PREFACE

Collecting the Mathematics tests from the contests choosing the best students is not only my favorite interest but also many different people's. This selected book is an adequate collection of the Math tests in the Mathematical Olympiads tests from 14 countries, from different regions and from the International Mathematical Olympiads tests as well.

I had a lot of effort to finish this book. Besides, I'm also grateful to all students who gave me much support in my collection. They include students in class 11 of specialized Chemistry – Biologry, class 10 specialized Mathematics and class $10A_2$ in the school year 2003 – 2004, Nguyen Binh Khiem specialized High School in Vinh Long town.

This book may be lack of some Mathematical Olympiads tests from different countries. Therefore, I would like to receive both your supplement and your supplementary ideas. Please write or mail to me.

- Address: Cao Minh Quang, Mathematic teacher, Nguyen Binh Khiem specialized High School, Vinh Long town.
- Email: kt13quang@yahoo.com

Vinh Long, April 2006

Cao Minh Quang

Abbreviations

AIME	**American Invitational Mathematics Examination**
ASU	**All Soviet Union Math Competitions**
BMO	**British Mathematical Olympiads**
CanMO	**Canadian Mathematical Olympiads**
INMO	**Indian National Mathematical Olympiads**
USAMO	**United States Mathematical Olympiads**
APMO	**Asian Pacific Mathematical Olympiads**
IMO	**International Mathematical Olympiads**

CONTENTS

PART I. National Olympiads

AIME (1983 – 2004)

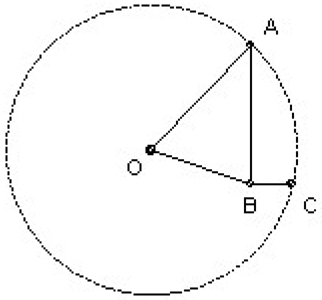
1st AIME 1983

1. x, y, z are real numbers greater than 1 and w is a positive real number. If $\log_x w = 24$, $\log_y w = 40$ and $\log_{xyz} w = 12$, find $\log_z w$.

2. Find the minimum value of $|x - p| + |x - 15| + |x - p - 15|$ for x in the range $p \le x \le 15$, where $0 < p < 15$.

3. Find the product of the real roots of the equation $x^2 + 18x + 30 = 2\sqrt{(x^2 + 18x + 45)}$.

4. A and C lie on a circle center O with radius $\sqrt{50}$. The point B inside the circle is such that $\angle ABC = 90^o$, AB = 6, BC = 2. Find OB.



5. w and z are complex numbers such that $w^2 + z^2 = 7$, $w^3 + z^3 = 10$. What is the largest possible real value of w + z?

6. What is the remainder on dividing $6^{83} + 8^{83}$ by 49?

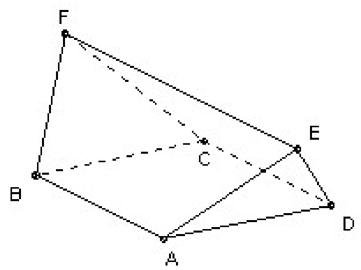
7. 25 knights are seated at a round table and 3 are chosen at random. Find the probability that at least two of the chosen 3 are sitting next to each other.

8. What is the largest 2-digit prime factor of the binomial coefficient 200C100?

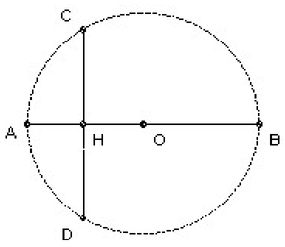
9. Find the minimum value of $(9x^2\sin^2 x + 4)/(x \sin x)$ for $0 < x < \pi$.

10. How many 4 digit numbers with first digit 1 have exactly two identical digits (like 1447, 1005 or 1231)?

11. ABCD is a square side $6\sqrt{2}$. EF is parallel to the square and has length $12\sqrt{2}$. The faces BCF and ADE are equilateral. What is the volume of the solid ABCDEF?

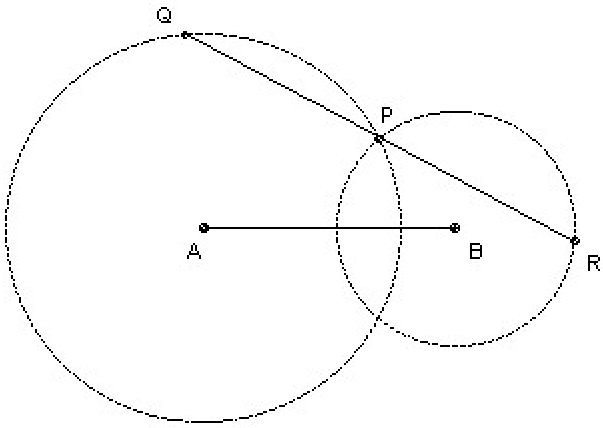


12. The chord CD is perpendicular to the diameter AB and meets it at H. The distances AB and CD are integral. The distance AB has 2 digits and the distance CD is obtained by reversing the digits of AB. The distance OH is a non-zero rational. Find AB.

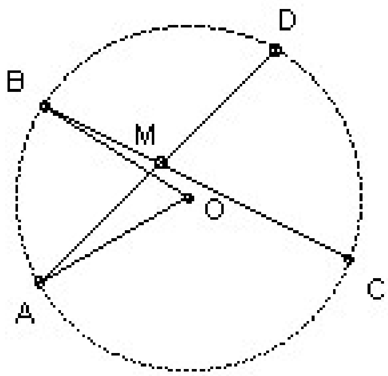


13. For each non-empty subset of {1, 2, 3, 4, 5, 6, 7} arrange the members in decreasing order with alternating signs and take the sum. For example, for the subset {5} we get 5. For {6, 3, 1} we get 6 - 3 + 1 = 4. Find the sum of all the resulting numbers.

14. The distance AB is 12. The circle center A radius 8 and the circle center B radius 6 meet at P (and another point). A line through P meets the circles again at Q and R (with Q on the larger circle), so that QP = PR. Find QP^2.



15. BC is a chord length 6 of a circle center O radius 5. A is a point on the circle closer to B than C such that there is just one chord AD which is bisected by BC. Find sin AOB.

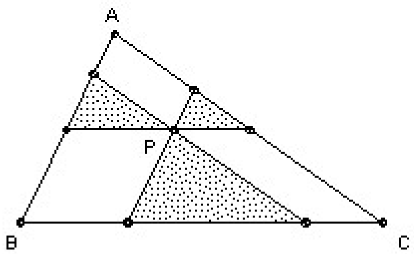


2nd AIME 1984

1. The sequence $a_1, a_2, \ldots, a_{98}$ satisfies $a_{n+1} = a_n + 1$ for n = 1, 2, ... , 97 and has sum 137. Find $a_2 + a_4 + a_6 + \ldots + a_{98}$.

2. Find the smallest positive integer n such that every digit of 15n is 0 or 8.

3. P is a point inside the triangle ABC. Lines are drawn through P parallel to the sides of the triangle. The areas of the three resulting triangles with a vertex at P have areas 4, 9 and 49. What is the area of ABC?



4. A sequence of positive integers includes the number 68 and has arithmetic mean 56. When 68 is removed the arithmetic mean of the remaining numbers is 55. What is the largest number than can occur in the sequence?

5. The reals x and y satisfy $\log_8 x + \log_4(y^2) = 5$ and $\log_8 y + \log_4(x^2) = 7$. Find xy.

6. Three circles radius 3 have centers at P (14, 92), Q (17, 76) and R (19, 84). The line L passes through Q and the total area of the parts of the circles in each half-plane (defined by L) is the same. What is the absolute value of the slope of L?

7. Let Z be the integers. The function $f : Z \to Z$ satisfies $f(n) = n - 3$ for $n > 999$ and $f(n) = f(f(n+5))$ for $n < 1000$. Find f(84).

8. $z^6 + z^3 + 1 = 0$ has a root $r e^{i\theta}$ with $90^o < \theta < 180^o$. Find θ.

9. The tetrahedron ABCD has AB = 3, area ABC = 15, area ABD = 12 and the angle between the faces ABC and ABD is 30^o. Find its volume.

10. An exam has 30 multiple-choice problems. A contestant who answers m questions correctly and n incorrectly (and does not answer 30 - m - n questions) gets a score of 30 + 4m - n. A contestant scores N > 80. A knowledge of N is sufficient to deduce how many questions the contestant scored correctly. That is not true for any score M satisfying 80 < M < N. Find N.

11. Three red counters, four green counters and five blue counters are placed in a row in random order. Find the probability that no two blue counters are adjacent.

12. Let R be the reals. The function $f : R \to R$ satisfies $f(0) = 0$ and $f(2 + x) = f(2 - x)$ and $f(7 + x) = f(7 - x)$ for all x. What is the smallest possible number of values x such that $|x| \le 1000$ and $f(x) = 0$?

13. Find $10 \cot(\cot^{-1}3 + \cot^{-1}7 + \cot^{-1}13 + \cot^{-1}21)$.

14. What is the largest even integer that cannot be written as the sum of two odd composite positive integers?

15. The real numbers x, y, z, w satisfy: $x^2/(n^2 - 1^2) + y^2/(n^2 - 3^2) + z^2/(n^2 - 5^2) + w^2/(n^2 - 7^2) = 1$ for n = 2, 4, 6 and 8. Find $x^2 + y^2 + z^2 + w^2$.

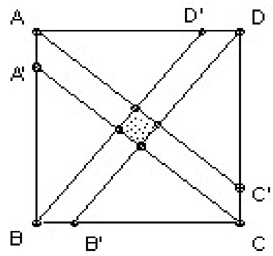
3rd AIME 1985

1. Let $x_1 = 97$, $x_2 = 2/x_1$, $x_3 = 3/x_2$, $x_4 = 4/x_3$, ... , $x_8 = 8/x_7$. Find $x_1x_2 ... x_8$.

2. The triangle ABC has angle B = 90°. When it is rotated about AB it gives a cone volume 800π. When it is rotated about BC it gives a cone volume 1920π. Find the length AC.

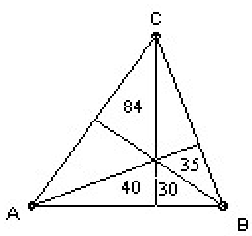
3. m and n are positive integers such that $N = (m + ni)^3 - 107i$ is a positive integer. Find N.

4. ABCD is a square side 1. Points A', B', C', D' are taken on the sides AB, BC, CD, DA respectively so that AA'/AB = BB'/BC = CC'/CD = DD'/DA = 1/n. The strip bounded by the lines AC' and A'C meets the strip bounded by the lines BD' and B'D in a square area 1/1985. Find n.



5. The integer sequence a_1, a_2, a_3, ... satisfies $a_{n+2} = a_{n+1} - a_n$ for n > 0. The sum of the first 1492 terms is 1985, and the sum of the first 1985 terms is 1492. Find the sum of the first 2001 terms.

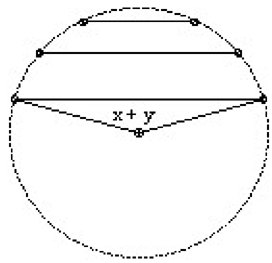
6. A point is taken inside a triangle ABC and lines are drawn through the point from each vertex, thus dividing the triangle into 6 parts. Four of the parts have the areas shown. Find area ABC.



7. The positive integers A, B, C, D satisfy $A^5 = B^4$, $C^3 = D^2$ and C = A + 19. Find D - B.

8. Approximate each of the numbers 2.56, 2.61, 2.65, 2.71, 2.79, 2.82, 2.86 by integers, so that the 7 integers have the same sum and the maximum absolute error E is as small as possible. What is 100E?

9. Three parallel chords of a circle have lengths 2, 3, 4 and subtend angles x, y, x + y at the center (where x + y < 180°). Find cos x.



10. How many of 1, 2, 3, ... , 1000 can be expressed in the form [2x] + [4x] + [6x] + [8x], for some real number x?

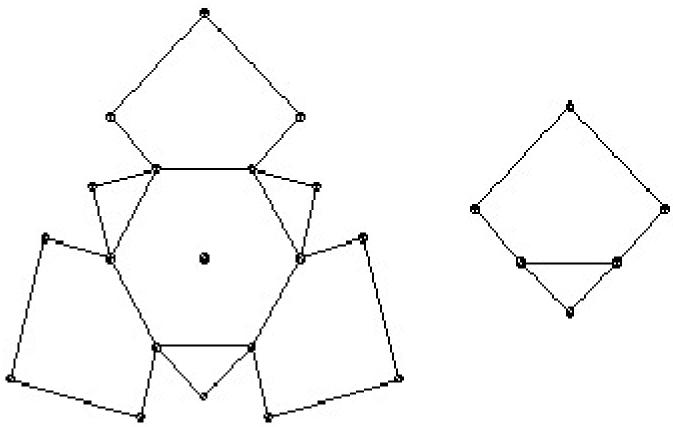
11. The foci of an ellipse are at (9, 20) and (49, 55), and it touches the x-axis. What is the length of its major axis?

12. A bug crawls along the edges of a regular tetrahedron ABCD with edges length 1. It starts at A and at each vertex chooses its next edge at random (so it has a 1/3 chance of going back along the edge it came on, and a 1/3 chance of going along each of the other two). Find the probability that after it has crawled a distance 7 it is again at A is p.

13. Let f(n) be the greatest common divisor of $100 + n^2$ and $100 + (n+1)^2$ for n = 1, 2, 3, ... What is the maximum value of f(n)?

14. In a tournament each two players played each other once. Each player got 1 for a win, 1/2 for a draw, and 0 for a loss. Let S be the set of the 10 lowest-scoring players. It is found that every player got exactly half his total score playing against players in S. How many players were in the tournament?

15. A 12 x 12 square is divided into two pieces by joining to adjacent side midpoints. Copies of the triangular piece are placed on alternate edges of a regular hexagon and copies of the other piece are placed on the other edges. The resulting figure is then folded to give a polyhedron with 7 faces. What is the volume of the polyhedron?



4th AIME 1986

1. Find the sum of the solutions to $x^{1/4} = 12/(7 - x^{1/4})$.

2. Find $(\sqrt{5} + \sqrt{6} + \sqrt{7})(\sqrt{5} + \sqrt{6} - \sqrt{7})(\sqrt{5} - \sqrt{6} + \sqrt{7})(-\sqrt{5} + \sqrt{6} + \sqrt{7})$.

3. Find tan(x+y) where tan x + tan y = 25 and cot x + cot y = 30.

4. $2x_1 + x_2 + x_3 + x_4 + x_5 = 6$

$x_1 + 2x_2 + x_3 + x_4 + x_5 = 12$

$x_1 + x_2 + 2x_3 + x_4 + x_5 = 24$

$x_1 + x_2 + x_3 + 2x_4 + x_5 = 48$

$x_1 + x_2 + x_3 + x_4 + 2x_5 = 96$

Find $3x_4 + 2x_5$.

5. Find the largest integer n such that n + 10 divides $n^3 + 100$.

6. For some n, we have (1 + 2 + ... + n) + k = 1986, where k is one of the numbers 1, 2, ... , n. Find k.

7. The sequence 1, 3, 4, 9, 10, 12, 13, 27, ... includes all numbers which are a sum of one or more distinct powers of 3. What is the 100th term?

8. Find the integral part of $\sum \log_{10}k$, where the sum is taken over all positive divisors of 1000000 except 1000000 itself.

9. A triangle has sides 425, 450, 510. Lines are drawn through an interior point parallel to the sides, the intersections of these lines with the interior of the triangle have the same length. What is it?

10. abc is a three digit number. If acb + bca + bac + cab + cba = 3194, find abc.

11. The polynomial $1 - x + x^2 - x^3 + ... - x^{15} + x^{16} - x^{17}$ can be written as a polynomial in y = x + 1. Find the coefficient of y^2.

12. Let X be a subset of {1, 2, 3, ... , 15} such that no two subsets of X have the same sum. What is the largest possible sum for X?

13. A sequence has 15 terms, each H or T. There are 14 pairs of adjacent terms. 2 are HH, 3 are HT, 4 are TH, 5 are TT. How many sequences meet these criteria?

14. A rectangular box has 12 edges. A long diagonal intersects 6 of them. The shortest distance of the other 6 from the long diagonal are $2\sqrt{5}$ (twice), $30/\sqrt{13}$ (twice), $15/\sqrt{10}$ (twice). Find the volume of the box.

15. The triangle ABC has medians AD, BE, CF. AD lies along the line y = x + 3, BE lies along the line y = 2x + 4, AB has length 60 and angle C = 90^o. Find the area of ABC.

5th AIME 1987

1. How many pairs of non-negative integers (m, n) each sum to 1492 without any carries?

2. What is the greatest distance between the sphere center (-2, -10, 5) radius 19, and the sphere center (12, 8, -16) radius 87?

3. A *nice* number equals the product of its proper divisors (positive divisors excluding 1 and the number itself). Find the sum of the first 10 nice numbers.

4. Find the area enclosed by the graph of $|x - 60| + |y| = |x/4|$.

5. m, n are integers such that $m^2 + 3m^2n^2 = 30n^2 + 517$. Find $3m^2n^2$.

6. ABCD is a rectangle. The points P, Q lie inside it with PQ parallel to AB. Points X, Y lie on AB (in the order A, X, Y, B) and W, Z on CD (in the order D, W, Z, C). The four parts AXPWD, XPQY, BYQZC, WPQZ have equal area. BC = 19, PQ = 87, XY = YB + BC + CZ = WZ = WD + DA + AX. Find AB.

7. How many ordered triples (a, b, c) are there, such that lcm(a, b) = 1000, lcm(b, c) = 2000, lcm(c, a) = 2000?

8. Find the largest positive integer n for which there is a unique integer k such that $8/15 < n/(n+k) < 7/13$.

9. P lies inside the triangle ABC. Angle B = 90^o and each side subtends an angle 120^o at P. If PA = 10, PB = 6, find PC.

10. A walks down an up-escalator and counts 150 steps. B walks up the same escalator and counts 75 steps. A takes three times as many steps in a given time as B. How many steps are visible on the escalator?

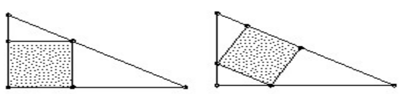
11. Find the largest k such that 3^{11} is the sum of k consecutive positive integers.

12. Let m be the smallest positive integer whose cube root is n + k, where n is an integer and $0 < k < 1/1000$. Find n.

13. Given distinct reals $x_1, x_2, x_3, \ldots, x_{40}$ we compare the first two terms x_1 and x_2 and swap them iff $x_2 < x_1$. Then we compare the second and third terms of the resulting sequence and swap them iff the later term is smaller, and so on, until finally we compare the 39th and 40th terms of the resulting sequence and swap them iff the last is smaller. If the sequence is initially in random order, find the probability that x_{20} ends up in the 30th place. [The original question asked for m+n if the prob is m/n in lowest terms.]

14. Let $m = (10^4 + 324)(22^4 + 324)(34^4 + 324)(46^4 + 324)(58^4 + 324)$ and $n = (4^4 + 324)(16^4 + 324)(28^4 + 324)(40^4 + 324)(52^4 + 324)$. Find m/n.

15. Two squares are inscribed in a right-angled triangle as shown. The first has area 441 and the second area 440. Find the sum of the two shorter sides of the triangle.



6th AIME 1988

1. A lock has 10 buttons. A combination is any subset of 5 buttons. It can be opened by pressing the buttons in the combination in any order. How many combinations are there? Suppose it is redesigned to allow a combination to be any subset of 1 to 9 buttons. How many combinations are there? [The original question asked for the difference.]

2. Let f(n) denote the square of the sum of the digits of n. Let $f^2(n)$ denote f(f(n)), $f^3(n)$ denote f(f(f(n))) and so on. Find $f^{1998}(11)$.

3. Given $\log_2(\log_8 x) = \log_8(\log_2 x)$, find $(\log_2 x)^2$.

4. x_i are reals such that $-1 < x_i < 1$ and $|x_1| + |x_2| + ... + |x_n| = 19 + |x_1 + ... + x_n|$. What is the smallest possible value of n?

5. Find the probability that a randomly chosen positive divisor of 10^{99} is divisible by 10^{88}. [The original question asked for m+n, where the prob is m/n in lowest terms.]

6. The vacant squares in the grid below are filled with positive integers so that there is an arithmetic progression in each row and each column. What number is placed in the square marked * ?

			*	
	74			
				186
		103		
0				

7. In the triangle ABC, the foot of the perpendicular from A divides the opposite side into parts length 3 and 17, and tan A = 22/7. Find area ABC.

8. f(m, n) is defined for positive integers m, n and satisfies f(m, m) = m, f(m, n) = f(n, m), f(m, m+n) = (1 + m/n) f(m, n). Find f(14, 52).

9. Find the smallest positive cube ending in 888.

10. The truncated cuboctahedron is a convex polyhedron with 26 faces: 12 squares, 8 regular hexagons and 6 regular octagons. There are three faces at each vertex: one square, one hexagon and one octagon. How many pairs of vertices have the segment joining them inside the polyhedron rather than on a face or edge?

11. A line L in the complex plane is a *mean line* for the points $w_1, w_2, ... , w_n$ if there are points $z_1, z_2, ... , z_n$ on L such that $(w_1 - z_1) + ... + (w_n - z_n) = 0$. There is a unique mean line for the points 32 + 170i, -7 + 64i, -9 + 200i, 1 + 27i, -14 + 43i which passes through the point 3i. Find its slope.

12. P is a point inside the triangle ABC. The line PA meets BC at D. Similarly, PB meets CA at E, and PC meets AB at F. If PD = PE = PF = 3 and PA + PB + PC = 43, find PA·PB·PC.

13. $x^2 - x - 1$ is a factor of $a x^{17} + b x^{16} + 1$ for some integers a, b. Find a.

14. The graph xy = 1 is reflected in y = 2x to give the graph $12x^2 + rxy + sy^2 + t = 0$. Find rs.

15. The boss places letter numbers 1, 2, ... , 9 into the typing tray one at a time during the day in that order. Each letter is placed on top of the pile. Every now and then the secretary takes the top letter from the pile and types it. She leaves for lunch remarking that letter 8 has already been typed. How many possible orders there are for the typing of the remaining letters. [For example, letters 1, 7 and 8 might already have been typed, and the remaining letters might be typed in the order 6, 5, 9, 4, 3, 2. So the sequence 6, 5, 9, 4, 3, 2 is one possibility. The empty sequence is another.]

7th AIME 1989

1. Find sqrt(1 + 28·29·30·31).

2. 10 points lie on a circle. How many distinct convex polygons can be formed by connected some or all of the points?

3. For some digit d we have 0.d25d25d25 ... = n/810, where n is a positive integer. Find n.

4. Given five consecutive positive integers whose sum is a cube and such that the sum of the middle three is a square, find the smallest possible middle integer.

5. A coin has probability p of coming up heads. If it is tossed five times, the probability of just two heads is the same as the probability of just one head. Find the probability of just three heads in five tosses. [The original question asked for m+n, where the probability is m/n in lowest terms.]

6. C and D are 100m apart. C runs in a straight line at 8m/s at an angle of 60^o to the ray towards D. D runs in a straight line at 7m/s at an angle which gives the earliest possible meeting with C. How far has C run when he meets D?

7. k is a positive integer such that 36 + k, 300 + k, 596 + k are the squares of three consecutive terms of an arithmetic progression. Find k

8. Given that:

$x_1 + 4 x_2 + 9 x_3 + 16 x_4 + 25 x_5 + 36 x_6 + 49 x_7 = 1$;

$4 x_1 + 9 x_2 + 16 x_3 + 25 x_4 + 36 x_5 + 49 x_6 + 64 x_7= 12$;

$9 x_1 + 16 x_2 + 25 x_3 + 36 x_4 + 49 x_5 + 64 x_6 + 81 x_7= 123$.

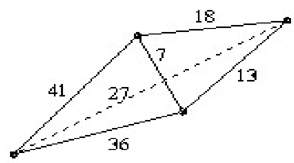
Find $16 x_1 + 25 x_2 + 36 x_3 + 49 x_4 + 64 x_5 + 81 x_6 + 100 x_7$.

9. Given that $133^5 + 110^5 + 84^5 + 27^5 = k^5$, with k an integer, find k.

10. The triangle ABC has AB = c, BC = a, CA = b as usual. Find cot C/(cot A + cot B) if $a^2 + b^2 = 1989 c^2$.

11. $a_1, a_2, ... , a_{121}$ is a sequence of positive integers not exceeding 1000. The value n occurs more frequently than any other, and m is the arithmetic mean of the terms of the sequence. What is the largest possible value of [m - n]?

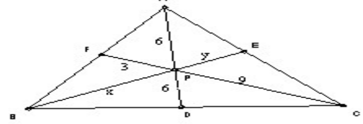
12. A tetrahedron has the edge lengths shown. Find the square of the distance between the midpoints of the sides length 41 and 13.



13. Find the largest possible number of elements of a subset of {1, 2, 3, ... , 1989} with the property that no two elements of the subset have difference 4 or 7.

14. Any number of the form M + Ni with M and N integers may be written in the complex base (i - n) as $a_m(i - n)^m + a_{m-1}(i - n)^{m-1} + ... + a_1(i - n) + a_0$ for some m >= 0, where the digits a_k lie in the range 0, 1, 2, ... , n^2. Find the sum of all ordinary integers which can be written to base i - 3 as 4-digit numbers.

15. In the triangle ABC, the segments have the lengths shown and x + y = 20. Find its area.



8th AIME 1990

1. The sequence 2, 3, 5, 6, 7, 10, 11, ... consists of all positive integers that are not a square or a cube. Find the 500th term.

2. Find $(52 + 6\sqrt{43})^{3/2} - (52 - 6\sqrt{43})^{3/2}$.

3. Each angle of a regular r-gon is 59/58 times larger than each angle of a regular s-gon. What is the largest possible value of s?

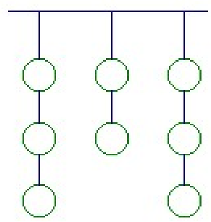
4. Find the positive solution to $1/(x^2 - 10x - 29) + 1/(x^2 - 10x - 45) = 2/(x^2 - 10x - 69)$.

5. n is the smallest positive integer which is a multiple of 75 and has exactly 75 positive divisors. Find n/75.

6. A biologist catches a random sample of 60 fish from a lake, tags them and releases them. Six months later she catches a random sample of 70 fish and finds 3 are tagged. She assumes 25% of the fish in the lake on the earlier date have died or moved away and that 40% of the fish on the later date have arrived (or been born) since. What does she estimate as the number of fish in the lake on the earlier date?

7. The angle bisector of angle A in the triangle A (-8, 5), B (-15, -19), C (1, -7) is $ax + 2y + c = 0$. Find a and c.

8. 8 clay targets are arranged as shown. In how many ways can they be shot (one at a time) if no target can be shot until the target(s) below it have been shot.



9. A fair coin is tossed 10 times. What is the chance that no two consecutive tosses are both heads.

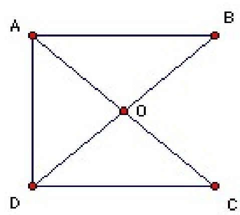
10. Given the two sets of complex numbers, $A = \{z : z^{18} = 1\}$, and $B = \{z : z^{48} = 1\}$, how many distinct elements are there in $\{zw : z \in A, w \in B\}$?

11. Note that $6! = 8\cdot 9\cdot 10$. What is the largest n such that n! is a product of n-3 consecutive positive integers.

12. A regular 12-gon has circumradius 12. Find the sum of the lengths of all its sides and diagonals.

13. How many powers 9^n with $0 \le n \le 4000$ have leftmost digit 9, given that 9^{4000} has 3817 digits and that its leftmost digit is 9.

14. ABCD is a rectangle with $AB = 13\sqrt{3}$, $AD = 12\sqrt{3}$. The figure is folded along OA and OD to form a tetrahedron. Find its volume.



15. The real numbers a, b, x, y satisfy $ax + by = 3$, $ax^2 + by^2 = 7$, $ax^3 + by^3 = 16$, $ax^4 + by^4 = 42$. Find $ax^5 + by^5$.

9th AIME 1991

1. m, n are positive integers such that $mn + m + n = 71$, $m^2n + mn^2 = 880$, find $m^2 + n^2$.

2. The rectangle ABCD has AB = 4, BC = 3. The side AB is divided into 168 equal parts by points $P_1, P_2, \ldots, P_{167}$ (in that order with P_1 next to A), and the side BC is divided into 168 equal parts by points $Q_{167}, Q_{166}, \ldots, Q_1$ (in that order with Q_1 next to C). The parallel segments $P_1Q_1, P_2Q_2, \ldots, P_{167}Q_{167}$ are drawn. Similarly, 167 segments are drawn between AD and DC, and finally the diagonal AC is drawn. Find the sum of the lengths of the 335 parallel segments.

3. Expand $(1 + 0.2)^{1000}$ by the binomial theorem to get $a^0 + a^1 + \ldots + a^{1000}$, where $a^i = 1000Ci\ (0.2)^i$. Which is the largest term?

4. How many real roots are there to $(1/5) \log_2 x = \sin(5\pi x)$?

5. How many fractions m/n, written in lowest terms, satisfy $0 < m/n < 1$ and $mn = 20!$?

6. The real number x satisfies $[x + 0.19] + [x + 0.20] + [x + 0.21] + \ldots + [x + 0.91] = 546$. Find $[100x]$.

7. Consider the equation $x = \sqrt{19} + 91/(\sqrt{19} + 91/(\sqrt{19} + 91/(\sqrt{19} + 91/(\sqrt{19} + 91/x))))$. Let k be the sum of the absolute values of the roots. Find k^2.

8. For how many reals b does $x^2 + bx + 6b$ have only integer roots?

9. If $\sec x + \tan x = 22/7$, find $\operatorname{cosec} x + \cot x$.

10. The letter string AAABBB is sent electronically. Each letter has 1/3 chance (independently) of being received as the other letter. Find the probability that using the ordinary text order the first three letters come rank strictly before the second three. (For example, ABA ranks before BAA, but after AAB.)

11. 12 equal disks are arranged without overlapping, so that each disk covers part of a circle radius 1 and between them they cover every point of the circle. Each disk touches two others. (Note that the disks are not required to cover every point inside the circle.) Find the total area of the disks.

12. ABCD is a rectangle. P, Q, R, S lie on the sides AB, BC, CD, DA respectively so that PQ = QR = RS = SP. PB = 15, BQ = 20, PR = 30, QS = 40. Find the perimeter of ABCD.

13. m red socks and n blue socks are in a drawer, where $m + n \leq 1991$. If two socks are taken out at random, the chance that they have the same color is 1/2. What is the largest possible value of m?

14. A hexagon is inscribed in a circle. Five sides have length 81 and the other side has length 31. Find the sum of the three diagonals from a vertex on the short side.

15. Let S_n be the minimum value of $\sum \sqrt{(2k-1)^2 + a_k^2}$ for positive reals $a_1, a_2, \ldots, a_n$ with sum 17. Find the values of n for which S_n is integral.

10th AIME 1992

1. Find the sum of all positive rationals a/30 (in lowest terms) which are < 10.

2. How many positive integers > 9 have their digits strictly increasing from left to right?

3. At the start of a weekend a player has won the fraction 0.500 of the matches he has played. After playing another four matches, three of which he wins, he has won more than the fraction 0.503 of his matches. What is the largest number of matches he could have won before the weekend?

4. The binomial coefficients nCm can be arranged in rows (with the nth row nC0, nC1, ... nCn) to form Pascal's triangle. In which row are there three consecutive entries in the ratio 3 : 4 : 5?

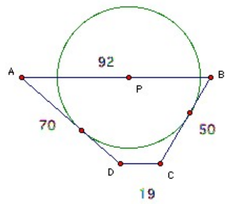
5. Let S be the set of all rational numbers which can be written as 0.abcabcabcabc... (where the integers a, b, c are not necessarily distinct). If the members of S are all written in the form r/s in lowest terms, how many different numerators r are required?

6. How many pairs of consecutive integers in the sequence 1000, 1001, 1002, ... , 2000 can be added without a carry? (For example, 1004 and 1005, but not 1005 and 1006.)

