that k colours are available to paint the squares of a $(k+1) \times n$ chessboard. What is the largest value of n, in terms of k, for which the board can be painted in such a way that there is no rectangle whose four corner squares have the same colour?

THE OLYMPIAD CORNER

No. 261

R.E. Woodrow

We begin this number with the problems of the XX Olimpiadi Italiane della Matematica. My thanks go to Christopher Small, Canadian Team Leader to the IMO in Athens, Greece for collecting them for our use.

XX OLIMPIADI ITALIANE DELLA MATEMATICA

Cesenatico, 7 May 2004

1. Reading the temperatures in Cesenatico for the months of December and January, Stefano notices an odd feature: on each day in that period, except for the first and the last, the lowest temperature was the sum of the lowest temperatures on the day before and the day after.

The lowest temperature was 5°C on December 3 and 2°C on January 31. Find the lowest temperature on December 25.

2. Let r and s be two parallel lines in the plane, and P and Q two points such that $P \in r$ and $Q \in s$. Consider circles C_P and C_Q such that C_P is tangent to r at P, C_Q is tangent to s at Q, and C_P and C_Q are tangent externally to each other at some point, say T. Find the locus of T when (C_P, C_Q) varies over all pairs of circles with the given properties.

3. (a) Determine whether the number 2005^{2004} can be written as the sum of the squares of two positive integers.

(b) Determine whether the number 2004^{2005} can be written as the sum of the squares of two positive integers.

4. Antonio and Bernardo play the following game: In the beginning there are two piles of tokens, one with m tokens and the other with n tokens. Each player in turn chooses one of the following moves:

- remove one token from one pile;
- remove one token from each of the two piles;
- move one token from one pile to the other.

The player with no possible moves loses.

Antonio always moves first. Depending on m and n, determine whether one of the two players has a winning strategy, and, if so, show who is the winning player.

5. Determine whether the following statement is true or false: For every sequence $x_1, x_2, x_3, \ldots$ of non-negative real numbers, there exist two sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ of non-negative real numbers such that

- $x_n = a_n + b_n$ for each n;
- $a_1 + \cdots + a_m \le m$ for infinitely many m; and
- $b_1 + \cdots + b_l \le \ell$ for infinitely many ℓ.

6. Let P be a point inside the triangle ABC. Say that the lines AP, BP, and CP meet the sides of ABC at A', B', and C', respectively. Let

$$x = \frac{AP}{PA'}, \quad y = \frac{BP}{PB'}, \quad z = \frac{CP}{PC'}.$$

Prove that $xyz = x + y + z + 2$.

Next we give the First and Second Papers of the Seventeenth Irish Mathematical Olympiad given in May 2004. Thanks again to Christopher Small, Canadian Team Leader to the IMO in Athens, for collecting them for our use.

17th IRISH MATHEMATICAL OLYMPIAD
First Paper — May 8, 2004

1. (a) For which positive integers n does $2n$ divide the sum of the first n positive integers?

(b) Determine, with proof, those positive integers n (if any) which have the property that $2n + 1$ divides the sum of the first n positive integers.

2. Each of the players in a tennis tournament played one match against each of the others. If every player won at least one match, show that there is a group A, B, C of three players for which A beat B, B beat C, and C beat A.

3. Let AB be a chord of length 6 of a circle of radius 5 centred at O. Let $PQRS$ denote the square inscribed in the sector OAB such that P is on the radius OA, S is on the radius OB, and Q and R are points on the arc of the circle between A and B. Find the area of $PQRS$.

4. Prove that there are only two real numbers x such that

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6) = 720.$$

5. Let $a, b \ge 0$. Prove that

$$\sqrt{2}\left(\sqrt{a(a+b)^3} + b\sqrt{a^2+b^2}\right) \le 3(a^2+b^2),$$

with equality if and only if $a = b$.

17^{th} IRISH MATHEMATICAL OLYMPIAD
Second Paper — May 8, 2004

1. Determine all pairs of prime numbers (p, q), with $2 \leq p, q < 100$, such that $p+6$, $p+10$, $q+4$, $q+10$, and $p+q+1$ are all prime numbers.

2. Let A and B be distinct points on a circle T. Let C be a point distinct from B such that $|AB| = |AC|$ and such that BC is tangent to T at B. Suppose that the bisector of $\angle ABC$ meets AC at a point D inside T. Show that $\angle ABC > 72^\circ$.

3. Suppose n is an integer ≥ 2. Determine the first digit after the decimal point in the decimal expansion of the number $\sqrt[3]{n^3+2n^2+n}$.

4. Define the function m of the three real variables x, y, and z by

$$m(x,y,z) \;=\; \max\{x^2,\, y^2,\, z^2\}\,.$$

Determine, with proof, the minimum value of m if x, y, and z vary in $\mathbb{R}$ subject to the restrictions $x+y+z=0$ and $x^2+y^2+z^2=1$.

5. Let p and q be distinct primes and let S be a subset of $\{1, 2, \ldots, p-1\}$. Let $N(S)$ denote the number of solutions of the equation

$$\sum_{i=1}^{q} x_i \;\equiv\; 0 \pmod{p}\;,$$

where $x_i \in S$, $i = 1, 2, \ldots, q$. Prove that $N(S)$ is a multiple of q.

Our last set of problems is the IMO Squad Selection Problems 2004 from the New Zealand Mathematical Olympiad. Thanks to Christopher Small, Canadian Team Leader to the IMO in Athens, for obtaining them for us.

NEW ZEALAND MATHEMATICAL OLYMPIAD
IMO Squad Selection Problems 2004

1. Let I be the incentre of triangle ABC, and let A', B', and C' be the reflections of I in BC, CA, and AB, respectively. The circle through A', B', and C' passes also through B. Find the angle $\angle ABC$.

2. Two players are taking turns to write integers on the blackboard in the range from 1 to 1000. The first player starts by writing the number 1. If the number a was already written on the board (please note that the numbers written at early stages are not erased), then the next number may be either $a+1$ or $2a$, provided that the last number does not exceed 1000. The player who writes 1000 wins. Which player, the first or the second, has a winning strategy?

3. For positive x_1, x_2, y_1, y_2, prove the inequality

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} \geq \frac{(x_1+x_2)^2}{y_1+y_2}.$$

4. For each positive integer n, let $w(n)$ be the number of positive prime divisors of n. Find the smallest positive integer k such that for all n

$$2^{w(n)} \leq k\sqrt[4]{n}.$$

5. Let I be the incentre of triangle ABC. Let points $A_1 \neq A_2$ lie on the line BC, points $B_1 \neq B_2$ lie on the line AC, and points $C_1 \neq C_2$ lie on the line AB so that $AI = A_1I = A_2I$, $BI = B_1I = B_2I$, $CI = C_1I = C_2I$. Prove that $A_1A_2 + B_1B_2 + C_1C_2 = P$, where P is the perimeter of $\triangle ABC$.

6. On each cell of a square 9×9 grid there is a trained beetle. Upon a whistle, each beetle moves to one of the neighbouring cells having a vertex but not an edge in common with the beetle's previous cell. The result is that some cells become empty and in some cells there are now several beetles. Find the minimal possible number of empty cells.

7. A function $f(x)$ is defined on the interval $[0,1]$, so that $f(0) = f(1) = 0$ and

$$f\left(\frac{a+b}{2}\right) \leq f(a) + f(b).$$

for all a and b from $[0,1]$.

(a) Show that the equation $f(x) = 0$ has infinitely many solutions on $[0,1]$.

(b) Are there functions on $[0,1]$ which satisfy the above conditions but are not identically zero?

8. Prove that any prime number $2^{2^n}+1$ cannot be represented as a difference of two fifth powers of integers.

I want to apologize for overlooking some solutions from Michel Bataille, Rouen, France, which were lost in my filing system. Bataille's name should have been added to the list of solvers for the following problems:

- Yugoslav Qualification 2nd Round, Problem 1 [2005 : 374; 2006 : 507];
- 27ième Olympiade Belge, Problem 4 [2005 : 375 ; 2006 : 509] ;
- Bosnia and Herzegovina, National Olympiad Selection Test, Problems 2 and 4 [2005 : 436; 2007 : 22];
- 15th Irish Olympiad, Problems 1, 4, 5, 8, and 9 [2005 : 437–439; 2007 : 28, 30, 33].

Now we turn to solutions from our readers to problems of the 10th Grade Romanian Mathematical Olympiad given [2006 : 85–86].

1. Let $OABC$ be a tetrahedron such that $OA \perp OB \perp OC \perp OA$, let r be the radius of its inscribed sphere, and let H be the orthocentre of triangle ABC. Prove that $OH \le r\big(\sqrt{3}+1\big)$.

Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

Let Δ_A, Δ_B, Δ_C, and Δ_O be the area of triangles OBC, OAC, OAB, and ABC, respectively. Taking into account that triangles OBC, OAC, and OAB are projections of ABC in mutually orthogonal directions, we have

$$\Delta_O^2 \;=\; \Delta_A^2+\Delta_B^2+\Delta_C^2\,.$$

Applying the AM–QM Inequality, we get

$$\Delta_O^2 \;=\; \Delta_A^2+\Delta_B^2+\Delta_C^2 \;\ge\; \tfrac{1}{3}(\Delta_A+\Delta_B+\Delta_C)^2\,,$$

and hence,

$$\frac{\Delta_A+\Delta_B+\Delta_C}{\Delta_O} \;\le \sqrt{3}\,. \tag{1}$$

Since OH is perpendicular to triangle ABC, the volume of the tetrahedron $OABC$ is

$$\tfrac{1}{3}OH\cdot\Delta_O \;=\; \tfrac{1}{3}r(\Delta_A+\Delta_B+\Delta_C+\Delta_O)\,,$$

from which we get

$$OH \;=\; r\,\frac{\Delta_A+\Delta_B+\Delta_C+\Delta_O}{\Delta_O} \;=\; r\left(1+\frac{\Delta_A+\Delta_B+\Delta_C}{\Delta_O}\right)\,.$$

Finally, using (1), we obtain $OH \le r\big(1+\sqrt{3}\big)$.

2. The complex numbers z_1, z_2, $\dots$, z_5 have the same non-zero modulus, and $\sum\limits_{i=1}^{5} z_i = \sum\limits_{i=1}^{5} z_i^2 = 0$. Prove that z_1, z_2, $\dots$, z_5 are the complex coordinates of the vertices of a regular pentagon.

Solved by Michel Bataille, Rouen, France; and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain. We give Bataille's solution.

Let r be the common modulus of z_1, z_2, $\dots$, z_5 and $p = z_1\cdot z_2\cdots z_5$ be their product. Note that $|p| = r^5 \ne 0$.

The complex numbers z_1, z_2, $\dots$, z_5 are the roots of the polynomial

$$\prod_{k=1}^{5}(z-z_k) \;=\; z^5-\sigma_1z^4+\sigma_2z^3-\sigma_3z^2+\sigma_4z-p\,,$$

where σ_k ($k = 1, 2, 3, 4$) are the usual symmetric functions of the roots $z_1, z_2, \ldots, z_5$. By hypothesis, $\sigma_1 = 0$. Since $2\sigma_2 = \sigma_1^2 - \sum_{i=1}^{5} z_i^2$, we also have $\sigma_2 = 0$. Using $z_i \cdot \overline{z_i} = r^2$, we compute:

$$\sigma_3 = \sum z_i z_j z_k = \sum_{i<j} \frac{p}{z_i z_j} = \frac{p}{r^4} \sum_{i<j} \overline{z_i z_j} = \frac{p}{r^4} \cdot \overline{\sigma_2} = 0$$

$$\sigma_4 = \sum z_i z_j z_k z_l = \sum_{i=1}^{5} \frac{p}{z_i} = \frac{p}{r^2} \sum_{i=1}^{5} \overline{z_i} = \frac{p}{r^2} \cdot \overline{\sigma_1} = 0\,.$$

It follows that $z_1, z_2, \ldots, z_5$ are the fifth roots of the non-zero complex number p and, as such, are the complex coordinates of the vertices of a regular pentagon.

The next block of solutions are for problems of the 15th Korean Mathematical Olympiad appearing [2006 : 86–87].

1. The computers in a computer lab are connected by cables as follows: Each computer is directly connected to exactly three other computers via cables. There is at most one cable joining two computers and any pair of computers in the lab can exchange data. (Two computers A and B can exchange data if there exists a sequence of computers starting from A and ending at B in which two computers next to each other in the sequence are directly joined by a cable.)

Let k be the smallest number of computers in the lab whose removal results in leaving just one computer in the lab or a pair of computers not able to exchange data any more. Let ℓ be the smallest number of cables whose deletion results in the existence of two computers that cannot exchange data any more. Show that $k = \ell$.

Solution by Joan P. Hutchison, Macalester College, St. Paul, Minnesota, USA.

In the language of graph theory, this problem asserts that vertex-connectivity k equals edge-connectivity l in a connected 3–regular graph G. Three-regularity implies that $l \leq 3$. Further, $k \leq l$ because if a set of edges disconnects the graph, then there is a set of vertices chosen one from each edge that disconnects or leaves one vertex. Therefore, we have equality if $k = 3$ and need only consider $k \in \{1, 2\}$.

Suppose $k = 1$. Then there is a vertex v whose removal leaves two or three components. Because v has degree 3, there must be an edge incident with v whose removal disconnects the graph.

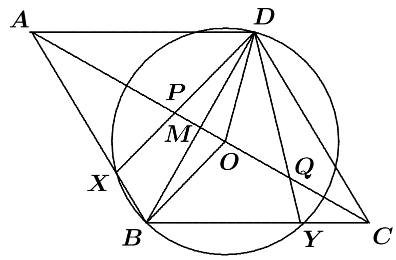
If $k = 2$, suppose the removal of vertices v and w disconnects the graph. If v is adjacent to w, then there are four edges joining $\{v, w\}$ to the rest of the graph. Since $k = 2$, $G\backslash\{v, w\}$ has two components, each with two edges to $\{v, w\}$. Either pair of edges disconnects the graph.

If v is not adjacent to w, then there are two or three components in $G\backslash\{v, w\}$. If three, there are six edges joining $\{v, w\}$ to the three components and they must be divided 2, 2, and 2. Each such pair of edges separates the graph. If there are two components A and B, then we may assume one edge at v connects to A, while the other two connect to B. Similarly, the edges incident with w are divided one and two. The removal of the two singleton edges disconnects the graph.

2. Let $ABCD$ be a rhombus with $\angle A < 90^\circ$. Let its two diagonals AC and BD meet at a point M. A point O on the line segment MC is selected such that $O \neq M$ and $OB < OC$. The circle centred at O passing through points B and D meets the line AB at point B and a point X (where $X = B$ when the line AB is tangent to the circle) and meets the line BC at point B and a point Y. Let the lines DX and DY meet the line segment AC at P and Q, respectively. Express the value of $\frac{OQ}{OP}$ in terms of t when $\frac{MA}{MO} = t$.

Solution by Mohammed Aassila, Strasbourg, France.

Since the quadrilateral $DXBY$ is cyclic, we have



$$\begin{aligned} \angle AXD &= \angle BYD \\ &= \angle BOA = \angle BOP; \end{aligned}$$

hence, quadrilateral $BOPX$ is cyclic. Consider the inversion I of pole O and power OB^2. Then the circle circumscribing $BOPX$ maps to the line AB. Thus, $I(P) = A$, which implies that $OP \cdot OA = OB^2$.

Similarly, we obtain $OQ \cdot OC = OB^2$. Therefore,

$$\frac{OQ}{OP} = \frac{OA}{OC} = \frac{MA + MO}{MA - MO} = \frac{\frac{MA}{MO} + 1}{\frac{MA}{MO} - 1} = \frac{t+1}{t-1}\,.$$

4. Suppose that the incircle of $\triangle ABC$ is tangent to the sides AB, BC, CA at points P, Q, R, respectively. Prove the following inequality:

$$\frac{BC}{PQ} + \frac{CA}{QR} + \frac{AB}{RP} \geq 6\,.$$

Solved by Mohammed Aassila, Strasbourg, France; and Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. We present the solution of Amengual Covas.

We use the standard notation a, b, c for the sides of triangle ABC, and s for the semiperimeter.

We have

$$\begin{aligned} PQ &= 2(s-b)\sin\frac{B}{2} = 2(s-b)\sqrt{\frac{(s-c)(s-a)}{ca}}\,, \\ QR &= 2(s-c)\sin\frac{C}{2} = 2(s-c)\sqrt{\frac{(s-a)(s-b)}{ab}}\,, \\ RP &= 2(s-a)\sin\frac{A}{2} = 2(s-a)\sqrt{\frac{(s-b)(s-c)}{bc}}\,. \end{aligned}$$

We apply the AM–GM Inequality and use the above expressions for PQ, QR, and RP to get

$$\frac{BC}{PQ}+\frac{CA}{QR}+\frac{AB}{RP} \geq 3\cdot\sqrt[3]{\frac{BC\cdot CA\cdot AB}{PQ\cdot QR\cdot RP}} = 3\cdot\sqrt[3]{\frac{(abc)^2}{8[(s-a)(s-b)(s-c)]^2}}\,.$$

Finally, we use the well-known inequality $abc \geq 8(s-a)(s-b)(s-c)$, which is equivalent to Euler's Inequality, to obtain

$$\frac{BC}{PQ}+\frac{CA}{QR}+\frac{AB}{RP} \geq 3\sqrt[3]{8} = 6\,.$$

Equality occurs only if $\triangle ABC$ is equilateral.

5. Answer the following where m is a positive integer.

(a) Prove that if $2^{m+1}+1$ divides $3^{2^m}+1$, then $2^{m+1}+1$ is a prime.

(b) Is the converse of (a) true?

Comment by Mohammed Aassila, Strasbourg, France.

Part (a) and its converse constitute Pepin's Test of primality for Fermat numbers (1877).

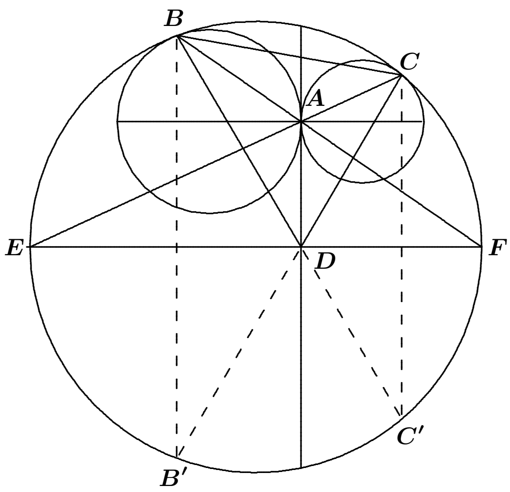
The next solutions are to problems of the X National Mathematical Olympiad of Turkey, given [2006 : 87–88].

2. Two circles are externally tangent to each other at a point A and internally tangent to a third circle Γ at points B and C. Let D be the mid-point of the secant of Γ which is tangent to the smaller circles at A. Show that A is the incentre of the triangle BCD if the centres of the circles are not collinear.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Let E and F be the end-points of the diameter of Γ which is perpendicular to AD. Since the perpendicular bisector of a chord of a circle is a diameter of this circle, EF passes through D. The diameters of the smaller circles through A are perpendicular to AD and hence are parallel to EF.

Therefore points B, A, and F are collinear, and C, A, and E are collinear. (*Proof:* (1) Proposition 1 of the book of Lemmas of Archimedes states: "two circles touch at P and if TU, VW be parallel diameters in them, PUW is a right line". (2) Points A and B are corresponding points in the inversion centred at F with the power of the inversion equal to $FD \cdot FE$, and points A and C are corresponding points in the inversion centred at F with the power of the inversion equal to $ED \cdot EF$.)



Let B' and C' be the reflections of B and C, respectively, across EF. Then B' and C' lie on Γ. Since $BB'C'C$ is an isosceles trapezoid, BC' and $B'C$ intersect at D. Thus,

$$\begin{aligned} \angle CBA &= \angle CBF = \angle CEF = \angle FBC' = \angle ABD \\ \text{and} \quad \angle ACB &= \angle ECB = \angle EFB = \angle B'CE = \angle DCA, \end{aligned}$$

making A the incentre of triangle ABC.

5. In an acute triangle ABC with $|BC| < |AC| < |AB|$, the points D on side AB and E on side AC satisfy the condition $|BD| = |BC| = |CE|$. Show that the circumradius of the triangle ADE is equal to the distance between the incentre and the circumcentre of the triangle ABC.

Solved by Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Smeenk's solution.

Let O be the circumcentre of $\triangle ABC$ and I its incentre. Let the projections of O and I onto AC be O_2 and I_2, respectively, and let the projections of O and I onto AB be O_3 and I_3, respectively. Let the projections of O onto II_2 and II_3 be D_1 and E_1, respectively.

We have

$OD_1 = O_2I_2 = \frac{1}{2}(c-a) = \frac{1}{2}AD$

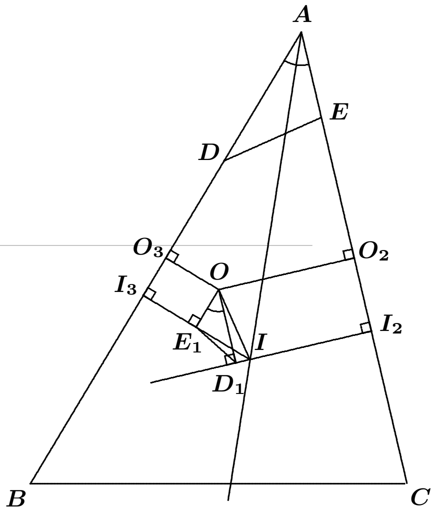
and

$OE_1 = O_3I_3 = \frac{1}{2}(b-a) = \frac{1}{2}AE$.

Also, $\angle D_1OE_1 = \angle DAE$. Hence, $\triangle OD_1E_1$ is similar to $\triangle ADE$, and the scale factor of the similarity is $\frac{1}{2}$. Since

$$\angle OD_1I = \angle OE_1I = 90^\circ,$$

we see that OI is the diameter of the circumcircle of $\triangle OD_1E_1$. Then OI is the circumradius of $\triangle ADE$.



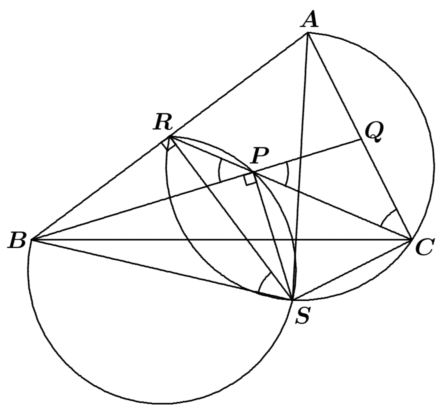
Remark. The points O, E_1, D_1, and I are concyclic if and only if $\angle OIE_1 = \angle OD_1E_1 = \angle ADE$. Since $IE_1 \perp AB$, it follows that $OI \perp DE$.

Next we turn to readers' solutions to problems given in the April 2007 number of the *Corner* and the Japan Mathematical Olympiad 2003 given at [2006 : 149–150].

1. A point P lies in a triangle ABC. The edge AC meets the line BP at Q, and AB meets CP at R. Suppose that $AR = RB = CP$ and $CQ = PQ$. Find $\angle BRC$.

Solved by Mohammed Aassila, Strasbourg, France; and Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. We give the solution of Amengual Covas.

Let S be the second point of intersection of the circumcircles of triangles BPR and RCA.



Since B, S, P, and R are concyclic, we have $\angle BSR = \angle BPR$; since A, R, S, and C are concyclic, $\angle RCA = \angle RSA$. Then

$$\angle BSR \;=\; \angle BPR \;=\; \angle QPC \;=\; \angle PCQ \;=\; \angle RCA \;=\; RSA\,.$$

Thus, SR bisects $\angle BSA$. Since $BR = RA$, the angle bisector theorem gives us $BS = SA$. Consequently, $\angle BRS = 90^\circ$. Then $\angle BPS = 90^\circ$ (because B, S, P, and R are concyclic).

We have

$$\begin{aligned} \angle CPS &= 90^\circ - \angle QPC \;=\; 90^\circ - \angle BSR \;=\; \angle RBS = \angle ABS \\ \text{and}\quad \angle SCP &= \angle SCR \;=\; \angle SAR \;=\; \angle SAB\,. \end{aligned}$$

Therefore, triangles ABS and CPS are similar. Since $\triangle ABS$ is isosceles with $BS = SA$, it follows that $\triangle CPS$ is isosceles with $PS = SC$. Also,

$$\frac{PS}{BS} \;=\; \frac{CP}{AB} \;=\; \frac{\frac{1}{2}AB}{AB} \;=\; \frac{1}{2}\,,$$

making $\angle SBP = 30^\circ$ in right triangle PBS. Therefore,

$$\begin{aligned} \angle BRC &= \angle BRS + \angle SRC \;=\; \angle BRS + \angle SRP \\ &= \angle BRS + \angle SBP \;=\; 90^\circ + 30^\circ \;=\; 120^\circ\,. \end{aligned}$$

3. Find the greatest real number k such that, for any positive a, b, c with $a^2 > bc$,

$$(a^2 - bc)^2 \;>\; k(b^2 - ca)(c^2 - ab)\,.$$

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's solution.

The greatest k is 4.

First suppose that $(a^2 - bc)^2 \;>\; k(b^2 - ca)(c^2 - ab)$ whenever a, b, $c > 0$ and $a^2 > bc$. Let $t \in (0,1)$. Since $1^2 > t \cdot t$, we have

$$(1 - t^2)^2 \;>\; k(t^2 - t)(t^2 - t)\,,$$

from which we deduce that

$$\left(\frac{1+t}{t}\right)^2 \;>\; k\,.$$

It follows that

$$k \;\le\; \lim_{t\to 1}\left(\frac{1+t}{t}\right)^2 \;=\; 4\,.$$

Now we will show that $(a^2 - bc)^2 > 4(b^2 - ca)(c^2 - ab)$ whenever a, b, $c > 0$ and $a^2 > bc$. Assume on the contrary that

$$(a^2 - bc)^2 \;\le\; 4(b^2 - ca)(c^2 - ab) \tag{1}$$

for some positive a, b, c such that $a^2 > bc$, and define

$$f(x) \;=\; (b^2 - ca)x^2 + (a^2 - bc)x + (c^2 - ab)\,.$$

From (1), either $f(x) \geq 0$ for all real x or $f(x) \leq 0$ for all real x. Actually, the former holds since $f(1) = a^2 + b^2 + c^2 - ab - bc - ca > 0$ (note that $a = b = c$ is excluded by $a^2 > bc$, and so $a^2 + b^2 + c^2 > ab + bc + ca$).

It follows that $b^2 - ca$ is positive. Now, write

$$f(x) \;=\; (bx - c)^2 - ag(x) - x(a^2 - bc)\,,$$

where $g(x) = cx^2 - 2ax + b$. Since $a^2 - bc > 0$ and

$$g\left(\frac{c}{b}\right) \;=\; \frac{c(c^2 - ab) + b(b^2 - ac)}{b^2} \;>\; 0$$

(since $c^2 - ab$ has the same sign as $b^2 - ca$ by (1)), we have $f\left(\frac{c}{b}\right) < 0$, a contradiction. This completes the proof.

Next we look at solutions to problems of the Hungarian Mathematical Olympiad 2002–2003 First Round given at [2006 : 150].

1. A rectangular brick has volume $V = x$ cm^3, and surface area $S = y$ cm^2. Find the minimal volume for which $x = 10y$.

Solution by Houda Anoun, Bordeaux, France.

Let a, b, and c be the dimensions of the brick. Then $x = abc$ and $y = 2a^2 + 2b^2 + 2c^2$. By the AM–GM Inequality, we have

$$x^{2/3} \;=\; (a^2b^2c^2)^{1/3} \;\leq\; \frac{a^2 + b^2 + c^2}{3} \;=\; \frac{y}{6}\,.$$

When $x = 10y$, we get $x^{2/3} \leq \frac{x}{60}$; that is, $x \geq 60^3 = 216000$. Moreover, when $a = b = c = 60$ we have $x = 10y = 216000$. Hence, the minimal volume for which $x = 10y$ is 216000.

3. Let ABC be a triangle. We drop a perpendicular from A to the internal bisectors starting from B and C, their feet being A_1 and A_2. In the same way we define B_1, B_2 and C_1, C_2. Prove that

$$2(A_1A_2 + B_1B_2 + C_1C_2) \;=\; AB + BC + CA\,.$$

Solution by Michel Bataille, Rouen, France.