7. ABCD is a tetrahedron. Area ABC = 120, area BCD = 80. BC = 10 and the faces ABC and BCD meet at an angle of 30^o. What is the volume of ABCD?

8. If A is the sequence $a_1, a_2, a_3, \dots$, define ΔA to be the sequence $a_2 - a_1, a_3 - a_2, a_4 - a_3, \dots$. If Δ(ΔA) has all terms 1 and $a_{19} = a_{92} = 0$, find a_1.

9. ABCD is a trapezoid with AB parallel to CD, AB = 92, BC = 50, CD = 19, DA = 70. P is a point on the side AB such that a circle center P touches AD and BC. Find AP.



10. A is the region of the complex plane {z : z/40 and 40/w have real and imaginary parts in (0, 1)}, where w is the complex conjugate of z (so if z = a + ib, then w = a - ib). (Unfortunately, there does not appear to be any way of writing z with a bar over it in HTML4). Find the area of A to the nearest integer.

11. L, L' are the lines through the origin that pass through the first quadrant (x, y > 0) and make angles π/70 and π/54 respectively with the x-axis. Given any line M, the line R(M) is obtained by reflecting M first in L and then in L'. $R^n(M)$ is obtained by applying R n times. If M is the line y = 19x/92, find the smallest n such that $R^n(M) = M$.

12. The game of Chomp is played with a 5 x 7 board. Each player alternately takes a bite out of the board by removing a square any and any other squares above and/or to the left of it. How many possible subsets of the 5 x 7 board (including the original board and the empty set) can be obtained by a sequence of bites?

13. The triangle ABC has AB = 9 and BC/CA = 40/41. What is the largest possible area for ABC?

14. ABC is a triangle. The points A', B', C' are on sides BC, CA, AB and AA', BB', CC' meet at O. Also AO/A'O + BO/B'O + CO/C'O = 92. Find (AO/A'O)(BO/B'O)(CO/C'O).

15. How many integers n in {1, 2, 3, ... , 1992} are such that m! never ends in exactly n zeros?

11th AIME 1993

1. How many even integers between 4000 and 7000 have all digits different?

2. Starting at the origin, an ant makes 40 moves. The nth move is a distance $n^2/2$ units. Its moves are successively due E, N, W, S, E, N ... . How far from the origin does it end up?

3. In a fish contest one contestant caught 15 fish. The other contestants all caught less. an contestants caught n fish, with $a_0 = 9$, $a_1 = 5$, $a_2 = 7$, $a_3 = 23$, $a_{13} = 5$, $a_{14} = 2$. Those who caught 3 or more fish averaged 6 fish each. Those who caught 12 or fewer fish averaged 5 fish each. What was the total number of fish caught in the contest?

4. How many 4-tuples (a, b, c, d) satisfy $0 < a < b < c < d < 500$, $a + d = b + c$, and $bc - ad = 93$?

5. Let $p_0(x) = x^3 + 313x^2 - 77x - 8$, and $p_n(x) = p_{n-1}(x-n)$. What is the coefficient of x in $p_{20}(x)$?

6. What is the smallest positive integer that can be expressed as a sum of 9 consecutive integers, and as a sum of 10 consecutive integers, and as a sum of 11 consecutive integers?

7. Six numbers are drawn at random, without replacement, from the set {1, 2, 3, ... , 1000}. Find the probability that a brick whose side lengths are the first three numbers can be placed inside a box with side lengths the second three numbers with the sides of the brick and the box parallel.

8. S has 6 elements. How many ways can we select two (possibly identical) subsets of S whose union is S?

9. Given 2000 points on a circle. Add labels 1, 2, ... , 1993 as follows. Label any point 1. Then count two points clockwise and label the point 2. Then count three points clockwise and label the point 3, and so on. Some points may get more than one label. What is the smallest label on the point labeled 1993?

10. A polyhedron has 32 faces, each of which has 3 or 5 sides. At each of it s V vertices it has T triangles and P pentagons. What is the value of 100P + 10T + V? You may assume Euler's formula (V + F = E + 2, where F is the number of faces and E the number of edges).

11. A and B play a game repeatedly. In each game players toss a fair coin alternately. The first to get a head wins. A starts in the first game, thereafter the loser starts the next game. Find the probability that A wins the sixth game.

12. A = (0, 0), B = (0, 420), C = (560, 0). P_1 is a point inside the triangle ABC. P_n is chosen at random from the midpoints of $P_{n-1}A$, $P_{n-1}B$, and $P_{n-1}C$. If P_7 is (14, 92), find the coordinates of P_1.

13. L, L' are straight lines 200 ft apart. A and A' start 200 feet apart, A on L and A' on L'. A circular building 100 ft in diameter lies midway between the paths and the line joining A and A' touches the building. They begin walking in the same direction (past the building). A walks at 1 ft/sec, A' walks at 3 ft/sec. Find the amount of time before they can see each other again.

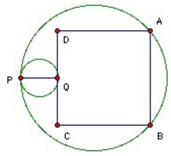
14. R is a 6 x 8 rectangle. R' is another rectangle with one vertex on each side of R. R' can be rotated slightly and still remain within R. Find the smallest perimeter that R' can have.

15. The triangle ABC has AB = 1995, BC = 1993, CA = 1994. CX is an altitude. Find the distance between the points at which the incircles of ACX and BCX touch CX.

12th AIME 1994

1. The sequence 3, 15, 24, 48, ... is those multiples of 3 which are one less than a square. Find the remainder when the 1994th term is divided by 1000.

2. The large circle has diameter 40 and the small circle diameter 10. They touch at P. PQ is a diameter of the small circle. ABCD is a square touching the small circle at Q. Find AB.



3. The function f satisfies $f(x) + f(x-1) = x^2$ for all x. If f(19) = 94, find the remainder when f(94) is divided by 1000.

4. Find n such that $[\log_2 1] + [\log_2 2] + [\log_2 3] + ... + [\log_2 n] = 1994$.

5. What is the largest prime factor of p(1) + p(2) + ... + p(999), where p(n) is the product of the non-zero digits of n?

6. How many equilateral triangles of side $2/\sqrt{3}$ are formed by the lines $y = k$, $y = x\sqrt{3} + 2k$, $y = -x\sqrt{3} + 2k$ for k = -10, -9, ... , 9, 10?

7. For how many ordered pairs (a, b) do the equations $ax + by = 1$, $x^2 + y^2 = 50$ have (1) at least one solution, and (2) all solutions integral?

8. Find ab if (0, 0), (a, 11), (b, 37) is an equilateral triangle.

9. A bag contains 12 tiles marked 1, 1, 2, 2, ... , 6, 6. A player draws tiles one at a time at random and holds them. If he draws a tile matching a tile he already holds, then he discards both. The game ends if he holds three unmatched tiles or if the bag is emptied. Find the probability that the bag is emptied.

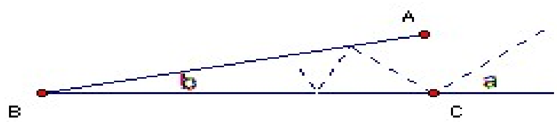
10. ABC is a triangle with $\angle C = 90^o$. CD is an altitude. $BD = 29^3$, and AC, AD, BC are all integers. Find cos B.

11. Given 94 identical bricks, each 4 x 10 x 19, how many different heights of tower can be built (assuming each brick adds 4, 10 or 19 to the height)?

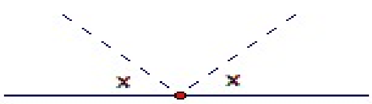
12. A 24 x 52 field is fenced. An additional 1994 of fencing is available. It is desired to divide the entire field into identical square (fenced) plots. What is the largest number that can be obtained?

13. The equation $x^{10} + (13x - 1)^{10} = 0$ has 5 pairs of complex roots $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5$. Each pair a_i, b_i are complex conjugates. Find $\sum 1/(a_i b_i)$.

14. AB and BC are mirrors of equal length. Light strikes BC at C and is reflected to AB. After several reflections it starts to move away from B and emerges again from between the mirrors. How many times is it reflected by AB or BC if $\angle b = 1.994^o$ and $\angle a = 19.94^o$?



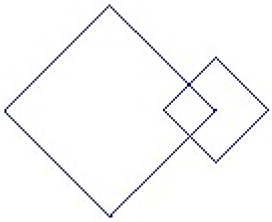
At each reflection the two angles x are equal:



15. ABC is a paper triangle with AB = 36, AC = 72 and $\angle B = 90^o$. Find the area of the set of points P inside the triangle such that if creases are made by folding (and then unfolding) each of A, B, C to P, then the creases do not overlap.

13th AIME 1995

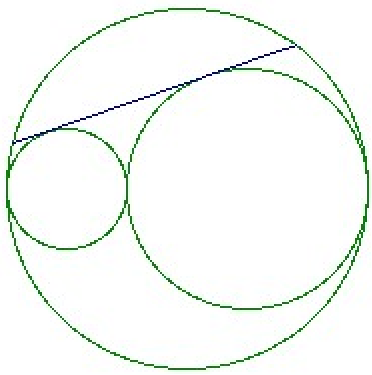
1. Starting with a unit square, a sequence of square is generated. Each square in the sequence has half the side-length of its predecessor and two of its sides bisected by its predecessor's sides as shown. Find the total area enclosed by the first five squares in the sequence.



2. Find the product of the positive roots of $\sqrt{1995}\, x^{\log_{1995} x} = x^2$.

3. A object moves in a sequence of unit steps. Each step is N, S, E or W with equal probability. It starts at the origin. Find the probability that it reaches (2, 2) in less than 7 steps.

4. Three circles radius 3, 6, 9 touch as shown. Find the length of the chord of the large circle that touches the other two.



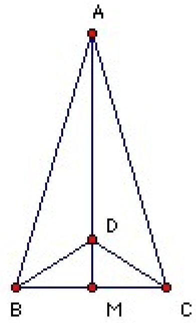
5. Find b if $x^4 + ax^3 + bx^2 + cx + d$ has 4 non-real roots, two with sum 3 + 4i and the other two with product 13 + i.

6. How many positive divisors of n^2 are less than n but do not divide n, if $n = 2^{31}3^{19}$?

7. Find (1 - sin t)(1 - cos t) if (1 + sin t)(1 + cos t) = 5/4.

8. How many ordered pairs of positive integers x, y have $y < x \le 100$ and x/y and (x+1)/(y+1) integers?

9. ABC is isosceles as shown with the altitude AM = 11. AD = 10 and ∠BDC = 3 ∠BAC. Find the perimeter of ABC.



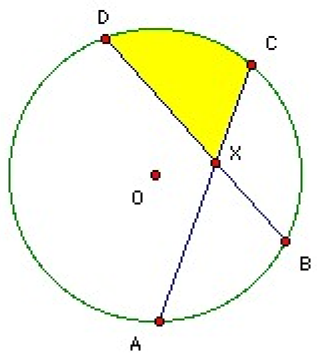
10. What is the largest positive integer that cannot be written as 42a + b, where a and b are positive integers and b is composite?

11. A rectangular block a x 1995 x c, with a ≤ 1995 ≤ c is cut into two non-empty parts by a plane parallel to one of the faces, so that one of the parts is similar to the original. How many possibilities are there for (a, c)?

12. OABCD is a pyramid, with ABCD a square, OA = OB = OC = OD, and $\angle$ AOB = 45°. Find cos θ, where θ is the angle between two adjacent triangular faces.

13. Find $\sum_1^{1995} 1/f(k)$, where f(k) is the closest integer to $k^{1/4}$.

14. O is the center of the circle. AC = BD = 78, OA = 42, OX = 18. Find the area of the shaded area.



15. A fair coin is tossed repeatedly. Find the probability of obtaining five consecutive heads before two consecutive tails.

14th AIME 1996

1. The square below is *magic*. It has a number in each cell. The sums of each row and column and of the two main diagonals are all equal. Find x.

x	19	96
1		

2. For how many positive integers $n < 1000$ is $[\log_2 n]$ positive and even?

3. Find the smallest positive integer n for which $(xy - 3x - 7y - 21)^n$ has at least 1996 terms.

4. A wooden unit cube rests on a horizontal surface. A point light source a distance x above an upper vertex casts a shadow of the cube on the surface. The area of the shadow (excluding the part under the cube) is 48. Find x.

5. The roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, c. The equation with roots a+b, b+c, c+a is $x^3 + rx^2 + sx + t = 0$. Find t.

6. In a tournament with 5 teams each team plays every other team once. Each game ends in a win for one of the two teams. Each team has ½ chance of winning each game. Find the probability that no team wins all its games or loses all its games.

7. 2 cells of a 7 x 7 board are painted black and the rest white. How many different boards can be produced (boards which can be rotated into each other do not count as different).

8. The harmonic mean of a, b > 0 is $2ab/(a + b)$. How many ordered pairs m, n of positive integer with $m < n$ have harmonic mean 6^{20}?

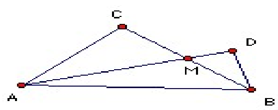
9. There is a line of lockers numbered 1 to 1024, initially all closed. A man walks down the line, opens 1, then alternately skips and opens each closed locker (so he opens 1, 3, 5, ... , 1023). At the end of the line he walks back, opens the first closed locker, then alternately skips and opens each closed locker (so he opens 1024, skips 1022 and so on). He continues to walk up and down the line until all the lockers are open. Which locker is opened last?

10. Find the smallest positive integer n such that $\tan 19n^o = (\cos 96^o + \sin 96^o)/(\cos 96^o - \sin 96^o)$.

11. Let the product of the roots of $z^6 + z^4 + z^3 + z^2 + 1 = 0$ with positive imaginary part be $r(\cos \theta^o + i \sin \theta^o)$. Find θ.

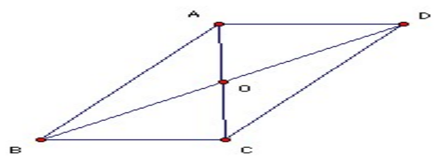
12. Find the average value of $|a_1 - a_2| + |a_3 - a_4| + |a_5 - a_6| + |a_7 - a_8| + |a_9 - a_{10}|$ for all permutations $a_1, a_2, ... , a_{10}$ of 1, 2, ... , 10.

13. $AB = \sqrt{30}$, $BC = \sqrt{15}$, $CA = \sqrt{6}$. M is the midpoint of BC. $\angle ADB = 90^o$. Find area ADB/area ABC.



14. A 150 x 324 x 375 block is made up of unit cubes. Find the number of cubes whose interior is cut by a long diagonal of the block.

15. ABCD is a parallelogram. $\angle BAC = \angle CBD = 2 \angle DBA$. Find $\angle ACB/\angle AOB$, where O is the intersection of the diagonals.



15th AIME 1997

1. How many of 1, 2, 3, ... , 1000 can be written as the difference of the squares of two non-negative integers?

2. The 9 horizontal and 9 vertical lines on an 8 x 8 chessboard form r rectangles including s squares. Find s/r in lowest terms.

3. M is a 2-digit number ab, and N is a 3-digit number cde. We have 9·M·N = abcde. Find M, N.

4. Circles radii 5, 5, 8, k are mutually externally tangent. Find k.

5. The closest approximation to r = 0.abcd (where any of a, b, c, d may be zero) of the form 1/n or 2/n is 2/7. How many possible values are there for r?

6. $A_1A_2...A_n$ is a regular polygon. An equilateral triangle A_1BA_2 is constructed outside the polygon. What is the largest n for which BA_1A_n can be consecutive vertices of a regular polygon?

7. A car travels at 2/3 mile/min due east. A circular storm starts with its center 110 miles due north of the car and travels southeast at $1/\sqrt{2}$ miles/min. The car enters the storm circle at time t_1 mins and leaves it at t_2. Find $(t_1 + t_2)/2$.

8. How many 4 x 4 arrays of 1s and -1s are there with all rows and all columns having zero sum?

9. The real number x has $2 < x^2 < 3$ and the fractional parts of 1/x and x^2 are the same. Find $x^{12} - 144/x$.

10. A card can be red, blue or green, have light, medium or dark shade, and show a circle, square or triangle. There are 27 cards, one for each possible combination. How many possible 3-card subsets are there such that for each of the three characteristics (color, shade, shape) the cards in the subset are all the same or all different?

11. Find $[100(\cos 1^o + \cos 2^o + ... + \cos 44^o)/(\sin 1^o + \sin 2^o + ... + \sin 44^o)]$.

12. a, b, c, d are non-zero reals and f(x) = (ax + b)/(cx + d). We have f(19) = 19, f(97) = 97 and f(f(x)) = x for all x (except -d/c). Find the unique y not in the range of f.

13. Let $S = \{(x, y) : | ||x| - 2| - 1| + | ||y| - 2| - 1| = 1$. If S is made out of wire, what is the total length of wire is required?

14. v, w are roots of $z^{1997} = 1$ chosen at random. Find the probability that $|v + w| >= \sqrt{(2 + \sqrt{3})}$.

15. Find the area of the largest equilateral triangle that can be inscribed in a rectangle with sides 10 and 11.

16th AIME 1998

1. For how many k is $lcm(6^6, 8^8, k) = 12^{12}$?

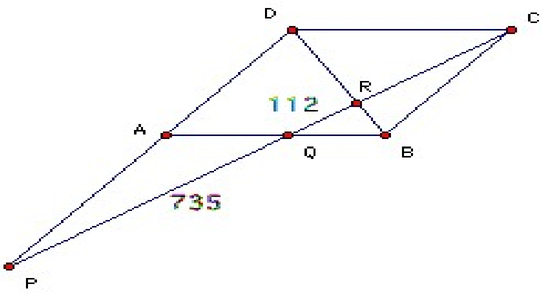
2. How many ordered pairs of positive integers m, n satisfy $m \le 2n \le 60$, $n \le 2m \le 60$?

3. The graph of $y^2 + 2xy + 40|x| = 400$ divides the plane into regions. Find the area of the bounded region.

4. Nine tiles labeled 1, 2, 3, ... , 9 are randomly divided between three players, three tiles each. Find the probability that the sum of each player's tiles is odd.

5. Find $|A_{19} + A_{20} + ... + A_{98}|$, where $A_n = ½n(n-1) \cos(n(n-1)½\pi)$.

6. ABCD is a parallelogram. P is a point on the ray DA such that PQ = 735, QR = 112. Find RC.



7. Find the number of ordered 4-tuples (a, b, c, d) of odd positive integers with sum 98.

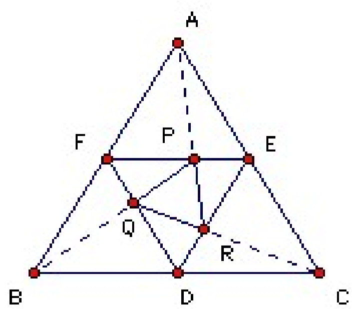
8. The sequence 1000, n, 1000-n, n-(1000-n), ... terminates with the first negative term (the n+2th term is the nth term minus the n+1th term). What positive integer n maximises the length of the sequence?

9. Two people arrive at a cafe independently at random times between 9am and 10am and each stay for m minutes. What is m if there is a 40% chance that they are in the cafe together at some moment.

10. 8 sphere radius 100 rest on a table with their centers at the vertices of a regular octagon and each sphere touching its two neighbors. A sphere is placed in the center so that it touches the table and each of the 8 spheres. Find its radius.

11. A cube has side 20. Two adjacent sides are UVWX and U'VWX'. A lies on UV a distance 15 from V, and F lies on VW a distance 15 from V. E lies on WX' a distance 10 from W. Find the area of intersection of the cube and the plane through A, F, E.

12. ABC is equilateral, D, E, F are the midpoints of its sides. P, Q, R lie on EF, FD, DE respectively such that A, P, R are collinear, B, Q, P, are collinear, and C, R, Q are collinear. Find area ABC/area PQR.



13. Let A be any set of positive integers, so the elements of A are $a_1 < a_2 < ... < a_n$. Let $f(A) = \sum a_k i^k$. Let $S_n = \sum f(A)$, where the sum is taken over all non-empty subsets A of {1, 2, ... , n}. Given that $S_8 = -176-64i$, find S_9.

14. An a x b x c box has half the volume of an (a+2) x (b+2) x (c+2) box, where $a \le b \le c$. What is the largest possible c?

15. D is the set of all 780 dominos [m,n] with $1 \le m < n \le 40$ (note that unlike the familiar case we cannot have m = n). Each domino [m,n] may be placed in a line as [m,n] or [n,m]. What is the longest possible line of dominos such that if [a,b][c,d] are adjacent then b = c?

17th AIME 1999

1. Find the smallest a_5, such that a_1, a_2, a_3, a_4, a_5 is a strictly increasing arithmetic progression with all terms prime.

2. A line through the origin divides the parallelogram with vertices (10, 45), (10, 114), (28, 153), (28, 84) into two congruent pieces. Find its slope.

3. Find the sum of all positive integers n for which $n^2 - 19n + 99$ is a perfect square.

4. Two squares side 1 are placed so that their centers coincide. The area inside both squares is an octagon. One side of the octagon is 43/99. Find its area.

5. For any positive integer n, let t(n) be the (non-negative) difference between the digit sums of n and n+2. For example t(199) = |19 - 3| = 16. How many possible values t(n) are less than 2000?

6. A map T takes a point (x, y) in the first quadrant to the point ($\sqrt{x}$, $\sqrt{y}$). Q is the quadrilateral with vertices (900, 300), (1800, 600), (600, 1800), (300, 900). Find the greatest integer not exceeding the area of T(Q).

7. A rotary switch has four positions A, B, C, D and can only be turned one way, so that it can be turned from A to B, from B to C, from C to D, or from D to A. A group of 1000 switches are all at position A. Each switch has a unique label $2^a3^b5^c$, where a, b, c = 0, 1, 2, ... , or 9. A 1000 step process is now carried out. At each step a different switch S is taken and all switches whose labels divide the label of S are turned one place. For example, if S was $2\cdot3\cdot5$, then the 8 switches with labels 1, 2, 3, 5, 6, 10, 15, 30 would each be turned one place. How many switches are in position A after the process has been completed?

8. T is the region of the plane x + y + z = 1 with x,y,z $\geq$0. S is the set of points (a, b, c) in T such that just two of the following three inequalities hold: $a \leq 1/2$, $b \leq 1/3$, $c \leq 1/6$. Find area S/area T.

9. f is a complex-valued function on the complex numbers such that function f(z) = (a + bi)z, where a and b are real and |a + ib| = 8. It has the property that f(z) is always equidistant from 0 and z. Find b.

10. S is a set of 10 points in the plane, no three collinear. There are 45 segments joining two points of S. Four distinct segments are chosen at random from the 45. Find the probability that three of these segments form a triangle (so they all involve two from the same three points in S).

11. Find $\sin 5^o + \sin 10^o + \sin 15^o + ... + \sin 175^o$. You may express the answer as tan(a/b).

12. The incircle of ABC touches AB at P and has radius 21. If AP = 23 and PB = 27, find the perimeter of ABC.

13. 40 teams play a tournament. Each team plays every other team just once. Each game results in a win for one team. If each team has a 50% chance of winning each game, find the probability that at the end of the tournament every team has won a different number of games.

14. P lies inside the triangle ABC, and angle PAB = angle PBC = angle PCA. If AB = 13, BC = 14, CA = 15, find tan PAB.

15. A paper triangle has vertices (0, 0), (34, 0), (16, 24). The midpoint triangle has as its vertices the midpoints of the sides. The paper triangle is folded along the sides of its midpoint triangle to form a pyramid. What is the volume of the pyramid?

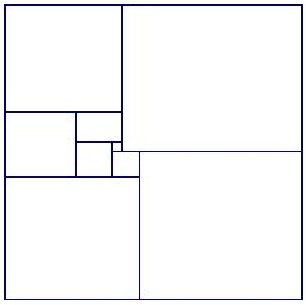
La18th AIME1 2000

1. Find the smallest positive integer n such that if $10^n = M \cdot N$, where M and N are positive integers, then at least one of M and N must contain the digit 0.

2. m, n are integers with $0 < n < m$. A is the point (m, n). B is the reflection of A in the line y = x. C is the reflection of B in the y-axis, D is the reflection of D in the x-axis, and E is the reflection of D in the y-axis. The area of the pentagon ABCDE is 451. Find u + v.

3. m, n are relatively prime positive integers. The coefficients of x^2 and x^3 in the expansion of $(mx + b)^{2000}$ are equal. Find m + n.

4. The figure shows a rectangle divided into 9 squares. The squares have integral sides and adjacent sides of the rectangle are coprime. Find the perimeter of the rectangle.



5. Two boxes contain between them 25 marbles. All the marbles are black or white. One marble is taken at random from each box. The probability that both marbles are black is 27/50. If the probability that both marbles are white is m/n, where m and n are relatively prime, find m + n.

6. How many pairs of positive integers m, n have $n < m < 1000000$ and their arithmetic mean equal to their geometric mean plus 2?

7. x, y, z are positive reals such that $xyz = 1$, $x + 1/z = 5$, $y + 1/x = 29$. Find $z + 1/y$.

8. A sealed conical vessel is in the shape of a right circular cone with height 12, and base radius 5. The vessel contains some liquid. When it is held point down with the base horizontal the liquid is 9 deep. How deep is it when the container is held point up and base horizontal?

9. Find the real solutions to: $\log_{10}(2000xy) - \log_{10}x \log_{10}y = 4$, $\log_{10}(2yz) - \log_{10}y \log_{10}z = 1$, $\log_{10}zx - \log_{10}z \log_{10}x = 0$.

10. The sequence $x_1, x_2, \ldots, x_{100}$ has the property that, for each k, x_k is k less than the sum of the other 99 numbers. Find x_{50}.

11. Find [S/10], where S is the sum of all numbers m/n, where m and n are relatively prime positive divisors of 1000.

12. The real-valued function f on the reals satisfies $f(x) = f(398 - x) = f(2158 - x) = f(3214 - x)$. What is the largest number of distinct values that can appear in f(0), f(1), f(2), ... , f(999)?

13. A fire truck is at the intersection of two straight highways in the desert. It can travel at 50mph on the highway and at 14mph over the desert. Find the area it can reach in 6 mins.

14. Triangle ABC has AB = AC. P lies on AC, and Q lies on AB. We have AP = PQ = QB = BC. Find angle ACB/angle APQ.

15. There are cards labeled from 1 to 2000. The cards are shuffled and placed in a pile. The top card is placed on the table, then the next card at the bottom of the pile. Then the next card is placed on the table to the right of the first card, and the next card is placed at the bottom of the pile. This process is continued until all the cards are on the table. The final order (from left to right) is 1, 2, 3, ... , 2000. In the original pile, how many cards were above card 1999?

18th AIME2 2000

1. Find $2/\log_4(2000^6) + 3/\log_5(2000^6)$.

2. How many lattice points lie on the hyperbola $x^2 - y^2 = 2000^2$?

3. A deck of 40 cards has four each of cards marked 1, 2, 3, ... 10. Two cards with the same number are removed from the deck. Find the probability that two cards randomly selected from the remaining 38 have the same number as each other.

4. What is the smallest positive integer with 12 positive even divisors and 6 positive odd divisors?

5. You have 8 different rings. Let n be the number of possible arrangements of 5 rings on the four fingers of one hand (each finger has zero or more rings, and the order matters). Find the three leftmost non-zero digits of n.

6. A trapezoid ABCD has AB parallel to DC, and DC = AB + 100. The line joining the midpoints of AD and BC divides the trapezoid into two regions with areas in the ratio 2 : 3. Find the length of the segment parallel to DC that joins AD and BC and divides the trapezoid into two regions of equal area.

7. Find $1/(2!\ 17!) + 1/(3!\ 16!) + ... + 1/(9!\ 10!)$.

8. The trapezoid ABCD has AB parallel to DC, BC perpendicular to AB, and AC perpendicular to BD. Also $AB = \sqrt{11}$, $AD = \sqrt{1001}$. Find BC.

9. z is a complex number such that $z + 1/z = 2 \cos 3^o$. Find $[z^{2000} + 1/z^{2000}] + 1$.

10. A circle radius r is inscribed in ABCD. It touches AB at P and CD at Q. AP = 19, PB = 26, CQ = 37, QD = 23. Find r.

11. The trapezoid ABCD has AB and DC parallel, and AD = BC. A, D have coordinates (20,100), (21,107) respectively. No side is vertical or horizontal, and AD is not parallel to BC. B and C have integer coordinates. Find the possible slopes of AB.

12. A, B, C lie on a sphere center O radius 20. AB = 13, BC = 14, CA = 15. `Find the distance of O from the triangle ABC.

13. The equation $2000x^6 + 100x^5 + 10x^3 + x - 2 = 0$ has just two real roots. Find them.

14. Every positive integer k has a unique *factorial expansion* $k = a_1\ 1! + a_2\ 2! + ... + a_m\ m!$, where $m+1 > a_m > 0$, and $i+1 > a_i \geq 0$. Given that $16! - 32! + 48! - 64! + ... + 1968! - 1984! + 2000! = a_1\ 1! + a_2\ 2! + ... + a_n\ n!$, find $a_1 - a_2 + a_3 - a_4 + ... + (-1)^{j+1}\ a_j$.

15. Find the least positive integer n such that $1/(\sin 45^o \sin 46^o) + 1/(\sin 47^o \sin 48^o) + ... + 1/(\sin 133^o \sin 134^o) = 2/\sin n^o$.

19th AIME1 2001

1. Find the sum of all positive two-digit numbers that are divisible by both their digits.

2. Given a finite set A of reals let m(A) denote the mean of its elements. S is such that $m(S \square \{1\}) = m(S) - 13$ and $m(S \square \{2001\}) = m(S) + 27$. Find m(S).

3. Find the sum of the roots of the polynomial $x^{2001} + (½ - x)^{2001}$.

4. The triangle ABC has $\angle A = 60^o$, $\angle B = 45^o$. The bisector of $\angle A$ meets BC at T where AT = 24. Find area ABC.

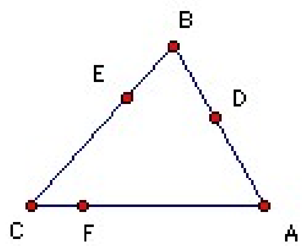
5. An equilateral triangle is inscribed in the ellipse $x^2 + 4y^2 = 4$, with one vertex at (0,1) and the corresponding altitude along the y-axis. Find its side length.

6. A fair die is rolled four times. Find the probability that each number is no smaller than the preceding number.

7. A triangle has sides 20, 21, 22. The line through the incenter parallel to the shortest side meets the other two sides at X and Y. Find XY.

8. A number n is called a *double* if its base-7 digits form the base-10 number 2n. For example, 51 is 102 in base 7. What is the largest double?

9. ABC is a triangle with AB = 13, BC = 15, CA = 17. Points D, E, F on AB, BC, CA respectively are such that AD/AB = α, BE/BC = β, CF/CA = γ, where $\alpha + \beta + \gamma = 2/3$, and $\alpha^2 + \beta^2 + \gamma^2 = 2/5$. Find area DEF/area ABC.



10. S is the array of lattice points (x, y, z) with x = 0, 1 or 2, y = 0, 1, 2, or 3 and z = 0, 1, 2, 3 or 4. Two distinct points are chosen from S at random. Find the probability that their midpoint is in S.

11. 5N points form an array of 5 rows and N columns. The points are numbered left to right, top to bottom (so the first row is 1, 2, ... , N, the second row N+1, ... , 2N, and so on). Five points, $P_1, P_2, \ldots, P_5$ are chosen, P_1 in the first row, P_2 in the second row and so on. P_i has number x_i. The points are now renumbered top to bottom, left to right (so the first column is 1, 2, 3, 4, 5 the second column 6, 7, 8, 9, 10 and so on). P_i now has number y_i. We find that $x_1 = y_2$, $x_2 = y_1$, $x_3 = y_4$, $x_4 = y_5$, $x_5 = y_3$. Find the smallest possible value of N.

12. Find the inradius of the tetrahedron vertices (6,0,0), (0,4,0), (0,0,2) and (0,0,0).

13. The chord of an arc of $\angle d$ (where $d < 120^o$) is 22. The chord of an arc of $\angle 2d$ is x+20, and the chord of an arc of $\angle 3d$ is x. Find x.