Let I be the incentre of ΔABC. Then,

$$\begin{aligned} \angle BAA_1 &= \frac{\pi}{2} - \frac{B}{2} = \frac{A+C}{2} > \frac{A}{2} = \angle BAI\,, \\ \angle CAA_2 &= \frac{\pi}{2} - \frac{C}{2} = \frac{A+B}{2} > \frac{A}{2} = \angle CAI\,. \end{aligned}$$

Thus, A_1 and A_2 are on opposite sides of the bisector AI. Moreover,

$$\angle IAA_1 = \frac{\pi}{2} - \angle AIA_1 = \frac{\pi}{2} - \left(\frac{A}{2} + \frac{B}{2}\right) = \frac{C}{2}$$

(since $\angle AIA_1 = \pi - \angle AIB$) and similarly, $\angle IAA_2 = \frac{B}{2}$. It follows that

$$IA_1 = AI \sin\left(\frac{C}{2}\right) \quad \text{and} \quad IA_2 = AI \sin\left(\frac{B}{2}\right)\,. \tag{1}$$

With the familiar notations $a = BC$, $b = CA$, and $c = AB$, we also have

$$AA_1 = c \sin\left(\frac{B}{2}\right) \quad \text{and} \quad AA_2 = b \sin\left(\frac{C}{2}\right)\,. \tag{2}$$

Now, observe that A, A_2, I, and A_1 are (in this order) on the circle with diameter AI. Ptolemy's Theorem gives $A_1A_2 \cdot AI = IA_2 \cdot AA_1 + IA_1 \cdot AA_2$, from which, using (1) and (2), we get

$$A_1A_2 = c \sin^2\left(\frac{B}{2}\right) + b \sin^2\left(\frac{C}{2}\right)\,.$$

There are analogous results for B_1B_2 and C_1C_2. Now we have

$$\begin{aligned} &2(A_1A_2 + B_1B_2 + C_1C_2) \\ &\quad = 2(a+b)\sin^2\left(\frac{C}{2}\right) + 2(b+c)\sin^2\left(\frac{A}{2}\right) + 2(c+a)\sin^2\left(\frac{B}{2}\right) \\ &\quad = (a+b)(1-\cos C) + (b+c)(1-\cos A) + (c+a)(1 \cos B) \\ &\quad = 2(a+b+c) - \big[(c\cos B + b\cos C) \\ &\qquad\qquad + (a\cos C + c\cos A) + (a\cos B + b\cos A)\big] \\ &\quad = 2(a+b+c) - (a+b+c) = a+b+c = AB + BC + CA\,. \end{aligned}$$

Next we look at solutions for the Hungarian Mathematical Olympiad 2002–2003, Final Round given at [2006 : 151].

2. We colour the vertices of a 2003-gon with red, blue, and green such that neighbours cannot have the same colour. In how many ways can we accomplish this?

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Consider the graph whose vertices are the vertices of the 2003–gon and whose edges are the edges of the 2003–gon. This graph is the cycle $\mathcal{C}_{2003}$, and the problem is to determine the number of its proper colourings.

It is a classical exercise ([1], [2]) to prove that, more generally, the number of proper colourings of $\mathcal{C}_n$ with q colours is

$$P_{\mathcal{C}_n}(q) \;=\; (q-1)^n + (-1)^n(q-1)$$

(called the chromatic polynomial of C_n). In particular, $P_{\mathcal{C}_{2003}}(3) = 2^{2003} - 2$.

References

[1] L. Lovász, *Combinatorial problems and exercises*, North-Holland, exercise 9–39.

[2] I. Tomescu, *Problems in combinatorics and graph theory*, Wiley, exercise 10–16.

3. Let t be a fixed positive integer. Let $f_t(n)$ denote the number of integers k such that $1 \le k \le n$ and $\binom{k}{t}$ is odd. (If $1 \le k < t$, then $\binom{k}{t} = 0$.) Prove that if n is a sufficiently great power of 2, then $\frac{f_t(n)}{n} = \frac{1}{2^r}$, where r is an integer determined by t and independent of n.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Let $n = 2^p$ with $2^p > 2t$. Let $t = \sum\limits_{i=0}^{p} t_i 2^i$, where $t_i \in \{0, 1\}$ for each i. Note that this is the binary expansion of t except that t_i can be zero for some $i = p, p-1, \ldots$. Let r be the number of i such that $t_i = 1$. Then r is the number of 1s in the binary expansion of t; thus, r is clearly independent of n.

For $1 \le k \le n$, let $k = \sum\limits_{i=0}^{p} k_i 2^i$, where $k_i \in \{0, 1\}$ for each i. Recall Lucas' Theorem (see [1]), which states that

$$\binom{k}{t} \;\equiv\; \prod_{i=0}^{p} \binom{k_i}{t_i} \pmod 2 \;.$$

Thus, $\binom{k}{t}$ is odd if and only if $k_i \ge t_i$ for each i. Therefore, $\binom{k}{t}$ is odd if and only if $k_i = 1$ when $t_i = 1$, and k_i can be either 0 or 1 otherwise. It follows from the definition of r that there are exactly 2^{p-r} integers k such that $1 \le k \le n$ and $\binom{k}{t}$ is odd (note that $r \ge 1$ since $t > 0$, which ensures that at least one of the k_is is non-zero, so that $k \ge 1$). Then

$$\frac{f_t(n)}{n} \;=\; \frac{2^{p-r}}{2^p} \;=\; \frac{1}{2^r}\,.$$

References

[1] T. Andreescu, R. Gelca, *Mathematical Olympiad Challenges*, Birkhäuser, p. 84.

To complete this number of the *Corner*, we look at solutions to the 2002 Kürschák Competition given at [2006 : 151].

2. The Fibonacci sequence is defined by the following recursion: $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 2$. Suppose that the positive integers a and b satisfy:

$$\min\left\{\frac{f_n}{f_{n-1}}, \frac{f_{n+1}}{f_n}\right\} \le \frac{a}{b} \le \max\left\{\frac{f_n}{f_{n-1}}, \frac{f_{n+1}}{f_n}\right\}.$$

Prove that $b \ge f_{n+1}$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

The statement is false, as we can see by choosing $n = 2$ and $a = b = 1$. The correct statement is the one with strict inequalities:

$$\min\left\{\frac{f_n}{f_{n-1}}, \frac{f_{n+1}}{f_n}\right\} < \frac{a}{b} < \max\left\{\frac{f_n}{f_{n-1}}, \frac{f_{n+1}}{f_n}\right\}.$$

Lemma 1. Let x, y, z, t, a, and b be positive integers such that $yz - xt = 1$ and $\frac{x}{y} < \frac{a}{b} < \frac{z}{t}$. Then $b \ge y + t$.

Proof: Since $xb < ay$ and all the numbers are integers, we deduce that $xb \le ay - 1$. Similarly, $ta \le bz - 1$. Therefore,

$$txb \le t(ay-1) = tay - t \le (bz-1)y - t = bzy - (y+t),$$

which gives $y + t \le b(yz - tx) = b$. ∎

Lemma 2. For each $n > 1$, we have $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$.

Proof: The proof is by induction on n.

We have $f_3f_1 - f_2^2 = 2 \cdot 1 - 1^2 = (-1)^2$. Hence, the result is true for $n = 2$.

Assume that the result holds for some given $n > 1$. Then

$$\begin{aligned} f_{n+2}f_n - f_{n+1}^2 &= (f_{n+1} + f_n)f_n - f_{n+1}(f_n + f_{n-1}) \\ &= -(f_{n+1}f_{n-1} - f_n^2) = -(-1)^n = (-1)^{n+1}, \end{aligned}$$

which completes the induction. ∎

Now assume that

$$\min\left\{\frac{f_n}{f_{n-1}}, \frac{f_{n+1}}{f_n}\right\} < \frac{a}{b} < \max\left\{\frac{f_n}{f_{n-1}}, \frac{f_{n+1}}{f_n}\right\}.$$

Case 1. n is even.

From Lemma 2, we have $f_{n+1}f_{n-1} - f_n^2 = 1 \geq 0$, and therefore, $\frac{f_{n+1}}{f_n} \geq \frac{f_n}{f_{n-1}}$. Thus,

$$\frac{f_n}{f_{n-1}} < \frac{a}{b} < \frac{f_{n+1}}{f_n}.$$

Then, from Lemma 1, we have $b \geq f_n + f_{n-1} = f_{n+1}$, as desired.

Case 2. n is odd.

Arguing as in Case 1, we obtain

$$\frac{f_{n+1}}{f_n} < \frac{a}{b} < \frac{f_n}{f_{n-1}},$$

and the desired conclusion follows once again from Lemma 1.

3. Prove that one can distribute all the sides and diagonals of a convex 3^n–gon into groups of three segments such that in each group the three segments form a triangle.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We shall prove that the result holds for any convex $3k$–gon, where k is an odd integer.

Let $A_1, \ldots, A_k, B_1, \ldots, B_k, C_1, \ldots, C_k$ be the vertices of the $3k$–gon, in any order. Since k is odd, it follows that 2 is invertible modulo k. Let $\frac{1}{2}$ be its inverse. All subscripts are considered modulo k.

Note that for $1 \leq i < j \leq k$, we have $\frac{1}{2}(i+j) \not\equiv i \pmod{k}$ and $\frac{1}{2}(i+j) \not\equiv j \pmod{k}$. Moreover, for a given i and $m \not\equiv i \pmod{k}$, there exists a unique $j \not\equiv i \pmod{k}$ such that $m \equiv \frac{1}{2}(i+j) \pmod{k}$.

For $1 \leq i \leq k$, we form the triangles $A_iB_iC_i$; for $1 \leq i < j \leq k$, we form the triangles $A_iA_jB_{\frac{1}{2}(i+j)}$, $B_iB_jC_{\frac{1}{2}(i+j)}$, and $C_iC_jA_{\frac{1}{2}(i+j)}$. Now a straightforward verification shows that these triangles use each side and diagonal of the $3k$–gon exactly once, as desired.

The backlog of solutions is now cleared. Please send me your nice solutions and generalizations soon for use in the *Corner*.

BOOK REVIEW

John Grant McLoughlin

Mathematical Journeys
By Peter D. Schumer, published by the Wiley InterScience, 2004.
ISBN 0-471-22066-3, paperback, 199 pages, CDN$70.99.
Reviewed by ***Georg Gunther****, Sir Wilfred Grenfell College, Memorial University of Newfoundland, Corner Brook, NL.*

The most common question that non-mathematicians have for mathematicians is "what is it you guys actually do?" This delightful book provides an answer to this question, illustrating the vastness of the subject, the elegance and beauty of the kinds of reasoning that define mathematics, and the intrinsic fascination of the kinds of problems that have driven the development of this discipline for millennia.

Peter Schumer is an award-winning professor of mathematics at Middlebury College in Vermont. This book grew out of a lecture series given within the context of a math seminar.

Whether you are a professional mathematician, an educator, or a student who is interested and wants to learn more, *Mathematical Journeys* has something of value to offer. The sixteen chapters cover a broad spectrum: number theory, combinatorics, geometry, graph theory—all presented with a deft touch and a clear awareness of an ever-present historical context.

In some of the chapters, the author presents a number of problems and lets the necessary mathematics develop naturally. Here is an example taken from Chapter 2 (*The Green Chicken Contest*): *Show that it is impossible to weight two coins such that the probability of the three outcomes, two heads, a tail and a head, or two tails, are equally likely*. The readers are invited to solve this problem on their own; however, having stated the problem, the author gives a brief discussion of the relevant mathematics (in this case, some elementary probability theory) and then provides an elementary solution.

Other chapters single out and develop one particular mathematical idea. For example, Chapter 6 (*The Harmonic Series . . . and Less*) gives a beautiful introduction to the thorny issues surrounding infinite series and the perplexing questions of convergence and divergence, all developed simply and systematically, with numerous historical references.

All the chapters conclude with a selection of interesting and challenging problems. An appendix provides comments and solutions to these problems. For readers interested in digging deeper, the author has included a brief but comprehensive bibliography.

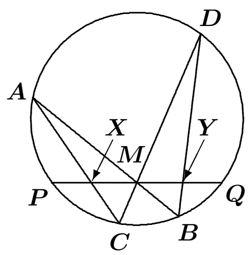
This book should appeal to a wide audience. High-school students should be able to follow the exposition. Teachers at all levels—high school, college, or university—will be able to use this volume as a source of problems or undergraduate research projects. Finally, students wishing to hone their problem-solving skills will find much here to delight them.

Butterfly Metamorphosis

Andy Liu

The Butterfly Theorem is a result which has acquired cult status. For two important surveys, see [1] and [5]. Much of this is later repeated in [3]. The setting of the Butterfly Theorem involves three concurrent chords in a circle.

Butterfly Theorem. Let PQ, AB, and CD be three chords of a circle concurrent at a point M, with A and D on one side of PQ and B and C on the other side. If $PM = QM$, then $XM = YM$, where X and Y are the points of intersection of PQ with AC and BD, respectively.

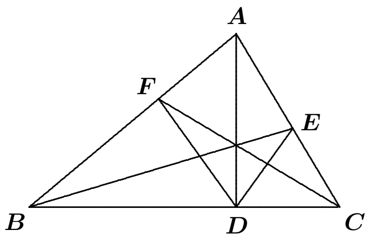


Our metamorphosis changes the setting to three concurrent cevians in a triangle. We will use techniques developed below to give a simple proof of the Butterfly Theorem.

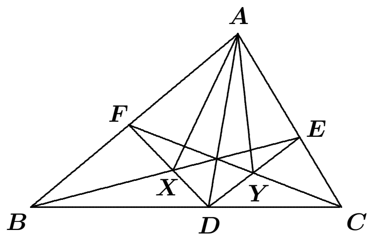
Theorem. Let AD, BE, and CF be three concurrent cevians in $\triangle ABC$.

(a) *First Metamorphosis:* If $\angle ADB = \angle ADC$, then $\angle ADF = \angle ADE$.

(b) *Second Metamorphosis:* If $\angle DAB = \angle DAC$, then $\angle DAX = \angle DAY$, where X is the point of intersection of FD and BE, and Y is the point of intersection of ED and CF.



(a)

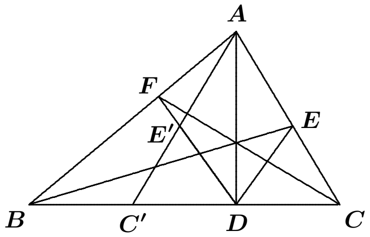


(b)

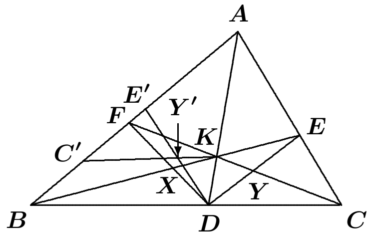
The condition $\angle ADB = \angle ADC$ in part (a) is, of course, just a clumsy way of saying that AD is an altitude. However, stating it this way highlights the relationship of this result to the Butterfly Theorem. This was, for instance, not observed in [4].

Proof: Our approach here is by symmetry.

(a) We fold $\angle BDC$ along its bisector AD, so that the image C' of C lies on BD and the image E' of E lies on AC'. The desired result is now equivalent to D, E', and F being collinear. By Ceva's Theorem, we have $\frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = 1$. Since $\frac{\overline{BD}}{\overline{DC}} = -\frac{\overline{BD}}{\overline{DC'}}$ while $\frac{\overline{CE}}{\overline{EA}} = \frac{\overline{C'E'}}{\overline{E'A}}$, we have $\frac{\overline{BD}}{\overline{DC'}} \cdot \frac{\overline{C'E'}}{\overline{E'A}} \cdot \frac{\overline{AF}}{\overline{FB}} = -1$. By the converse of Menelaus' Theorem, D, E', and F are indeed collinear.



(a)



(b)

(b) Let K be the point of concurrency of AD, BE, and CF. This time, we fold $\angle BAC$ along its bisector AD, so that the image C' of C and the image E' of E lie on AB, while the image Y' of Y is the point of intersection of DE' and KC'. The desired result is now equivalent to A, X, and Y' being collinear. By Menelaus' Theorem, we have

$$\frac{\overline{EK}}{\overline{KB}} \cdot \frac{\overline{BC}}{\overline{CD}} \cdot \frac{\overline{DY}}{\overline{YE}} = -1, \qquad \frac{\overline{BK}}{\overline{KE}} \cdot \frac{\overline{EA}}{\overline{AC}} \cdot \frac{\overline{CD}}{\overline{DB}} = -1,$$

$$\frac{\overline{CK}}{\overline{KF}} \cdot \frac{\overline{FX}}{\overline{XD}} \cdot \frac{\overline{DB}}{\overline{BC}} = -1, \qquad \frac{\overline{FK}}{\overline{KC}} \cdot \frac{\overline{CD}}{\overline{DB}} \cdot \frac{\overline{BA}}{\overline{AF}} = -1.$$

Multiplication and cancellation yields

$$\frac{\overline{FX}}{\overline{XD}} \cdot \frac{\overline{DY}}{\overline{YE}} \cdot \frac{\overline{EA}}{\overline{AC}} \cdot \frac{\overline{CD}}{\overline{DB}} \cdot \frac{\overline{BA}}{\overline{AF}} = 1.$$

Because AD bisects $\angle CAB$, we have $\frac{BA}{AC} = \frac{BD}{DC}$. It follows that

$$\frac{\overline{EA}}{\overline{AC}} \cdot \frac{\overline{CD}}{\overline{DB}} \cdot \frac{\overline{BA}}{\overline{AF}} = -\frac{\overline{E'A}}{\overline{AF}},$$

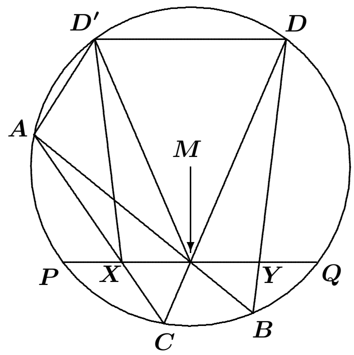
so that $\frac{\overline{FX}}{\overline{XD}} \cdot \frac{\overline{DY'}}{\overline{Y'E'}} \cdot \frac{\overline{E'A}}{\overline{AF}} = -1$. By the converse of Menelaus' Theorem, A, X, and Y' are collinear. ■

The First Metamorphosis later became Problem 5 of the 1994 Canadian Mathematical Olympiad. Our approach is different from all known proofs.

The Second Metamorphosis appeared as Problem 6 in the Spring 2006 Senior Advanced Level Paper of the International Mathematics of the Towns. Our approach is different from the official solution provided.

The approaches we have used so far provide a plausible motivation to perhaps the simplest proof of the Butterfly Theorem. We give the argument in [2] (repeated in [1]) in this light.

We fold PQ along its perpendicular bisector so that the image D' of D is the point of intersection of the circle and the line through D parallel to PQ. What we want to prove is that X coincides with the image Y' of Y. This will follow if we can prove that triangles DMY and $D'MX$ are congruent.



We have $DM = D'M$. Hence,

$$\angle DMY = \angle MDD' = \angle MD'D = \angle D'MX .$$

We will now prove that $\angle MDY = \angle MD'X$. Since $ACBD$ is a cyclic quadrilateral, $\angle MDY = \angle CAB$. We will have $\angle CAB = \angle MD'X$ if we can prove that $AD'MX$ is also a cyclic quadrilateral. Since $ACDD'$ is cyclic, $\angle D'AX + \angle MDD' = 180^\circ$. However, we have already proved that $\angle MDD' = \angle D'MX$, so that $\angle D'AX + \angle D'MX = 180^\circ$ too. Hence, $AD'MX$ is indeed cyclic, and it follows that $MX = MY$.

Acknowledgment. The author would like to thank the anonymous referee for some critical comments and useful references.

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[1] L. Bankoff, The Metamorphosis of the Butterfly Theorem, *Math. Mag.* **60** (1987) 195–210.

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[3] R. Honsberger, The Butterfly Problem and Other Delicacies from the Noble Art of Euclidean Geometry — Part I, *Two-Year Coll. Math. J.* **14** (1983) 2–8. (Journal later renamed *Coll. Math. J.*)

[4] R. Honsberger, The Butterfly Problem and Other Delicacies from the Noble Art of Euclidean Geometry — Part II, *Two-Year Coll. Math. J.* **14** (1983) 154–158. (Journal later renamed *Coll. Math. J.*)

[5] L. Sauvé, The Celebrated Butterfly Problem, *Eureka* **2** (1976) 3–5. (Journal later renamed *Crux Math.*)

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PROBLEMS

Solutions to problems in this issue should arrive no later than **1 October 2007**. *An asterisk* ($\star$) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

3226. *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

Let ABC be a triangle. Let $S = \sum_{\text{cyclic}} \cos\frac{A}{2}$ and $P = \prod_{\text{cyclic}} \cos\frac{A}{2}$. Prove that

(a) $\frac{S}{P} \leq 2\sqrt{3}\max\left\{\sec\frac{A}{2},\ \sec\frac{B}{2},\ \sec\frac{C}{2}\right\}$;

(b) $\frac{S}{P} \geq 4\max\left\{\sec^2\frac{B-C}{4},\ \sec^2\frac{A-B}{4},\ \sec^2\frac{C-A}{4}\right\}$.

3227. *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

Let $\alpha \in [0,1]$ and define

$$x_n = \left(\frac{\zeta(2)+\cdots+\zeta(n+1)}{n}\right)^{n^\alpha},$$

where ζ is the Riemann Zeta Function, defined by $\zeta(k) = \sum_{p=1}^{\infty}\frac{1}{p^k}$. Prove that

$$\lim_{n\to\infty} x_n = \begin{cases} 1, & \text{if } \alpha \in [0,1), \\ e, & \text{if } \alpha = 1. \end{cases}$$

3228. *Proposed by Mihály Bencze, Brasov, Romania.*

For $x \in \left(0, \frac{\pi}{2}\right)$, prove that

$$\frac{(n+1)!}{2\prod_{k=2}^{n}(k+\cos x)} \leq \left(\frac{x}{\sin x}\right)^{n-1} \leq \left(\frac{\pi}{2}\right)^{n-1} \cdot \frac{n!}{\prod_{k=2}^{n}(k+\cos x)}.$$

3229. *Proposed by Mihály Bencze, Brasov, Romania.*

(a) Let x and y be positive real numbers, and let n be a positive integer. Prove that

$$(x+y)^n \sum_{k=0}^{n} \frac{1}{\binom{n}{k} x^{n-k} y^k} \;\geq\; n+1+2\sum_{i=1}^{n}\sum_{k=0}^{n-i} \frac{\binom{n}{k}}{\binom{n}{k+i}} \;\geq\; (n+1)^2 .$$

(b)★ Let x_1, x_2, ..., x_k be positive real numbers, and let n be a positive integer. Determine the minimum value of

$$(x_1+x_2+\cdots+x_k)^n \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\,\ldots,\,i_k\,\geq\,0}} \frac{i_1!i_2!\cdots i_k!}{n!x_1^{i_1}x_2^{i_2}\cdots x_k^{i_k}} .$$

3230. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a, x, and y be positive real numbers. Prove that

$$\left(x^{a+1}+x+y\right)\left(y^{a+1}+y+x\right)\left(x^{a+1}+(x^a+1)y\right)\left(y^{a+1}+(y^a+1)x\right) \geq (xy)^a\left(x+\sqrt{xy}+y\right)^4 .$$

3231★. *Proposed by Ignotus, Tauramena, Casanare, Colombia.*

(a) A flea lives on the real number line at the number 1. One fine day it decides to take an n–day vacation. On the first day it jumps forward one unit landing at the number 2. Thereafter, for the remaining $n-1$ days, it jumps forward a number of units of its choice, as long as the number of units is a proper divisor of the number it is currently visiting. A sample 11–day vacation is

$$1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 15,\ 20,\ 30,\ 45,\ 60.$$

What is the furthest away from home the flea can get during its n–day vacation? Note that the 11–day vacation above does not get the flea as far as possible; here is one that gets the flea further:

$$1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 18,\ 27,\ 36,\ 54,\ 81.$$

(b) Suppose the flea wishes to visit, under the same rules as in (a), a certain number n. What is the least number, $V(n)$, of vacation days it will need to get there? For example, here is a scheme to get the flea to the number 100 in 13 days:

$$1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 18,\ 27,\ 36,\ 54,\ 72,\ 96,\ 100.$$

3232. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let P be a point in the interior of $\angle QOR$. Find the segment AB of minimum length which contains P with A on the ray OQ and B on the ray OR.

3233. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let $A_1A_2A_3$ be a triangle, and let P be an interior point. The cevian A_iP intersects the opposite side at A_i' for $1 \leq i \leq 3$. If $[XYZ]$ denotes the area of triangle XYZ, set $\Delta_1 = [PA_2A_1']$, $\Delta_2 = [PA_3A_2']$, $\Delta_3 = [PA_1A_3']$, and $\Delta = [A_1A_2A_3]$. Find the locus of P if $\Delta_1 + \Delta_2 + \Delta_3 = \frac{1}{2}\Delta$.

3234. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*

Let ABC be an equilateral triangle, and let P be an interior point. The lines AP, BP, and CP intersect the opposite sides at the points A', B', and C', respectively. Determine the position of the point P if

$$AC' + CB' + BA' \;=\; A'C + C'B + B'A\,.$$

3235. *Proposed by Geoffrey A. Kandall, Hamden, CT, USA.*

Let ABC be a triangle, and let A_1, B_1, C_1 be points on the sides BC, CA, AB, respectively, such that

$$\frac{BA_1}{A_1C} \;=\; \frac{CB_1}{B_1A} \;=\; \frac{AC_1}{C_1B} \;=\; k\,.$$

Let $\alpha = AA_1$, $\beta = BB_1$, $\gamma = CC_1$, and $\lambda = \dfrac{k^2+k+1}{(k+1)^2}$. Prove that

(a) $\alpha^2 + \beta^2 + \gamma^2 = \lambda(a^2+b^2+c^2)$;

(b) $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = \lambda^2\big(a^2b^2+b^2c^2+c^2a^2\big)$;

(c) $\alpha^4 + \beta^4 + \gamma^4 = \lambda^2\big(a^4+b^4+c^4\big)$.

3236. *Proposed by Todor Mitev, University of Rousse, Rousse, Bulgaria.*

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \;\leq\; 3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\,.$$

3237. *Proposed by Michel Bataille, Rouen, France.*

Find all integers n such that

$$\frac{7n-12}{2^n}+\frac{2n-14}{3^n}+\frac{24n}{6^n} \;=\; 1\,.$$

3238. *Proposed by Michel Bataille, Rouen, France.*

Let $\mathcal{T} = DBC$ be a triangle with $DB = DC$, and let A be a variable point in the interior of $\mathcal{T}$. The perpendiculars to BC through the mid-points of AB and AC meet DB and DC at P and Q, respectively. Find the locus of A for which P, A, and Q are collinear.

3226. *Proposé par Ovidiu Furdui, étudiant, Western Michigan University, Kalamazoo, MI, É-U.*

Soit un triangle ABC. Soit $S = \sum_{\text{cyclique}} \cos\frac{A}{2}$ et $P = \prod_{\text{cyclique}} \cos\frac{A}{2}$. Montrer que

(a) $\frac{S}{P} \leq 2\sqrt{3}\max\left\{\sec\frac{A}{2},\ \sec\frac{B}{2},\ \sec\frac{C}{2}\right\}$;

(b) $\frac{S}{P} \geq 4\max\left\{\sec^2\frac{B-C}{4},\ \sec^2\frac{A-B}{4},\ \sec^2\frac{C-A}{4}\right\}$.