14. How many different 19-digit binary sequences do not contain the subsequences 11 or 000?

15. The labels 1, 2, ... , 8 are randomly placed on the faces of an octahedron (one per face). Find the probability that no two adjacent faces (sharing an edge) have adjacent numbers, where 1 and 8 are also considered adjacent.

19th AIME2 2001

1. Find the largest positive integer such that each pair of consecutive digits forms a perfect square (eg 364).

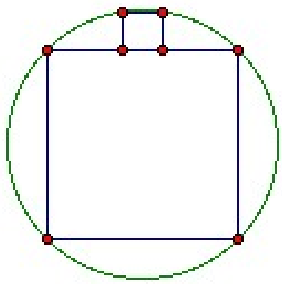
2. A school has 2001 students. Between 80% and 85% study Spanish, between 30% and 40% study French, and no one studies neither. Find m be the smallest number who could study both, and M the largest number.

3. The sequence a_1, a_2, a_3, ... is defined by $a_1 = 211$, $a_2 = 375$, $a_3 = 420$, $a_4 = 523$, $a_n = a_{n-1} - a_{n-2} + a_{n-3} - a_{n-4}$. Find $a_{531} + a_{753} + a_{975}$.

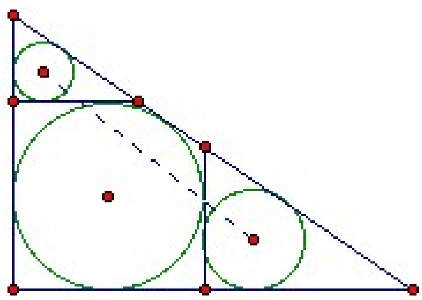
4. P lies on $8y = 15x$, Q lies on $10y = 3x$ and the midpoint of PQ is (8,6). Find the distance PQ.

5. A set of positive numbers has the *triangle property* if it has three elements which are the side lengths of a non-degenerate triangle. Find the largest n such that every 10-element subset of {4, 5, 6, ... , n} has the triangle property.

6. Find the area of the large square divided by the area of the small square.



7. The triangle is right-angled with sides 90, 120, 150. The common tangents inside the triangle are parallel to the two sides Find the length of the dashed line joining the centers of the two small circles.



8. The function f(x) satisfies $f(3x) = 3f(x)$ for all real x, and $f(x) = 1 - |x-2|$ for $1 \le x \le 3$. Find the smallest positive x for which $f(x) = f(2001)$.

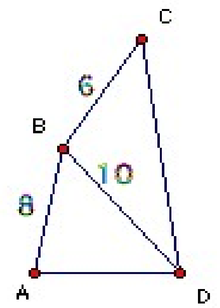
9. Each square of a 3 x 3 board is colored either red or blue at random (each with probability ½). Find the probability that there is no 2 x 2 red square.

10. How many integers $10^i - 10^j$ where $0 \le j < i \le 99$ are multiples of 1001?

11. In a tournament club X plays each of the 6 other sides once. For each match the probabilities of a win, draw and loss are equal. Find the probability that X finishes with more wins than losses.

12. The midpoint triangle of a triangle is that obtained by joining the midpoints of its sides. A regular tetrahedron has volume 1. On the outside of each face a small regular tetrahedron is placed with the midpoint triangle as its base, thus forming a new polyhedron. This process is carried out twice more (three times in all). Find the volume of the resulting polyhedron.

13. ABCD is a quadrilateral with AB = 8, BC = 6, BD = 10, $\angle A = \angle D$ and $\angle ABD = \angle C$. Find CD.



14. Find all the values $0 \le \theta < 360^o$ for which the complex number $z = \cos\theta + i\sin\theta$ satisfies $z^{28} - z^8 - 1 = 0$.

15. A cube has side 8. A hole with triangular cross-section is bored along a long diagonal. At one vertex it removes the last 2 units of each of the three edges at that vertex. The three sides of the hole are parallel to the long diagonal. Find the surface area of the part of the cube that is left (including the area of the inside of the hole).

20th AIME1 2002

1. A licence plate is 3 letters followed by 3 digits. If all possible licence plates are equally likely, what is the probability that a plate has either a letter palindrome or a digit palindrome (or both)?

2. 20 equal circles are packed in honeycomb fashion in a rectangle. The outer rows have 7 circles, and the middle row has 6. The outer circles touch the sides of the rectangle. Find the long side of the rectangle divided by the short side.

3. Jane is 25. Dick's age is d > 25. In n years both will have two-digit ages which are obtained by transposing digits (so if Jane will be 36, Dick will be 63). How many possible pairs (d, n) are there?

4. The sequence $x_1, x_2, x_3, \ldots$ is defined by $x_k = 1/(k^2 + k)$. A sum of consecutive terms $x_m + x_{m+1} + \ldots + x_n = 1/29$. Find m and n.

5. D is a regular 12-gon. How many squares (in the plane of D) have two or more of their vertices as vertices of D?

6. The solutions to $\log_{225}x + \log_{64}y = 4$, $\log_x 225 - \log_y 64 = 1$ are $(x, y) = (x_1, y_1)$ and (x_2, y_2). Find $\log_{30}(x_1y_1x_2y_2)$.

7. What are the first three digits after the decimal point in $(10^{2002} + 1)^{10/7}$? You may use the extended binomial theorem: $(x + y)^r = x^r(1 + 4 (y/x) + r(r-1)/2! (y/x)^2 + r(r-1)(r-2)/3! (y/x)^3 + \ldots)$ for r real and $|x/y| < 1$.

8. Find the smallest integer k for which there is more than one non-decreasing sequence of positive integers $a_1, a_2, a_3, \ldots$ such that $a_9 = k$ and $a_{n+2} = a_{n+1} + a_n$.

9. A, B, C paint a long line of fence-posts. A paints the first, then every ath, B paints the second then every bth, C paints the third, then every cth. Every post gets painted just once. Find all possible triples (a, b, c).

10. ABC is a triangle with angle $B = 90^o$. AD is an angle bisector. E lies on the side AB with AE = 3, EB = 9, and F lies on the side AC with AF = 10, FC = 27. EF meets AD at G. Find the nearest integer to area GDCF.

11. A cube with two faces ABCD, BCEF, has side 12. The point P is on the face BCEF a perpendicular distance 5 from the edge BC and from the edge CE. A beam of light leaves A and travels along AP, at P it is reflected inside the cube. Each time it strikes a face it is reflected. How far does it travel before it hits a vertex?

12. The complex sequence $z_0, z_1, z_2, \ldots$ is defined by $z_0 = i + 1/137$ and $z_{n+1} = (z_n + i)/(z_n - i)$. Find z_{2002}.

13. The triangle ABC has AB = 24. The median CE is extended to meet the circumcircle at F. CE = 27, and the median AD = 18. Find area ABF.

14. S is a set of positive integers containing 1 and 2002. No elements are larger than 2002. For every n in S, the arithmetic mean of the other elements of S is an integer. What is the largest possible number of elements of S?

15. ABCDEFGH is a polyhedron. Face ABCD is a square side 12. Face ABFG is a trapezoid with GF parallel to AB and GF = 6, AG = BF = 8. Face CDE is an isosceles triangle with ED = EC = 14. E is a distance 12 from the plane ABCD. The other faces are EFG, ADEG and BCEF. Find EG^2.

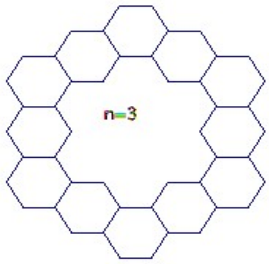
20th AIME2 2002

1. n is an integer between 100 and 999 inclusive, and so is n' the integer formed by reversing its digits. How many possible values are there for |n-n'|?

2. P (7,12,10), Q (8,8,1) and R (11,3,9) are three vertices of a cube. What is its surface area?

3. a, b, c are positive integers forming an increasing geometric sequence, b-a is a square, and $\log_6 a + \log_6 b + \log_6 c = 6$. Find a + b + c.

4. Hexagons with side 1 are used to form a large hexagon. The diagram illustrates the case n = 3 with three unit hexagons on each side of the large hexagon. Find the area enclosed by the unit hexagons in the case n = 202.



5. Find the sum of all positive integers $n = 2^a 3^b$ $(a, b \geq 0)$ such that n^6 does not divide 6^n.

6. Find the integer closest to $1000 \sum_3^{10000} 1/(n^2-4)$.

7. Find the smallest n such that $\sum_1^n k^2$ is a multiple of 200. You may assume $\sum_1^n k^2 = n(n+1)(2n+1)/6$.

8. Find the smallest positive integer n for which there are no integer solutions to [2002/x] = n.

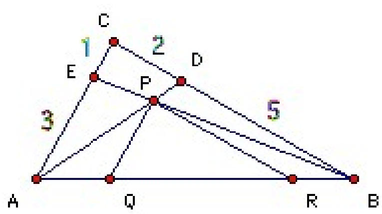
9. Let S = {1, 2, ... , 10}. Find the number of unordered pairs A, B, where A and B are disjoint non-empty subsets of S.

10. Find the two smallest positive values of x for which $\sin(x^o) = \sin(x \text{ rad})$.

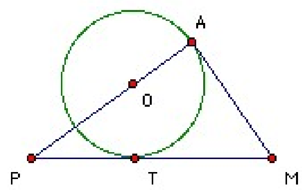
11. Two different geometric progressions both have sum 1 and the same second term. One has third term 1/8. Find its second term.

12. An unfair coin is tossed 10 times. The probability of heads on each toss is 0.4. Let a_n be the number of heads in the first n tosses. Find the probability that $a_n/n \leq 0.4$ for n = 1, 2, ... , 9 and $a_{10}/10 = 0.4$.

13. ABC is a triangle, D lies on the side BC and E lies on the side AC. AE = 3, EC = 1, CD = 2, DB = 5, AB = 8. AD and BE meet at P. The line parallel to AC through P meets AB at Q, and the line parallel to BC through P meets AB at R. Find area PQR/area ABC.



14. Triangle APM has $\angle A = 90^o$ and perimeter 152. A circle center O (on AP) has radius 19 and touches AM at A and PM at T. Find OP.



15. Two circles touch the x-axis and the line $y = mx$ ($m > 0$). They meet at (9,6) and another point and the product of their radii is 68. Find m.

21st AIME1 2003

1. Find positive integers k, n such that $k \cdot n! = (((3!)!)!/3!$ and n is as large as possible.

2. Concentric circles radii 1, 2, 3, ... , 100 are drawn. The interior of the smallest circle is colored red and the annular regions are colored alternately green and red, so that no two adjacent regions are the same color. Find the total area of the green regions divided by the area of the largest circle.

3. S = {1, 2, 3, 5, 8, 13, 21, 34}. Find $\sum$ max(A) where the sum is taken over all 28 two-element subsets A of S.

4. Find n such that $\log_{10}\sin x + \log_{10}\cos x = -1$, $\log_{10}(\sin x + \cos x) = (\log_{10}n - 1)/2$.

5. Find the volume of the set of points that are inside or within one unit of a rectangular 3 x 4 x 5 box.

6. Let S be the set of vertices of a unit cube. Find the sum of the areas of all triangles whose vertices are in S.

7. The points A, B, C lie on a line in that order with AB = 9, BC = 21. Let D be a point not on AC such that AD = CD and the distances AD and BD are integral. Find the sum of all possible n, where n is the perimeter of triangle ACD.

8. $0 < a < b < c < d$ are integers such that a, b, c is an arithmetic progression, b, c, d is a geometric progression, and d - a = 30. Find a + b + c + d.

9. How many four-digit integers have the sum of their two leftmost digits equals the sum of their two rightmost digits?

10. Triangle ABC has AC = BC and $\angle ACB = 106^o$. M is a point inside the triangle such that $\angle MAC = 7^o$ and $\angle MCA = 23^o$. Find $\angle CMB$.

11. The angle x is chosen at random from the interval $0^o < x < 90^o$. Find the probability that there is no triangle with side lengths $\sin^2x$, $\cos^2x$ and sin x cos x.

12. ABCD is a convex quadrilateral with AB = CD = 180, perimeter 640, AD $\neq$ BC, and $\angle A = \angle C$. Find cos A.

13. Find the number of 1, 2, ... , 2003 which have more 1s than 0s when written in base 2.

14. When written as a decimal, the fraction m/n (with m < n) contains the consecutive digits 2, 5, 1 (in that order). Find the smallest possible n.

15. AB = 360, BC = 507, CA = 780. M is the midpoint of AC, D is the point on AC such that BD bisects $\angle ABC$. F is the point on BC such that BD and DF are perpendicular. The lines FD and BM meet at E. Find DE/EF.

La21st AIME2 2003

1. The product N of three positive integers is 6 times their sum. One of the integers is the sum of the other two. Find the sum of all possible values of N.

2. N is the largest multiple of 8 which has no two digits the same. What is N mod 1000?

3. How many 7-letter sequences are there which use only A, B, C (and not necessarily all of those), with A never immediately followed by B, B never immediately followed by C, and C never immediately followed by A?

4. T is a regular tetrahedron. T' is the tetrahedron whose vertices are the midpoints of the faces of T. Find vol T'/vol T.

5. A log is in the shape of a right circular cylinder diameter 12. Two plane cuts are made, the first perpendicular to the axis of the log and the second at a 45o angle to the first, so that the line of intersection of the two planes touches the log at a single point. The two cuts remove a wedge from the log. Find its volume.

6. A triangle has sides 13, 14, 15. It is rotated through 180^o about its centroid to form an overlapping triangle. Find the area of the union of the two triangles.

7. ABCD is a rhombus. The circumradii of ABD, ACD are 12.5, 25. Find the area of the rhombus.

8. Corresponding terms of two arithmetic progressions are multiplied to give the sequence 1440, 1716, 1848, ... Find the eighth term.

9. The roots of $x^4 - x^3 - x^2 - 1 = 0$ are a, b, c, d. Find p(a) + p(b) + p(c) + p(d), where $p(x) = x^6 - x^5 - x^3 - x^2 - x$.

10. Find the largest possible integer n such that $\sqrt{n} + \sqrt{(n+60)} = \sqrt{m}$ for some non-square integer m.

11. ABC has AC = 7, BC = 24, angle C = 90^o. M is the midpoint of AB, D lies on the same side of AB as C and had DA = DB = 15. Find area CDM.

12. n people vote for one of 27 candidates. Each candidate's percentage of the vote is at least 1 less than his number of votes. What is the smallest possible value of n? (So if a candidate gets m votes, then $100m/n \le m-1$.)

13. A bug moves around a wire triangle. At each vertex it has 1/2 chance of moving towards each of the other two vertices. What is the probability that after crawling along 10 edges it reaches its starting point?

14. ABCDEF is a convex hexagon with all sides equal and opposite sides parallel. Angle FAB = 120^o. The y-coordinates of A, B are 0, 2 respectively, and the y-coordinates of the other vertices are 4, 6, 8, 10 in some order. Find its area.

15. The distinct roots of the polynomial $x^{47} + 2x^{46} + 3x^{45} + ... + 24x^{24} + 23x^{23} + 22x^{22} + ... + 2x^2 + x$ are $z_1, z_2, ... , z_n$. Let z_k^2 have imaginary part b_ki. Find $|b_1| + |b_2| + ... + |b_n|$.

22nd AIME1 2004

1. n has 4 digits, which are consecutive integers in decreasing order (from left to right). Find the sum of the possible remainders when n is divided by 37.

2. The set A consists of m consecutive integers with sum 2m. The set B consists of 2m consecutive integers with sum m. The difference between the largest elements of A and B is 99. Find m.

3. P is a convex polyhedron with 26 vertices, 60 edges and 36 faces. 24 of the faces are triangular and 12 are quadrilaterals. A *space diagonal* is a line segment connecting two vertices which do not belong to the same face. How many space diagonals does P have?

4. A square X has side 2. S is the set of all segments length 2 with endpoints on adjacent sides of X. The midpoints of the segments in S enclose a region with area A. Find 100A to the nearest whole number.

5. A and B took part in a two-day maths contest. At the end both had attempted questions worth 500 points. A scored 160 out of 300 attempted on the first day and 140 out of 200 attempted on the second day, so his two-day success ratio was 300/500 = 3/5. B's attempted figures were different from A's (but with the same two-day total). B had a positive integer score on each day. For each day B's success ratio was less than A's. What is the largest possible two-day success ratio that B could have achieved?

6. An integer is *snakelike* if its decimal digits $d_1d_2...d_k$ satisfy $d_i < d_{i+1}$ for i odd and $d_i > d_{i+1}$ for i even. How many snakelike integers between 1000 and 9999 have four distinct digits?

7. Find the coefficient of x^2 in the polynomial (1-x)(1+2x)(1-3x)...(1+14x)(1-15x).

8. A *regular n-star* is the union of n equal line segments P_1P_2, P_2P_3, ... , P_nP_1 in the plane such that the angles at P_i are all equal and the path $P_1P_2...P_nP_1$ turns counterclockwise through an angle less than 180° at each vertex. There are no regular 3-stars, 4-stars or 6-stars, but there are two non-similar regular 7-stars. How many non-similar regular 1000-stars are there?

9. ABC is a triangle with sides 3, 4, 5 and DEFG is a 6 x 7 rectangle. A line divides ABC into a triangle T_1 and a trapezoid R_1. Another line divides the rectangle into a triangle T_2 and a trapezoid R_2, so that T_1 and T_2 are similar, and R_1 and R_2 are similar. Find the smallest possible value of area T_1.

10. A circle radius 1 is randomly placed so that it lies entirely inside a 15 x 36 rectangle ABCD. Find the probability that it does not meet the diagonal AC.

11. The surface of a right circular cone is painted black. The cone has height 4 and its base has radius 3. It is cut into two parts by a plane parallel to the base, so that the volume of the top part (the small cone) divided by the volume of the bottom part (the frustrum) equals k and painted area of the top part divided by the painted are of the bottom part also equals k. Find k.

12. Let S be the set of all points (x,y) such that x, y ∈ (0,1], and $[\log_2(1/x)]$ and $[\log_5(1/y)]$ are both even. Find area S.

13. The roots of the polynomial $(1 + x + x^2 + ... + x^{17})^2 - x^{17}$ are $r_k e^{i2\pi a_k}$, for k = 1, 2, ... , 34 where $0 < a_1 \le a_2 \le ... \le a_{34} < 1$ and r_k are positive. Find $a_1 + a_2 + a_3 + a_4 + a_5$.

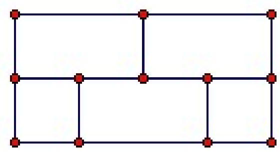
14. A unicorn is tethered by a rope length 20 to the base of a cylindrical tower. The rope is attached to the tower at ground level and to the unicorn at height 4 and pulled tight. The unicorn's end of the rope is a distance 4 from the nearest point of the tower. Find the length of the rope which is in contact with the tower.

15. Define f(1) = 1, f(n) = n/10 if n is a multiple of 10 and f(n) = n+1 otherwise. For each positive integer m define the sequence a_1, a_2, a_3, ... by a_1 = m, $a_{n+1} = f(a_n)$. Let g(m) be the smallest n such that a_n = 1. For example, g(100) = 3, g(87) = 7. Let N be the number of positive integers m such that g(m) = 20. How many distinct prime factors does N have?

ASU (1961 – 2002)

1st ASU 1961 problems

1. Given 12 vertices and 16 edges arranged as follows:



Draw any curve which does not pass through any vertex. Prove that the curve cannot intersect each edge just once. Intersection means that the curve crosses the edge from one side to the other. For example, a circle which had one of the edges as tangent would not intersect that edge.

2. Given a rectangle ABCD with AC length e and four circles centers A, B, C, D and radii a, b, c, d respectively, satisfying $a+c=b+d<e$. Prove you can inscribe a circle inside the quadrilateral whose sides are the two outer common tangents to the circles center A and C, and the two outer common tangents to the circles center B and D.

3. Prove that any 39 successive natural numbers include at least one whose digit sum is divisible by 11.

4. (a) Arrange 7 stars in the 16 places of a 4 x 4 array, so that no 2 rows and 2 columns contain all the stars.

(b) Prove this is not possible for <7 stars.

5. (a) Given a quadruple (a, b, c, d) of positive reals, transform to the new quadruple (ab, bc, cd, da). Repeat arbitarily many times. Prove that you can never return to the original quadruple unless a=b=c=d=1.

(b) Given n a power of 2, and an n-tuple $(a_1, a_2, \dots, a_n)$ transform to a new n-tuple $(a_1a_2, a_2a_3, \dots, a_{n-1}a_n, a_na_1)$. If all the members of the original n-tuple are 1 or -1, prove that with sufficiently many repetitions you obtain all 1s.

6. (a) A and B move clockwise with equal angular speed along circles center P and Q respectively. C moves continuously so that AB=BC=CA. Establish C's locus and speed.

(b) ABC is an equilateral triangle and P satisfies AP=2, BP=3. Establish the maximum possible value of CP.

7. Given an m x n array of real numbers. You may change the sign of all numbers in a row or of all numbers in a column. Prove that by repeated changes you can obtain an array with all row and column sums non-negative.

8. Given n<1 points, some pairs joined by an edge (an edge never joins a point to itself). Given any two distinct points you can reach one from the other in just one way by moving along edges. Prove that there are n-1 edges.

9. Given any natural numbers m, n and k. Prove that we can always find relatively prime natural numbers r and s such that rm+sn is a multiple of k.

10. A and B play the following game with N counters. A divides the counters into 2 piles, each with at least 2 counters. Then B divides each pile into 2 piles, each with at least one counter. B then takes 2 piles according to a rule which both of them know, and A takes the remaining 2 piles. Both A and B make their choices in order to end up with as many counters as possible. There are 3 possibilities for the rule:

R1 B takes the biggest heap (or one of them if there is more than one) and the smallest heap (or one of them if there is more than one).

R2 B takes the two middling heaps (the two heaps that A would take under *R1*).

R3 B has the choice of taking either the biggest and smallest, or the two middling heaps. For each rule, how many counters will A get if both players play optimally?

11. Given three arbitary infinite sequences of natural numbers, prove that we can find unequal natural numbers m, n such that for each sequence the mth member is not less than the nth member.

***12.** 120 unit squares are arbitarily arranged in a 20 x 25 rectangle (both position and orientation is arbitary). Prove that it is always possible to place a circle of unit diameter inside the rectangle without intersecting any of the squares.

2nd ASU 1962 problems

1. ABCD is any convex quadrilateral. Construct a new quadrilateral as follows. Take A' so that A is the midpoint of DA'; similarly, B' so that B is the midpoint of AB'; C' so that C is the midpoint of BC'; and D' so that D is the midpoint of CD'. Show that the area of A'B'C'D' is five times the area of ABCD.

2. Given a fixed circle C and a line L throught the center O of C. Take a variable point P on L and let K be the circle center P through O. Let T be the point where a common tangent to C and K meets K. What is the locus of T?

3. Given integers $a_0, a_1, \ldots, a_{100}$, satisfying $a_1>a_0$, $a_1>0$, and $a_{r+2}=3\, a_{r+1} - 2\, a_r$ for r=0, 1, ... , 98.

Prove $a_{100} > 2^{99}$

4. Prove that there are no integers a, b, c, d such that the polynomial ax^3+bx^2+cx+d equals 1 at x=19 and 2 at x=62.

5. Given an n x n array of numbers. n is odd and each number in the array is 1 or -1. Prove that the number of rows and columns containing an odd number of -1s cannot total n.

6. Given the lengths AB and BC and the fact that the medians to those two sides are perpendicular, construct the triangle ABC.

7. Given four positive real numbers a, b, c, d such that abcd=1, prove that $a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd \geq 10$.

8. Given a fixed regular pentagon ABCDE with side 1. Let M be an arbitary point inside or on it. Let the distance from M to the closest vertex be r_1, to the next closest be r_2 and so on, so that the distances from M to the five vertices satisfy $r_1 \leq r_2 \leq r_3 \leq r_4 \leq r_5$. Find (a) the locus of M which gives r_3 the minimum possible value, and (b) the locus of M which gives r_3 the maximum possible value.

9. Given a number with 1998 digits which is divisible by 9. Let x be the sum of its digits, let y be the sum of the digits of x, and z the sum of the digits of y. Find z.

10. AB=BC and M is the midpoint of AC. H is chosen on BC so that MH is perpendicular to BC. P is the midpoint of MH. Prove that AH is perpendicular to BP.

11. The triangle ABC satisfies $0 \leq AB \leq 1 \leq BC \leq 2 \leq CA \leq 3$. What is the maximum area it can have?

12. Given unequal integers x, y, z prove that $(x-y)^5 + (y-z)^5 + (z-x)^5$ is divisible by $5(x-y)(y-z)(z-x)$.

13. Given $a_0, a_1, \ldots, a_n$, satisfying $a_0 = a_n = 0$, and and $a_{k-1} - 2a_k + a_{k+1} \geq 0$ for k=0, 1, ... , n-1. Prove that all the numbers are negative or zero.

14. Given two sets of positive numbers with the same sum. The first set has m numbers and the second n. Prove that you can find a set of less than m+n positive numbers which can be arranged to part fill an m x n array, so that the row and column sums are the two given sets.

3rd ASU 1963 problems

1. Given 5 circles. Every 4 have a common point. Prove that there is a point common to all 5.

2. 8 players compete in a tournament. Everyone plays everyone else just once. The winner of a game gets 1, the loser 0, or each gets 1/2 if the game is drawn. The final result is that everyone gets a different score and the player placing second gets the same as the total of the four bottom players. What was the result of the game between the player placing third and the player placing seventh?

3. (a) The two diagonals of a quadrilateral each divide it into two parts of equal area. Prove it is a parallelogram.

(b) The three main diagonals of a hexagon each divide it into two parts of equal area. Prove they have a common point. [If ABCDEF is a hexagon, then the main diagonals are AD, BE and CF.]

4. The natural numbers m and n are relatively prime. Prove that the greatest common divisor of m+n and m^2+n^2 is either 1 or 2.

5. Given a circle c and two fixed points A, B on it. M is another point on c, and K is the midpoint of BM. P is the foot of the perpendicular from K to AM.

(a) prove that KP passes through a fixed point (as M varies);

(b) find the locus of P.

6. Find the smallest value x such that, given any point inside an equilateral triangle of side 1, we can always choose two points on the sides of the triangle, collinear with the given point and a distance x apart.

7. (a) A 6 x 6 board is tiled with 2 x 1 dominos. Prove that we can always divide the board into two rectangles each of which is tiled separately (with no domino crossing the dividing line).
(b) Is this true for an 8 x 8 board?

8. Given a set of n different positive reals $\{a_1, a_2, \dots, a_n\}$. Take all possible non-empty subsets and form their sums. Prove we get at least n(n+1)/2 different sums.

9. Given a triangle ABC. Let the line through C parallel to the angle bisector of B meet the angle bisector of A at D, and let the line through C parallel to the angle bisector of A meet the angle bisector of B at E. Prove that if DE is parallel to AB, then CA=CB.

10. An infinite arithmetic progression contains a square. Prove it contains infinitely many squares.

11. Can we label each vertex of a 45-gon with one of the digits 0, 1, ... , 9 so that for each pair of distinct digits i, j one of the 45 sides has vertices labeled i, j?

12. Find all real p, q, a, b such that we have $(2x-1)^{20} - (ax+b)^{20} = (x^2+px+q)^{10}$ for all x.

13. We place labeled points on a circle as follows. At step 1, take two points at opposite ends of a diameter and label them both 1. At step n>1, place a point at the midpoint of each arc created at step n-1 and label it with the sum of the labels at the two adjacent points. What is the total sum of the labels after step n?

For example, after step 4 we have: 1, 4, 3, 5, 2, 5, 3, 4, 1, 4, 3, 5, 2, 5, 3, 4.

14. Given an isosceles triangle, find the locus of the point P inside the triangle such that the distance from P to the base equals the geometric mean of the distances to the sides.

4th ASU 1964 problems

1. In the triangle ABC, the length of the altitude from A is not less than BC, and the length of the altitude from B is not less than AC. Find the angles.

2. If m, k, n are natural numbers and n>1, prove that we cannot have $m(m+1) = k^n$.

3. Reduce each of the first billion natural numbers (billion = 10^9) to a single digit by taking its digit sum repeatedly. Do we get more 1s than 2s?

4. Given n odd and a set of integers $a_1, a_2, \ldots , a_n$, derive a new set $(a_1 + a_2)/2, (a_2 + a_3)/2, \ldots , (a_{n-1} + a_n)/2, (a_n + a_1)/2$. However many times we repeat this process for a particular starting set we always get integers. Prove that all the numbers in the starting set are equal.

For example, if we started with 5, 9, 1, we would get 7, 5, 3, and then 6, 4, 5, and then 5, 4.5, 5.5. The last set does not consist entirely of integers.

5. (a) The convex hexagon ABCDEF has all angles equal. Prove that AB - DE = EF - BC = CD - FA.
(b) Given six lengths $a_1, \ldots , a_6$ satisfying $a_1 - a_4 = a_5 - a_2 = a_3 - a_6$, show that you can construct a hexagon with sides $a_1, \ldots , a_6$ and equal angles.

6. Find all possible integer solutions for $\sqrt{(x + \sqrt{(x \ldots (x + \sqrt{(x)}) \ldots)})} = y$, where there are 1998 square roots.

7. ABCD is a convex quadrilateral. A' is the foot of the perpendicular from A to the diagonal BD, B' is the foot of the perpendicular from B to the diagonal AC, and so on. Prove that A'B'C'D' is similar to ABCD.

8. Find all natural numbers n such that n^2 does not divide n!.

9. Given a lattice of regular hexagons. A bug crawls from vertex A to vertex B along the edges of the hexagons, taking the shortest possible path (or one of them). Prove that it travels a distance at least AB/2 in one direction. If it travels exactly AB/2 in one direction, how many edges does it traverse?

10. A circle center O is inscribed in ABCD (touching every side). Prove that $\angle AOB + \angle COD = 180^o$.

11. The natural numbers a, b, n are such that for every natural number k not equal to b, b - k divides $a - k^n$. Prove that $a = b^n$.

12. How many (algebraically) different expressions can we obtain by placing parentheses in $a_1/a_2/ \ldots /a_n$?

he smallest number of tetrahedrons into which a cube can be partitioned?

14. (a) Find the smallest square with last digit not 0 which becomes another square by the deletion of its last two digits. (b) Find all squares, not containing the digits 0 or 5, such that if the second digit is deleted the resulting number divides the original one.

15. A circle is inscribed in ABCD. AB is parallel to CD, and BC = AD. The diagonals AC, BD meet at E. The circles inscribed in ABE, BCE, CDE, DAE have radius r_1, r_2, r_3, r_4 respectively. Prove that $1/r_1 + 1/r_3 = 1/r_2 + 1/r_4$.

5th ASU 1965 problems

1. (a) Each of $x_1, ... , x_n$ is -1, 0 or 1. What is the minimal possible value of the sum of all x_ix_j with $1 \le i < j \le n$? (b) Is the answer the same if the x_i are real numbers satisfying $0 \le |x_i| \le 1$ for $1 \le i \le n$?

2. Two players have a 3 x 3 board. 9 cards, each with a different number, are placed face up in front of the players. Each player in turn takes a card and places it on the board until all the cards have been played. The first player wins if the sum of the numbers in the first and third rows is greater than the sum in the first and third columns, loses if it is less, and draws if the sums are equal. Which player wins and what is the winning strategy?

3. A circle is circumscribed about the triangle ABC. X is the midpoint of the arc BC (on the opposite side of BC to A), Y is the midpoint of the arc AC, and Z is the midpoint of the arc AB. YZ meets AB at D and YX meets BC at E. Prove that DE is parallel to AC and that DE passes through the center of the inscribed circle of ABC.

4. Bus numbers have 6 digits, and leading zeros are allowed. A number is considered lucky if the sum of the first three digits equals the sum of the last three digits. Prove that the sum of all lucky numbers is divisible by 13.

5. The beam of a lighthouse on a small rock penetrates to a fixed distance d. As the beam rotates the extremity of the beam moves with velocity v. Prove that a ship with speed at most v/8 cannot reach the rock without being illuminated.

6. A group of 100 people is formed to patrol the local streets. Every evening 3 people are on duty. Prove that you cannot arrange for every pair to meet just once on duty.