3227. *Proposé par Ovidiu Furdui, étudiant, Western Michigan University, Kalamazoo, MI, É-U.*

Soit $\alpha \in [0,1]$. On définit

$$x_n = \left(\frac{\zeta(2) + \cdots + \zeta(n+1)}{n}\right)^{n^\alpha},$$

où ζ est la fonction zéta de Riemann, defini par $\zeta(k) = \sum_{p=1}^{\infty} \frac{1}{p^k}$. Montrer que

$$\lim_{n\to\infty} x_n = \begin{cases} 1, & \text{si } \alpha \in [0,1), \\ e, & \text{si } \alpha = 1. \end{cases}$$

3228. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Pour $x \in \left(0, \frac{\pi}{2}\right)$, montrer que

$$\frac{(n+1)!}{2\prod_{k=2}^{n}(k+\cos x)} \leq \left(\frac{x}{\sin x}\right)^{n-1} \leq \left(\frac{\pi}{2}\right)^{n-1} \cdot \frac{n!}{\prod_{k=2}^{n}(k+\cos x)}.$$

3229. *Proposé par Mihály Bencze, Brasov, Roumanie.*

(a) Soit x et y deux nombres réels positifs, et soit n un entier positif. Montrer que

$$(x+y)^n \sum_{k=0}^{n} \frac{1}{\binom{n}{k} x^{n-k} y^k} \geq n + 1 + 2\sum_{i=1}^{n}\sum_{k=0}^{n-i} \frac{\binom{n}{k}}{\binom{n}{k+i}} \geq (n+1)^2.$$

(b)★ Soit $x_1, x_2, \ldots, x_k$ k nombres réels positifs, et soit n un entier positif. Déterminer la valeur minimale de

$$(x_1 + x_2 + \cdots + x_k)^n \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k \geq 0}} \frac{i_1! i_2! \cdots i_k!}{n! x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}}.$$

3230. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a, x et y trois nombres réels positifs. Montrer que

$$(x^{a+1}+x+y)(y^{a+1}+y+x)(x^{a+1}+(x^a+1)y)(y^{a+1}+(y^a+1)x) \geq (xy)^a(x+\sqrt{xy}+y)^4 .$$

3231★. *Proposé par Ignotus, Tauramena, Casanare, La Colombie.*

(a) On imagine une puce vivant sur la droite réelle au nombre 1. Un beau jour, elle décide de prendre n jours de vacances. Pour son premier jour, elle saute vers la droite pour se poser au nombre 2. Pour les $n-1$ jours suivants, elle continue à sauter vers la droite d'un nombre d'unités de son choix, pourvu que ce nombre d'unités soit un diviseur strict du nombre où elle séjourne présentement. Voici un exemple possible pour des vacances de 11 jours :

$$1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 15,\ 20,\ 30,\ 45,\ 60.$$

Quel est l'éloignement maximal que peut atteindre la puce pendant n jours de vacances ? Il faut noter que l'exemple ci-dessus ne répond pas à la question, car voici un meilleur choix, toujours pour 11 jours de vacances :

$$1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 18,\ 27,\ 36,\ 54,\ 81.$$

(b) Supposons que, aux mêmes conditions qu'en (a), la puce veuille visiter un certain nombre n. Quel est le plus petit nombre $V(n)$ de jours de vacances nécessaires pour qu'elle puisse s'y rendre ? Par exemple, voici le plan à suivre pour se rendre au nombre 100 en 13 jours :

$$1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 18,\ 27,\ 36,\ 54,\ 72,\ 96,\ 100.$$

3232. *Proposé par George Tsintsifas, Thessalonique, Grèce.*

Soit P un point intérieur de l'angle QOR. Trouver le segment AB, de longueur minimale, et qui contienne P, avec A sur le rayon OQ et B sur le rayon OR.

3233. *Proposé par George Tsintsifas, Thessalonique, Grèce.*

Soit P un point intérieur d'un triangle $A_1A_2A_3$. La cévienne A_iP coupe le côté opposé en A_i' pour $1 \leq i \leq 3$. Si $[XYZ]$ désigne l'aire du triangle XYZ, posons $\Delta_1 = [PA_2A_1']$, $\Delta_2 = [PA_3A_2']$, $\Delta_3 = [PA_1A_3']$ et $\Delta = [A_1A_2A_3]$. Trouver le lieu des points P tels que $\Delta_1+\Delta_2+\Delta_3 = \frac{1}{2}\Delta$.

3234. *Proposé par George Tsintsifas, Thessalonique, Grèce.*

Soit P un point intérieur d'un triangle équilatéral ABC. Les droites AP, BP et CP coupent respectivement les côtés opposés aux points A', B' et C'. Déterminer la position du point P si on a

$$AC'+CB'+BA' = A'C+C'B+B'A .$$

3235. *Proposé par Geoffrey A. Kandall, Hamden, CT, É-U.*

Soit respectivement A_1, B_1 et C_1 des points sur les côtés BC, CA et AB d'un triangle ABC, de sorte que

$$\frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{AC_1}{C_1B} = k\,.$$

Soit $\alpha = AA_1$, $\beta = BB_1$, $\gamma = CC_1$ et $\lambda = \dfrac{k^2+k+1}{(k+1)^2}$. Montrer que

(a) $\alpha^2 + \beta^2 + \gamma^2 = \lambda(a^2 + b^2 + c^2)$;

(b) $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = \lambda^2\big(a^2b^2 + b^2c^2 + c^2a^2\big)$;

(c) $\alpha^4 + \beta^4 + \gamma^4 = \lambda^2\big(a^4 + b^4 + c^4\big)$.

3236. *Proposé par Todor Mitev, Université de Rousse, Rousse, Bulgarie.*

Soit a, b et c trois nombres réels positifs tels que $abc = 1$. Montrer que

$$a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \le 3 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a}\,.$$

3237. *Proposé par Michel Bataille, Rouen, France.*

Trouver tous les entiers n tels que

$$\frac{7n-12}{2^n} + \frac{2n-14}{3^n} + \frac{24n}{6^n} = 1\,.$$

3238. *Proposé par Michel Bataille, Rouen, France.*

Soit $\mathcal{T} = DBC$ un triangle avec $DB = DC$, et soit A un point variable dans l'intérieur de $\mathcal{T}$. Les perpendiculaires à BC passant par les milieux de AB et de AC coupent respectivement DB et DC en P et Q. Trouver le lieu de A pour lequel P, A et Q sont colinéaires.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologize for omitting the name of Peter Y. Woo, Biola University, La Mirada, CA, USA from the list of solvers of 3102.

3125. [2006 : 111] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let m_a, h_a, and w_a denote the lengths of the median, the altitude, and the internal angle bisector, repectively, to side a in $\triangle ABC$. Define m_b, m_c, h_b, h_c, w_b, and w_c similarly. Let R be circumradius of $\triangle ABC$.

(a) Show that

$$\sum_{\text{cyclic}} \frac{b^2+c^2}{m_a} \leq 12R\,.$$

(b) Show that

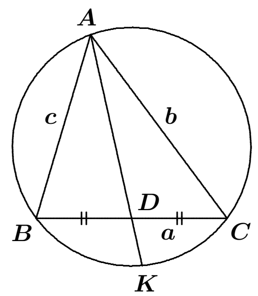
$$\sum_{\text{cyclic}} \frac{b^2+c^2}{h_a} \geq 12R\,.$$

(c)⋆ Determine the range of

$$\frac{1}{R}\sum_{\text{cyclic}} \frac{b^2+c^2}{w_a}\,.$$

Solution by C.R. Pranesachar, Indian Institute of Science, Bangalore, India, modified by the editor.

(a) Let the median through A meet BC at its mid-point D and meet the circumcircle of triangle ABC at the point K, as shown in the diagram. By Apollonius' Theorem,



$$\begin{aligned} b^2+c^2 &= 2(AD^2+BD^2) \\ &= 2(AD^2+BD\cdot DC) \\ &= 2(AD^2+AD\cdot DK) \\ &= 2AD\cdot(AD+DK) \\ &= 2m_a\cdot AK \\ &\leq 2m_a\cdot 2R \;=\; 4Rm_a\,. \end{aligned}$$

Thus, $\frac{b^2+c^2}{m_a} \leq 4R$; whence, $\sum_{\text{cyclic}} \frac{b^2+c^2}{m_a} \leq 3(4R) = 12R$.

(b) Since $h_a = \dfrac{2[ABC]}{a}$ and $R = \dfrac{abc}{4[ABC]}$, we obtain

$$\frac{b^2+c^2}{h_a} \;=\; 4R\left(\frac{b^2+c^2}{2bc}\right) \;\geq\; 4R\,,$$

which implies that $\sum\limits_{\text{cyclic}} \dfrac{b^2+c^2}{h_a} \geq 3(4R) = 12R$.

(c) Let $s_w = \dfrac{1}{R}\sum\limits_{\text{cyclic}} \dfrac{b^2+c^2}{w_a}$. We will prove that the range of s_w is $(4,\infty)$.

Using the formula $w_a = \dfrac{2bc}{b+c}\cdot\cos\dfrac{A}{2}$, we obtain

$$\frac{b^2+c^2}{Rw_a} \;=\; \frac{b+c}{R\cos\frac{A}{2}}\left(\frac{b^2+c^2}{2bc}\right) \;\geq\; \frac{b+c}{R\cos\frac{A}{2}}\,.$$

Since $b = 2R\sin B$ and $c = 2R\sin C$ (by the extended Law of Sines), we get

$$\frac{b^2+c^2}{Rw_a} \;\geq\; \frac{2(\sin B+\sin C)}{\cos\frac{A}{2}} \;=\; \frac{4\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\cos\frac{A}{2}} \;=\; 4\cos\frac{B-C}{2}\,.$$

Hence,

$$s_w \;\geq\; 4\sum_{\text{cyclic}}\cos\frac{B-C}{2}\,.$$

Since $\cos x \geq 1 - 2x/\pi$ for $0 \leq x \leq \pi/2$, we have

$$\sum_{\text{cyclic}}\cos\frac{B-C}{2} \;=\; \sum_{\text{cyclic}}\cos\frac{|B-C|}{2} \;\geq\; \sum_{\text{cyclic}}\left(1-\frac{1}{\pi}|B-C|\right)\,.$$

Without loss of generality, assume that $C \leq B \leq A$. Then

$$\begin{aligned}\sum_{\text{cyclic}}\cos\frac{B-C}{2} \;&\geq\; 3-\frac{1}{\pi}(B-C-C+A+A-B)\\ &=\; 3-\frac{2}{\pi}(A-C) \;>\; 3-\frac{2}{\pi}\pi \;=\; 1\,.\end{aligned}$$

Thus $s_w > 4$.

Taking $a = 2$ and $b = c = x+1$, where $x \in (0,\infty)$, we obtain

$$s_w \;=\; 4+\frac{(x^2+2x+5)(x+3)\sqrt{2x}}{(x+1)^{\frac{5}{2}}}\,.$$

This is a continuous function of x on $(0,\infty)$ and has the limits 4 and ∞ as x tends to 0 and ∞, respectively. Therefore, the range of s_w is $(4,\infty)$.

Also solved by ARKADY ALT, San Jose, CA, USA (parts (a) and (b)); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (parts (a) and (b)); MICHEL BATAILLE, Rouen, France (parts (a) and (b)); FRANCISCO BELLOT ROSADO, *I.B. Emilio Ferrari, Valladolid, Spain* (*parts* (a) *and* (b))*; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain* (*parts* (a), (b), *and* (c))*; DRAGOLJUB MILOŠEVIĆ and G. MILANOVAC, Serbia* (*part* (b) *only*)*; VEDULA N. MURTY, Dover, PA, USA* (*parts* (a) *and* (b))*; JOEL SCHLOSBERG, Bayside, NY, USA* (*part* (b) *only*)*; PETER Y. WOO, Biola University, La Mirada, CA, USA* (*parts* (a) *and* (b))*; LI ZHOU, Polk Community College, Winter Haven, FL, USA* (*parts* (a) *and* (b))*; and the proposer* (*parts* (a) *and* (b))*. There were also three incomplete solutions to part* (c) *of the problem.*

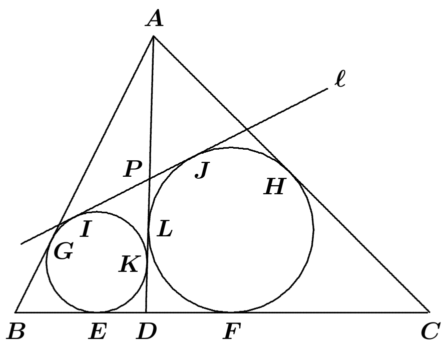
Pranesachar's solution contained some additional detail which has not been included in the modified version above. He proved that the range of the sum $\frac{1}{R}\sum_{\text{cyclic}}\frac{b^2+c^2}{h_a}$ *in part* (a) *is* $(0,12]$ *and the range of* $\frac{1}{R}\sum_{\text{cyclic}}\frac{b^2+c^2}{w_a}$ *in part* (b) *is* $[12,\infty)$.

3126. [2006 : 171, 174; corrected 2006 : 303, 306] *Proposed by Hidetoshi Fukugawa, Kani, Gifu, Japan.*

Let D be any point on the side BC of triangle ABC. Let Γ_1 and Γ_2 be the incircles of $\triangle ABD$ and $\triangle ACD$, respectively. Let ℓ be the common external tangent to Γ_1 and Γ_2 which is different from BC. If P is the point of intersection of AD and ℓ, show that $2AP = AB + AC - BC$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Suppose that Γ_1 is tangent to BC, AB, ℓ, and AD, at E, G, I, and K, respectively, and that Γ_2 is tangent to BC, AC, ℓ, and AD, at F, H, J, and L, respectively. Then,



$$\begin{aligned} & AB + AC - BC \\ &= AG + AH - EF \\ &= AK + AL - IJ \\ &= 2AP + PK + PL - PI - PJ \\ &= 2AP\,. \end{aligned}$$

Also solved by CLAUDIO ARCONCHER, Jundiaí, Brazil; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; JOHN G. HEUVER, Grande Prairie, AB; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; BIN ZHAO, student, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China; and the proposer.

This problem generated many different solutions, but Zhou's was the neatest. Konečný noted that a very similar problem was in the 20th USA Mathematical Olympiad. Three readers did not see the printed correction to the problem and simply pointed out that there was something wrong.

3127. [2004–075] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let H be the foot of the altitude from A to BC, where BC is the longest side of $\triangle ABC$. Let R, R_1, and R_2 be the circumradii of $\triangle ABC$, $\triangle ABH$, and $\triangle ACH$, respectively. Similarly, let r, r_1, r_2 be the inradii of these triangles. Prove that

(a) $R_1^2+R_2^2-R^2$ is positive, negative, or zero according as angle A is acute, obtuse, or right-angled.

(b) $r_1^2+r_2^2-r^2$ is positive, negative, or zero according as angle A is obtuse, acute, or right-angled.

Solution by Michel Bataille, Rouen, France.

(a) Let $\Delta = R_1^2+R_2^2-R^2$. Since ABH and ACH are right triangles, we have $R_1 = \frac{1}{2}c = R\sin C$ and $R_2 = \frac{1}{2}b = R\sin B$ (using the familiar notation for $\triangle ABC$). Then

$$\begin{aligned}\Delta &= R^2(\sin^2 C+\sin^2 B-1) = R^2(\sin^2 B-\cos^2 C)\\ &= R^2\left(\sin B-\sin(\tfrac{\pi}{2}-C)\right)\left(\sin B+\sin(\tfrac{\pi}{2}-C)\right).\end{aligned}$$

Using the identity $(\sin\alpha-\sin\beta)(\sin\alpha+\sin\beta) = \sin(\alpha-\beta)\sin(\alpha+\beta)$, we obtain

$$\Delta = R^2\sin\left(B+C-\tfrac{\pi}{2}\right)\sin\left(B-C+\tfrac{\pi}{2}\right) = R^2\cos A\cos(B-C).$$

Since A is the largest angle of $\triangle ABC$, angles B and C are acute; hence, $\cos(B-C)>0$. Thus, Δ has the same sign as $\cos A$, and the result follows.

(b) Let $\delta = r_1^2+r_2^2-r^2$ and $h = AH$. Since the inradius equals the semiperimeter minus the hypotenuse for right triangles, we calculate

$$\delta = \tfrac{1}{4}(h+HB-c)^2+\tfrac{1}{4}(h+HC-b)^2-r^2.$$

Since H is between B and C, we have $HB+HC=a$. Also, $HB=c\cos B$, $HC=b\cos C$, and $2h^2+HB^2+HC^2=b^2+c^2$; hence,

$$\delta = \tfrac{1}{2}b^2(1-\cos C)+\tfrac{1}{2}c^2(1-\cos B)-\tfrac{1}{2}h(b+c-a)-r^2.$$

Noting that $h=\dfrac{bc\sin A}{a}=\dfrac{bc}{2R}$ and using the Law of Sines, we obtain

$$\begin{aligned}\delta = {}& 2R^2\sin^2 B(1-\cos C)+2R^2\sin^2 C(1-\cos B)\\ &-2R^2\sin B\sin C(\sin B+\sin C-\sin A)-r^2.\end{aligned}$$

Now, using the usual half-angle formulas, along with the identity

$$\sin B+\sin C-\sin A = 4\cos\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C$$

and the known relation $r = 4R\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$, we obtain

$$\begin{aligned}\delta &= 16R^2\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C\left(\cos^2\tfrac{1}{2}B+\cos^2\tfrac{1}{2}C\right.\\ &\qquad \left.-\ 2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C-\sin^2\tfrac{1}{2}A\right)\\ &= 8R^2\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C\left(1+\cos A+\cos B+\cos C\right.\\ &\qquad \left.-\ 4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C\right)\\ &= 8R^2\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C\big(1+\cos A+\cos B+\cos C\\ &\qquad -(\sin A+\sin B+\sin C)\big)\,.\end{aligned}$$

Thus, δ has the same sign as

$$\delta' = (1+\cos A-\sin A)+(\cos B+\cos C)-(\sin B+\sin C)\,.$$

But

$$\begin{aligned}\delta' &= 2\cos\tfrac{1}{2}A\left(\cos\tfrac{1}{2}A-\sin\tfrac{1}{2}A\right)+2\cos\left(\tfrac{1}{2}(B-C)\right)\left(\sin\tfrac{1}{2}A-\cos\tfrac{1}{2}A\right)\\ &= 4\left(\sin\tfrac{1}{2}A-\cos\tfrac{1}{2}A\right)\sin\left(\tfrac{\pi}{4}-\tfrac{1}{2}C\right)\sin\left(\tfrac{\pi}{4}-\tfrac{1}{2}B\right)\end{aligned}$$

has the same sign as $\sin\frac{1}{2}A-\cos\frac{1}{2}A$. The result follows.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOHN G. HEUVER, Grande Prairie, AB; JOEL SCHLOSBERG, Bayside, NY, USA (part (a) *only*)*; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.*

In part (a) *of the solution above, Bataille uses the trigonometric identity*

$$(\sin\alpha-\sin\beta)(\sin\alpha+\sin\beta) = \sin(\alpha-\beta)\sin(\alpha+\beta)\,,$$

which is new to this editor. Perhaps we should not tell our students about this one, lest they jump to the conclusion that there is a universal distributive law $\sin(\alpha+\beta)=\sin\alpha+\sin\beta$ *at work here. Ed Barbeau has made similar observations about this identity in* Fallacies, Flaws, and Flimflam, *Mathematical Association of America, 2000, pages 32–33.*

Three of the solvers deduced that δ' (in the notation of the featured solution above) satisfies $\delta' = (2R+r-s)/R$ and then obtained the desired connection between the size of angle A and the sign of δ' by a suitable reference: Heuver to page 232 of [D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989]*; Janous to item 11.27 in* [O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff Publ., Groningen, 1969]*; and Romero to problem 1088* [1985 : 289; 1987 : 124–125].

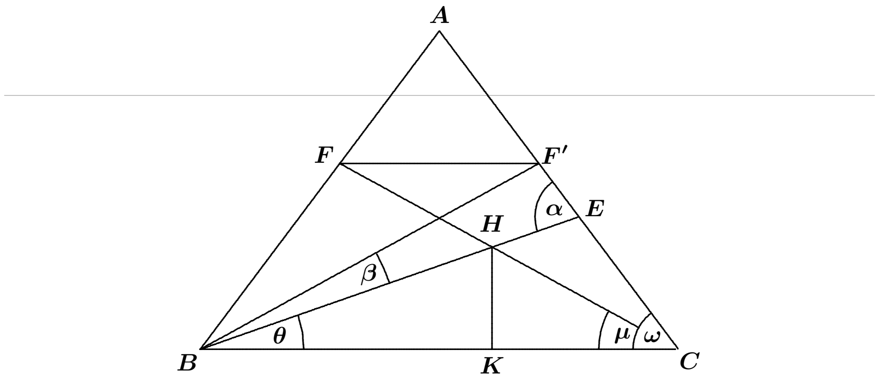
3128. *Proposed by K.R.S. Sastry, Bangalore, India.*

In triangle ABC, we have $AB = AC = 5$, $BC = 6$. Let E be a point on AC and F a point on AB such that $BE = CF$, $\angle EBC \neq \angle FCB$, and $\sin\theta = 5/13$, where $\theta = \angle EBC$. Let H be the point of intersection of BE and CF, and let K be the point on BC such that $HK \perp BC$.

Find the length of HK.

Solution by Roy Barbara, University of Beirut, Beirut, Lebanon.

Let F' be the mirror image of F in the altitude to BC through A. Let $\mu = \angle FCB$, $\omega = \angle ACB$, $\alpha = \angle AEB$, and $\beta = \angle EBF'$. Since $\theta \neq \mu$, we have $F' \neq E$. Note that $\triangle BEF'$ is isosceles, since $BF' = CF = BE$.



From $\sin\theta = 5/13$ and $\cos\omega = 3/5$, we obtain $\tan\theta = 5/12$ and $\tan\omega = 4/3$. We have $\alpha = \theta + \omega$. Thus, $\tan\alpha = \tan(\theta+\omega) = 63/16$. Since $\tan\alpha > 0$, we see that α is an acute angle. Thus, E lies between F' and C (since $\triangle BEF'$ is isosceles). Hence, $\mu = \angle FCB = \angle F'BC = \beta + \theta$.

Next, since $\beta + 2\alpha = \pi$ (summing the angles in $\triangle BEF'$), we calculate

$$\tan\beta \;=\; \tan(\pi - 2\alpha) \;=\; -\tan(2\alpha) \;=\; \frac{2016}{3713}\,.$$

We then deduce that

$$\tan\mu \;=\; \tan(\beta+\theta) \;=\; \frac{253}{204}\,.$$

Since $\tan\theta = \dfrac{HK}{BK}$, $\tan\mu = \dfrac{HK}{KC}$, and $BK + KC = BC = 6$, we obtain

$$HK \;=\; \frac{6\tan\theta\tan\mu}{\tan\theta + \tan\mu} \;=\; \frac{1265}{676}\,.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; TAICHI MAEKAWA, Takatsuki City, Osaka, Japan; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; JOEL SCHLOSBERG, Bayside, NY, USA; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. Our solvers found many different ways to get to the solution. Two submissions were incorrect.

Konečný commented that he expected a "nicer" answer and asked what the value of $\tan\theta$ *would have to be to get* $HK = 2$. *This editor asks whether there are any pairs of "nice" numbers for* HK *and* $\tan\theta$, *leaving the meaning of "nice" to our readers.*

3129. [2006 : 172, 174] *Proposed by K.R.S. Sastry, Bangalore, India.*

In $\triangle ABC$, the adjacent internal trisectors of the angles B and C meet at the point P, and the adjacent internal trisectors of the angles A and C meet at the point Q.

Characterize those triangles in which $AQ + BP = AB$.

Solution by Roy Barbara, University of Beirut, Beirut, Lebanon, modified by the editor.

The triangles in which $AQ+BP = AB$ are precisely those with a right angle at C.

Denote by 3α, 3β, and 3γ the angles at A, B, and C, respectively, and by R the circumradius of $\triangle ABC$. Recall that for any real number θ,

$$\begin{aligned} \sin 3\theta &= \sin\theta(3-4\sin^2\theta) = \sin\theta(1+2\cos 2\theta) \\ &= 4\sin\theta\sin\left(\frac{\pi}{3}-\theta\right)\sin\left(\frac{\pi}{3}+\theta\right) . \end{aligned}$$

These identities will be used without comment.

We have

$$\frac{AB}{\sin\gamma} = \frac{2R\sin 3\gamma}{\sin\gamma} = 2R(1+2\cos 2\gamma) . \tag{1}$$

The Law of Sines in $\triangle AQC$ gives

$$\begin{aligned} \frac{AQ}{\sin\gamma} &= \frac{AC}{\sin(\alpha+\gamma)} = \frac{AC}{\sin(\frac{\pi}{3}-\beta)} = \frac{2R\sin 3\beta}{\sin(\frac{\pi}{3}-\beta)} \\ &= 8R\sin\beta\sin(\tfrac{\pi}{3}+\beta) = 4R\Big(\cos\tfrac{\pi}{3}-\cos(\tfrac{\pi}{3}+2\beta)\Big) \\ &= 2R\Big(1-2\cos(\tfrac{\pi}{3}+2\beta)\Big) . \end{aligned} \tag{2}$$

Similarly,

$$\frac{BP}{\sin\gamma} = 2R\Big(1-2\cos(\tfrac{\pi}{3}+2\alpha)\Big) . \tag{3}$$

Using (1), (2), and (3), we see that $AQ+BP = AB$ if and only if

$$1-2\cos(\tfrac{\pi}{3}+2\beta)+1-2\cos(\tfrac{\pi}{3}+2\alpha) = 1+2\cos 2\gamma ;$$

that is,

$$\cos 2\gamma - \tfrac{1}{2} = -\cos(\tfrac{\pi}{3}+2\alpha)-\cos(\tfrac{\pi}{3}+2\beta) . \tag{4}$$

Now we rewrite both sides of (4):

$$\begin{aligned} \cos 2\gamma-\tfrac{1}{2} &= \cos 2\gamma+\cos\tfrac{2\pi}{3} = 2\cos(\tfrac{\pi}{3}-\gamma)\cos(\tfrac{\pi}{3}+\gamma) , \\ -\cos(\tfrac{\pi}{3}&+2\alpha)-\cos(\tfrac{\pi}{3}+2\beta) \\ &= -2\cos(\alpha-\beta)\cos(\tfrac{\pi}{3}+\alpha+\beta) \\ &= -2\cos(\alpha-\beta)\cos(\tfrac{2\pi}{3}-\gamma) = 2\cos(\alpha-\beta)\cos(\tfrac{\pi}{3}+\gamma) . \end{aligned}$$

Using these expressions in(4), we see that $AQ+BP = AB$ if and only if

$$\cos(\tfrac{\pi}{3}-\gamma)\cos(\tfrac{\pi}{3}+\gamma) = \cos(\alpha-\beta)\cos(\tfrac{\pi}{3}+\gamma) . \tag{5}$$

If there is a right angle at C (that is, if $\gamma = \frac{\pi}{6}$), then both sides of equation (5) are zero. If C is not a right angle ($\gamma \neq \frac{\pi}{6}$), then (5) reduces to

$\cos|\alpha-\beta| = \cos\left(\frac{\pi}{3}-\gamma\right)$. In this case (5) is never satisfied, because the cosine function is injective on $[0,\pi]$ and $0 \le |\alpha-\beta| < \alpha+\beta = \frac{\pi}{3}-\gamma < \pi$.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3130. [2006 : 172, 174] *Proposed by Michel Bataille, Rouen, France.*

Let A, B, C be the angles of a triangle. Show that

$$\left(\cos\tfrac{1}{2}A+\cos\tfrac{1}{2}B+\cos\tfrac{1}{2}C\right)\left(\csc\tfrac{1}{2}A+\csc\tfrac{1}{2}B+\csc\tfrac{1}{2}C\right) - \left(\cot\tfrac{1}{2}A+\cot\tfrac{1}{2}B+\cot\tfrac{1}{2}C\right) \ge 6\sqrt{3}\,.$$

Essentially the same proof by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; D.M. Milošević, Pranjani, Yugoslavia; and D.J. Smeenk, Zaltbommel, the Netherlands.