7. A tangent to the inscribed circle of a triangle drawn parallel to one of the sides meets the other two sides at X and Y. What is the maximum length XY, if the triangle has perimeter p?

8. The n^2 numbers x_{ij} satisfy the n^3 equations: $x_{ij} + x_{jk} + x_{ki} = 0$. Prove that we can find numbers $a_1, ... , a_n$ such that $x_{ij} = a_i - a_j$.

9. Can 1965 points be arranged inside a square with side 15 so that any rectangle of unit area placed inside the square with sides parallel to its sides must contain at least one of the points?

10. Given n real numbers $a_1, a_2, ... , a_n$, prove that you can find n integers $b_1, b_2, ... , b_n$, such that the sum of any subset of the original numbers differs from the sum of the corresponding b_i by at most $(n + 1)/4$.

11. A tourist arrives in Moscow by train and wanders randomly through the streets on foot. After supper he decides to return to the station along sections of street that he has traversed an odd number of times. Prove that this is always possible. [In other words, given a path over a graph from A to B, find a path from B to A consisting of edges that are used an odd number of times in the first path.]

12. (a) A committee has met 40 times, with 10 members at every meeting. No two people have met more than once at committee meetings. Prove that there are more than 60 people on the committee. (b) Prove that you cannot make more than 30 subcommittees of 5 members from a committee of 25 members with no two subcommittees having more than one common member.

13. Given two relatively prime natural numbers r and s, call an integer *good* if it can be represented as mr + ns with m, n non-negative integers and *bad* otherwise. Prove that we can find an integer c, such that just one of k, c - k is good for any k. How many bad numbers are there?

14. A spy-plane circles point A at a distance 10 km with speed 1000 km/h. A missile is fired towards the plane from A at the same speed and moves so that it is always on the line between A and the plane. How long does it take to hit?

15. Prove that the sum of the lengths of the edges of a polyhedron is at least 3 times the greatest distance between two points of the polyhedron.

16. An alien moves on the surface of a planet with speed not exceeding u. A spaceship searches for the alien with speed v. Prove the spaceship can always find the alien if $v>10u$.

6th ASU 1966 problems

1. There are an odd number of soldiers on an exercise. The distance between every pair of soldiers is different. Each soldier watches his nearest neighbour. Prove that at least one soldier is not being watched.

2. (a) B and C are on the segment AD with AB = CD. Prove that for any point P in the plane: PA + PD ≥ PB + PC.

(b) Given four points A, B, C, D on the plane such that for any point P on the plane we have PA + PD ≥ PB + PC. Prove that B and C are on the segment AD with AB = CD.

3. Can both $x^2 + y$ and $x + y^2$ be squares for x and y natural numbers?

4. A group of children are arranged into two equal rows. Every child in the back row is taller than the child standing in front of him in the other row. Prove that this remains true if each row is rearranged so that the children increase in height from left to right.

5. A rectangle ABCD is drawn on squared paper with its vertices at lattice points and its sides lying along the gridlines. AD = k AB with k an integer. Prove that the number of shortest paths from A to C starting out along AD is k times the number starting out along AB.

6. Given non-negative real numbers $a_1, a_2, \dots, a_n$, such that $a_{i-1} \le a_i \le 2a_{i-1}$ for i = 2, 3, ... , n. Show that you can form a sum $s = b_1a_1 + \dots + b_na_n$ with each b_i +1 or -1, so that $0 \le s \le a_1$.

7. Prove that you can always draw a circle radius A/P inside a convex polygon with area A and perimeter P.

8. A graph has at least three vertices. Given any three vertices A, B, C of the graph we can find a path from A to B which does not go through C. Prove that we can find two disjoint paths from A to B.

[A graph is a finite set of vertices such that each pair of distinct vertices has either zero or one edges joining the vertices. A path from A to B is a sequence of vertices $A_1, A_2, \dots, A_n$ such that $A=A_1$, $B=A_n$ and there is an edge between A_i and A_{i+1} for i = 1, 2, ... , n-1. Two paths from A to B are disjoint if the only vertices they have in common are A and B.]

9. Given a triangle ABC. Suppose the point P in space is such that PH is the smallest of the four altitudes of the tetrahedron PABC. What is the locus of H for all possible P?

10. Given 100 points on the plane. Prove that you can cover them with a collection of circles whose diameters total less than 100 and the distance between any two of which is more than 1. [The distance between circles radii r and s with centers a distance d apart is the greater of 0 and d - r - s.]

11. The distance from A to B is d kilometers. A plane P is flying with constant speed, height and direction from A to B. Over a period of 1 second the angle PAB changes by α degrees and the angle PBA by β degrees. What is the minimal speed of the plane?

12. Two players alternately choose the sign for one of the numbers 1, 2, ... , 20. Once a sign has been chosen it cannot be changed. The first player tries to minimize the final absolute value of the total and the second player to maximize it. What is the outcome (assuming both players play perfectly)? Example: the players might play successively: 1, 20, -19, 18, -17, 16, -15, 14, -13, 12, -11, 10, -9, 8, -7, 6, -5, 4, -3, 2. Then the outcome is 12. However, in this example the second player played badly!

1st ASU 1967 problems

1. In the acute-angled triangle ABC, AH is the longest altitude (H lies on BC), M is the midpoint of AC, and CD is an angle bisector (with D on AB).

(a) If AH ≤ BM, prove that the angle ABC ≤ 60.

(b) If AH = BM = CD, prove that ABC is equilateral.

2. (a) The digits of a natural number are rearranged and the resultant number is added to the original number. Prove that the answer cannot be 99 ... 9 (1999 nines).

(b) The digits of a natural number are rearranged and the resultant number is added to the original number to give 10^{10}. Prove that the original number was divisible by 10.

3. Four lighthouses are arbitarily placed in the plane. Each has a stationary lamp which illuminates an angle of 90 degrees. Prove that the lamps can be rotated so that at least one lamp is visible from every point of the plane.

4. (a) Can you arrange the numbers 0, 1, ... , 9 on the circumference of a circle, so that the difference between every pair of adjacent numbers is 3, 4 or 5? For example, we can arrange the numbers 0, 1, ... , 6 thus: 0, 3, 6, 2, 5, 1, 4.

(b) What about the numbers 0, 1, ... , 13?

5. Prove that there exists a number divisible by 5^{1000} with no zero digit.

6. Find all integers x, y satisfying $x^2 + x = y^4 + y^3 + y^2 + y$.

7. What is the maximum possible length of a sequence of natural numbers $x_1, x_2, x_3, \ldots$ such that $x_i \le 1998$ for $i \ge 1$, and $x_i = |x_{i-1} - x_{i-2}|$ for $i \ge 3$.

8. 499 white rooks and a black king are placed on a 1000 x 1000 chess board. The rook and king moves are the same as in ordinary chess, except that taking is not allowed and the king is allowed to remain in check. No matter what the initial situation and no matter how white moves, the black king can always:

(a) get into check (after some finite number of moves);

(b) move so that apart from some initial moves, it is always in check after its move;

(c) move so that apart from some initial moves, it is always in check (even just after white has moved).
Prove or disprove each of (a) - (c).

9. ABCD is a unit square. One vertex of a rhombus lies on side AB, another on side BC, and a third on side AD. Find the area of the set of all possible locations for the fourth vertex of the rhombus.

10. A natural number k has the property that if k divides n, then the number obtained from n by reversing the order of its digits is also divisible by k. Prove that k is a divisor of 99.

2nd ASU 1968 problems

1. An octagon has equal angles. The lengths of the sides are all integers. Prove that the opposite sides are equal in pairs.

2. Which is greater: 31^{11} or 17^{14}? [No calculators allowed!]

3. A circle radius 100 is drawn on squared paper with unit squares. It does not touch any of the grid lines or pass through any of the lattice points. What is the maximum number of squares it can pass through?

4. In a group of students, 50 speak English, 50 speak French and 50 speak Spanish. Some students speak more than one language. Prove it is possible to divide the students into 5 groups (not necessarily equal), so that in each group 10 speak English, 10 speak French and 10 speak Spanish.

5. Prove that: $2/(x^2 - 1) + 4/(x^2 - 4) + 6/(x^2 - 9) + ... + 20/(x^2 - 100) = 11/((x - 1)(x + 10)) + 11/((x - 2)(x + 9)) + ... + 11/((x - 10)(x + 1))$.

6. The difference between the longest and shortest diagonals of the regular n-gon equals its side. Find all possible n.

7. The sequence a_n is defined as follows: $a_1 = 1$, $a_{n+1} = a_n + 1/a_n$ for $n \geq 1$. Prove that $a_{100} > 14$.

8. Given point O inside the acute-angled triangle ABC, and point O' inside the acute-angled triangle A'B'C'. D, E, F are the feet of the perpendiculars from O to BC, CA, AB respectively, and D', E', F' are the feet of the perpendiculars from O' to B'C', C'A', A'B' respectively. OD is parallel to O'A', OE is parallel to O'B' and OF is parallel to O'C'. Also OD·O'A' = OE·O'B' = OF·O'C'. Prove that O'D' is parallel to OA, O'E' to OB and O'F' to OC, and that O'D'·OA = O'E'·OB = O'F'·OC.

9. Prove that any positive integer not exceeding n! can be written as a sum of at most n distinct factors of n!.

10. Given a triangle ABC, and D on the segment AB, E on the segment AC, such that AD = DE = AC, BD = AE, and DE is parallel to BC. Prove that BD equals the side of a regular 10-gon inscribed in a circle with radius AC.

11. Given a regular tetrahedron ABCD, prove that it is contained in the three spheres on diameters AB, BC and AD. Is this true for any tetrahedron?

12. (a) Given a 4 x 4 array with + signs in each place except for one non-corner square on the perimeter which has a - sign. You can change all the signs in any row, column or diagonal. A diagonal can be of any length down to 1. Prove that it is not possible by repeated changes to arrive at all + signs.

(b) What about an 8 x 8 array?

13. The medians divide a triangle into 6 smaller triangles. 4 of the circles inscribed in the smaller triangles have equal radii. Prove that the original triangle is equilateral.

14. Prove that we can find positive integers x, y satisfying $x^2 + x + 1 = py$ for an infinite number of primes p.

15. 9 judges each award 20 competitors a rank from 1 to 20. The competitor's score is the sum of the ranks from the 9 judges, and the winner is the competitor with the lowest score. For each competitor the difference between the highest and lowest ranking (from different judges) is at most 3. What is the highest score the winner could have obtained?

16. $\{a_i\}$ and $\{b_i\}$ are permutations of $\{1/1, 1/2, \ldots, 1/n\}$. $a_1 + b_1 \geq a_2 + b_2 \geq \ldots \geq a_n + b_n$. Prove that for every m ($1 \leq m \leq n$) $a_m + a_n \geq 4/m$.

17. There is a set of scales on the table and a collection of weights. Each weight is on one of the two pans. Each weight has the name of one or more pupils written on it. All the pupils are outside the room. If a pupil enters the room then he moves the weights with his name on them to the other pan. Show that you can let in a subset of pupils one at a time, so that the scales change position after the last pupil has moved his weights.

18. The streets in a city are on a rectangular grid with m east-west streets and n north-south streets. It is known that a car will leave some (unknown) junction and move along the streets at an unknown and possibly variable speed, eventually returning to its starting point without ever moving along the same block twice. Detectors can be positioned anywhere except at a junction to record the time at which the car passes and it direction of travel. What is the minimum number of detectors needed to ensure that the car's route can be reconstructed?

19. The circle inscribed in the triangle ABC touches the side AC at K. Prove that the line joining the midpoint of AC with the center of the circle bisects the segment BK.

20. The sequence $a_1, a_2, \ldots, a_n$ satisfies the following conditions: $a_1 = 0$, $|a_i| = |a_{i-1} + 1|$ for $i = 2, 3, \ldots, n$. Prove that $(a_1 + a_2 + \ldots + a_n)/n \geq -1/2$.

21. The sides and diagonals of ABCD have rational lengths. The diagonals meet at O. Prove that the length AO is also rational.

3rd ASU 1969 problems

1. In the quadrilateral ABCD, BC is parallel to AD. The point E lies on the segment AD and the perimeters of ABE, BCE and CDE are equal. Prove that BC = AD/2.

2. A wolf is in the center of a square field and there is a dog at each corner. The wolf can run anywhere in the field, but the dogs can only run along the sides. The dogs' speed is 3/2 times the wolf's speed. The wolf can kill a single dog, but two dogs together can kill the wolf. Prove that the dogs can prevent the wolf escaping.

3. A finite sequence of 0s and 1s has the following properties: (1) for any $i < j$, the sequences of length 5 beginning at position i and position j are different; (2) if you add an additional digit at either the start or end of the sequence, then (1) no longer holds. Prove that the first 4 digits of the sequence are the same as the last 4 digits.

4. Given positive numbers a, b, c, d prove that at least one of the inequalities does not hold: $a + b < c + d$; $(a + b)(c + d) < ab + cd$; $(a + b)cd < ab(c + d)$.

5. What is the smallest positive integer a such that we can find integers b and c so that $ax^2 + bx + c$ has two distinct positive roots less than 1?

6. n is an integer. Prove that the sum of all fractions 1/rs, where r and s are relatively prime integers satisfying $0 < r < s \le n$, $r + s > n$, is 1/2.

7. Given n points in space such that the triangle formed from any three of the points has an angle greater than 120 degrees. Prove that the points can be labeled 1, 2, 3, ... , n so that the angle defined by i, i+1, i+2 is greater than 120 degrees for i = 1, 2, ... , n-2.

8. Find 4 different three-digit numbers (in base 10) starting with the same digit, such that their sum is divisible by 3 of the numbers.

9. Every city in a certain state is directly connected by air with at most three other cities in the state, but one can get from any city to any other city with at most one change of plane. What is the maximum possible number of cities?

10. Given a pentagon with equal sides.

(a) Prove that there is a point X on the longest diagonal such that every side subtends an angle at most 90 degrees at X.

(b) Prove that the five circles with diameter one of the pentagon's sides do not cover the pentagon.

11. Given the equation $x^3 + ax^2 + bx + c = 0$, the first player gives one of a, b, c an integral value. Then the second player gives one of the remaining coefficients an integral value, and finally the first player gives the remaining coefficient an integral value. The first player's objective is to ensure that the equation has three integral roots (not necessarily distinct). The second player's objective is to prevent this. Who wins?

12. 20 teams compete in a competition. What is the smallest number of games that must be played to ensure that given any three teams at least two play each other?

13. A regular n-gon is inscribed in a circle radius R. The distance from the center of the circle to the center of a side is h_n. Prove that $(n+1)h_{n+1} - nh_n > R$.

14. Prove that for any positive numbers $a_1, a_2, \dots , a_n$ we have:

$$a_1/(a_2+a_3) + a_2/(a_3+a_4) + \dots + a_{n-1}/(a_n+a_1) + a_n/(a_1+a_2) > n/4.$$

4th ASU 1970 problems

1. Given a circle, diameter AB and a point C on AB, show how to construct two points X and Y on the circle such that (1) Y is the reflection of X in the line AB, (2) YC is perpendicular to XA.

2. The product of three positive numbers is 1, their sum is greater than the sum of their inverses. Prove that just one of the numbers is greater than 1.

3. What is the greatest number of sides of a convex polygon that can equal its longest diagonal?

4. n is a 17 digit number. m is derived from n by taking its decimal digits in the reverse order. Show that at least one digit of n + m is even.

5. A room is an equilateral triangle side 100 meters. It is subdivided into 100 rooms, all equilateral triangles with side 10 meters. Each interior wall between two rooms has a door. If you start inside one of the rooms and can only pass through each door once, show that you cannot visit more than 91 rooms. Suppose now the large triangle has side k and is divided into k^2 small triangles by lines parallel to its sides. A chain is a sequence of triangles, such that a triangle can only be included once and consecutive triangles have a common side. What is the largest possible number of triangles in a chain?

6. Given 5 segments such that any 3 can be used to form a triangle. Show that at least one of the triangles is acute-angled.

7. ABC is an acute-angled triangle. The angle bisector AD, the median BM and the altitude CH are concurrent. Prove that angle A is more than 45 degrees.

8. Five n-digit binary numbers have the property that every two numbers have the same digits in just m places, but no place has the same digit in all five numbers. Show that $2/5 \le m/n \le 3/5$.

9. Show that given 200 integers you can always choose 100 with sum a multiple of 100.

10. ABC is a triangle with incenter I. M is the midpoint of BC. IM meets the altitude AH at E. Show that AE = r, the radius of the inscribed circle.

11. Given any positive integer n, show that we can find infinitely many integers m such that m has no zeros (when written as a decimal number) and the sum of the digits of m and mn is the same.

12. Two congruent rectangles of area A intersect in eight points. Show that the area of the intersection is more than A/2.

13. If the numbers from 11111 to 99999 are arranged in an arbitrary order show that the resulting 444445 digit number is not a power of 2.

14. S is the set of all positive integers with n decimal digits or less and with an even digit sum. T is the set of all positive integers with n decimal digits or less and an odd digit sum. Show that the sum of the kth powers of the members of S equals the sum for T if $1 \le k < n$.

15. The vertices of a regular n-gon are colored (each vertex has only one color). Each color is applied to at least three vertices. The vertices of any given color form a regular polygon. Show that there are two colors which are applied to the same number of vertices.

5th ASU 1971 problems

1. Prove that we can find a number divisible by 2^n whose decimal representation uses only the digits 1 and 2.

2. (1) $A_1A_2A_3$ is a triangle. Points B_1, B_2, B_3 are chosen on A_1A_2, A_2A_3, A_3A_1 respectively and points D_1, D_2 D_3 on A_3A_1, A_1A_2, A_2A_3 respectively, so that if parallelograms $A_iB_iC_iD_i$ are formed, then the lines A_iC_i concur. Show that $A_1B_1 \cdot A_2B_2 \cdot A_3B_3 = A_1D_1 \cdot A_2D_2 \cdot A_3D_3$.

(2) $A_1A_2 \ldots A_n$ is a convex polygon. Points B_i are chosen on A_iA_{i+1} (where we take A_{n+1} to mean A_1), and points D_i on $A_{i-1}A_i$ (where we take A_0 to mean A_n) such that if parallelograms $A_iB_iC_iD_i$ are formed, then the n lines A_iC_i concur. Show that $\prod A_iB_i = \prod A_iD_i$.

3. (1) Player A writes down two rows of 10 positive integers, one under the other. The numbers must be chosen so that if a is under b and c is under d, then a + d = b + c. Player B is allowed to ask for the identity of the number in row i, column j. How many questions must he ask to be sure of determining all the numbers?

(2) An m x n array of positive integers is written on the blackboard. It has the property that for any four numbers a, b, c, d with a and b in the same row, c and d in the same row, a above c (in the same column) and b above d (in the same column) we have a + d = b + c. If some numbers are wiped off, how many must be left for the table to be accurately restored?

4. Circles, each with radius less than R, are drawn inside a square side 1000R. There are no points on different circles a distance R apart. Show that the total area covered by the circles does not exceed 340,000 R^2.

5. You are given three positive integers. A move consists of replacing $m \le n$ by 2m, n-m. Show that you can always make a series of moves which results in one of the integers becoming zero. [For example, if you start with 4, 5, 10, then you could get 8, 5, 6, then 3, 10, 6, then 6, 7, 6, then 0, 7, 12.]

6. The real numbers a, b, A, B satisfy $(B - b)^2 < (A - a)(Ba - Ab)$. Show that the quadratics $x^2 + ax + b = 0$ and $x^2 + Ax + B = 0$ have real roots and between the roots of each there is a root of the other.

7. The projections of a body on two planes are circles. Show that the circles have the same radius.

8. An integer is written at each vertex of a regular n-gon. A move is to find four adjacent vertices with numbers a, b, c, d (in that order), so that $(a - d)(b - c) < 0$, and then to interchange b and c. Show that only finitely many moves are possible. For example, a possible sequence of moves is shown below:

1 7 2 3 5 4

1 2 7 3 5 4

1 2 3 7 5 4

1 2 3 5 7 4

2 1 3 5 7 4

9. A polygon P has an inscribed circle center O. If a line divides P into two polygons with equal areas and equal perimeters, show that it must pass through O.

10. Given any set S of 25 positive integers, show that you can always find two such that none of the other numbers equals their sum or difference.

11. A and B are adjacent vertices of a 12-gon. Vertex A is marked - and the other vertices are marked +. You are allowed to change the sign of any n adjacent vertices. Show that by a succession of moves of this type with n = 6 you cannot get B marked - and the other vertices marked +. Show that the same is true if all moves have n = 3 or if all moves have n = 4.

12. Equally spaced perpendicular lines divide a large piece of paper into unit squares. N squares are colored black. Show that you can always cut out a set of disjoint square pieces of paper, so that all the black squares are removed and the black area of each piece is between 1/5 and 4/5 of its total area.

13. n is a positive integer. S is the set of all triples (a, b, c) such that $1 \le a, b, c, \le n$. What is the smallest subset X of triples such that for every member of S one can find a member of X which differs in only one position. [For example, for n = 2, one could take X = { (1, 1, 1), (2, 2, 2) }.]

14. Let $f(x, y) = x^2 + xy + y^2$. Show that given any real x, y one can always find integers m, n such that f(x-m, y-n) <= 1/3. What is the corresponding result if $f(x, y) = x^2 + axy + y^2$ with $0 \le a \le 2$?

15. A switch has two inputs 1, 2 and two outputs 1, 2. It either connects 1 to 1 and 2 to 2, or 1 to 2 and 2 to 1. If you have three inputs 1, 2, 3 and three outputs 1, 2, 3, then you can use three switches, the first across 1 and 2, then the second across 2 and 3, and finally the third across 1 and 2. It is easy to check that this allows the output to be any permutation of the inputs and that at least three switches are required to achieve this. What is the minimum number of switches required for 4 inputs, so that by suitably setting the switches the output can be any permutation of the inputs?

6th ASU 1972 problems

1. ABCD is a rectangle. M is the midpoint of AD and N is the midpoint of BC. P is a point on the ray CD on the opposite side of D to C. The ray PM intersects AC at Q. Show that MN bisects the angle PNQ.

2. Given 50 segments on a line show that you can always find either 8 segments which are disjoint or 8 segments with a common point.

3. Find the largest integer n such that $4^{27} + 4^{1000} + 4^{n}$ is a square.

4. a, m, n are positive integers and a > 1. Show that if $a^m + 1$ divides $a^n + 1$, then m divides n. The positive integer b is relatively prime to a, show that if $a^m + b^m$ divides $a^n + b^n$ then m divides n.

5. A sequence of finite sets of positive integers is defined as follows. $S_0 = \{m\}$, where m > 1. Then given S_n you derive S_{n+1} by taking k^2 and k+1 for each element k of S_n. For example, if $S_0 = \{5\}$, then $S_2 = \{7, 26, 36, 625\}$. Show that S_n always has 2^n distinct elements.

6. Prove that a collection of squares with total area 1 can always be arranged inside a square of area 2 without overlapping.

7. O is the point of intersection of the diagonals of the convex quadrilateral ABCD. Prove that the line joining the centroids of ABO and CDO is perpendicular to the line joining the orthocenters of BCO and ADO.

8. 9 lines each divide a square into two quadrilaterals with areas 2/5 and 3/5 that of the square. Show that 3 of the lines meet in a point.

9. A 7-gon is inscribed in a circle. The center of the circle lies inside the 7-gon. A, B, C are adjacent vertices of the 7-gon show that the sum of the angles at A, B, C is less than 450 degrees.

10. Two players play the following game. At each turn the first player chooses a decimal digit, then the second player substitutes it for one of the stars in the subtraction | **** - **** |. The first player tries to end up with the largest possible result, the second player tries to end up with the smallest possible result. Show that the first player can always play so that the result is at least 4000 and that the second player can always play so that the result is at most 4000.

11. For positive reals x, y let f(x, y) be the smallest of x, 1/y, y + 1/x. What is the maximum value of f(x, y)? What are the corresponding x, y?

12. P is a convex polygon and X is an interior point such that for every pair of vertices A, B, the triangle XAB is isosceles. Prove that all the vertices of P lie on some circle center X.

13. Is it possible to place the digits 0, 1, 2 into unit squares of 100 x 100 cross-lined paper such that every 3 x 4 (and every 4 x 3) rectangle contains three 0s, four 1s and five 2s?

14. $x_1, x_2, \ldots, x_n$ are positive reals with sum 1. Let s be the largest of $x_1/(1 + x_1)$, $x_2/(1 + x_1 + x_2)$, ... , $x_n/(1 + x_1 + \ldots + x_n)$. What is the smallest possible value of s? What are the corresponding x_i?

15. n teams compete in a tournament. Each team plays every other team once. In each game a team gets 2 points for a win, 1 for a draw and 0 for a loss. Given any subset S of teams, one can find a team (possibly in S) whose total score in the games with teams in S was odd. Prove that n is even.

7th ASU 1973 problems

1. You are given 14 coins. It is known that genuine coins all have the same weight and that fake coins all have the same weight, but weigh less than genuine coins. You suspect that 7 particular coins are genuine and the other 7 fake. Given a balance, how can you prove this in three weighings (assuming that you turn out to be correct)?

2. Prove that a 9 digit decimal number whose digits are all different, which does not end with 5 and or contain a 0, cannot be a square.

3. Given $n > 4$ points, show that you can place an arrow between each pair of points, so that given any point you can reach any other point by travelling along either one or two arrows in the direction of the arrow.

4. OA and OB are tangent to a circle at A and B. The line parallel to OB through A meets the circle again at C. The line OC meets the circle again at E. The ray AE meets the line OB at K. Prove that K is the midpoint of OB.

5. $p(x) = ax^2 + bx + c$ is a real quadratic such that $|p(x)| \le 1$ for all $|x| \le 1$. Prove that $|cx^2 + bx + a| \le 2$ for $|x| \le 1$.

6. Players numbered 1 to 1024 play in a knock-out tournament. There ar no draws, the winner of a match goes through to the next round and the loser is knocked-out, so that there are 512 matches in the first round, 256 in the second and so on. If m plays n and $m < n-2$ then m always wins. What is the largest possible number for the winner?

7. Define $p(x) = ax^2 + bx + c$. If $p(x) = x$ has no real roots, prove that $p(p(x)) = 0$ has no real roots.

8. At time 1, n unit squares of an infinite sheet of paper ruled in squares are painted black, the rest remain white. At time k+1, the color of each square is changed to the color held at time k by a majority of the following three squares: the square itself, its northern neighbour and its eastern neighbour. Prove that all the squares are white at time n+1.

9. ABC is an acute-angled triangle. D is the reflection of A in BC, E is the reflection of B in AC, and F is the reflection of C in AB. Show that the circumcircles of DBC, ECA, FAB meet at a point and that the lines AD, BE, CF meet at a point.

10. n people are all strangers. Show that you can always introduce some of them to each other, so that afterwards each person has met a different number of the others. [problem: this is false as stated. Each person must have 0, 1, ... or n-1 meetings,so all these numbers must be used. But if one person has met no one, then another cannot have met everyone.]

11. A king moves on an 8 x 8 chessboard. He can move one square at a time, diagonally or orthogonally (so away from the borders he can move to any of eight squares). He makes a complete circuit of the board, starting and finishing on the same square and visiting every other square just once. His trajectory is drawn by joining the center of the squares he moves to and from for each move. The trajectory does not intersect itself. Show that he makes at least 28 moves parallel to the sides of the board (the others being diagonal) and that a circuit is possible with exactly 28 moves parallel to the sides of the board. If the board has side length 8, what is the maximum and minimum possible length for such a trajectory.

12. A triangle has area 1, and sides $a \ge b \ge c$. Prove that $b^2 \ge 2$.

13. A convex n-gon has no two sides parallel. Given a point P inside the n-gon show that there are at most n lines through P which bisect the area of the n-gon.

14. a, b, c, d, e are positive reals. Show that $(a + b + c + d + e)^2 \geq 4(ab + bc + cd + de + ea)$.

15. Given 4 points which do not lie in a plane, how many parallelepipeds have all 4 points as vertices?

8th ASU 1974 problems

1. A collection of n cards is numbered from 1 to n. Each card has either 1 or -1 on the back. You are allowed to ask for the product of the numbers on the back of any three cards. What is the smallest number of questions which will allow you to determine the numbers on the backs of all the cards if n is (1) 30, (2) 31, (3) 32? If 50 cards are arranged in a circle and you are only allowed to ask for the product of the numbers on the backs of three adjacent cards, how many questions are needed to determine the product of the numbers on the backs of all 50 cards?

2. Find the smallest positive integer which can be represented as $36^m - 5^n$.

3. Each side of a convex hexagon is longer than 1. Is there always a diagonal longer than 2? If each of the main diagonals of a hexagon is longer than 2, is there always a side longer than 1?

4. Circles radius r and R touch externally. AD is parallel to BC. AB and CD touch both circles. AD touches the circle radius r, but not the circle radius R, and BC touches the circle radius R, but not the circle radius r. What is the smallest possible length for AB?

5. Given n unit vectors in the plane whose sum has length less than one. Show that you can arrange them so that the sum of the first k has length less than 2 for every $1 < k < n$.

6. Find all real a, b, c such that $|ax + by + cz| + |bx + cy + az| + |cx + ay + bz| = |x + y + z|$ for all real x,y,z.

7. ABCD is a square. P is on the segment AB and Q is on the segment BC such that BP = BQ. H lies on PC such that BHC is a right angle. Show that DHQ is a right angle.

8. The n points of a graph are each colored red or blue. At each move we select a point which differs in color from more than half of the points to which it it is joined and we change its color. Prove that this process must finish after a finite number of moves.

9. Find all positive integers m, n such that n^n has m decimal digits and m^m has n decimal digits.

10. In the triangle ABC, angle C is 90 deg and AC = BC. Take points D on CA and E on CB such that CD = CE. Let the perpendiculars from D and C to AE meet AB at K and L respectively. Show that KL = LB.

11. One rat and two cats are placed on a chess-board. The rat is placed first and then the two cats choose positions on the border squares. The rat moves first. Then the cats and the rat move alternately. The rat can move one square to an adjacent square (but not diagonally). If it is on a border square, then it can also move off the board. On a cat move, both cats move one square. Each must move to an adjacent square, and not diagonally. The cats win if one of them moves onto the same square as the rat. The rat wins if it moves off the board. Who wins? Suppose there are three cats (and all three cats move when it is the cats' turn), but that the rat gets an extra initial turn. Prove that the rat wins.

12. Arrange the numbers 1, 2, ... , 32 in a sequence such that the arithmetic mean of two numbers does not lie between them. (For example, ... 3, 4, 5, 2, 1, ... is invalid, because 2 lies between 1 and 3.) Can you arrange the numbers 1, 2, ... , 100 in the same way?

13. Find all three digit decimal numbers $a_1a_2a_3$ which equal the mean of the six numbers $a_1a_2a_3$, $a_1a_3a_2$, $a_2a_1a_3$, $a_2a_3a_1$, $a_3a_1a_2$, $a_3a_2a_1$.

14. No triangle of area 1 can be fitted inside a convex polygon. Show that the polygon can be fitted inside a triangle of area 4.

15. f is a function on the closed interval [0, 1] with non-negative real values. $f(1) = 1$ and $f(x + y) \geq f(x) + f(y)$ for all x, y. Show that $f(x) \leq 2x$ for all x. Is it necessarily true that $f(x) \leq 1.9x$ for all x.

16. The triangle ABC has area 1. D, E, F are the midpoints of the sides BC, CA, AB. P lies in the segment BF, Q lies in the segment CD, R lies in the segment AE. What is the smallest possible area for the intersection of triangles DEF and PQR?

9th ASU 1975 problems

1. (1) O is the circumcenter of the triangle ABC. The triangle is rotated about O to give a new triangle A'B'C'. The lines AB and A'B' intersect at C'', BC and B'C' intersect at A'', and CA and C'A' intersect at B''. Show that A''B''C'' is similar to ABC.

(2) O is the center of the circle through ABCD. ABCD is rotated about O to give the quadrilateral A'B'C'D'. Prove that the intersection points of corresponding sides form a parallelogram.

2. A triangle ABC has unit area. The first player chooses a point X on side AB, then the second player chooses a point Y on side BC, and finally the first player chooses a point Z on side CA. The first player tries to arrange for the area of XYZ to be as large as possible, the second player tries to arrange for the area to be as small as possible. What is the optimum strategy for the first player and what is the best he can do (assuming the second player plays optimally)?