Since $A+B+C=\pi$, and $0 < A, B, C < \pi$, we have

$$\begin{aligned} 0 &= \cos\tfrac{\pi}{2} = \cos\left(\tfrac{1}{2}(A+B+C)\right) \\ &= \cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C - \cos\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C \\ &\qquad - \sin\tfrac{1}{2}A\cos\tfrac{1}{2}B\sin\tfrac{1}{2}C - \sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\cos\tfrac{1}{2}C\,. \end{aligned}$$

We divide by $\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$ and re-arrange the terms to obtain

$$\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C = \cot\tfrac{1}{2}A+\cot\tfrac{1}{2}B+\cot\tfrac{1}{2}C\,.$$

Since $\cot\frac{1}{2}A$, $\cot\frac{1}{2}B$, and $\cot\frac{1}{2}C$ are each positive, we apply the AM–GM Inequality to get

$$\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C \ge 3\sqrt[3]{\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C}\,,$$

which implies that

$$\sqrt[3]{\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C} \ge \sqrt{3}\,. \tag{1}$$

By another application of the AM–GM Inequality, we have

$$\begin{aligned} &\left(\cos\tfrac{1}{2}A+\cos\tfrac{1}{2}B+\cos\tfrac{1}{2}C\right)\left(\csc\tfrac{1}{2}A+\csc\tfrac{1}{2}B+\csc\tfrac{1}{2}C\right) \\ &\qquad - \left(\cot\tfrac{1}{2}A+\cot\tfrac{1}{2}B+\cot\tfrac{1}{2}C\right) \\ &= \frac{\cos\frac{1}{2}A+\cos\frac{1}{2}C}{\sin\frac{1}{2}B} + \frac{\cos\frac{1}{2}B+\cos\frac{1}{2}C}{\sin\frac{1}{2}A} + \frac{\cos\frac{1}{2}A+\cos\frac{1}{2}B}{\sin\frac{1}{2}C} \\ &\ge 6\sqrt[3]{\left(\frac{\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}{\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}\right)^2} = 6\sqrt[3]{\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C} \ge 6\sqrt{3}\,, \end{aligned}$$

where we have used (1) to obtain the last inequality. Equality holds if and only if $\cot\frac{1}{2}A = \cot\frac{1}{2}B = \cot\frac{1}{2}C$. Since $\cot t$ is injective on $\left(0, \frac{\pi}{2}\right)$, we see that equality holds if and only if $\triangle ABC$ is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; SCOTT BROWN, Auburn University, Montgomery, AL, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOHN G. HEUVER, Grande Prairie, AB; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Janous showed that the proposed inequality is a special case of more general inequalities for functions f, $g : (0,\pi) \to \mathbb{R}^+$ satisfying $\prod_{\text{cyclic}} f(A) \ge \lambda$ and either $\prod_{\text{cyclic}} g(A) \ge \mu$ or $\prod_{\text{cyclic}} f(A)g(A) \ge \nu$. In the first case,

$$\left(\sum_{\text{cyclic}} f(A)\right)\left(\sum_{\text{cyclic}} g(A)\right) - \sum_{\text{cyclic}} f(A)g(A) \;\ge\; 6(\lambda\mu)^{1/3}\,,$$

and in the second case,

$$\left(\sum_{\text{cyclic}} f(A)\right)\left(\sum_{\text{cyclic}} g(A)\right) - \sum_{\text{cyclic}} f(A)g(A) \;\ge\; 6\nu^{1/3}\,.$$

In both cases, equality holds for equilateral triangles. Janous also noted that the application of inequalities 2.42, 2.32, 2.12, 2.28 in [1] yields more general inequalities of the form

$$\left(\sum_{\text{cyclic}} (\cos(A/2))^p\right)\left(\sum_{\text{cyclic}} (\csc(A/2))^p\right) - \sum_{\text{cyclic}} (\cot(A/2))^p \;\ge\; 6\cdot 3^{p/2}$$

for any positive real number p.

References

[1] O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff Publ., Groningen, 1969.

3131. [2006 : 172, 175] *Proposed by Michel Bataille, Rouen, France.*

The normal at M to a conic with focus F meets the focal axis at N. Let H and K be points on MF such that $HN \perp MF$ and $KN \perp MN$. If $\frac{1}{HN} = \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$ and $NK = \sqrt{ab}$ (where $a > b > 0$), show that $KI = (a+b)/2$ for some significant point I on MN.

I. *Solution by D.J. Smeenk, Zaltbommel, the Netherlands.*

One possible position for I is the point where the line MN intersects the line perpendicular to MK at K: the right triangles HNK and NKI are similar; whence, $IK/KN = KN/NH$, and

$$KI \;=\; \frac{\sqrt{ab}\cdot\sqrt{ab}}{2ab/(a+b)} \;=\; \frac{a+b}{2}\,.$$

[*Ed.:* We shall see below that the point I is more "significant" than can be seen from Smeenk's solution. It is, in fact, the centre of curvature of the conic at the point M (that is, the centre of the circle whose curvature is the same as the conic's at their shared point M).]

II. *Solution for the case of an ellipse by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

We assume that the conic is the ellipse $x^2/\alpha^2 + y^2/\beta^2 = 1$; the focus F is the point $(c, 0)$, where c satisfies $c^2 + \beta^2 = \alpha^2$. For some value of t, $0 < t \leq \pi/2$, let M be the point $(\alpha\cos t, \beta\sin t)$. Using standard calculus, one determines the radius of curvature at M to be $k^3/(\alpha\beta)$, where

$$k^2 \;=\; \alpha^2\sin^2 t + \beta^2\cos^2 t\,.$$

The slope of the tangent at M is $-\beta\cos t/\alpha\sin t$, so that the slope of the normal there must be $\alpha\sin t/\beta\cos t$. Therefore, MN has the equation $y - \beta\sin t = \dfrac{\alpha\sin t}{\beta\cos t}(x - \alpha\cos t)$, or

$$(\beta\cos t)y - (\alpha\sin t)x + c^2\sin t\cos t \;=\; 0\,.$$

This line meets the x–axis at $N\left(\dfrac{c^2\cos t}{\alpha}, 0\right)$; whence,

$$MN^2 \;=\; \beta^2\sin^2 t + \left(\frac{\beta^2\cos t}{\alpha}\right)^2 \;=\; \frac{\beta^2k^2}{\alpha^2}\,. \qquad (1)$$

Since the equation of the line MF is $y = -(x - c)\beta\sin t/(c - \alpha\cos t)$, the distance from N to MF is

$$NH \;=\; \frac{\left|\dfrac{\beta c^2\sin t\cos t}{a} - \beta c\sin t\right|}{\sqrt{\beta^2\sin^2 t + (c - \alpha\cos t)^2}}\,.$$

The square-root term in the denominator simplifies as follows:

$$\begin{aligned}
\beta^2\sin^2 t + (c - \alpha\cos t)^2 &= \beta^2\sin^2 t + c^2 - 2\alpha c\cos t + \alpha^2\cos^2 t \\
&= \beta^2\sin^2 t + c^2 - 2\alpha c\cos t + (\beta^2 + c^2)\cos^2 t \\
&= \alpha^2 - 2\alpha c\cos t + c^2\cos^2 t \;=\; (\alpha - c\cos t)^2\,;
\end{aligned}$$

thus, $NH = \beta c\sin t/\alpha$, and

$$\begin{aligned}
MH^2 &= MN^2 - NH^2 \;=\; \frac{\beta^2k^2}{\alpha^2} - \frac{\beta^2c^2\sin^2 t}{\alpha^2} \\
&= \frac{\beta^2(\alpha^2\sin^2 t + (\alpha^2 - c^2)\cos^2 t) - \beta^2c^2\sin^2 t}{\alpha^2} \\
&= \frac{\beta^2(\alpha^2 - c^2\cos^2 t - c^2\sin^2 t)}{\alpha^2} \;=\; \frac{\beta^4}{\alpha^2}\,.
\end{aligned}$$

Hence,

$$MH \;=\; \beta^2/\alpha \qquad (2)$$

(which, incidentally, is both the latus rectum of the ellipse and the radius of curvature at $(\alpha, 0)$).

From the similar right triangles MNK and MHN, we see that $MK/MN = MN/MH$. Combining this proportion with equations (1) and (2), we see that

$$MK \;=\; \frac{MN^2}{MH} \;=\; \frac{k^2}{\alpha}\,.$$

Finally, from the similar right triangles IMK and NMH,

$$IM \;=\; \frac{MN \cdot MK}{MH} \;=\; \frac{k^3}{\alpha\beta}\,,$$

so that IM is the radius of curvature at M, as claimed.

Complete solutions came from APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; and the proposer.

The solutions of Bataille and Demis used a general form for a conic that produces the centre of curvature for an arbitrary conic—Bataille's is based on the latus rectum while Demis' is based on the eccentricity. We have presented Woo's solution even though it is valid only for an ellipse, because it has the virtue of using familiar formulas. The corresponding formulas for the hyperbola and the parabola may be obtained easily.

This problem provides a simple construction for the centre of curvature of a conic at any point M that is not on the major axis: Join M to the foci F and F'; call N the point where the bisector of $\angle FMF'$ meets FF'. (This is the normal to the conic at M; in the case of a parabola, MF' is taken to be the line through M that is parallel to the axis.) Define K to be the point where the perpendicular to MN at N meets MF; then I is the point where the perpendicular to MF at K meets the normal MN. Clearly, the numbers a and b that appear in the problem are a function of M; they are only indirectly related to the lengths of the semi-major and semi-minor axes that are usually denoted by these letters. The notation seems to have been chosen by Bataille for its nice relationship to the construction of the point I.

3132. [2006 : 172, 174] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $F(n)$ be the number of ones in the binary expression of the positive integer n. For example,

$$F(5) \;=\; F\left(101_{(2)}\right) \;=\; 2\,,$$
$$F(15) \;=\; F\left(1111_{(2)}\right) \;=\; 4\,.$$

Let $S_k = \sum\limits_{n=1}^{\infty} \dfrac{F^k(n)}{n(n+1)}$, where $F^k(n)$ is defined recursively by $F^1 = F$ and $F^k = F \circ F^{k-1}$ for $k \geq 2$.

(a) Prove that $S_1 = 2\ln 2$.

(b) Prove that $\frac{18}{5}\ln 2 - \frac{1}{15} \leq S_2 \leq 4\ln 2$.

(c) Prove that $\frac{218}{25}\ln 2 - \frac{7}{25} \leq S_3 \leq 11\ln 2$.

(d)★ Compute S_k.

[*Ed:* In this problem, the expression $F^k(n)$ means $\left(F(n)\right)^k$, rather than the k^{th} iterate of F, as stated above.]

Solution to part (a) *by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For any positive integer n, denote the binary representation of n by $\sum_{j=0}^{\infty} z_j(n)2^j$. Then $F(n) = \sum_{j=0}^{\infty} z_j(n)$. Now

$$\begin{aligned} S_1 &= \sum_{n=1}^{\infty} \frac{F(n)}{n(n+1)} = \sum_{n=1}^{\infty}\sum_{j=0}^{\infty} \frac{z_j(n)}{n(n+1)} = \sum_{j=0}^{\infty}\sum_{n=1}^{\infty} \frac{z_j(n)}{n(n+1)} \\ &= \sum_{j=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{u=0}^{2^j-1} \frac{1}{(\nu 2^{j+1}+2^j+u)(\nu 2^{j+1}+2^j+u+1)} \\ &= \sum_{j=0}^{\infty}\sum_{\nu=0}^{\infty} \left(\frac{1}{\nu 2^{j+1}+2^j} - \frac{1}{\nu 2^{j+1}+2^j+2^j}\right) \\ &= \sum_{j=0}^{\infty}\sum_{\nu=0}^{\infty} \left(\frac{1}{(2\nu+1)2^j} - \frac{1}{(2\nu+2)2^j}\right) \\ &= \sum_{j=0}^{\infty}\frac{1}{2^j}\sum_{\nu=0}^{\infty} \left(\frac{1}{2\nu+1} - \frac{1}{2\nu+2}\right) = \sum_{j=0}^{\infty}\frac{1}{2^j}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 2\ln 2\,. \end{aligned}$$

Solution to parts (b) *and* (c) *by the proposer.*

We have $F(2m) = F(m)$ and $F(2m+1) = 1 + F(2m) = 1 + F(m)$. Let $a_k \leq S_k \leq b_k$. Then

$$\begin{aligned} S_k &= \sum_{n=1}^{\infty} \frac{F^k(n)}{n(n+1)} = \sum_{m=0}^{\infty} \frac{\big(1+F(m)\big)^k}{(2m+1)(2m+2)} + \sum_{m=1}^{\infty} \frac{F^k(m)}{2m(2m+1)} \\ &= \sum_{m=0}^{\infty} \frac{1}{(2m+1)(2m+2)} + \sum_{m=0}^{\infty} \frac{\sum_{p=1}^{k-1}\binom{k}{p}F^p(m)}{(2m+1)(2m+2)} \\ &\qquad + \sum_{m=0}^{\infty} \frac{F^k(m)}{(2m+1)(2m+2)} + \sum_{m=1}^{\infty} \frac{F^k(2m)}{2m(2m+1)} \\ &= \ln 2 + \sum_{m=1}^{\infty} \frac{F^k(m)}{2(2m+1)}\left(\frac{1}{m+1} + \frac{1}{m}\right) + \frac{\sum_{p=1}^{k-1}\binom{k}{p}F^p(m)}{(2m+1)(2m+2)} \\ &= \ln 2 + \frac{1}{2}S_k + \sum_{m=1}^{\infty} \frac{\sum_{p=1}^{k-1}\binom{k}{p}F^p(m)}{(2m+1)(2m+2)}\,. \end{aligned}$$

Hence,

$$S_k = 2\ln 2 + 2\sum_{p=1}^{k-1}\binom{k}{p}\sum_{m=1}^{\infty} \frac{F^p(m)}{(2m+1)(2m+2)}\,. \tag{1}$$

Since $\dfrac{1}{(2m+1)(2m+2)} \le \dfrac{1}{4m(m+1)}$, from (1) we have

$$S_k \;\le\; 2\ln 2 + \frac{1}{2}\sum_{p=1}^{k-1}\binom{k}{p}\sum_{m=1}^{\infty}\frac{F^p(m)}{m(m+1)} \;\le\; 2\ln 2 + \frac{1}{2}\sum_{p=1}^{k-1}\binom{k}{p} b_p\,.$$

From part (a), we have $b_1 = 2\ln 2$ and, for $k = 2$, we conclude that

$$S_2 \;\le\; 2\ln 2 + \frac{1}{2}\binom{2}{1} b_1 \;=\; 4\ln 2\,;$$

thus, $b_2 = 4\ln 2$. For $k = 3$, we have

$$S_3 \;\le\; 2\ln 2 + \frac{1}{2}\left[\binom{3}{1} + \binom{3}{2}\right] \;=\; 11\ln 2\,,$$

which means that $b_3 = 11\ln 2$.

On the other hand, $\dfrac{1}{(2m+1)(2m+2)} \;\ge\; \dfrac{1}{5m(m+1)}$ for $m \ge 2$, and from (1) we have

$$\begin{aligned} S_k \;&\ge\; 2\ln 2 + \frac{2}{5}\sum_{p=1}^{k-1}\binom{k}{p}\sum_{m=2}^{\infty}\frac{F^p(m)}{m(m+1)} + \frac{1}{6}\sum_{p=1}^{k-1}\binom{k}{p} \\ &\ge\; 2\ln 2 + \frac{2}{5}\left(\sum_{p=1}^{k-1}\binom{k}{p}\sum_{m=2}^{\infty}\frac{F^p(m)}{m(m+1)} - \frac{1}{2}\sum_{p=1}^{k-1}\binom{k}{p}\right) + \frac{2(2^k-2)}{12} \\ &\ge\; 2\ln 2 - \frac{2^k-2}{30} + \frac{2}{5}\sum_{p=1}^{k-1}\binom{k}{p} a_p\,. \end{aligned}$$

For $k = 2$, we conclude from part (a) that

$$S_2 \;\ge\; 2\ln 2 - \frac{1}{15} + \frac{2}{5}\binom{2}{1} a_1 \;=\; \frac{18}{5}\ln 2 - \frac{1}{15}\,.$$

Since $a_2 = \frac{18}{5}\ln 2 - \frac{1}{15}$, we have

$$S_3 \;\ge\; 2\ln 2 - \frac{1}{5} + \frac{2}{5}\left[\binom{3}{1} a_1 + \binom{3}{2} a_2\right] \;\ge\; \frac{118}{25}\ln 2 - \frac{7}{25}\,.$$

Part (a) *also solved by JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer. Part* (d) *remains open.*

3133. [2006 : 172, 175] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be any triangle. Show that

$$\sum_{\text{cyclic}} \frac{1 + 2\sin A - \cos 2A}{8 + 3\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right) + \cos\left(\frac{3A}{2}\right)\cos\left(\frac{3(B-C)}{2}\right)} \;\le\; 1\,.$$

Composite of almost identical solutions by Michel Bataille, Rouen, France; and the proposer.

We first make the following observations:

$$1 + 2\sin A - \cos 2A \;=\; 2\sin A + 2\sin^2 A\,, \qquad (1)$$

$$\begin{aligned} 2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right) &= \cos\left(\frac{A+B-C}{2}\right) + \cos\left(\frac{A+C-B}{2}\right) \\ &= \cos\left(\frac{\pi-2C}{2}\right) + \cos\left(\frac{\pi-2B}{2}\right) \\ &= \sin C + \sin B\,, \qquad (2) \end{aligned}$$

$$\begin{aligned} &2\cos\left(\frac{3A}{2}\right)\cos\left(\frac{3(B-C)}{2}\right) \\ &\quad= \cos\left(\frac{3(A+B-C)}{2}\right) + \cos\left(\frac{3(A+C-B)}{2}\right) \\ &\quad= \cos\left(\frac{3(\pi-2C)}{2}\right) + \cos\left(\frac{3(\pi-2B)}{2}\right) \\ &\quad= -\sin 3C - \sin 3B\,. \qquad (3) \end{aligned}$$

Using (2), (3), and the formula $\sin 3x = 3\sin x - 4\sin^3 x$, we get

$$\begin{aligned} &8 + 3\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right) + \cos\left(\frac{3A}{2}\right)\cos\left(\frac{3(B-C)}{2}\right) \\ &= 8 + \tfrac{3}{2}(\sin B + \sin C) - \tfrac{1}{2}(\sin 3B + \sin 3C) \\ &= 2(4 + \sin^3 B + \sin^3 C) \;\geq\; 2(3 + \sin^3 A + \sin^3 B + \sin^3 C)\,. \qquad (4) \end{aligned}$$

Let L denote the left side of the inequality in the problem statement. Using (1) and (4), we obtain

$$L \;\leq\; \sum_{\text{cyclic}} \frac{2\sin A + 2\sin^2 A}{2(3 + \sin^3 A + \sin^3 B + \sin^3 C)} \;=\; \frac{\sum\limits_{\text{cyclic}} (\sin A + \sin^2 A)}{\sum\limits_{\text{cyclic}} (1 + \sin^3 A)}\,.$$

Hence, it suffices to show that $\sum\limits_{\text{cyclic}} (\sin A + \sin^2 A) \leq \sum\limits_{\text{cyclic}} (1 + \sin^3 A)$, which is equivalent in succession to

$$\begin{aligned} \sum_{\text{cyclic}} (1 + \sin^3 A - \sin A - \sin^2 A) &\;\geq\; 0\,, \\ \sum_{\text{cyclic}} (1 + \sin A)(1 - \sin A)^2 &\;\geq\; 0\,. \end{aligned}$$

Since the last inequality is clearly true, our proof is complete.

Also solved by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

From the last inequality in the proof featured above, it is clear that equality cannot hold in the given inequality. This was explicitly pointed out by Janous, who believes that the best upper bound λ for L is $\lambda = (90 + 69\sqrt{3})/229 \approx 0.915$, attained when $A = B = C = \frac{\pi}{3}$.

3134. [2006 : 173, 175] *Proposed by Mihály Bencze, Brasov, Romania.*

Let O be the circumcentre of $\triangle ABC$. Let D, E, and F be the mid-points of BC, CA, and AB, respectively; let K, M, and N be the mid-points of OA, OB, and OC, respectively. Denote the circumradius, inradius, and semiperimeter of $\triangle ABC$ by R, r, and s, respectively. Prove that

$$2(KD + ME + NF) \;\geq\; R + 3r + \frac{s^2 + r^2}{2R}\,.$$

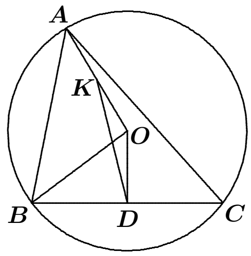
Solution by Michel Bataille, Rouen, France, modified by the editor.

First, we prove that

$$\frac{KD^2}{R^2} \;=\; \frac{1}{4} + \cos^2 A + \cos A \cos(B - C)\,. \qquad (1)$$

We can assume that B is closer to A than to C.

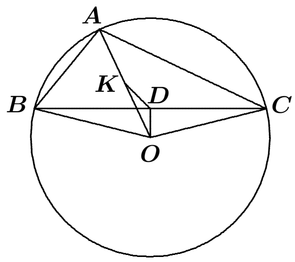
If $\angle A$ is a right angle, then the points D and O coincide and (1) is equivalent to $R = 2KO$, which is clearly true.



If $\angle A$ is acute, then

$$\begin{aligned} \angle KOD &= \angle AOB + \angle BOD \\ &= 2C + A \\ &= \pi - (B - C) \end{aligned}$$

and $OD = R\cos A$. The Law of Cosines applied in $\triangle KOD$ yields (1).



If $\angle A$ is obtuse, then

$$\begin{aligned} \angle KOD &= \angle BOD - \angle AOB \\ &= (\pi - A) - 2C \\ &= B - C \end{aligned}$$

and $OD = R\cos(\pi - A) = R\cos A$. Again the Law of Cosines in $\triangle KOD$ yields (1).

From (1), we get

$$\begin{aligned} \frac{KD^2}{R^2} &= \left[\cos A + \frac{1}{2}\cos(B - C)\right]^2 + \frac{1}{4}\sin^2(B - C) \\ &\geq \left[\cos A + \frac{1}{2}\cos(B - C)\right]^2\,, \end{aligned}$$

and therefore,

$$KD \;\geq\; R\left|\cos A + \tfrac{1}{2}\cos(B - C)\right| \;\geq\; R\left[\cos A + \tfrac{1}{2}\cos(B - C)\right]\,.$$

Similar inequalities hold for ME and NF. Adding these and using the well-known formulas

$$\sum_{\text{cyclic}} \cos A = 1 + \frac{r}{R},$$

$$\sum_{\text{cyclic}} \cos B \cos C = \frac{r^2 + s^2 - 4R^2}{4R^2},$$

and $$\sum_{\text{cyclic}} \sin B \sin C = \frac{r^2 + s^2 + 4Rr}{4R^2},$$

we obtain

$$\begin{aligned} 2(KD + ME + NF) & \\ \geq\ & 2R\Bigg[\sum_{\text{cyclic}} \cos A + \tfrac{1}{2}\sum_{\text{cyclic}} \cos(B - C)\Bigg] \\ =\ & R\left[2\left(1 + \frac{r}{R}\right) + \frac{r^2 + s^2 - 4R^2}{4R^2} + \frac{r^2 + s^2 + 4Rr}{4R^2}\right] \\ =\ & R + 3r + \frac{s^2 + r^2}{2R}. \end{aligned}$$

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3135. [2006 : 173, 176], corrected [2006 : 303, 306], corrected again [2006 : 514, 516] *Proposed by Marian Marinescu, Monbonnot, France.*

Let $\mathbb{R}^+$ be the set of non-negative real numbers. For all a, b, $c \in \mathbb{R}^+$, let $H(a, b, c)$ be the set of all functions $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$h(x) \geq h\big(h(ax)\big) + h(bx) + cx$$

for all $x \in \mathbb{R}^+$. Prove that $H(a, b, c)$ is non-empty if and only if $b \leq 1$ and $4ac \leq (1 - b)^2$.

[*Ed:* The version of this problem that was originally printed in Crux [2006 : 173, 176] was revised twice in later issues to correct typographical errors. Above is the final corrected version [2006 : 514, 516], as submitted by the proposer. Unfortunately, it is still not quite right, as can be seen by observing that the zero function $h(x) \equiv 0$ is always in $H(a, b, c)$ if $c = 0$. The stated equivalence also fails in the case where $c > 0$, $a = 0$ and $b = 1$. To avoid dealing with these special cases, we will correct the problem (once more!) by requiring $c > 0$ instead of just $c \geq 0$ and then asking for a proof that $H(a, b, c)$ is non-empty if and only if $b < 1$ and $4ac \leq (1 - b)^2$. The proposer's solution is then essentially correct.]

Solution by the proposer, modified by the editor.

First suppose that $b < 1$ and $4ac \leq (1-b)^2$. Let p be any real number such that $ap^2 - (1-b)p + c \leq 0$. For example, choose $p = (1-b)/(2a)$ if $a > 0$ and $p = c/(1-b)$ if $a = 0$. Then the function $h(x) = px$ satisfies the functional inequality in the problem. Thus, $H(a,b,c)$ is non-empty.

Now suppose $H(a,b,c)$ is non-empty. Choose $h \in H(a,b,c)$, and let $r = \inf\{h(x)/x : x > 0\}$. Then $r \geq 0$ and $h(x) \geq rx$ for all $x \geq 0$. Then $h(bx) \geq rbx$ and $h\big(h(ax)\big) \geq rh(ax) \geq r^2ax$ for all $x \geq 0$. Using the given functional inequality (which is satisfied by h), we get $h(x) \geq (ar^2 + br + c)x$ for all $x \geq 0$. Then, by the definition of r, we must have $ar^2 + br + c \leq r$; that is, $ar^2 - (1-b)r + c \leq 0$. Since r is real and non-negative, it follows that $4ac \leq (1-b)^2$ and $b < 1$.

Also solved by ROY BARBARA, University of Beirut, Beirut, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; APOSTOLIS K. DEMIS, Varvakeio High School, Athens, Greece; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

Most solutions were for the first corrected version of the problem [2006 : 303, 306]. These solutions have been accepted as correct, since they contained the main ideas needed to solve the problem in the form given above.

3136. [2006 : 173, 176] *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be a triangle with circumcircle Γ; let ℓ be a transversal which meets the line BC at L, the line CA at M, and the line AB at N. Let Γ_1 be the circle through A which is tangent to BC at L, and let Γ_2 and Γ_3 be similarly defined with respect to B and C. Let QR, RP, and PQ be the common chords of Γ and Γ_1, Γ and Γ_2, and Γ and Γ_3, respectively. Prove that AP, BQ, and CR are concurrent.

Solution by Apostolis K. Demis, Varvakeio High School, Athens, Greece.

We denote the intersection point of BC and QR by A', of CA and RP by B', and of AB and PQ by C'. Define K to be the point where BQ and AP intersect; we are to prove that K lies also on CR. Also, let us denote the second points of intersection of Γ with Γ_1, Γ_2, Γ_3 by D, E, F, respectively.

Applying the Theorem of Menelaus to triangle ABC with respect to the transversal ℓ, we get

$$\frac{AM}{MC} \cdot \frac{CL}{LB} \cdot \frac{BN}{NA} = -1\,.$$

The power of the point C' with respect to Γ and Γ_3 is $C'A \cdot C'B = C'F \cdot C'C = C'N^2$; whence,

$$\frac{C'N}{C'A} = \frac{C'B}{C'N} = \frac{C'B - C'N}{C'N - C'A}\,.$$

Thus, $(C'N/C'A) \cdot (C'B/C'N) = (C'B - C'N)^2/(C'N - C'A)^2$, or (in terms of oriented line segments)

$$\frac{BC'}{C'A} = -\frac{BN^2}{NA^2}.$$

Similarly,

$$\frac{CA'}{A'B} = -\frac{CL^2}{LB^2} \quad \text{and} \quad \frac{AB'}{B'C} = -\frac{AM^2}{MC^2}.$$

Therefore,

$$\frac{AB'}{B'C} \cdot \frac{CA'}{A'B} \cdot \frac{BC'}{C'A} = -\frac{AM^2}{MC^2} \cdot \frac{CL^2}{LB^2} \cdot \frac{BN^2}{NA^2} = -1,$$

and, consequently, the points A', B', and C' are collinear. The triangles APB' and BQA' are therefore perspective from C', so that, by Desargues' Theorem, we deduce that the points $K = AP \cap BQ$, $R = PB' \cap QA'$, and $C = B'A \cap A'B$ are collinear. Thus, K is the common point of the lines CR, BQ, and AP, as desired.

Comment. There are other interesting incidences to be discovered in this rich configuration. For example, C' lies on the line joining the points $AP \cap BC$ and $BQ \cap AC$. Also, C' belongs to the Pascal line defined by the hexagon $ABECFD$.

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.