3. What is the smallest perimeter for a convex 32-gon whose vertices are all lattice points?

4. Given a 7 x 7 square subdivided into 49 unit squares, mark the center of n unit squares, so that no four marks form a rectangle with sides parallel to the square. What is the largest n for which this is possible? What about a 13 x 13 square?

5. Given a convex hexagon, take the midpoint of each of the six diagonals joining vertices which are separated by a single vertex (so if the vertices are in order A, B, C, D, E, F, then the diagonals are AC, BD, CE, DF, EA, FB). Show that the midpoints form a convex hexagon with a quarter the area of the original.

6. Show that there are 2^{n+1} numbers each with 2^n digits, all 1 or 2, so that every two numbers differ in at least half their digits.

7. There are finitely many polygons in the plane. Every two have a common point. Prove that there is a straight line intersecting all the polygons.

8. a, b, c are positive reals. Show that $a^3 + b^3 + c^3 + 3abc > ab(a + b) + bc(b + c) + ca(c + a)$.

9. Three flies crawl along the perimeter of a triangle. At least one fly makes a complete circuit of the perimeter. For the entire period the center of mass of the flies remains fixed. Show that it must be at the centroid of the triangle. [You may not assume, without proof, that the flies have the same mass, or that they crawl at the same speed, or that any fly crawls at a constant speed.]

10. The finite sequence a_n has each member 0, 1 or 2. A *move* involves replacing any two unequal members of the sequence by a single member different from either. A series of moves results in a single number. Prove that no series of moves can terminate in a (single) different number.

11. S is a horizontal strip in the plane. n lines are drawn so that no three are collinear and every pair intersects within the strip. A *path* starts at the bottom border of the strip and consists of a sequence of segments from the n lines. The path must change line at each intersection and must always move upwards. Show that: (1) there are at least n/2 disjoint paths; (2) there is a path of at least n segments; (3) there is a path involving not more than n/2 + 1 of the lines; and (4) there is a path that involves segments from all n lines.

12. For what n can we color the unit cubes in an n x n x n cube red or green so that every red unit cube has just two red neighbouring cubes (sharing a face) and every green unit cube has just two green neighbouring cubes.

13. p(x) is a polynomial with integral coefficients. f(n) = the sum of the (decimal) digits in the value p(n). Show that f(n) some value m infinitely many times.

14. 20 teams each play one game with every other team. Each game results in a win or loss (no draws). k of the teams are European. A separate trophy is awarded for the best European team on the basis of the k(k-1)/2 games in which both teams are European. This trophy is won by a single team. The same team comes last in the overall competition (winning fewer games than any other team). What is the largest possible value of k? If draws are allowed and a team scores 2 for a win and 1 for a draw, what is the largest possible value of k?

15. Given real numbers a_i, b_i and positive reals c_i, d_i, let $e_{ij} = (a_i+b_j)/(c_i+d_j)$. Let $M_i = \max_{0\le j\le n} e_{ij}$, $m_j = \min_{1\le i\le n} e_{ij}$. Show that we can find an e_{ij} with $1 \le i, j \le n$ such that $e_{ij} = M_i = m_j$.

10th ASU 1976 problems

1. 50 watches, all keeping perfect time, lie on a table. Show that there is a moment when the sum of the distances from the center of the table to the center of each dial equals the sum of the distances from the center of the table to the tip of each minute hand.

2. 1000 numbers are written in line 1, then further lines are constructed as follows. If the number m occurs in line n, then we write under it in line n+1, each time it occurs, the number of times that m occurs in line n. Show that lines 11 and 12 are identical. Show that we can choose numbers in line 1, so that lines 10 and 11 are not identical.

3. (1) The circles C_1, C_2, C_3 with equal radius all pass through the point X. C_i and C_j also intersect at the point Y_{ij}. Show that angle XO_1Y_{12} + angle XO_2Y_{23} + angle XO_3Y_{31} = 180 deg, where O_i is the center of circle C_i.

4. a_1 and a_2 are positive integers less than 1000. Define $a_n = \min\{ |a_i - a_j|: 0 < i < j < n\}$. Show that $a_{21} = 0$.

5. Can you label each vertex of a cube with a different three digit binary number so that the numbers at any two adjacent vertices differ in at least two digits?

6. a, b, c, d are vectors in the plane such that a + b + c + d = 0. Show that $|a| + |b| + |c| + |d| \geq |a + d| + |b + d| + |c + d|$.

7. S is a set of 1976 points which form a regular 1976-gon. T is the set of all points which are the midpoint of at least one pair of points in S. What is the greatest number of points of T which lie on a single circle?

8. n rectangles are drawn on a rectangular sheet of paper. Each rectangle has its sides parallel to the sides of the paper. No pair of rectangles has an interior point in common. If the rectangles were removed show that the rest of the sheet would be in at most n+1 parts.

9. There are three straight roads. On each road a man is walking at constant speed. At time t = 0, the three men are not collinear. Prove that they will be collinear for t > 0 at most twice.

10. Initially, there is one beetle on each square in the set S. Suddenly each beetle flies to a new square, subject to the following conditions: (1) the new square may be the same as the old or different; (2) more than one beetle may choose the same new square; (3) if two beetles are initially in squares with a common vertex, then after the flight they are either in the same square or in squares with a common vertex. Suppose S is the set of all squares in the middle row and column of a 99 x 99 chess board, is it true that there must always be a beetle whose new square shares a vertex with its old square (or is identical with it)? What if S also includes all the border squares (so S is rows 1, 50 and 99 and columns 1, 50 and 99)? What if S is all squares of the board?

11. Call a triangle *big* if each side is longer than 1. Show that we can draw 100 big triangles inside an equilateral triangle with side length 5 so that all the triangles are disjoint. Show that you can draw 100 big triangles with every vertex inside or on an equilateral triangle with side 3, so that they cover the equilateral triangle, and any two big triangles either (1) are disjoint, or (2) have as intersection a common vertex, or (3) have as intersection a common side.

12. n is a positive integer. A universal sequence of length m is a sequence of m integers each between 1 and n such that one can obtain any permutation of 1, 2, ... , n by deleting suitable members of the sequence. For example, 1, 2, 3, 1, 2, 1, 3 is a universal sequence of length 7 for n = 3. But 1, 2, 3, 2, 1, 3, 1 is not universal, because one cannot obtain the permutation 3, 1, 2. Show that one can always obtain a universal sequence for n of length $n^2 - n + 1$. Show that a universal sequence for n must have length at least $n(n + 1)/2$. Show that the shortest sequence for n = 4 has 12 members. [You are told, but do not have to prove, that there is a universal sequence for n of length $n^2 - 2n + 4$.]

13. n real numbers are written around a circle. One of the numbers is 1 and the sum of the numbers is 0. Show that there are two adjacent numbers whose difference is at least n/4. Show that there is a number which differs from the arithmetic mean of its two neighbours by at least $8/n^2$. Improve this result to some k/n^2 with k > 8. Show that for n = 30, we can take k = 1800/113. Give an example of 30 numbers such that no number differs from the arithmetic mean of its two neighbours by more than 2/113.

14. You are given a regular n-gon. Each vertex is marked +1 or -1. A move consists of changing the sign of all the vertices which form a regular k-gon for some 1 < k <= n. [A regular 2-gon means two vertices which have the center of the n-gon as their midpoint.]. For example, if we label the vertices of a regular 6-gon 1, 2, 3, 4, 5, 6, then you can change the sign of {1, 4}, {2, 5}, {3, 6}, {1, 3, 5}, {2, 4, 6} or {1, 2, 3, 4, 5, 6}. Show that for (1) n = 15, (2) n = 30, (3) any n > 2, we can find some initial marking which cannot be changed to all +1 by any series of moves. Let f(n) be the largest number of markings, so that no one can be obtained from any other by any series of moves. Show that $f(200) = 2^{80}$.

15. S is a sphere with unit radius. P is a plane through the center. For any point x on the sphere f(x) is the perpendicular distance from x to P. Show that if x, y, z are the ends of three mutually perpendicular radii, then $f(x)^2 + f(y)^2 + f(z)^2 = 1$ (*). Now let g(x) be any function on the points of S taking non-negative real values and satisfying (*). Regard the intersection of P and S as the equator, the poles as the points with f(x) = 1 and lines of longitude as semicircles through both poles. (1) If x and y have the same longitude and both lie on the same side of the equator with x closer to the pole, show that g(x) > g(y). (2) Show that for any x, y on the same side of the equator with x closer to the pole than y we have g(x) > g(y). (3) Show that if x and y are the same distance from the pole then g(x) = g(y). (4) Show that g(x) = f(x) for all x.

11th ASU 1977 problems

1. P is a polygon. Its sides do not intersect except at its vertices, and no three vertices lie on a line. The pair of sides AB, PQ is called *special* if (1) AB and PQ do not share a vertex and (2) either the *line* AB intersects the *segment* PQ or the line PQ intersects the segment AB. Show that the number of special pairs is even.

2. n points lie in the plane, not all on a single line. A real number is assigned to each point. The sum of the numbers is zero for all the points lying on any line. Show that all the assigned numbers must be zero.

3. (1) The triangle ABC is inscribed in a circle. D is the midpoint of the arc BC (not containing A), similarly E and F. Show that the hexagon formed by the intersection of ABC and DEF has its main diagonals parallel to the sides of ABC and intersecting in a single point.

(2) EF meets AB at X and AC at Y. Prove that AXIY is a rhombus, where I is the center of the circle inscribed in ABC.

4. Black and white tokens are placed around a circle. First all the black tokens with one or two white neighbors are removed. Then all white tokens with one or two black neighbors are removed. Then all black tokens with one or two white neighbors and so on until all the tokens have the same color. Is it possible to arrange 40 tokens so that only one remains after 4 moves? What is the minimum possible number of moves to go from 1000 tokens to one?

5. a_n is an infinite sequence such that $(a_{n+1} - a_n)/2$ tends to zero. Show that a_n tends to zero.

6. There are direct routes between every two cities in a country. The fare between each pair of cities is the same in both directions. Two travellers decide to visit all the cities. The first traveller starts at a city and travels to the city with the most expensive fare (or if there are several such, any one of them). He then repeats this process, never visiting a city twice, until he has been to all the cities (so he ends up in a different city from the one he starts from). The second traveller has a similar plan, except that he always chooses the cheapest fare, and does not necessarily start at the same city. Show that the first traveller spends at least as much on fares as the second.

7. Each vertex of a convex polyhedron has three edges. Each face is a cyclic polygon. Show that its vertices all lie on a sphere.

8. Given a polynomial $x_{10} + a_9x^9 + ... + a_1x + 1$. Two players alternately choose one of the coefficients a_1 to a_9 (which has not been chosen before) and assign a real value to it. The first player wins iff the resulting polynomial has no real roots. Who wins?

9. Seven elves sit at a table. Each elf has a cup. In total the cups contain 3 liters of milk. Each elf in turn gives all his milk to the others in equal shares. At the end of the process each elf has the same amount of milk as at the start. What was that?

10. We call a number *doubly square* if (1) it is a square with an even number 2n of (decimal) digits, (2) its first n digits form a square, (3) its last n digits form a non-zero square. For example, 1681 is doubly square, but 2500 is not. (1) find all 2-digit and 4-digit doubly square numbers. (2) Is there a 6-digit doubly square number? (3) Show that there is a 20-digit doubly square number. (4) Show that there are at least ten 100-digit doubly square numbers. (5) Show that there is a 30-digit doubly square number.

11. Given a sequence $a_1, a_2, ... , a_n$ of positive integers. Let S be the set of all sums of one or more members of the sequence. Show that S can be divided into n subsets such that the smallest member of each subset is at least half the largest member.

12. You have 1000 tickets numbered 000, 001, ... , 999 and 100 boxes numbered 00, 01, ... , 99. You may put each ticket into any box whose number can be obtained from the ticket number by deleting one digit. Show that you can put every ticket into 50 boxes, but not into less than 50. Show that if you have 10000 4-digit tickets and you are allowed to delete two digits, then you can put every ticket into 34 boxes. For n+2 digit tickets, where you delete n digits, what is the minimum number of boxes required?

13. Given a 100 x 100 square divided into unit squares. Several paths are drawn. Each path is drawn along the sides of the unit squares. Each path has its endpoints on the sides of the big square, but does not contain any other points which are vertices of unit squares and lie on the big square sides. No path intersects itself or any other path. Show that there is a vertex apart from the four corners of the big square that is not on any path.

14. The positive integers $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n$ satisfy: $(a_1 + a_2 + \ldots + a_m) = (b_1 + b_2 + \ldots + b_n) < mn$. Show that we can delete some (but not all) of the numbers so that the sum of the remaining a's equals to the sum of the remaining b's.

15. Given 1000 square plates in the plane with their sides parallel to the coordinate axes (but possibly overlapping and possibly of different sizes). Let S be the set of points covered by the plates. Show that you can choose a subset T of plates such that every point of S is covered by at least one and at most four plates in T.

16. You are given a set of scales and a set of n different weights. R represents the state in which the right pan is heavier, L represents the state in which the left pan is heavier and B represents the state in which the pans balance. Show that given any n-letter string of Rs and Ls you can put the weights onto the scales one at a time so that the string represents the successive states of the scales. For example, if the weights were 1, 2 and 3 and the string was LRL, then you would place 1 in the left pan, then 2 in the right pan, then 3 in the left pan.

17. A polynomial is *monic* if its leading coefficient is 1. Two polynomials p(x) and q(x) *commute* if p(q(x)) = q(p(x)).

(1) Find all monic polynomials of degree 3 or less which commute with $x^2 - k$.

(2) Given a monic polynomial p(x), show that there is at most one monic polynomial of degree n which commutes with $p(x)^2$.

(3) Find the polynomials described in (2) for n = 4 and n = 8.

(4) If q(x) and r(x) are monic polynomials which both commute with $p(x)^2$, show that q(x) and r(x) commute.

(5) Show that there is a sequence of polynomials $p_2(x), p_3(x), \ldots$ such that $p_2(x) = x^2 - 2$, $p_n(x)$ has degree n and all polynomials in the sequence commute.

12th ASU 1978 problems

1. a_n is the nearest integer to $\sqrt{n}$. Find $1/a_1 + 1/a_2 + ... + 1/a_{1980}$.

2. ABCD is a quadrilateral. M is a point inside it such that ABMD is a parallelogram. $\angle$CBM = $\angle$CDM. Show that $\angle$ACD = $\angle$BCM.

3. Show that there is no positive integer n for which $1000^n - 1$ divides $1978^n - 1$.

4. If P, Q are points in space the point [PQ] is the point on the line PQ on the opposite side of Q to P and the same distance from Q. K_0 is a set of points in space. Given K_n we derive K_{n+1} by adjoining all the points [PQ] with P and Q in K_n.

(1) K_0 contains just two points A and B, a distance 1 apart, what is the smallest n for which K_n contains a point whose distance from A is at least 1000?

(2) K_0 consists of three points, each pair a distance 1 apart, find the area of the smallest convex polygon containing K_n.

(3) K_0 consists of four points, forming a regular tetrahedron with volume 1. Let H_n be the smallest convex polyhedron containing K_n. How many faces does H_1 have? What is the volume of H_n?

5. Two players play a game. There is a heap of m tokens and a heap of n < m tokens. Each player in turn takes one or more tokens from the heap which is larger. The number he takes must be a multiple of the number in the smaller heap. For example, if the heaps are 15 and 4, the first player may take 4, 8 or 12 from the larger heap. The first player to clear a heap wins. Show that if m > 2n, then the first player can always win. Find all k such that if m > kn, then the first player can always win.

6. Show that there is an infinite sequence of reals $x_1, x_2, x_3, ...$ such that $|x_n|$ is bounded and for any m > n, we have $|x_m - x_n| > 1/(m - n)$.

7. Let $p(x) = x^2 + x + 1$. Show that for every positive integer n, the numbers n, p(n), p(p(n)), p(p(p(n))), ... are relatively prime.

8. Show that for some k, you can find 1978 different sizes of square with all its vertices on the graph of the function y = k sin x.

9. The set S_0 has the single member (5, 19). We derive the set S_{n+1} from S_n by adjoining a pair to S_n. If S_n contains the pair (2a, 2b), then we may adjoin the pair (a, b). If S contains the pair (a, b) we may adjoin (a+1, b+1). If S contains (a, b) and (b, c), then we may adjoin (a, c). Can we obtain (1, 50)? (1, 100)? If We start with (a, b), with a < b, instead of (5, 19), for which n can we obtain (1, n)?

10. An n-gon area A is inscribed in a circle radius R. We take a point on each side of the polygon to form another n-gon. Show that it has perimeter at least 2A/R.

11. Two players play a game by moving a piece on an n x n chessboard. The piece is initially in a corner square. Each player may move the piece to any adjacent square (which shares a side with its current square), except that the piece may never occupy the same square twice. The first player who is unable to move loses. Show that for even n the first player can always win, and for odd n the second player can always win. Who wins if the piece is initially on a square adjacent to the corner?

12. Given a set of n non-intersecting segments in the plane. No two segments lie on the same line. Can we successively add n-1 additional segments so that we end up with a single non-intersecting path? Each segment we add must have as its endpoints two existing segment endpoints.

13. a and b are positive real numbers. x_i are real numbers lying between a and b. Show that $(x_1 + x_2 + ... + x_n)(1/x_1 + 1/x_2 + ... + 1/x_n) \le n^2(a + b)^2/4ab$.

14. n > 3 is an integer. Let S be the set of lattice points (a, b) with $0 \le a, b < n$. Show that we can choose n points of S so that no three chosen points are collinear and no four chosen points from a parallelogram.

15. Given any tetrahedron, show that we can find two planes such that the areas of the projections of the tetrahedron onto the two planes have ratio at least √2.

16. $a_1, a_2, ... , a_n$ are real numbers. Let $b_k = (a_1 + a_2 + ... + a_k)/k$ for k = 1, 2, ... , n. Let $C = (a_1 - b_1)^2 + (a_2 - b_2)^2 + ... + (a_n - b_n)^2$, and $D = (a_1 - b_n)^2 + (a_2 - b_n)^2 + ... + (a_n - b_n)^2$. Show that $C \le D \le 2C$.

17. Let $x_n = (1 + \sqrt{2} + \sqrt{3})^n$. We may write $x_n = a_n + b_n\sqrt{2} + c_n\sqrt{3} + d_n\sqrt{6}$, where a_n, b_n, c_n, d_n are integers. Find the limit as n tends to infinity of b_n/a_n, c_n/a_n, d_n/a_n.

13th ASU 1979 problems

1. T is an isosceles triangle. Another isosceles triangle T' has one vertex on each side of T. What is the smallest possible value of area T'/area T?

2. A grasshopper hops about in the first quadrant (x, y >= 0). From (x, y) it can hop to (x+1, y-1) or to (x-5, y+7), but it can never leave the first quadrant. Find the set of points (x, y) from which it can never get further than a distance 1000 from the origin.

3. In a group of people every person has less than 4 enemies. Assume that A is B's enemy iff B is A's enemy. Show that we can divide the group into two parts, so that each person has at most one enemy in his part.

4. Let S be the set {0, 1}. Given any subset of S we may add its arithmetic mean to S (provided it is not already included - S never includes duplicates). Show that by repeating this process we can include the number 1/5 in S. Show that we can eventually include any rational number between 0 and 1.

5. The real sequence $x_1 \geq x_2 \geq x_3 \geq \ldots$ satisfies $x_1 + x_4/2 + x_9/3 + x_{16}/4 + \ldots + x_N/n \leq 1$ for every square $N = n^2$. Show that it also satisfies $x_1 + x_2/2 + x_3/3 + \ldots + x_n/n \leq 3$.

6. Given a finite set X of points in the plane. S is a set of vectors **AB** where (A, B) are some pairs of points in X. For every point A the number of vectors **AB** (starting at A) in S equals the number of vectors **CA** (ending at A) in S. Show that the sum of the vectors in S is zero.

7. What is the smallest number of pieces that can be placed on an 8 x 8 chessboard so that every row, column and diagonal has at least one piece? [A diagonal is any line of squares parallel to one of the two main diagonals, so there are 30 diagonals in all.] What is the smallest number for an n x n board?

8. a and b are real numbers. Find real x and y satisfying: $(x - y (x^2 - y^2)^{1/2} = a(1 - x^2 + y^2)^{1/2}$ and $(y - x (x^2 - y^2)^{1/2} = b(1 - x^2 + y^2)^{1/2}$.

9. A set of square carpets have total area 4. Show that they can cover a unit square.

10. x_i are real numbers between 0 and 1. Show that $(x_1 + x_2 + \ldots + x_n + 1)^2 \geq 4(x_1^2 + x_2^2 + \ldots + x_n^2)$.

11. m and n are relatively prime positive integers. The interval [0, 1] is divided into m + n equal subintervals. Show that each part except those at each end contains just one of the numbers 1/m, 2/m, 3/m, ... , (m-1)/m, 1/n, 2/n, ... , (n-1)/n.

12. Given a point P in space and 1979 lines $L_1, L_2, \ldots, L_{1979}$ containing it. No two lines are perpendicular. P_1 is a point on L_1. Show that we can find a point A_n on L_n (for n = 2, 3, ... , 1979) such that the following 1979 pairs of lines are all perpendicular: $A_{n-1}A_{n+1}$ and L_n for n = 1, ... , 1979. [We regard A_{-1} as A_{1979} and A_{1980} as A_1.]

13. Find a sequence $a_1, a_2, \ldots, a_{25}$ of 0s and 1s such that the following sums are all odd:

$a_1a_1 + a_2a_2 + \ldots + a_{25}a_{25}$

$a_1a_2 + a_2a_3 + \ldots + a_{24}a_{25}$

$a_1a_3 + a_2a_4 + \ldots + a_{23}a_{25}$

...

$a_1a_{24} + a_2a_{25}$

a_1a_{25}

Show that we can find a similar sequence of n terms for some n > 1000.

14. A convex quadrilateral is divided by its diagonals into four triangles. The incircles of each of the four are equal. Show that the quadrilateral has all its sides equal.

14th ASU 1980 problems

1. All two digit numbers from 19 to 80 inclusive are written down one after the other as a single number N = 192021...7980. Is N divisible by 1980?

2. A square is divided into n parallel strips (parallel to the bottom side of the square). The width of each strip is integral. The total width of the strips with odd width equals the total width of the strips with even width. A diagonal of the square is drawn which divides each strip into a left part and a right part. Show that the sum of the areas of the left parts of the odd strips equals the sum of the areas of the right parts of the even strips.

3. 35 containers of total weight 18 must be taken to a space station. One flight can take any collection of containers weighing 3 or less. It is possible to take any subset of 34 containers in 7 flights. Show that it must be possible to take all 35 containers in 7 flights.

4. ABCD is a convex quadrilateral. M is the midpoint of BC and N is the midpoint of CD. If k = AM + AN show that the area of ABCD is less than $k^2/2$.

5. Are there any solutions in positive integers to $a^4 = b^3 + c^2$?

6. Given a point P on the diameter AC of the circle K, find the chord BD through P which maximises the area of ABCD.

7. There are several settlements around Big Lake. Some pairs of settlements are directly connected by a regular shipping service. For all A ≠ B, settlement A is directly connected to X iff B is not directly connected to Y, where B is the next settlement to A counterclockwise and Y is the next settlement to X counterclockwise. Show that you can move between any two settlements with at most 3 trips.

8. A six digit (decimal) number has six different digits, none of them 0, and is divisible by 37. Show that you can obtain at least 23 other numbers which are divisible by 37 by permuting the digits.

9. Find all real solutions to:

$$\sin x + 2\sin(x+y+z) = 0$$

$$\sin y + 3\sin(x+y+z) = 0$$

$$\sin z + 4\sin(x+y+z) = 0$$

10. Given 1980 vectors in the plane. The sum of every 1979 vectors is a multiple of the other vector. Not all the vectors are multiples of each other. Show that the sum of all the vectors is zero.

11. Let f(n) be the sum of n and its digits. For example, f(34) = 41. Is there an integer such that f(n) = 1980? Show that given any positive integer m we can find n such that f(n) = m or m+1.

12. Some unit squares in an infinite sheet of squared paper are colored red so that every 2 x 3 and 3 x 2 rectangle contains exactly two red squares. How many red squares are there in a 9 x 11 rectangle?

13. There is a flu epidemic in elf city. The course of the disease is always the same. An elf is infected one day, he is sick the next, recovered and immune the third, recovered but not immune thereafter. Every day every elf who is not sick visits all his sick friends. If he is not immune he is sure to catch flu if he visits a sick elf. On day 1 no one is immune and one or more elves are infected from some external source. Thereafter there is no further external infection and the epidemic spreads as described above. Show that it is sure to die out (irrespective of the number of elves, the number of friends each has, and the number infected on day 1). Show that if one or more elves is immune on day 1, then it is possible for the epidemic to continue indefinitely.

14. Define the sequence a_n of positive integers as follows. $a_1 = m$. $a_{n+1} = a_n$ plus the product of the digits of a_n. For example, if m = 5, we have 5, 10, 10, ... Is there an m for which the sequence is unbounded?

15. ABC is equilateral. A line parallel to AC meets AB at M and BC at P. D is the center of the equilateral triangle BMP. E is the midpoint of AP. Find the angles of DEC.

16. A rectangular box has sides $x < y < z$. Its perimeter is $p = 4(x + y + z)$, its surface area is $s = 2(xy + yz + zx)$ and its main diagonal has length $d = \sqrt{(x^2 + y^2 + z^2)}$. Show that $3x < (p/4 - \sqrt{(d^2 - s/2)}$ and $3z > (p/4 + \sqrt{(d^2 - s/2)}$.

17. S is a set of integers. Its smallest element is 1 and its largest element is 100. Every element of S except 1 is the sum of two distinct members of the set or double a member of the set. What is the smallest possible number of integers in S?

18. Show that there are infinitely many positive integers n such that $[a^{3/2}] + [b^{3/2}] = n$ has at least 1980 integer solutions.

19. ABCD is a tetrahedron. Angles ACB and ADB are 90 deg. Let k be the angle between the lines AC and BD. Show that $\cos k < CD/AB$.

20. x_0 is a real number in the interval (0, 1) with decimal representation $0.d_1d_2d_3...$. We obtain the sequence x_n as follows. x_{n+1} is obtained from x_n by rearranging the 5 digits d_{n+1}, d_{n+2}, d_{n+3}, d_{n+4}, d_{n+5}. Show that the sequence x_n converges. Can the limit be irrational if x_0 is rational? Find a number x_0 so that every member of the sequence is irrational, no matter how the rearrangements are carried out.

15th ASU 1981 problems

1. A chess board is placed on top of an identical board and rotated through 45 degrees about its center. What is the area which is black in both boards?

2. AB is a diameter of the circle C. M and N are any two points on the circle. The chord MA' is perpendicular to the line NA and the chord MB' is perpendicular to the line NB. Show that AA' and BB' are parallel.

3. Find an example of m and n such that m is the product of n consecutive positive integers and also the product of n+2 consecutive positive integers. Show that we cannot have n = 2.

4. Write down a row of arbitrary integers (repetitions allowed). Now construct a second row as follows. Suppose the integer n is in column k in the first row. In column k in the second row write down the number of occurrences of n in row 1 in columns 1 to k inclusive. Similarly, construct a third row under the second row (using the values in the second row), and a fourth row. An example follows:

7 1 2 1 7 1 1

1 1 1 2 2 3 4

1 2 3 1 2 1 1

1 1 1 2 2 3 4

Show that the fourth row is always the same as the second row.

5. Let S be the set of points (x, y) given by $y \le -x^2$ and $y \ge x^2 - 2x + a$. Find the area of the rectangle with sides parallel to the axes and the smallest possible area which encloses S.

6. ABC, CDE, EFG are equilateral triangles (not necessarily the same size). The vertices are counter-clockwise in each case. A, D, G are collinear and AD = DG. Show that BFD is equilateral.

7. 1000 people live in a village. Every evening each person tells his friends all the news he heard during the day. All news eventually becomes known (by this process) to everyone. Show that one can choose 90 people, so that if you give them some news on the same day, then everyone will know in 10 days.

8. The reals a and b are such that $a \cos x + b \cos 3x > 1$ has no real solutions. Show that $|b| \le 1$.

9. ABCD is a convex quadrilateral. K is the midpoint of AB and M is the midpoint of CD. L lies on the side BC and N lies on the side AD. KLMN is a rectangle. Show that its area is half that of ABCD.

10. The sequence a_n of positive integers is such that (1) $a_n \le n^{3/2}$ for all n, and (2) m-n divides $k_m - k_n$ (for all m > n). Find a_n.

11. Is it possible to color half the cells in a rectangular array white and half black so that in each row and column more than 3/4 of the cells are the same color?

12. ACPH, AMBE, AHBT, BKXM and CKXP are parallelograms. Show that ABTE is also a parallelogram (vertices are labeled anticlockwise).

13. Find all solutions (x, y) in positive integers to $x^3 - y^3 = xy + 61$.

14. Eighteen teams are playing in a tournament. So far, each team has played exactly eight games, each with a different opponent. Show that there are three teams none of which has yet played the other.

15. ABC is a triangle. A' lies on the side BC with BA'/BC = 1/4. Similarly, B' lies on the side CA with CB'/CA = 1/4, and C' lies on the side AB with AC'/AB = 1/4. Show that the perimeter of A'B'C' is between 1/2 and 3/4 of the perimeter of ABC.

16. The positive reals x, y satisfy $x^3 + y^3 = x - y$. Show that $x^2 + y^2 < 1$.

17. A convex polygon is drawn inside the unit circle. Someone makes a copy by starting with one vertex and then drawing each side successively. He copies the angle between each side and the previous side accurately, but makes an error in the length of each side of up to a factor $1 \pm p$. As a result the last side ends up a distance d from the starting point. Show that $d < 4p$.

18. An integer is initially written at each vertex of a cube. A move is to add 1 to the numbers at two vertices connected by an edge. Is it possible to equalise the numbers by a series of moves in the following cases? (1) The initial numbers are (1) 0, except for one vertex which is 1. (2) The initial numbers are 0, except for two vertices which are 1 and diagonally opposite on a face of the cube. (3) Initially, the numbers going round the base are 1, 2, 3, 4. The corresponding vertices on the top are 6, 7, 4, 5 (with 6 above the 1, 7 above the 2 and so on).

19. Find 21 consecutive integers, each with a prime factor less than 17.

20. Each of the numbers from 100 to 999 inclusive is written on a separate card. The cards are arranged in a pile in random order. We take cards off the pile one at a time and stack them into 10 piles according to the last digit. We then put the 1 pile on top of the 0 pile, the 2 pile on top of the 1 pile and so on to get a single pile. We now take them off one at a time and stack them into 10 piles according to the middle digit. We then consolidate the piles as before. We then take them off one at a time and stack them into 10 piles according to the first digit and finally consolidate the piles as before. What can we say about the order in the final pile?

21. Given 6 points inside a 3 x 4 rectangle, show that we can find two points whose distance does not exceed $\sqrt{5}$.

22. What is the smallest value of $4 + x^2y^4 + x^4y^2 - 3x^2y^2$ for real x, y? Show that the polynomial cannot be written as a sum of squares. [Note the candidates did not know calculus.]

23. ABCDEF is a prism. Its base ABC and its top DEF are congruent equilateral triangles. The side edges are AD, BE and CF. Find all points on the base wich are equidistant from the three lines AE, BF and CD.

16th ASU 1982 problems

1. The circle C has center O and radius r and contains the points A and B. The circle C' touches the rays OA and OB and has center O' and radius r'. Find the area of the quadrilateral OAO'B.

2. The sequence a_n is defined by $a_1 = 1$, $a_2 = 2$, $a_{n+2} = a_{n+1} + a_n$. The sequence b_n is defined by $b_1 = 2$, $b_2 = 1$, $b_{n+2} = b_{n+1} + b_n$. How many integers belong to both sequences?

3. N is a sum of n powers of 2. If N is divisible by $2^m - 1$, prove that $n \geq m$. Does there exist a number divisible by 11...1 (m 1s) which has the sum of its digits less than m?

4. A non-negative real is written at each vertex of a cube. The sum of the eight numbers is 1. Two players choose faces of the cube alternately. A player cannot choose a face already chosen or the one opposite, so the first player plays twice, the second player plays once. Can the first player arrange that the vertex common to all three chosen faces is $\leq 1/6$?

5. A library is open every day except Wednesday. One day three boys, A, B, C visit the library together for the first time. Thereafter they visit the library many times. A always makes his next visit two days after the previous visit, unless the library is closed on that day, in which case he goes the following day. B always makes his next visit three days after the previous visit (or four if the library is closed). C always makes his next visit four days after the previous visit (or five if the library is closed). For example, if A went first on Monday, his next visit would be Thursday, then Saturday. If B went first on Monday, his next visit would be on Thursday. All three boys are subsequently in the library on a Monday. What day of the week was their first visit?

6. ABCD is a parallelogram and AB is not equal to BC. M is chosen so that (1) $\angle MAC = \angle DAC$ and M is on the opposite side of AC to D, and (2) $\angle MBD = \angle CBD$ and M is on the opposite side of BD to C. Find AM/BM in terms of k = AC/BD.

7. 3n points divide a circle into 3n arcs. One third of the arcs have length 1, one third have length 2 and one third have length 3. Show that two of the points are at opposite ends of a diameter.

8. M is a point inside a regular tetrahedron. Show that we can find two vertices A, B of the tetrahedron such that $\cos AMB \leq -1/3$.

9. $0 < x, y, z < \pi/2$. We have $\cos x = x$, $\sin(\cos y) = y$, $\cos(\sin z) = z$. Which of x, y, z is the largest and which the smallest?

10. P is a polygon with 2n+1 sides. A new polygon is derived by taking as its vertices the midpoints of the sides of P. This process is repeated. Show that we must eventually reach a polygon which is homothetic to P.

11. $a_1, a_2, \dots, a_{1982}$ is a permutation of 1, 2, ... , 1982. If $a_1 > a_2$, we swap a_1 and a_2. Then if (the new) $a_2 > a_3$ we swap a_2 and a_3. And so on. After 1981 potential swaps we have a new permutation $b_1, b_2, \dots, b_{1982}$. We then compare b_{1982} and b_{1981}. If $b_{1981} > b_{1982}$, we swap them. We then compare b_{1980} and (the new) b_{1981}. So we arrive finally at $c_1, c_2, \dots, c_{1982}$. We find that $a_{100} = c_{100}$. What value is a_{100}?

12. Cucumber River has parallel banks a distance 1 meter apart. It has some islands with total perimeter 8 meters. It is claimed that it is always possible to cross the river (starting from an arbitrary point) by boat in at most 3 meters. Is the claim always true for any arrangement of islands? [Neglect the current.]

13. The parabola $y = x^2$ is drawn and then the axes are deleted. Can you restore them using ruler and compasses?

14. An integer is put in each cell of an n x n array. The difference between the integers in cells which share a side is 0 or 1. Show that some integer occurs at least n times.

15. x is a positive integer. Put $a = x^{1/12}$, $b = x^{1/4}$, $c = x^{1/6}$. Show that $2^a + 2^b \geq 2^{1+c}$.

16. What is the largest subset of {1, 2, ... , 1982} with the property that no element is the product of two other distinct elements.

17. A real number is assigned to each unit square in an infinite sheet of squared paper. Show that some cell contains a number that is less than or equal to at least four of its eight neighbors.

18. Given a real sequence $a_1, a_2, \ldots, a_n$, show that it is always possible to choose a subsequence such that (1) for each $i \leq n-2$ at least one and at most two of a_i, a_{i+1}, a_{i+2} are chosen and (2) the sum of the absolute values of the numbers in the subsequence is at least 1/6 $\sum_1^n |a_i|$.

19. An n x n array has a cross in n - 1 cells. A move consists of moving a row to a new position or moving a column to a new position. For example, one might move row 2 to row 5, so that row 1 remained in the same position, row 3 became row 2, row 4 became row 3, row 5 became row 4, row 2 became row 5 and the remaining rows remained in the same position. Show that by a series of moves one can end up with all the crosses below the main diagonal.

20. Let {a} denote the difference between a and the nearest integer. For example {3.8} = 0.2, {-5.4} = 0.4. Show that $|a|\,|a-1|\,|a-2| \ldots |a-n| >= \{a\}\, n!/2^n$.

21. Do there exist polynomials p(x), q(x), r(x) such that $p(x-y+z)^3 + q(y-z-1)^3 + r(z-2x+1)^3 = 1$ for all x, y, z? Do there exist polynomials p(x), q(x), r(x) such that $p(x-y+z)^3 + q(y-z-1)^3 + r(z-x+1)^3 = 1$ for all x, y, z?

22. A tetrahedron T' has all its vertices inside the tetrahedron T. Show that the sum of the edge lengths of T' is less than 4/3 times the corresponding sum for T.

17th ASU 1983 problems

1. A 4 x 4 array of unit cells is made up of a grid of total length 40. Can we divide the grid into 8 paths of length 5? Into 5 paths of length 8?

2. Three positive integers are written on a blackboard. A move consists of replacing one of the numbers by the sum of the other two less one. For example, if the numbers are 3, 4, 5, then one move could lead to 4, 5, 8 or 3, 5, 7 or 3, 4, 6. After a series of moves the three numbers are 17, 1967 and 1983. Could the initial set have been 2, 2, 2? 3, 3, 3?

3. C_1, C_2, C_3 are circles, none of which lie inside either of the others. C_1 and C_2 touch at Z, C_2 and C_3 touch at X, and C_3 and C_1 touch at Y. Prove that if the radius of each circle is increased by a factor $2/\sqrt{3}$ without moving their centers, then the enlarged circles cover the triangle XYZ.

4. Find all real solutions x, y to $y^2 = x^3 - 3x^2 + 2x$, $x^2 = y^3 - 3y^2 + 2y$.

5. The positive integer k has n digits. It is rounded to the nearest multiple of 10, then to the nearest multiple of 100 and so on (n-1 roundings in all). Numbers midway between are rounded up. For example, 1474 is rounded to 1470, then to 1500, then to 2000. Show that the final number is less than 18k/13.

6. M is the midpoint of BC. E is any point on the side AC and F is any point on the side AB. Show that area MEF ≤ area BMF + area CME.

7. a_n is the last digit of $[10^{n/2}]$. Is the sequence a_n periodic? b_n is the last digit of $[2^{n/2}]$. Is the sequence b_n periodic?

8. A and B are acute angles such that $\sin^2 A + \sin^2 B = \sin(A + B)$. Show that $A + B = \pi/2$.

9. The projection of a tetrahedron onto the plane P is ABCD. Can we find a distinct plane P' such that the projection of the tetrahedron onto P' is A'B'C'D' and AA', BB', CC' and DD' are all parallel?

10. Given a quadratic equation $ax^2 + bx + c$. If it has two real roots $A \le B$, transform the equation to $x^2 + Ax + B$. Show that if we repeat this process we must eventually reach an equation with complex roots. What is the maximum possible number of transformations before we reach such an equation?

11. a, b, c are positive integers. If a^b divides b^a and c^a divides a^c, show that c^b divides b^c.

12. A *word* is a finite string of As and Bs. Can we find a set of three 4-letter words, ten 5-letter words, thirty 6-letter words and five 7-letter words such that no word is the beginning of another word. [For example, if ABA was a word, then ABAAB could not be a word.]

13. Can you place an integer in every square of an infinite sheet of squared paper so that the sum of the integers in every 4 x 6 (or 6 x 4) rectangle is (1) 10, (2) 1?

14. A point is chosen on each of the three sides of a triangle and joined to the opposite vertex. The resulting lines divide the triangle into four triangles and three quadrilaterals. The four triangles all have area A. Show that the three quadrilaterals have equal area. What is it (in terms of A)?

15. A group of children form two equal lines side-by-side. Each line contains an equal number of boys and girls. The number of mixed pairs (one boy in one line next to one girl in the other line) equals the number of unmixed pairs (two girls side-by-side or two boys side-by-side). Show that the total number of children in the group is a multiple of 8.

16. A 1 x k rectangle can be divided by two perpendicular lines parallel to the sides into four rectangles, each with area at least 1 and one with area at least 2. What is the smallest possible k?

17. O is a point inside the triangle ABC. a = area OBC, b = area OCA, c = area OAB. Show that the vector sum a**OA** + b**OB** + c**OC** is zero.

18. Show that given any 2m+1 different integers lying between -(2m-1) and 2m-1 (inclusive) we can always find three whose sum is zero.

19. Interior points D, E, F are chosen on the sides BC, CA, AB (not at the vertices). Let k be the length of the longest side of DEF. Let a, b, c be the lengths of the longest sides of AFE, BDF, CDE respectively. Show that $k \geq \sqrt{3}\min(a, b, c)/2$. When do we have equality?

20. X is a union of k disjoint intervals of the real line. It has the property that for any h < 1 we can find two points of X which are a distance h apart. Show that the sum of the lengths of the intervals in X is at least 1/k.

21. x is a real. The decimal representation of x includes all the digits at least once. Let f(n) be the number of distinct n-digit segments in the representation. Show that if for some n we have $f(n) \leq n+8$, then x is rational.

18th ASU 1984 problems

1. Show that we can find n integers whose sum is 0 and whose product is n iff n is divisible by 4.

2. Show that $(a + b)^2/2 + (a + b)/4 \geq a\sqrt{b} + b\sqrt{a}$ for all positive a and b.

3. ABC and A'B'C' are equilateral triangles and ABC and A'B'C' have the same sense (both clockwise or both counter-clockwise). Take an arbitrary point O and points P, Q, R so that OP is equal and parallel to AA', OQ is equal and parallel to BB', and OR is equal and parallel to CC'. Show that PQR is equilateral.

4. Take a large number of unit squares, each with one edge red, one edge blue, one edge green, and one edge yellow. For which m, n can we combine mn squares by placing similarly colored edges together to get an m x n rectangle with one side entirely red, another entirely bue, another entirely green, and the fourth entirely yellow.

5. Let $A = \cos^2 a$, $B = \sin^2 a$. Show that for all real a and positive x, y we have $x^A y^B < x + y$.

6. Two players play a game. Each takes it in turn to paint three unpainted edges of a cube. The first player uses red paint and the second blue paint. So each player has two moves. The first player wins if he can paint all edges of some face red. Can the first player always win?

7. $n > 3$ positive integers are written in a circle. The sum of the two neighbours of each number divided by the number is an integer. Show that the sum of those integers is at least 2n and less than 3n. For example, if the numbers were 3, 7, 11, 15, 4, 1, 2 (with 2 also adjacent to 3), then the sum would be $14/7 + 22/11 + 15/15 + 16/4 + 6/1 + 4/2 + 9/3 = 20$ and $14 \leq 20 < 21$.

8. The incircle of the triangle ABC has center I and touches BC, CA, AB at D, E, F respectively. The segments AI, BI, CI intersect the circle at D', E', F' respectively. Show that DD', EE', FF' are collinear.

9. Find all integers m, n such that $(5 + 3\sqrt{2})^m = (3 + 5\sqrt{2})^n$.

10. $x_1 < x_2 < x_3 < ... < x_n$. y_i is a permutation of the x_i. We have that $x_1 + y_1 < x_2 + y_2 < ... < x_n + y_n$. Prove that $x_i = y_i$.

11. ABC is a triangle and P is any point. The lines PA, PB, PC cut the circumcircle of ABC again at A'B'C' respectively. Show that there are at most eight points P such that A'B'C' is congruent to ABC.

12. The positive reals x, y, z satisfy $x^2 + xy + y^2/3 = 25$, $y^2/3 + z^2 = 9$, $z^2 + zx + x^2 = 16$. Find the value of $xy + 2yz + 3zx$.

13. Starting with the polynomial $x^2 + 10x + 20$, a move is to change the coefficient of x by 1 or to change the coefficient of x^0 by 1 (but not both). After a series of moves the polynomial is changed to $x^2 + 20x + 10$. Is it true that at some intermediate point the polynomial had integer roots?

14. The center of a coin radius r traces out a polygon with perimeter p which has an incircle radius $R > r$. What is the area of the figure traced out by the coin?

15. Each weight in a set of n has integral weight and the total weight of the set is 2n. A balance is initially empty. We then place the weights onto a pan of the balance one at a time. Each time we place the heaviest weight not yet placed. If the pans balance, then we place the weight onto the left pan. Otherwise, we place the weight onto the lighter pan. Show that when all the weights have been placed, the scales will balance. [For example, if the weights are 2, 2, 1, 1. Then we must place 2 in the left pan, followed by 2 in the right pan, followed by 1 in the left pan, followed by 1 in the right pan.]

16. A number is prime however we order its digits. Show that it cannot contain more than three different digits. For example, 337 satisfies the conditions because 337, 373 and 733 are all prime.

17. Find all pairs of digits (b, c) such that the number b ... b6c ... c4, where there are n bs and n cs is a square for all positive integers n.

18. A, B, C and D lie on a line in that order. Show that if X does not lie on the line then |XA| + |XD| + | |AB| - |CD| | > |XB| + |XC|.

19. The real sequence x_n is defined by $x_1 = 1$, $x_2 = 1$, $x_{n+2} = x_{n+1}^2 - x_n/2$. Show that the sequence converges and find the limit.

20. The squares of a 1983 x 1984 chess board are colored alternately black and white in the usual way. Each white square is given the number 1 or the number -1. For each black square the product of the numbers in the neighbouring white squares is 1. Show that all the numbers must be 1.

21. A 3 x 3 chess board is colored alternately black and white in the usual way with the center square white. Each white square is given the number 1 or the number -1. A move consists of simultaneously changing each number to the product of the adjacent numbers. So the four corner squares are each changed to the number previously in the center square and the center square is changed to the product of the four numbers in the corners. Show that after finitely many moves all numbers are 1.

22. Is ln 1.01 greater or less than 2/201?

23. C_1, C_2, C_3 are circles with radii r_1, r_2, r_3 respectively. The circles do not intersect and no circle lies inside any other circle. C_1 is larger than the other two. The two outer common tangents to C_1 and C_2 meet at A ("outer" means that the points where the tangent touches the two circles lie on the same side of the line of centers). The two outer common tangents to C_1 and C_3 intersect at B. The two tangents from A to C_3 and the two tangents from B to C_2 form a quadrangle. Show that it has an inscribed circle and find its radius.

24. Show that any cross-section of a cube through its center has area not less than the area of a face.

19th ASU 1985 problems

1. ABC is an acute angled triangle. The midpoints of BC, CA and AB are D, E, F respectively. Perpendiculars are drawn from D to AB and CA, from E to BC and AB, and from F to CA and BC. The perpendiculars form a hexagon. Show that its area is half the area of the triangle.

2. Is there an integer n such that the sum of the (decimal) digits of n is 1000 and the sum of the squares of the digits is 1000^2?

3. An 8 x 8 chess-board is colored in the usual way. What is the largest number of pieces can be placed on the black squares (at most one per square), so that each piece can be taken by at least one other? A piece A can take another piece B if they are (diagonally) adjacent and the square adjacent to B and opposite to A is empty.

4. Call a side or diagonal of a regular n-gon a *segment*. How many colors are required to paint all the segments of a regular n-gon, so that each segment has a single color and every two segments with a vertex in common have different colors.

5. Given a line L and a point O not on the line, can we move an arbitrary point X to O using only rotations about O and reflections in L?

6. The quadratic $p(x) = ax^2 + bx + c$ has $a > 100$. What is the maximum possible number of integer values x such that $|p(x)| < 50$?

7. In the diagram below a, b, c, d, e, f, g, h, i, j are distinct positive integers and each (except a, e, h and j) is the sum of the two numbers to the left and above. For example, b = a + e, f = e + h, i = h + j. What is the smallest possible value of d?

j

h i

e f g

a b c d

8. $a_1 < a_2 < ... < a_n < ...$ is an unbounded sequence of positive reals. Show that there exists k such that $a_1/a_2 + a_2/a_3 + ... + a_h/a_{h+1} < h-1$ for all $h > k$. Show that we can also find a k such that $a_1/a_2 + a_2/a_3 + ... + a_h/a_{h+1} < h-1985$ for all $h > k$.

9. Find all pairs (x, y) such that $|\sin x - \sin y| + \sin x \sin y <= 0$.

10. ABCDE is a convex pentagon. A' is chosen so that B is the midpoint of AA', B' is chosen so that C is the midpoint of BB' and so on. Given A', B', C', D', E', how do we construct ABCDE using ruler and compasses?

11. The sequence $a_1, a_2, a_3, ...$ satisfies $a_{4n+1} = 1$, $a_{4n+3} = 0$, $a_{2n} = a_n$. Show that it is not periodic.

12. n lines are drawn in the plane. Some of the resulting regions are colored black, no pair of painted regions have a boundary line in common (but they may have a common vertex). Show that at most $(n^2 + n)/3$ regions are black.

13. Each face of a cube is painted a different color. The same colors are used to paint every face of a cubical box a different color. Show that the cube can always be placed in the box, so that every face is a different color from the box face it is in contact with.

14. The points A, B, C, D, E, F are equally spaced on the circumference of a circle (in that order) and AF is a diameter. The center is O. OC and OD meet BE at M and N respectively. Show that MN + CD = OA.

15. A move replaces the real numbers a, b, c, d by a-b, b-c, c-d, d-a. If a, b, c, d are not all equal, show that at least one of the numbers can exceed 1985 after a finite number of moves.

16. $a_1 < a_2 < ... < a_n$ and $b_1 > b_2 > ... > b_n$. Taken together the a_i and b_i constitute the numbers 1, 2, ... , 2n. Show that $|a_1 - b_1| + |a_2 - b_2| + ... + |a_n - b_n| = n^2$.

17. An r x s x t cuboid is divided into rst unit cubes. Three faces of the cuboid, having a common vertex, are colored. As a result exactly half the unit cubes have at least one face colored. What is the total number of unit cubes?

18. ABCD is a parallelogram. A circle through A and B has radius R. A circle through B and D has radius R and meets the first circle again at M. Show that the circumradius of AMD is R.

19. A regular hexagon is divided into 24 equilateral triangles by lines parallel to its sides. 19 different numbers are assigned to the 19 vertices. Show that at least 7 of the 24 triangles have the property that the numbers assigned to its vertices increase counterclockwise.

20. x is a real number. Define $x_0 = 1 + \sqrt{(1 + x)}$, $x_1 = 2 + x/x_0$, $x_2 = 2 + x/x_1$, ... , $x_{1985} = 2 + x/x_{1984}$. Find all solutions to $x_{1985} = x$.

21. A regular pentagon has side 1. All points whose distance from every vertex is less than 1 are deleted. Find the area remaining.

22. Given a large sheet of squared paper, show that for n > 12 you can cut along the grid lines to get a rectangle of more than n unit squares such that it is impossible to cut it along the grid lines to get a rectangle of n unit squares from it.

23. The cube ABCDA'B'C'D' has unit edges. Find the distance between the circle circumscribed about the base ABCD and the circumcircle of AB'C.

20th ASU 1986 problems

1. The quadratic $x^2 + ax + b + 1$ has roots which are positive integers. Show that $(a^2 + b^2)$ is composite.

2. Two equal squares, one with blue sides and one with red sides, intersect to give an octagon with sides alternately red and blue. Show that the sum of the octagon's red side lengths equals the sum of its blue side lengths.

3. ABC is acute-angled. What point P on the segment BC gives the minimal area for the intersection of the circumcircles of ABP and ACP?

4. Given n points can one build n-1 roads, so that each road joins two points, the shortest distance between any two points along the roads belongs to $\{1, 2, 3, \ldots, n(n-1)/2\}$, and given any element of $\{1, 2, 3, \ldots, n(n-1)/2\}$ one can find two points such that the shortest distance between them along the roads is that element?

5. Prove that there is no convex quadrilateral with vertices at lattice points so that one diagonal has twice the length of the other and the angle between them is 45 degrees.

6. Prove that we can find an m x n array of squares so that the sum of each row and the sum of each column is also a square.

7. Two circles intersect at P and Q. A is a point on one of the circles. The lines AP and AQ meet the other circle at B and C respectively. Show that the circumradius of ABC equals the distance between the centers of the two circles. Find the locus of the circumcircle as A varies.

8. A regular hexagon has side 1000. Each side is divided into 1000 equal parts. Let S be the set of the vertices and all the subdividing points. All possible lines parallel to the sides and with endpoints in S are drawn, so that the hexagon is divided into equilateral triangles with side 1. Let X be the set of all vertices of these triangles. We now paint any three unpainted members of X which form an equilateral triangle (of any size). We then repeat until every member of X except one is painted. Show that the unpainted vertex is not a vertex of the original hexagon.

9. Let d(n) be the number of (positive integral) divisors of n. For example, d(12) = 6. Find all n such that $n = d(n)^2$.

10. Show that for all positive reals x_i we have $1/x_1 + 1/(x_1 + x_2) + \ldots + n/(x_1 + \ldots + x_n) < 4/a_1 + 4/a_2 + \ldots + 4/a_n$.

11. ABC is a triangle with AB ≠ AC. Show that for each line through A, there is at most one point X on the line (excluding A, B, C) with $\angle ABX = \angle ACX$. Which lines contain no such points X?

12. An n x n x n cube is divided into n^3 unit cubes. Show that we can assign a different integer to each unit cube so that the sum of each of the $3n^2$ rows parallel to an edge is zero.

13. Find all positive integers a, b, c so that $a^2 + b = c$ and a has n > 1 decimal digits all the same, b has n decimal digits all the same, and c has 2n decimal digits all the same.

14. Two points A and B are inside a convex 12-gon. Show that if the sum of the distances from A to each vertex is a and the sum of the distances from B to each vertex is b, then $|a - b| < 10\,|AB|$.

15. There are 30 cups each containing milk. An elf is able to transfer milk from one cup to another so that the amount of milk in the two cups is equalised. Is there an initial distribution of milk so that the elf cannot equalise the amount in all the cups by a finite number of such transfers?

16. A 99 x 100 chess board is colored in the usual way with alternate squares black and white. What fraction of the main diagonal is black? What if the board is 99 x 101?

17. $A_1A_2 \ldots A_n$ is a regular n-gon and P is an arbitrary point in the plane. Show that if n is even we can choose signs so that the vector sum $\pm PA_1 \pm PA_2 \pm \ldots \pm PA_n = 0$, but if n is odd, then this is only possible for finitely many points P.

18. A 1 or a -1 is put into each cell of an n x n array as follows. A -1 is put into each of the cells around the perimeter. An unoccupied cell is then chosen arbitrarily. It is given the product of the four cells which are closest to it in each of the four directions. For example, if the cells below containing a number or letter (except x) are filled and we decide to fill x next, then x gets the product of a, b, c and d.

-1	-1	-1	-1	-1
-1	a	1		-1
c	x		d	-1
-1				
-1	b	-1	-1	-1

What is the minimum and maximum number of 1s that can be obtained?

19. Prove that $|\sin 1| + |\sin 2| + \ldots + |\sin 3n| > 8n/5$.

20. Let S be the set of all numbers which can be written as 1/mn, where m and n are positive integers not exceeding 1986. Show that the sum of the elements of S is not an integer.

21. The incircle of a triangle has radius 1. It also lies inside a square and touches each side of the square. Show that the area inside both the square and the triangle is at least 3.4. Is it at least 3.5?

22. How many polynomials p(x) have all coefficients 0, 1, 2 or 3 and take the value n at x = 2?

23. A and B are fixed points outside a sphere S. X and Y are chosen so that S is inscribed in the tetrahedron ABXY. Show that the sum of the angles AXB, XBY, BYA and YAX is independent of X and Y.

21st ASU 1987 problems

1. Ten players play in a tournament. Each pair plays one match, which results in a win or loss. If the ith player wins a_i matches and loses b_i matches, show that $\sum a_i^2 = \sum b_i^2$.

2. Find all sets of 6 weights such that for each of n = 1, 2, 3, ... , 63, there is a subset of weights weighing n.

3. ABCDEFG is a regular 7-gon. Prove that 1/AB = 1/AC + 1/AD.

4. Your opponent has chosen a 1 x 4 rectangle on a 7 x 7 board. At each move you are allowed to ask whether a particular square of the board belongs to his rectangle. How many questions do you need to be certain of identifying the rectangle. How many questions are needed for a 2 x 2 rectangle?

5. Prove that $1^{1987} + 2^{1987} + ... + n^{1987}$ is divisible by n+2.

6. An L is an arrangement of 3 adjacent unit squares formed by deleting one unit square from a 2 x 2 square. How many Ls can be placed on an 8 x 8 board (with no interior points overlapping)? Show that if any one square is deleted from a 1987 x 1987 board, then the remaining squares can be covered with Ls (with no interior points overlapping).

7. Squares ABC'C", BCA'A", CAB'B" are constructed on the outside of the sides of the triangle ABC. The line A'A" meets the lines AB and AC at P and P'. Similarly, the line B'B" meets the lines BC and BA at Q and Q', and the line C'C" meets the lines CA and CB at R and R'. Show that P, P', Q, Q', R and R' lie on a circle.

8. $A_1, A_2, ... , A_{2m+1}$ and $B_1, B_2, ... , B_{2n+1}$ are points in the plane such that the 2m+2n+2 lines $A_1A_2, A_2A_3, ... , A_{2m}A_{2m+1}, A_{2m+1}A_1, B_1B_2, B_2B_3, ... , B_{2n}B_{2n+1}, B_{2n+1}B_1$ are all different and no three of them are concurrent. Show that we can find i and j such that A_iA_{i+1}, B_jB_{j+1} are opposite sides of a convex quadrilateral (if i = 2m+1, then we take A_{i+1} to be A_1. Similarly for j = 2n+1).

9. Find 5 different relatively prime numbers, so that the sum of any subset of them is composite.

10. ABCDE is a convex pentagon with $\angle ABC = \angle ADE$ and $\angle AEC = \angle ADB$. Show that $\angle BAC = \angle DAE$.

11. Show that there is a real number x such that all of cos x, cos 2x, cos 4x, ... $\cos(2^n x)$ are negative.

12. The positive reals a, b, c, x, y, z satisfy a + x = b + y = c + z = k. Show that ax + by + cz $\le k^2$.

13. A real number with absolute value at most 1 is put in each square of a 1987 x 1987 board. The sum of the numbers in each 2 x 2 square is 0. Show that the sum of all the numbers does not exceed 1987.

14. AB is a chord of the circle center O. P is a point outside the circle and C is a point on the chord. The angle bisector of APC is perpendicular to AB and a distance d from O. Show that BC = 2d.

15. Players take turns in choosing numbers from the set {1, 2, 3, ... , n}. Once m has been chosen, no divisor of m may be chosen. The first player unable to choose a number loses. Who has a winning strategy for n = 10? For n = 1000?

16. What is the smallest number of subsets of S = {1, 2, ... , 33}, such that each subset has size 9 or 10 and each member of S belongs to the same number of subsets?

17. Some lattice points in the plane are marked. S is a set of non-zero vectors. If you take any one of the marked points P and add place each vector in S with its beginning at P, then more vectors will have their ends on marked points than not. Show that there are an infinite number of points.

18. A convex pentagon is cut along all its diagonals to give 11 pieces. Show that the pieces cannot all have equal areas.

19. The set S_0 = {1, 2!, 4!, 8!, 16!, ... }. The set S_{n+1} consists of all finite sums of distinct elements of S_n. Show that there is a positive integer not in S_{1987}.

20. If the graph of the function f = f(x) is rotated through 90 degrees about the origin, then it is not changed. Show that there is a unique solution to f(b) = b. Give an example of such a function.

21. A convex polyhedron has all its faces triangles. Show that it is possible to color some edges red and the others blue so that given any two vertices one can always find a path between them along the red edges and another path between them along the blue edges.

22. Show that $(2n+1)^n \geq (2n)^n + (2n-1)^n$ for every positive integer n.

22nd ASU 1988 problems

1. A book contains 30 stories. Each story has a different number of pages under 31. The first story starts on page 1 and each story starts on a new page. What is the largest possible number of stories that can begin on odd page numbers?

2. ABCD is a convex quadrilateral. The midpoints of the diagonals and the midpoints of AB and CD form another convex quadrilateral Q. The midpoints of the diagonals and the midpoints of BC and CA form a third convex quadrilateral Q'. The areas of Q and Q' are equal. Show that either AC or BD divides ABCD into two parts of equal area.

3. Show that there are infinitely many triples of distinct positive integers a, b, c such that each divides the product of the other two and $a + b = c + 1$.

4. Given a sequence of 19 positive integers not exceeding 88 and another sequence of 88 positive integers not exceeding 19. Show that we can find two subsequences of consecutive terms, one from each sequence, with the same sum.

5. The quadrilateral ABCD is inscribed in a fixed circle. It has AB parallel to CD and the length AC is fixed, but it is otherwise allowed to vary. If h is the distance between the midpoints of AC and BD and k is the distance between the midpoints of AB and CD, show that the ratio h/k remains constant.

6. The numbers 1 and 2 are written on an empty blackboard. Whenever the numbers m and n appear on the blackboard the number $m + n + mn$ may be written. Can we obtain (1) 13121, (2) 12131?

7. If rationals x, y satisfy $x^5 + y^5 = 2x^2y^2$ show that $1 - xy$ is the square of a rational.

8. There are 21 towns. Each airline runs direct flights between every pair of towns in a group of five. What is the minimum number of airlines needed to ensure that at least one airline runs direct flights between every pair of towns?

9. Find all positive integers n satisfying $(1 + 1/n)^{n+1} = (1 + 1/1998)^{1998}$.

10. A, B, C are the angles of a triangle. Show that $2(\sin A)/A + 2(\sin B)/B + 2(\sin C)/C \le (1/B + 1/C)\sin A + (1/C + 1/A)\sin B + (1/A + 1/B)\sin C$.

11. Form 10A has 29 students who are listed in order on its duty roster. Form 10B has 32 students who are listed in order on its duty roster. Every day two students are on duty, one from form 10A and one from form 10B. Each day just one of the students on duty changes and is replaced by the following student on the relevant roster (when the last student on a roster is replaced he is replaced by the first). On two particular days the same two students were on duty. Is it possible that starting on the first of these days and ending the day before the second, every pair of students (one from 10A and one from 10B) shared duty exactly once?

12. In the triangle ABC, the angle C is obtuse and D is a fixed point on the side BC, different from B and C. For any point M on the side BC, different from D, the ray AM intersects the circumcircle S of ABC at N. The circle through M, D and N meets S again at P, different from N. Find the location of the point M which minimises MP.

13. Show that there are infinitely many odd composite numbers in the sequence $1^1, 1^1 + 2^2, 1^1 + 2^2 + 3^3, 1^1 + 2^2 + 3^3 + 4^4, \ldots$.

14. ABC is an acute-angled triangle. The tangents to the circumcircle at A and C meet the tangent at B at M and N. The altitude from B meets AC at P. Show that BP bisects the angle MPN.

15. What is the minimal value of b/(c + d) + c/(a + b) for positive real numbers b and c and non-negative real numbers a and d such that $b + c \geq a + d$?

16. n^2 real numbers are written in a square n x n table so that the sum of the numbers in each row and column equals zero. A move is to add a row to one column and subtract it from another (so if the entries are a_{ij} and we select row i, column h and column k, then column h becomes $a_{1h} + a_{i1}, a_{2h} + a_{i2}, \ldots, a_{nh} + a_{in}$, column k becomes $a_{1k} - a_{i1}, a_{2k} - a_{i2}, \ldots, a_{nk} - a_{in}$, and the other entries are unchanged). Show that we can make all the entries zero by a series of moves.

17. In the acute-angled triangle ABC, the altitudes BD and CE are drawn. Let F and G be the points of the line ED such that BF and CG are perpendicular to ED. Prove that EF = DG.

18. Find the minimum value of xy/z + yz/x + zx/y for positive reals x, y, z with $x^2 + y^2 + z^2 = 1$.

19. A polygonal line connects two opposite vertices of a cube with side 2. Each segment of the line has length 3 and each vertex lies on the faces (or edges) of the cube. What is the smallest number of segments the line can have?

20. Let m, n, k be positive integers with $m \geq n$ and $1 + 2 + \ldots + n = mk$. Prove that the numbers 1, 2, ... , n can be divided into k groups in such a way that the sum of the numbers in each group equals m.

21. A polygonal line with a finite number of segments has all its vertices on a parabola. Any two adjacent segments make equal angles with the tangent to the parabola at their point of intersection. One end of the polygonal line is also on the axis of the parabola. Show that the other vertices of the polygonal line are all on the same side of the axis.

22. What is the smallest n for which there is a solution to $\sin x_1 + \sin x_2 + \ldots + \sin x_n = 0$, $\sin x_1 + 2 \sin x_2 + \ldots + n \sin x_n = 100$?

23. The sequence of integers a_n is given by $a_0 = 0$, $a_n = p(a_{n-1})$, where p(x) is a polynomial whose coefficients are all positive integers. Show that for any two positive integers m, k with greatest common divisor d, the greatest common divisor of a_m and a_k is a_d.

24. Prove that for any tetrahedron the radius of the inscribed sphere r < ab/(2(a + b)), where a and b are the lengths of any pair of opposite edges.

23rd ASU 1989 problems

1. 7 boys each went to a shop 3 times. Each pair met at the shop. Show that 3 must have been in the shop at the same time.

2. Can 77 blocks each 3 x 3 x 1 be assembled to form a 7 x 9 x 11 block?

3. The incircle of ABC touches AB at M. N is any point on the segment BC. Show that the incircles of AMN, BMN, ACN have a common tangent.

4. A positive integer n has exactly 12 positive divisors $1 = d_1 < d_2 < d_3 < ... < d_{12} = n$. Let $m = d_4 - 1$. We have $d_m = (d_1 + d_2 + d_4) d_8$. Find n.

5. Eight pawns are placed on a chessboard, so that there is one in each row and column. Show that an even number of the pawns are on black squares.

6. ABC is a triangle. A', B', C' are points on the segments BC, CA, AB respectively. Angle B'A'C' = angle A and AC'/C'B = BA'/A'C = CB'/B'A. Show that ABC and A'B'C' are similar.

7. One bird lives in each of n bird-nests in a forest. The birds change nests, so that after the change there is again one bird in each nest. Also for any birds A, B, C, D (not necessarily distinct), if the distance AB < CD before the change, then AB > CD after the change. Find all possible values of n.

8. Show that the 120 five digit numbers which are permutations of 12345 can be divided into two sets with each set having the same sum of squares.

9. We are given 1998 normal coins, 1 heavy coin and 1 light coin, which all look the same. We wish to determine whether the average weight of the two abnormal coins is less than, equal to, or greater than the weight of a normal coin. Show how to do this using a balance 4 times or less.

10. A triangle with perimeter 1 has side lengths a, b, c. Show that $a^2 + b^2 + c^2 + 4abc < 1/2$.

11. ABCD is a convex quadrilateral. X lies on the segment AB with AX/XB = m/n. Y lies on the segment CD with CY/YD = m/n. AY and DX intersect at P, and BY and CX intersect at Q. Show that area XQYP/area ABCD $< mn/(m^2 + mn + n^2)$.

12. A 23 x 23 square is tiled with 1 x 1, 2 x 2 and 3 x 3 squares. What is the smallest possible number of 1 x 1 squares?

13. Do there exist two reals whose sum is rational, but the sum of their nth powers is irrational for all n > 1? Do there exist two reals whose sum is irrational, but the sum of whose nth powers is rational for all n > 1?

14. An insect is on a square ceiling side 1. The insect can jump to the midpoint of the segment joining it to any of the four corners of the ceiling. Show that in 8 jumps it can get to within 1/100 of any chosen point on the ceiling.

15. ABCD has AB = CD, but AB not parallel to CD, and AD parallel to BC. The triangle is ABC is rotated about C to A'B'C. Show that the midpoints of BC, B'C and A'D are collinear.

16. Show that for each integer n > 0, there is a polygon with vertices at lattice points and all sides parallel to the axes, which can be dissected into 1 x 2 (and/or 2 x 1) rectangles in exactly n ways.

17. Find the smallest positive integer n for which we can find an integer m such that $[10^n/m] = 1989$.

18. ABC is a triangle. Points D, E, F are chosen on BC, CA, AB such that B is equidistant from D and F, and C is equidistant from D and E. Show that the circumcenter of AEF lies on the bisector of EDF.

19. S and S' are two intersecting spheres. The line BXB' is parallel to the line of centers, where B is a point on S, B' is a point on S' and X lies on both spheres. A is another point on S, and A' is another point on S' such that the line AA' has a point on both spheres. Show that the segments AB and A'B' have equal projections on the line AA'.

20. Two walkers are at the same altitude in a range of mountains. The path joining them is piecewise linear with all its vertices above the two walkers. Can they each walk along the path until they have changed places, so that at all times their altitudes are equal?

21. Find the least possible value of $(x + y)(y + z)$ for positive reals satisfying $(x + y + z)\, xyz = 1$.

22. A polyhedron has an even number of edges Show that we can place an arrow on each edge so that each vertex has an even number of arrows pointing towards it (on adjacent edges).

23. N is the set of positive integers. Does there exist a function $f: N \to N$ such that $f(n+1) = f(f(n)) + f(f(n+2))$ for all n.

24. A convex polygon is such that any segment dividing the polygon into two parts of equal area which has at least one end at a vertex has length < 1. Show that the area of the polygon is $< \pi/4$.

24th ASU 1990 problems

1. Show that $x^4 > x - 1/2$ for all real x.

2. The line joining the midpoints of two opposite sides of a convex quadrilateral makes equal angles with the diagonals. Show that the diagonals are equal.

3. A graph has 30 points and each point has 6 edges. Find the total number of triples such that each pair of points is joined or each pair of points is not joined.

4. Does there exist a rectangle which can be dissected into 15 congruent polygons which are not rectangles? Can a square be dissected into 15 congruent polygons which are not rectangles?

5. The point P lies inside the triangle ABC. A line is drawn through P parallel to each side of the triangle. The lines divide AB into three parts length c, c', c" (in that order), and BC into three parts length a, a', a" (in that order), and CA into three parts length b, b', b" (in that order). Show that abc = a'b'c' = a"b"c".

6. Find three non-zero reals such that all quadratics with those numbers as coefficients have two distinct rational roots.

7. What is the largest possible value of $| \dots | |a_1 - a_2| - a_3| - \dots - a_{1990}|$, where $a_1, a_2, \dots , a_{1990}$ is a permutation of 1, 2, 3, ... , 1990?

8. An equilateral triangle of side n is divided into n^2 equilateral triangles of side 1. A path is drawn along the sides of the triangles which passes through each vertex just once. Prove that the path makes an acute angle at at least n vertices.

9. Can the squares of a 1990 x 1990 chessboard be colored black or white so that half the squares in each row and column are black and cells symmetric with respect to the center are of opposite color?

10. Let $x_1, x_2, \dots , x_n$ be positive reals with sum 1. Show that $x_1^2/(x_1 + x_2) + x_2^2/(x_2 + x_3) + \dots + x_{n-1}^2/(x_{n-1} + x_n) + x_n^2/(x_n + x_1) \ge 1/2$.

11. ABCD is a convex quadrilateral. X is a point on the side AB. AC and DX intersect at Y. Show that the circumcircles of ABC, CDY and BDX have a common point.

12. Two grasshoppers sit at opposite ends of the interval [0, 1]. A finite number of points (greater than zero) in the interval are marked. A move is for a grasshopper to select a marked point and jump over it to the equidistant point the other side. This point must lie in the interval for the move to be allowed, but it does not have to be marked. What is the smallest n such that if each grasshopper makes n moves or less, then they end up with no marked points between them?

13. Find all integers n such that $[n/1!] + [n/2!] + \dots + [n/10!] = 1001$.

14. A, B, C are adjacent vertices of a regular 2n-gon and D is the vertex opposite to B (so that BD passes through the center of the 2n-gon). X is a point on the side AB and Y is a point on the side BC so that angle XDY = $\pi/2n$. Show that DY bisects angle XYC.

15. A graph has n points and n(n-1)/2 edges. Each edge is colored with one of k colors so that there are no closed monochrome paths. What is the largest possible value of n (given k)?

16. Given a point X and n vectors $\mathbf{x}_i$ with sum zero in the plane. For each permutation of the vectors we form a set of n points, by starting at X and adding the vectors in order. For example, with the original ordering we get X_1 such that $XX_1 = \mathbf{x}_1$, X_2 such that $X_1X_2 = \mathbf{x}_2$ and so on. Show that for some permutation we can find two points Y, Z with angle YXZ = 60 deg, so that all the points lie inside or on the triangle XYZ.

17. Two unequal circles intersect at X and Y. Their common tangents intersect at Z. One of the tangents touches the circles at P and Q. Show that ZX is tangent to the circumcircle of PXQ.

18. Given 1990 piles of stones, containing 1, 2, 3, ... , 1990 stones. A move is to take an equal number of stones from one or more piles. How many moves are needed to take all the stones?

19. A quadratic polynomial p(x) has positive real coefficients with sum 1. Show that given any positive real numbers with product 1, the product of their values under p is at least 1.

20. A cube side 100 is divided into a million unit cubes with faces parallel to the large cube. The edges form a lattice. A prong is any three unit edges with a common vertex. Can we decompose the lattice into prongs with no common edges?

21. For which positive integers n is $3^{2n+1} - 2^{2n+1} - 6^n$ composite?

22. If every altitude of a tetrahedron is at least 1, show that the shortest distance between each pair of opposite edges is more than 2.

23. A game is played in three moves. The first player picks any real number, then the second player makes it the coefficient of a cubic, except that the coefficient of x^3 is already fixed at 1. Can the first player make his choices so that the final cubic has three distinct integer roots?

24. Given 2n genuine coins and 2n fake coins. The fake coins look the same as genuine coins but weigh less (but all fake coins have the same weight). Show how to identify each coin as genuine or fake using a balance at most 3n times.

25th ASU 1991 problems

1. Find all integers a, b, c, d such that ab - 2cd = 3, ac + bd = 1.

2. n numbers are written on a blackboard. Someone then repeatedly erases two numbers and writes half their arithmetic mean instead, until only a single number remains. If all the original numbers were 1, show that the final number is not less than 1/n.

3. Four lines in the plane intersect in six points. Each line is thus divided into two segments and two rays. Is it possible for the eight segments to have lengths 1, 2, 3, ... , 8? Can the lengths of the eight segments be eight distinct integers?

4. A lottery ticket has 50 cells into which one must put a permutation of 1, 2, 3, ... , 50. Any ticket with at least one cell matching the winning permutation wins a prize. How many tickets are needed to be sure of winning a prize?

5. Find unequal integers m, n such that mn + n and mn + m are both squares. Can you find such integers between 988 and 1991?

6. ABCD is a rectangle. Points K, L, M, N are chosen on AB, BC, CD, DA respectively so that KL is parallel to MN, and KM is perpendicular to LN. Show that the intersection of KM and LN lies on BD.

7. An investigator works out that he needs to ask at most 91 questions on the basis that all the answers will be yes or no and all will be true. The questions may depend upon the earlier answers. Show that he can make do with 105 questions if at most one answer could be a lie.

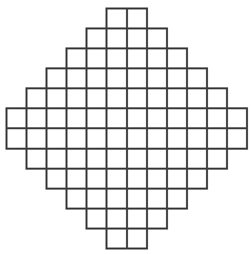
8. A minus sign is placed on one square of a 5 x 5 board and plus signs are placed on the remaining squares. A move is to select a 2 x 2, 3 x 3, 4 x 4 or 5 x 5 square and change all the signs in it. Which initial positions allow a series of moves to change all the signs to plus?

9. Show that $(x + y + z)^2/3 \geq x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy}$ for all non-negative reals x, y, z.

10. Does there exist a triangle in which two sides are integer multiples of the median to that side? Does there exist a triangle in which every side is an integer multiple of the median to that side?

11. The numbers 1, 2, 3, ... , n are written on a blackboard (where $n \geq 3$). A move is to replace two numbers by their sum and non-negative difference. A series of moves makes all the numbers equal k. Find all possible k.

12. The figure below is cut along the lines into polygons (which need not be convex). No polygon contains a 2 x 2 square. What is the smallest possible number of polygons?



13. ABC is an acute-angled triangle with circumcenter O. The circumcircle of ABO intersects AC and BC at M and N. Show that the circumradii of ABO and MNC are the same.

14. A polygon can be transformed into a new polygon by making a straight cut, which creates two new pieces each with a new edge. One piece is then turned over and the two new edges are reattached. Can repeated transformations of this type turn a square into a triangle?

15. An *h x k minor* of an n x n table is the hk cells which lie in h rows and k columns. The *semiperimeter* of the minor is h + k. A number of minors each with semiperimeter at least n together include all the cells on the main diagonal. Show that they include at least half the cells in the table.

16. (1) $r_1, r_2, \dots, r_{100}, c_1, c_2, \dots, c_{100}$ are distinct reals. The number $r_i + c_j$ is written in position i, j of a 100 x 100 array. The product of the numbers in each column is 1. Show that the product of the numbers in each row is -1. (2) $r_1, r_2, \dots, r_{2n}, c_1, c_2, \dots, c_{2n}$ are distinct reals. The number $r_i + c_j$ is written in position i, j of a 2n x 2n array. The product of the numbers in each column is the same. Show that the product of the numbers in each row is also the same.

17. A sequence of positive integers is constructed as follows. If the last digit of a_n is greater than 5, then a_{n+1} is $9a_n$. If the last digit of a_n is 5 or less and a_n has more than one digit, then a_{n+1} is obtained from a_n by deleting the last digit. If a_n has only one digit, which is 5 or less, then the sequence terminates. Can we choose the first member of the sequence so that it does not terminate?

18. p(x) is the cubic $x^3 - 3x^2 + 5x$. If h is a real root of p(x) = 1 and k is a real root of p(x) = 5, find h + k.

19. The chords AB and CD of a sphere intersect at X. A, C and X are equidistant from a point Y on the sphere. Show that BD and XY are perpendicular.

20. Do there exist 4 vectors in the plane so that none is a multiple of another, but the sum of each pair is perpendicular to the sum of the other two? Do there exist 91 non-zero vectors in the plane such that the sum of any 19 is perpendicular to the sum of the others?

21. ABCD is a square. The points X on the side AB and Y on the side AD are such that $AX \cdot AY = 2\, BX \cdot DY$. The lines CX and CY meet the diagonal BD in two points. Show that these points lie on the circumcircle of AXY.

22. X is a set with 100 members. What is the smallest number of subsets of X such that every pair of elements belongs to at least one subset and no subset has more than 50 members? What is the smallest number if we also require that the union of any two subsets has at most 80 members?

23. The real numbers $x_1, x_2, \dots, x_{1991}$ satisfy $|x_1 - x_2| + |x_2 - x_3| + \dots + |x_{1990} - x_{1991}| = 1991$. What is the maximum possible value of $|s_1 - s_2| + |s_2 - s_3| + \dots + |s_{1990} - s_{1991}|$, where $s_n = (x_1 + x_2 + \dots + x_n)/n$?

1st CIS 1992 problems

1. Show that $x^4 + y^4 + z^2 \geq xyz\sqrt{8}$ for all positive reals x, y, z.

2. E is a point on the diagonal BD of the square ABCD. Show that the points A, E and the circumcenters of ABE and ADE form a square.

3. A country contains n cities and some towns. There is at most one road between each pair of towns and at most one road between each town and each city, but all the towns and cities are connected, directly or indirectly. We call a route between a city and a town a gold route if there is no other route between them which passes through fewer towns. Show that we can divide the towns and cities between n republics, so that each belongs to just one republic, each republic has just one city, and each republic contains all the towns on at least one of the gold routes between each of its towns and its city.

4. Given an infinite sheet of square ruled paper. Some of the squares contain a piece. A move consists of a piece jumping over a piece on a neighbouring square (which shares a side) onto an empty square and removing the piece jumped over. Initially, there are no pieces except in an m x n rectangle (m, n > 1) which has a piece on each square. What is the smallest number of pieces that can be left after a series of moves?

5. Does there exist a 4-digit integer which cannot be changed into a multiple of 1992 by changing 3 of its digits?

6. A and B lie on a circle. P lies on the minor arc AB. Q and R (distinct from P) also lie on the circle, so that P and Q are equidistant from A, and P and R are equidistant from B. Show that the intersection of AR and BQ is the reflection of P in AB.

7. Find all real x, y such that $(1 + x)(1 + x^2)(1 + x^4) = 1+ y^7$, $(1 + y)(1 + y^2)(1 + y^4) = 1+ x^7$?

8. An m x n rectangle is divided into mn unit squares by lines parallel to its sides. A gnomon is the figure of three unit squares formed by deleting one unit square from a 2 x 2 square. For what m, n can we divide the rectangle into gnomons so that no two gnomons form a rectangle and no vertex is in four gnomons?

9. Show that for any real numbers x, y > 1, we have $x^2/(y - 1) + y^2/(x - 1) \geq 8$.

10. Show that if 15 numbers lie between 2 and 1992 and each pair is coprime, then at least one is prime.

11. A cinema has its seats arranged in n rows x m columns. It sold mn tickets but sold some seats more than once. The usher managed to allocate seats so that every ticket holder was in the correct row or column. Show that he could have allocated seats so that every ticket holder was in the correct row or column and at least one person was in the correct seat. What is the maximum k such that he could have always put every ticket holder in the correct row or column and at least k people in the correct seat?

12. Circles C and C' intersect at O and X. A circle center O meets C at Q and R and meets C' at P and S. PR and QS meet at Y distinct from X. Show that $\angle YXO = 90^o$.

13. Define the sequence $a_1 = 1, a_2, a_3, \ldots$ by $a_{n+1} = a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2 + n$. Show that 1 is the only square in the sequence.

14. ABCD is a parallelogram. The excircle of ABC opposite A has center E and touches the line AB at X. The excircle of ADC opposite A has center F and touches the line AD at Y. The line FC meets the line AB at W, and the line EC meets the line AD at Z. Show that WX = YZ.

15. Half the cells of a 2m x n board are colored black and the other half are colored white. The cells at the opposite ends of the main diagonal are different colors. The center of each black cell is connected to the center of every other black cell by a straight line segment, and similarly for the white cells. Show that we can place an arrow on each segment so that it becomes a vector and the vectors sum to zero.

16. A graph has 17 points and each point has 4 edges. Show that there are two points which are not joined and which are not both joined to the same point.

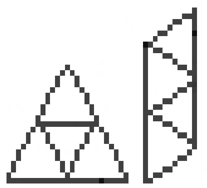
17. Let f(x) = a cos(x + 1) + b cos(x + 2) + c cos(x + 3), where a, b, c are real. Given that f(x) has at least two zeros in the interval (0, π), find all its real zeros.

18. A plane intersects a sphere in a circle C. The points A and B lie on the sphere on opposite sides of the plane. The line joining A to the center of the sphere is normal to the plane. Another plane p intersects the segment AB and meets C at P and Q. Show that BP·BQ is independent of the choice of p.

19. If you have an algorithm for finding all the real zeros of any cubic polynomial, how do you find the real solutions to x = p(y), y = p(x), where p is a cubic polynomial?

20. Find all integers k > 1 such that for some distinct positive integers a, b, the number $k^a + 1$ can be obtained from $k^b + 1$ by reversing the order of its (decimal) digits.

21. An equilateral triangle side 10 is divided into 100 equilateral triangles of side 1 by lines parallel to its sides. There are m equilateral tiles of 4 unit triangles and 25 - m straight tiles of 4 unit triangles (as shown below). For which values of m can they be used to tile the original triangle. [The straight tiles may be turned over.]



22. 1992 vectors are given in the plane. Two players pick unpicked vectors alternately. The winner is the one whose vectors sum to a vector with larger magnitude (or they draw if the magnitudes are the same). Can the first player always avoid losing?

23. If a > b > c > d > 0 are integers such that ad = bc, show that $(a - d)^2 \geq 4d + 8$.

21st Russian 1995 problems

1. A goods train left Moscow at x hrs y mins and arrived in Saratov at y hrs z mins. The journey took z hrs x mins. Find all possible values of x.

2. The chord CD of a circle center O is perpendicular to the diameter AB. The chord AE goes through the midpoint of the radius OC. Prove that the chord DE goes through the midpoint of the chord BC.

3. f(x), g(x), h(x) are quadratic polynomials. Can f(g(h(x))) = 0 have roots 1, 2, 3, 4, 5, 6, 7, 8?

4. Can the integers 1 to 81 be arranged in a 9 x 9 array so that the sum of the numbers in every 3 x 3 subarray is the same?

5. Solve cos(cos(cos(cos x))) = sin(sin(sin(sin x))).

6. Does there exist a sequence of positive integers such that every positive integer occurs exactly once in the sequence and for each k the sum of the first k terms is divisible by k?

7. A convex polygon has all angles equal. Show that at least two of its sides are not longer than their neighbors.

8. Can we find 12 geometrical progressions whose union includes all the numbers 1, 2, 3, ... , 100?

9. R is the reals. f: R → R is any function. Show that we can find functions g: R → R and h: R → R such that f(x) = g(x) + h(x) and the graphs of g and h both have an axial symmetry.

10. Given two points in a plane a distance 1 apart, one wishes to construct two points a distance n apart using only a compass. One is allowed to draw a circle whose center is any point constructed so far (or given initially) and whose radius is the distance between any two points constructed so far (or given initially). One is also allowed to mark the intersection of any two circles. Let C(n) be the smallest number of circles which must be drawn to get two points a distance n apart. One can also carry out the construction with rule and compass. In this case one is also allowed to draw the line through any two points constructed so far (or given initially) and to mark the intersection of any two lines or of any line and a circle. Let R(n) be the smallest number of circles and lines which must be drawn in this case to get two points a distance n apart (starting with just two points, which are a distance 1 apart). Show that C(n)/R(n) → ∞.

11. Show that we can find positive integers A, B, C such that (1) A, B, C each have 1995 digits, none of them 0, (2) B and C are each formed by permuting the digits of A, and (3) A + B = C.

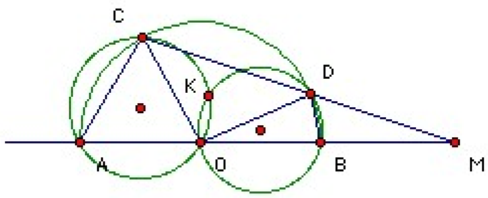
12. ABC is an acute-angled triangle. A_2, B_2, C_2 are the midpoints of the altitudes AA_1, BB_1, CC_1 respectively. Find $\angle B_2A_1C_2 + \angle C_2B_1A_2 + \angle A_2C_1B_2$.

13. There are three heaps of stones. Sisyphus moves stones one at a time. If he takes a stone from one pile, leaving A behind, and adds it to a pile containing B before the move, then Zeus pays him B - A. (If B - A is negative, then Sisyphus pays Zeus A - B.) After some moves the three piles all have the same number of stones that they did originally. What is the maximum net amount that Zeus can have paid Sisyphus?

14. The number 1 or -1 is written in each cell of a 2000 x 2000 array. The sum of all the numbers in the array is non-negative. Show that there are 1000 rows and 1000 columns such that the sum of the numbers at their intersections is at least 1000.

15. A sequence $a_1, a_2, a_3, \ldots$ of positive integers is such that for all $i \neq j$, $\gcd(a_i, a_j) = \gcd(i, j)$. Prove that $a_i = i$ for all i.

16. C, D are points on the semicircle diameter AB, center O. CD meets the line AB at M (with MB < MA, MD < MC). The circumcircles of AOC and DOB meet again at K. Show that $\angle MKO = 90^o$.



17. p(x) and q(x) are non-constant polynomials with leading coefficient 1. Prove that the sum of the squares of the coefficients of the polynomial p(x)q(x) is at least p(0) + q(0).

18. Given any positive integer k, show that we can find $a_1 < a_2 < a_3 < \ldots$ such that $a_1 = k$ and $(a_1^2 + a_2^2 + \ldots + a_n^2)$ is divisible by $(a_1 + a_2 + \ldots + a_n)$ for all n.

19. For which n can we find n-1 numbers $a_1, a_2, \ldots, a_{n-1}$ all non-zero mod n such that 0, a_1, a_1+a_2, $a_1+a_2+a_3$, ... , $a_1+a_2+\ldots+a_{n-1}$ are all distinct mod n.

20. ABCD is a tetrahedron with altitudes AA', BB', CC', DD'. The altitudes all pass through the point X. B" is a point on BB' such that BB"/B"B' = 2. C" and D" are similar points on CC', DD' respectively. Prove that X, A', B", C", D" lie on a sphere.

22nd Russian 1996 problems

1. Can a majority of the numbers from 1 to a million be represented as the sum of a square and a (non-negative) cube?

2. Non-intersecting circles of equal radius are drawn centered on each vertex of a triangle. From each vertex a tangent is drawn to the other circles which intersects the opposite side of the triangle. The six resulting lines enclose a hexagon. Color alternate sides of the hexagon red and blue. Show that the sum of the blue sides equals the sum of the red sides.

3. $a^n + b^n = p^k$ for positive integers a, b and k, where p is an odd prime and $n > 1$ is an odd integer. Show that n must be a power of p.

4. The set X has 1600 members. P is a collection of 16000 subsets of X, each having 80 members. Show that there must be two members of P which have 3 or less members in common.

5. Show that the arithmetic progression 1, 730, 1459, 2188, ... contains infinitely many powers of 10.

6. The triangle ABC has CA = CB, circumcenter O and incenter I. The point D on BC is such that DO is perpendicular to BI. Show that DI is parallel to AC.

7. Two piles of coins have equal weight. There are m coins in the first pile and n coins in the second pile. For any $0 < k \le \min(m, n)$, the sum of the weights of the k heaviest coins in the first pile is not more than the sum of the weights of the k heaviest coins in the second pile. Show that if h is a positive integer and we replace every coin (in either pile) whose weight is less than h by a coin of weight h, then the first pile will weigh at least as much as the second.

8. An L is formed from three unit squares, so that it can be joined to a unit square to form a 2 x 2 square. Can a 5 x 7 board be covered with several layers of Ls (each covering 3 unit squares of the board), so that each square is covered by the same number of Ls?

9. ABCD is a convex quadrilateral. Points D and F are on the side BC so that the points on BC are in the order B, E, F, C. $\angle BAE = \angle CDF$ and $\angle EAF = \angle EDF$. Show that $\angle CAF = \angle BDE$.

10. Four pieces A, B, C, D are placed on the plane lattice. A move is to select three pieces and to move the first by the vector between the other two. For example, if A is at (1, 2), B at (-3, 4) and C at (5, 7), then one could move A to (9, 5). Show that one can always make a series of moves which brings A and B onto the same node.

11. Find powers of 3 which can be written as the sum of the kth powers ($k > 1$) of two relatively prime integers.

12. $a_1, a_2, \dots, a_m$ are non-zero integers such that $a_1 + a_2 2^k + a_3 3^k + \dots + a_m m^k = 0$ for k = 0, 1, 2, ... , n (where $n < m - 1$). Show that the sequence a_i has at least n+1 pairs of consecutive terms with opposite signs.

13. A different number is placed at each vertex of a cube. Each edge is given the greatest common divisor of the numbers at its two endpoints. Can the sum of the edge numbers equal the sum of the vertex numbers?

14. Three sergeants A, B, C and some soldiers serve in a platoon. The first day A is on duty, the second day B is on duty, the third day C, the fourth day A, the fifth B, the sixth C, the seventh A and so on. There is an infinite list of tasks. The commander gives the following orders: (1) the duty sergeant must issue at least one task to a soldier every day, (2) no soldier may have three or more tasks, (3) no soldier may be given more than one new task on any one day, (4) the set of soldiers receiving tasks must be different every day, (5) the first sergeant to violate any of (1) to (4) will be jailed. Can any of the sergeants be sure to avoid going to jail (strategies that involve collusion are not allowed)?

15. No two sides of a convex polygon are parallel. For each side take the angle subtended by the side at the point whose perpendicular distance from the line containing the side is the largest. Show that these angles add up to 180^o.

16. Two players play a game. The first player writes ten positive real numbers on a board. The second player then writes another ten. All the numbers must be distinct. The first player then arranges the numbers into 10 ordered pairs (a, b). The first player wins iff the ten quadratics $x^2 + ax + b$ have 11 distinct real roots between them. Which player wins?

17. The numbers from 1 to $n > 1$ are written down without a break. Can the resulting number be a palindrome (the same read left to right and right to left)? For example, if n was 4, the result would be 1234, which is not a palindrome.

18. n people move along a road, each at a fixed (but possibly different) speed. Over some period the sum of their pairwise distances decreases. Show that we can find a person such that the sum of his distances to the other people is decreasing throughout the period. [Note that people may pass each other during the period.]

19. $n > 4$. Show that no cross-section of a pyramid whose base is a regular n-gon (and whose apex is directly above the center of the n-gon) can be a regular (n+1)-gon.

20. Do there exist three integers each greater than one such that the square of each less one is divisible by both the others?

21. ABC is a triangle with circumcenter O and AB = AC. The line through O perpendicular to the angle bisector CD meets BC at E. The line through E parallel to the angle bisector meets AB at F. Show that DF = BE.

22. Do there exist two finite sets such that we can find polynomials of arbitrarily large degree with all coefficients in the first set and all roots real and in the second set?

23. The integers from 1 to 100 are permuted in an unknown way. One may ask for the order of any 50 integers. How many such questions are needed to deduce the permutation?

23rd Russian 1997 problems

1. p(x) is a quadratic polynomial with non-negative coefficients. Show that $p(xy)^2 \le p(x^2)p(y^2)$.

2. A convex polygon is invariant under a 90^o rotation. Show that for some R there is a circle radius R contained in the polygon and a circle radius $R\sqrt{2}$ which contains the polygon.

3. A rectangular box has integral sides a, b, c, with c odd. Its surface is covered with pieces of rectangular cloth. Each piece contains an even number of unit squares and has its sides parallel to edges of the box. The pieces may be bent along box edges length c (but not along the edges length a or b), but there must be no gaps and no part of the box may be covered by more than one thickness of cloth. Prove that the number of possible coverings is even.

4. The members of the Council of the Wise are arranged in a column. The king gives each sage a black or a white cap. Each sage can see the color of the caps of all the sages in front of him, but he cannot see his own or the colors of those behind him. Every minute a sage guesses the color of his cap. The king immediately executes those sages who are wrong. The Council agree on a strategy to minimise the number of executions. What is the best strategy? Suppose there are three colors of cap?

5. Find all integral solutions to $(m^2 - n^2)^2 = 1 + 16n$.

6. An n x n square grid is glued to make a cylinder. Some of its cells are colored black. Show that there are two parallel horizontal, vertical or diagonal lines (of n cells) which contain the same number of black cells.

7. Two circles meet at A and B. A line through A meets the circles again at C and D. M, N are the midpoints of the arcs BC, BD which do not contain A. K is the midpoint of the segment CD. Prove that $\angle MKN = 90^o$.

8. A polygon can be divided into 100 rectangles, but not into 99 rectangles. Prove that it cannot be divided into 100 triangles.

9. A cube side n is divided into unit cubes. A closed broken line without self-intersections is given. Each segment of the broken line connects the centers of two unit cubes with a common face. Show that we can color the edges of the unit cubes with two colors, so that each face of a small cube which is intersected by the broken line has an odd number of edges of each color, and each face which is not intersected by the broken line has an even number of edges of each color.

10. Do there exist reals b, c so that $x^2 + bx + c = 0$ and $x^2 + (b+1)x + (c+1) = 0$ both have two integral roots?

11. There are 33 students in a class. Each is asked how many other students share is first name and how many share his last name. The answers include all numbers from 0 to 10. Show that two students must have the same first name and the same last name.

12. The incircle of ABC touches AB, BC, CA at M, N, K respectively. The line through A parallel to NK meets the line MN at D. The line through A parallel to MN meets the line NK at E. Prove that the line DE bisects AB and AD.

13. The numbers 1, 2, 3, ... , 100 are arranged in the cells of a 10 x 10 square so that given any two cells with a common side the sum of their numbers does not exceed N. Find the smallest possible value of N.

14. The incircle of ABC touches the sides AC, AB, BC at K, M, N respectively. The median BB' meeets MN at D. Prove that the incenter lies on the line DK.

15. Find all solutions in positive integers to $a + b = \gcd(a,b)^2$, $b + c = \gcd(b,c)^2$, $c + a = \gcd(c,a)^2$.

16. Some stones are arranged in an infinite line of pots. The pots are numbered ... -3, -2, -1, 0, 1, 2, 3, ... Two moves are allowed: (1) take a stone from pot n-1 and a stone from pot n and put a stone into pot n+1 (for any n); (2) take two stones from pot n and put one stone into each of pots n+1 and n-2. Show that any sequence of moves must eventually terminate (so that no more moves are possible) and that the final state depends only on the initial state.

17. Consider all quadratic polynomials x2 + ax + b with integral coefficients such that $1 \le a, b \le 1997$. Let A be the number with integral roots and B the number with no real roots. Which of A, B is larger?

18. P is a polygon. L is a line, and X is a point on L, such that the lines containing the sides of P meet L in distinct points different from X. We color a vertex of P red iff its the lines containing its two sides meet L on opposite sides of X. Show that X is inside P iff there are an odd number of red vertices.

19. A sphere is inscribed in a tetrahedron. It touches one face at its incenter, another face at its orthocenter, and a third face at its centroid. Show that the tetrahedron must be regular.

20. 2 x 1 dominos are used to tile an m x n square, except for a single 1 x 1 hole at a corner. A domino which borders the hole along its short side may be slid one unit along its long side to cover the hole and open a new hole. Show that the hole may be moved to any other corner by moves of this type.

24th Russian 1998 problems

1. a and b are such that there are two arcs of the parabola $y = x^2 + ax + b$ lying between the ray $y = x$, $x > 0$ and $y = 2x$, $x > 0$. Show that the projection of the left-hand arc onto the x-axis is smaller than the projection of the right-hand arc by 1.

2. A convex polygon is partitioned into parallelograms, show that at least three vertices of the polygon belong to only one parallelogram.

3. Can you find positive integers a, b, c, so that the sum of any two has digit sum less than 5, but the sum of all three has digit sum more than 50?

4. A maze is a chessboard with walls between some squares. A piece responds to the commands left, right, up, down by moving one square in the indicated direction (parallel to the sides of the board), unless it meets a wall or the edge of the board, in which case it does not move. Is there a universal sequence of moves so that however the maze is constructed and whatever the initial position of the piece, by following the sequence it will visit every square of the board. You should assume that a maze must be constructed, so that some sequence of commands would allow the piece to visit every square.

5. Five watches each have the conventional 12 hour faces. None of them work. You wish to move forward the time on some of the watches so that they all show the same time and so that the sum of the times (in minutes) by which each watch is moved forward is as small as possible. How should the watches be set to maximise this minimum sum?

6. In the triangle ABC, AB > BC, M is the midpoint of AC and BL is the angle bisector of B. The line through L parallel to BC meets BM at E and the line through M parallel to AB meets BL at D. Show that ED is perpendicular to BL.

7. A chain has $n > 3$ numbered links. A customer asks for the order of the links to be changed to a new order. The jeweller opens the smallest possible number of links, but the customer chooses the new order in order to maximise this number. How many links have to be opened?

8. There are two unequal rational numbers $r < s$ on a blackboard. A move is to replace r by $rs/(s - r)$. The numbers on the board are initially positive integers and a sequence of moves is made, at the end of which the two numbers are equal. Show that the final numbers are positive integers.

9. A, B, C, D, E, F are points on the graph of $y = ax^3 + bx^2 + cx + d$ such that ABC and DEF are both straight lines parallel to the x-axis (with the points in that order from left to right). Show that the length of the projection of BE onto the x-axis equals the sum of the lengths of the projections of AB and CF onto the x-axis.

10. Two polygons are such that the distance between any two vertices of the same polygon is at most 1 and the distance between any vertex of one polygon and any vertex of the other is more than $1/\sqrt{2}$. Show that the interiors of the two polygons are disjoint.

11. The point A' on the incircle of ABC is chosen so that the tangent at A' passes through the foot of the bisector of angle A, but A' does not lie on BC. The line L_A is the line through A' and the midpint of BC. The lines L_B and L_C are defined similarly. Show that L_A, L_B and L_C all pass through a single point on the incircle.

12. X is a set. P is a collection of subsets of X, each of which have exactly 2k elements. Any subset of X with at most $(k+1)^2$ elements either has no subsets in P or is such that all its subsets which are in P have a common element. Show that every subset in P has a common element.

13. The numbers 19 and 98 are written on a blackboard. A move is to take a number n on the blackboard and replace it by n+1 or by n^2. Is it possible to obtain two equal numbers by a series of moves?

14. A binary operation * is defined on the real numbers such that (a * b) * c = a + b + c for all a, b, c. Show that * is the same as +.

15. Given a convex n-gon with no 4 vertices lying on a circle, show that the number of circles through three adjacent vertices of the n-gon such that all the other vertices lie inside the circle exceeds by two the number of circles through three vertices, no two of which are adjacent, such that all other vertices lie inside the circle.

16. Find the number of ways of placing a 1 or -1 into each cell of a $(2^n - 1)$ by $(2^n - 1)$ board, so that the number in each cell is the product the numbers in its neighbours (a neighbour is a cell which shares a side).

17. The incircle of the triangle ABC touches the sides BC, CA, AB at D, E, F respectively. D' is the midpoint of the arc BC that contains A, E' is the midpoint of the arc CA that contains B, and F' is the midpoint of the arc AB that contains C. Show that DD', EE', FF' are concurrent.

18. Given a collection of solid equilateral triangles in the plane, each of which is a translate of the others, such that every two have a common point. Show that there are three points, so that every triangle contains at least one of the points.

19. A connected graph has 1998 points and each point has degree 3. If 200 points, no two of them joined by an edge, are deleted, show that the result is a connected graph.

20. C_1 is the circle center (0, 1/2), diameter 1 which touches the parabola $y = x^2$ at the point (0, 0). The circle C_{n+1} has its center above C_n on the y axis, touches C_n and touches the parabola at two symmetrically placed points. Find the diameter of C_{1998}.

21. Do there exist 1998 different positive integers such that the product of any two is divisible by the square of their difference?

22. The tetrahedron ABCD has all edges less than 100 and contains two disjoint spheres of diameter 1. Show that it contains a sphere of diameter 1.01.

23. A figure is made out of unit squares joined along their sides. It has the property that if the squares of an m x n rectangle are filled with real numbers with positive sum, then the figure can be placed over the rectangle (possibly after being rotated, but with each square of the figure coinciding with a square of the rectangle) so that the sum of the numbers under each square is positive. Prove that a number of copies of the figure can be placed over an m x n rectangle so that each square of the rectangle is covered by the same number of figures.

25th Russian 1999 problems

1. The digits of n strictly increase from left to right. Find the sum of the digits of 9n.

2. Each edge of a finite connected graph is colored with one of N colors in such a way that there is just one edge of each color at each point. One edge of each color but one is deleted. Show that the graph remains connected.

3. ABC is a triangle. A' is the midpoint of the arc BC not containing A, and C' is the midpoint of the arc AB not containing C. S is the circle center A' touching BC and S' is the circle center C' touching AB. Show that the incenter of ABC lies on a common external tangent to S and S'.

4. The numbers from 1 to a million are colored black or white. A move consists of choosing a number and changing the color of the number and every other number which is not coprime to it. If the numbers are initially all black, can they all be changed to white by a series of moves?

5. An equilateral triangle side n is divided into n^2 equilateral triangles of side 1 by lines parallel to its sides, thus giving a network of nodes connected by line segments of length 1. What is the maximum number of segments that can be chosen so that no three chosen segments form a triangle?

6. Let {x} denote the fractional part of x. Show that $\{\sqrt{1}\} + \{\sqrt{2}\} + \{\sqrt{3}\} + ... + \{\sqrt{(n^2)}\} \le (n^2 - 1)/2$.

7. ABC is a triangle. A circle through A and B meets BC again at D, and a circle through B and C meets AB again at E, so that A, E, D, C lie on a circle center O. The two circles meet at B and F. Show that $\angle$ BFO = 90 deg.

8. A graph has 2000 points and every two points are joined by an edge. Two people play a game. The first player always removes one edge. The second player removes one or three edges. The player who first removes the last edge from a point wins. Does the first or second player have a winning strategy?

9. There are three empty bowls X, Y and Z on a table. Three players A, B and C take turns playing a game. A places a piece into bowl Y or Z, B places a piece into bowl Z or X, and C places a piece into bowl X or Y. The first player to place the 1999th piece into a bowl loses. Show that irrespective of who plays first and second (thereafter the order of play is determined) A and B can always conspire to make C lose.

10. The sequence $a_1, a_2, a_3, ...$ of positive integers is determined by its first two members and the rule $a_{n+2} = (a_{n+1} + a_n)/\gcd(a_n, a_{n+1})$. For which values of a_1 and a_2 is it bounded?

11. The incircle of the triangle ABC touches the sides BC, CA, AB at D, E, F respectively. Each pair from the incircles of AEF, DBF, DEC has two common external tangents, one of which is a side of the triangle ABC. Show that the other three tangents are concurrent.

12. A piece is placed in each unit square of an n x n square on an infinite board of unit squares. A move consists of finding two adjacent pieces (in squares which have a common side) so that one of the pieces can jump over the other onto an empty square. The piece jumped over is removed. Moves are made until no further moves are possible. Show that at least $n^2/3$ moves are made.

13. A number n has sum of digits 100, whilst 44n has sum of digits 800. Find the sum of the digits of 3n.

14. The positive reals x, y satisfy $x^2 + y^3 \ge x^3 + y^4$. Show that $x^3 + y^3 \le 2$.

15. A graph of 12 points is such that every 9 points contain a complete subgraph of 5 points. Show that the graph has a complete subgraph of 6 points. [A complete graph has all possible edges.]

16. Do there exist 19 distinct positive integers whose sum is 1999 and each of which has the same digit sum?

17. The function f assigns an integer to each rational. Show that there are two distinct rationals r and s, such that $f(r) + f(s) \le 2\ f(r/2 + s/2)$.

18. A quadrilateral has an inscribed circle C. For each vertex, the incircle is drawn for the triangle formed by the vertex and the two points at which C touches the adjacent sides. Each pair of adjacent incircles has two common external tangents, one a side of the quadrilateral. Show that the other four form a rhombus.

19. Four positive integers have the property that the square of the sum of any two is divisible by the product of the other two. Show that at least three of the integers are equal.

20. Three convex polygons are drawn in the plane. We say that one of the polygons, P, can be separated from the other two if there is a line which meets none of the polygons such that the other two polygons are on the opposite side of the line to P. Show that there is a line which intersects all three polygons iff one of the polygons cannot be separated from the other two.

21. Let A be a vertex of a tetrahedron and let p be the tangent plane at A to the circumsphere of the tetrahedron. Let L, L', L" be the lines in which p intersects the three sides of the tetrahedron through A. Show that the three lines form six angles of 60^o iff the product of each pair of opposite sides of the tetrahedron is equal.

26th Russian 2000 problems

1. The equations $x^2 + ax + 1 = 0$ and $x^2 + bx + c = 0$ have a common real root, and the equations $x^2 + x + a = 0$ and $x^2 + cx + b = 0$ have a common real root. Find a + b + c.

2. A chooses a positive integer $X \le 100$. B has to find it. B is allowed to ask 7 questions of the form "What is the greatest common divisor of X + m and n?" for positive integers m, n < 100. Show that he can find X.

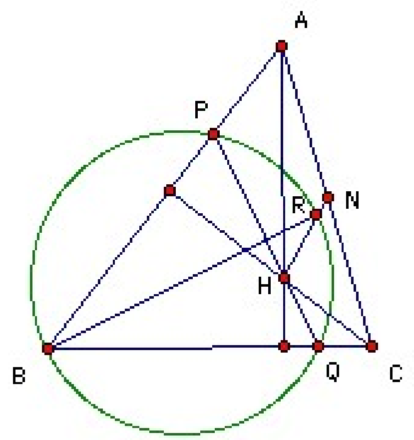
3. O is the circumcenter of the obtuse-angled triangle ABC. K is the circumcenter of AOC. The lines AB, BC meet the circumcircle of AOC again at M, N respectively. L is the reflection of K in the line MN. Show that the lines BL and AC are perpendicular.

4. Some pairs of towns are connected by a road. At least 3 roads leave each town. Show that there is a cycle containing a number of towns which is not a multiple of 3.

5. Find $[1/3] + [2/3] + [2^2/3] + [2^3/3] + ... + [2^{1000}/3]$.

6. We have $-1 < x_1 < x_2 < ... < x_n < 1$ and $y_1 < y_2 < ... < y_n$ such that $x_1 + x_2 + ... + x_n = x_1^{13} + x_2^{13} + ... + x_n^{13}$. Show that $x_1^{13}y_1 + x_2^{13}y_2 + ... + x_n^{13}y_n < x_1y_1 + ... + x_ny_n$.

7. ABC is acute-angled and is not isosceles. The bisector of the acute angle between the altitudes from A and C meets AB at P and BC at Q. The angle bisector of B meets the line joining HN at R, where H is the orthocenter and N is the midpoint of AC. Show that BPRQ is cyclic.



8. We wish to place 5 stones with distinct weights in increasing order of weight. The stones are indistinguisable (apart from their weights). Nine questions of the form "Is it true that A < B < C?" are allowed (and get a yes/no answer). Is that sufficient?

9. R is the reals. Find all functions f: R → R which satisfy $f(x+y) + f(y+z) + f(z+x) \ge 3f(x+2y+3z)$ for all x, y, z.

10. Show that it is possible to partition the positive integers into 100 non-empty sets so that if a + 99b = c for integers a, b, c, then a, b, c are not all in different sets.

11. ABCDE is a convex pentagon whose vertices are all lattice points. A'B'C'D'E' is the pentagon formed by the diagonals. Show that it must have a lattice point on its boundary or inside it.

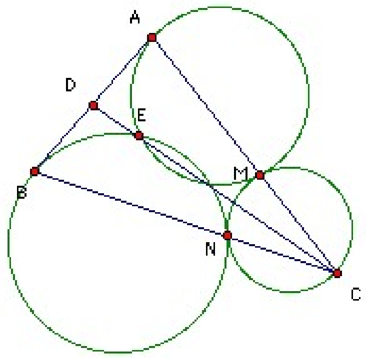
12. $a_1, a_2, ... , a_n$ are non-negative integers not all zero. Put $m_1 = a_1$, $m_2 = \max(a_2, (a_1+a_2)/2)$, $m_3 = \max(a_3, (a_2+a_3)/2 + (a_1+a_2+a_3)/3)$, $m_4 = \max(a_4, (a_3+a_4)/2, (a_2+a_3+a_4)/3, (a_1+a_2+a_3+a_4)/4)$,

... , $m_n = \max(a_n, (a_{n-1}+a_n)/2, (a_{n-2}+a_{n-1}+a_n)/3, ... , (a_1+a_2+...+a_n)/n)$. Show that for any $\alpha > 0$ the number of $m_i > \alpha$ is $< (a_1+a_2+...+a_n)/\alpha$.

13. The sequence $a_1, a_2, a_3, ...$ is constructed as follows. $a_1 = 1$. $a_{n+1} = a_n - 2$ if $a_n - 2$ is a positive integer which has not yet appeared in the sequence, and $a_n + 3$ otherwise. Show that if a_n is a square, then $a_n > a_{n-1}$.

14. Some cells of a 2n x 2n board contain a white token or a black token. All black tokens which have a white token in the same column are removed. Then all white tokens which have one of the remaining black tokens in the same row are removed. Show that we cannot end up with more than n^2 black tokens and more than n^2 white tokens.

15. ABC is a triangle. E is a point on the median from C. A circle through E touches AB at A and meets AC again at M. Another circle through E touches AB at B and meets BC again at N. Show that the circumcircle of CMN touches the two circles.

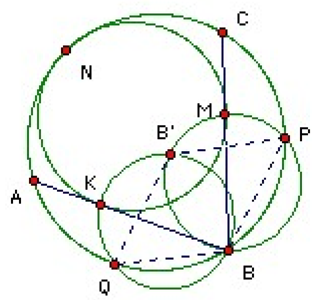


16. 100 positive integers are arranged around a circle. The greatest common divisor of the numbers is 1. An allowed operation is to add to a number the greatest common divisor of its two neighbors. Show that by a sequence of such operations we can get 100 numbers, every two of which are relatively prime.

17. S is a finite set of numbers such that given any three there are two whose sum is in S. What is the largest number of elements that S can have?

18. A perfect number is equal to the sum of all its positive divisors other than itself. Show that if a perfect number > 6 is divisible by 3, then it is divisible by 9. Show that a perfect number > 28 divisible by 7 must be divisible by 49.

19. A larger circle contains a smaller circle and touches it at N. Chords BA, BC of the larger circle touch the smaller circle at K, M respectively. The midpoints of the arcs BC, BA (not containing N) are P, Q respectively. The circumcircles of BPM, BQK meet again at B'. Show that BPB'Q is a parallelogram.



20. Several thin unit cardboard squares are put on a rectangular table with sides parallel to the sides of the table. The squares may overlap. Each square is colored with one of k colors.

Given any k squares of different colors, we can find two that overlap. Show that for one of the colors we can nail all the squares of that color to the table with 2k-2 nails.

21. Show that $\sin^n 2x + (\sin^n x - \cos^n x)^2 \le 1$.

22. ABCD has an inscribed circle center O. The lines AB and CD meet at X. The incircle of XAD touches AD at L. The excircle of XBC opposite X touches BC at K. X, K, L are collinear. Show that O lies on the line joining the midpoints of AD and BC.

23. Each cell of a 100 x 100 board is painted with one of four colors, so that each row and each column contains exactly 25 cells of each color. Show that there are two rows and two columns whose four intersections are all different colors.

27th Russian 2001 problems

1. Are there more positive integers under a million for which the nearest square is odd or for which it is even?

2. A monic quartic and a monic quadratic both have real coefficients. The quartic is negative iff the quadratic is negative and the set of values for which they are negative is an interval of length more than 2. Show that at some point the quartic has a smaller value than the quadratic.

3. ABCD is parallelogram and P a point inside, such that the midpoint of AD is equidistant from P and C, and the midpoint of CD is equidistant from P and A. Let Q be the midpoint of PB. Show that $\angle PAQ = \angle PCQ$.

4. No three diagonals of a convex 2000-gon meet at a point. The diagonals (but not the sides) are each colored with one of 999 colors. Show that there is a triangle whose sides are on three diagonals of the same color.

5. 2001 coins, each value 1, 2 or 3 are arranged in a row. Between any two coins of value 1 there is at least one coin, between any two of value 2 there are at least two coins, and between any two of value 3 there are at least three coins. What is the largest number of value 3 coins that could be in the row?

6. Given a graph of 2n+1 points, given any set of n points, there is another point joined to each point in the set. Show that there is a point joined to all the other points.

7. N is any point on AC is the longest side of the triangle ABC, such that the perpendicular bisector of AN meets the side AB at K and the perpendicular bisector of NC meets the side BC at M. Prove that BKOM is cyclic, where O is the circumcenter of ABC.

8. Find all odd positive integers $n > 1$ such that if a and b are relatively prime divisors of n, then $a + b - 1$ divides n.

9. Let $A_1, A_2, \dots, A_{100}$ be subsets of a line, each a union of 100 disjoint closed segments. Prove that the intersection of all hundred sets is a union of at most 9901 disjoint closed segments. [A single point is considered to be a closed segment.]

10. The circle C' is inside the circle C and touches it at N. A tangent at the point X of C' meets C at A and B. M is the midpoint of the arc AB which does not contain N. Show that the circumradius of BMX is independent of the position of X.

11. Some pairs of towns in a country are joined by roads, so that there is a unique route from any town to another which does not pass through any town twice. Exactly 100 of the towns have only one road. Show that it is possible to construct 50 new roads so that there will still be a route between any two towns even if any one of the roads (old or new) is closed for maintenance.

12. $x^3 + ax^2 + bx + c$ has three distinct real roots, but $(x^2 + x + 2001)^3 + a(x^2 + x + 2001)^2 + b(x^2 + x + 2001) + c$ has no real roots. Show that $2001^3 + a\,2001^2 + b\,2001 + c > 1/64$.

13. An n x n *Latin square* has the numbers from 1 to n^2 arranged in its cells (one per cell) so that the sum of every row and column is the same. For every pair of cells in a Latin square the centers of the cells are joined by an arrow pointing to the cell with the larger number. Show that the sum of these vectors is zero.

14. The altitudes AD, BE, CF of the triangle ABC meet at H. Points P, Q, R are taken on the segments AD, BE, CF respectively, so that the sum of the areas of triangles ABR, AQC and PBC equals the area of ABC. Show that P, Q, R, H are cyclic.

15. S is a set of 100 stones. f(S) is the set of integers n such that we can find n stones in the collection weighing half the total weight of the set. What is the maximum possible number of integers in f(S)?

16. There are two families of convex polygons in the plane. Each family has a pair of disjoint polygons. Any polygon from one family intersects any polygon from the other family. Show that there is a line which intersects all the polygons.

17. N contestants answered an exam with n questions. a_i points are awarded for a correct answer to question i and nil for an incorrect answer. After the questions had been marked it was noticed that by a suitable choice of positive numbers a_i any desired ranking of the contestants could be achieved. What is the largest possible value of N?

18. The quadratics $x^2 + ax + b$ and $x^2 + cx + d$ have real coefficients and take negative values on disjoint intervals. Show that there are real numbers h, k such that $h(x^2 + ax + b) + k(x^2 + cx + d) > 0$ for all x.

19. $m > n$ are positive integers such that $m^2 + mn + n^2$ divides $mn(m + n)$. Show that $(m - n)^3 > mn$.

20. A country has 2001 towns. Each town has a road to at least one other town. If a subset of the towns is such that any other town has a road to at least one member of the subset, then it has at least $k > 1$ towns. Show that the country may be partitioned into 2001 - k republics so that no two towns in the same republic are joined by a road.

21. ABCD is a tetrahedron. O is the circumcenter of ABC. The sphere center O through A, B, C meets the edges DA, DB, DC again at A', B', C'. Show that the tangent planes to the sphere at A', B', C' pass through the center of the sphere through A', B', C', D.

28th Russian 2002 problems

1. Can the cells of a 2002 x 2002 table be filled with the numbers from 1 to 2002^2 (one per cell) so that for any cell we can find three numbers a, b, c in the same row or column (or the cell itself) with a = bc?

2. ABC is a triangle. D is a point on the side BC. A is equidistant from the incenter of ABD and the excenter of ABC which lies on the internal angle bisector of B. Show that AC = AD.

3. Given 18 points in the plane, no three collinear, so that they form 816 triangles. The sum of the area of these triangles is A. Six are colored red, six green and six blue. Show that the sum of the areas of the triangles whose vertices are the same color does not exceed A/4.

4. A graph has n points and 100 edges. A move is to pick a point, remove all its edges and join it to any points which it was not joined to immediately before the move. What is the smallest number of moves required to get a graph which has two points with no path between them?

5. The real polynomials p(x), q(x), r(x) have degree 2, 3, 3 respectively and satisfy $p(x)^2 + q(x)^2 = r(x)^2$. Show that either q(x) or r(x) has all its roots real.

6. ABCD is a cyclic quadrilateral. The tangent at A meets the ray CB at K, and the tangent at B meets the ray DA at M, so that BK = BC and AM = AD. Show that the quadrilateral has two sides parallel.

7. Show that for any integer n > 10000, there are integers a, b such that $n < a^2 + b^2 < n + 3\ n^{1/4}$.

8. A graph has 2002 points. Given any three distinct points A, B, C there is a path from A to B that does not involve C. A move is to take any cycle (a set of distinct points $P_1, P_2, \ldots, P_n$ such that P_1 is joined to P_2, P_2 is joined to P_3, ... , P_{n-1} is joined to P_n, and P_n is joined to P_1) remove its edges and add a new point X and join it to each point of the cycle. After a series of moves the graph has no cycles. Show that at least 2002 points have only one edge.

9. n points in the plane are such that for any three points we can find a cartesian coordinate system in which the points have integral coordinates. Show that there is a cartesian coordinate system in which all n points have integral coordinates.

10. Show that for n > m > 0 and 0 < x < π/2 we have $| \sin^n x - \cos^n x | \le 3/2\ | \sin^m x - \cos^m x |$.

11. [unclear]

12. Eight rooks are placed on an 8 x 8 chessboard, so that there is just one rook in each row and column. Show that we can find four rooks, A, B, C, D, so that the distance between the centers of the squares containing A and B equals the distance between the centers of the squares containing C and D.

13. Given k+1 cells. A stack of 2n cards, numbered from 1 to 2n, is in arbitrary order on one of the cells. A move is to take the top card from any cell and place it either on an unoccupied cell or on top of the top card of another cell. The latter is only allowed if the card being moved has number m and it is placed on top of card m+1. What is the largest n for which it is always possible to make a series of moves which result in the cards ending up in a single stack on a different cell.

14. O is the circumcenter of ABC. Points M, N are taken on the sides AB, BC respectively so that ∠MON = ∠B. Show that the perimeter of MBN is at least AC.

15. 2^{2n-1} odd numbers are chosen from $\{2^{2n} + 1, 2^{2n} + 2, 2^{2n} + 3, \ldots , 2^{3n}\}$. Show that we can find two of them such that neither has its square divisible by any of the other chosen numbers.

16. Show that $\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + yz + zx$ for positive reals x, y, z with sum 3.

17. In the triangle ABC, the excircle touches the side BC at A' and a line is drawn through A' parallel to the internal bisector of angle A. Similar lines are drawn for the other two sides. Show that the three lines are concurrent.

18. There are a finite number of red and blue lines in the plane, no two parallel. There is always a third line of the opposite color through the point of intersection of two lines of the same color. Show that all the lines have a common point.

19. Find the smallest positive integer which can be represented both as a sum of 2002 positive integers each with the same sum of digits, and as a sum of 2003 positive integers each with the same sum of digits.

20. ABCD is a cyclic quadrilateral. The diagonals AC and BD meet at X. The circumcircles of ABX and CDX meet again at Y. Z is taken so that the triangles BZC and AYD are similar. Show that if BZCY is convex, then it has an inscribed circle.

21. Show that for infinitely many n the if $1 + 1/2 + 1/3 + \ldots + 1/n = r/s$ in lowest terms, then r is not a prime power.

BMO (1965 – 2004)

1st BMO 1965

1. Sketch $f(x) = (x^2 + 1)/(x + 1)$. Find all points where $f'(x) = 0$ and describe the behaviour when x or f(x) is large.

2. X, at the centre a circular pond. Y, at the edge, cannot swim, but can run at speed 4v. X can run faster than 4v and can swim at speed v. Can X escape?

3. Show that $n^p - n$ is divisible by p for p = 3, 7, 13 and any integer n.

4. What is the largest power of 10 dividing 100 x 99 x 98 x ... x 1?

5. Show that $n(n + 1)(n + 2)(n + 3) + 1$ is a square for n = 1, 2, 3, ...

6. The fractional part of a real is the real less the largest integer not exceeding it. Show that we can find n such that the fractional part of $(2 + \sqrt{2})^n > 0.999$.

7. What is the remainder on dividing $x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$ by $x - 1$? By $x^2 - 1$?

8. For what real b can we find x satisfying: $x^2 + bx + 1 = x^2 + x + b = 0$?

9. Show that for any real, positive x, y, z, not all equal, we have: $(x + y)(y + z)(z + x) > 8xyz$.

10. A chord length $\sqrt{3}$ divides a circle C into two arcs. R is the region bounded by the chord and the shorter arc. What is the largest area of rectangle than can be drawn in R?

2nd BMO 1966

1. Find the greatest and least values of $f(x) = (x^4 + x^2 + 5)/(x^2 + 1)^2$ for real x.

2. For which distinct, real a, b, c are all the roots of $\pm\sqrt{(x - a)} \pm\sqrt{(x - b)} \pm\sqrt{(x - c)} = 0$ real?

3. Sketch $y^2 = x^2(x + 1)/(x - 1)$. Find all stationary values and describe the behaviour for large x.

4. A_1, A_2, A_3, A_4 are consecutive vertices of a regular n-gon. $1/A_1A_2 = 1/A_1A_3 + 1/A_1A_4$. What are the possible values of n?

5. A spanner has an enclosed hole which is a regular hexagon side 1. For what values of s can it turn a square nut side s?

6. Find the largest interval over which $f(x) = \sqrt{(x - 1)} + \sqrt{(x + 24 - 10\sqrt{(x - 1)})}$ is real and constant.

7. Prove that $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ cannot be terms in an arithmetic progression.

8. Given 6 different colours, how many ways can we colour a cube so that each face has a different colour? Show that given 8 different colours, we can colour a regular octahedron in 1680 ways so that each face has a different colour.

9. The angles of a triangle are A, B, C. Find the smallest possible value of tan A/2 + tan B/2 + tan C/2 and the largest possible value of tan A/2 tan B/2 tan C/2.

10. One hundred people of different heights are arranged in a 10 x 10 array. X, the shortest of the 10 people who are the tallest in their row, is a different height from Y, the tallest of the 10 people who are the shortest in their column. Is X taller or shorter than Y?

11. (a) Show that given any 52 integers we can always find two whose sum or difference is a multiple of 100.

(b) Show that given any set 100 integers, we can find a non-empty subset whose sum is a multiple of 100.

3rd BMO 1967

1. a, b are the roots of $x^2 + Ax + 1 = 0$, and c, d are the roots of $x^2 + Bx + 1 = 0$. Prove that $(a - c)(b - c)(a + d)(b + d) = B^2 - A^2$.

2. Graph $x^8 + xy + y^8 = 0$, showing stationary values and behaviour for large values. [Hint: put $z = y/x$.]

3. (a) The triangle ABC has altitudes AP, BQ, CR and AB > BC. Prove that AB + CR ≥ BC + AP. When do we have equality?

(b) Prove that if the inscribed and circumscribed circles have the same centre, then the triangle is equilateral.

4. We are given two distinct points A, B and a line *l* in the plane. Can we find points (in the plane) equidistant from A, B and *l*? How do we construct them?

5. Show that $(x - \sin x)(\pi - x - \sin x)$ is increasing in the interval $(0, \pi/2)$.

6. Find all x in $[0, 2\pi]$ for which $2 \cos x \le |\sqrt{(1 + \sin 2x)} - \sqrt{(1 - \sin 2x)}| \le \sqrt{2}$.

7. Find all reals a, b, c, d such that $abc + d = bcd + a = cda + b = dab + c = 2$.

8. For which positive integers n does 61 divide $5^n - 4^n$?

9. None of the angles in the triangle ABC are zero. Find the greatest and least values of $\cos^2 A + \cos^2 B + \cos^2 C$ and the values of A, B, C for which they occur.

10. A collects pre-1900 British stamps and foreign stamps. B collects post-1900 British stamps and foreign special issues. C collects pre-1900 foreign stamps and British special issues. D collects post-1900 foreign stamps and British special issues. What stamps are collected by (1) no one, (2) everyone, (3) A and D, but not B?

11. The streets for a rectangular grid. B is h blocks north and k blocks east of A. How many shortest paths are there from A to B?

4th BMO 1968

1. C is the circle center the origin and radius 2. Another circle radius 1 touches C at (2, 0) and then rolls around C. Find equations for the locus of the point P of the second circle which is initially at (2, 0) and sketch the locus.

2. Cows are put in a field when the grass has reached a fixed height, any cow eats the same amount of grass a day. The grass continues to grow as the cows eat it. If 15 cows clear 3 acres in 4 days and 32 cows clear 4 acres in 2 days, how many cows are needed to clear 6 acres in 3 days?

3. The distance between two points (x, y) and (x', y') is defined as $|x - x'| + |y - y'|$. Find the locus of all points with non-negative x and y which are equidistant from the origin and the point (a, b) where $a > b$.

4. Two balls radius a and b rest on a table touching each other. What is the radius of the largest sphere which can pass between them?

5. If reals x, y, z satisfy $\sin x + \sin y + \sin z = \cos x + \cos y + \cos z = 0$. Show that they also satisfy $\sin 2x + \sin 2y + \sin 2z = \cos 2x + \cos 2y + \cos 2z = 0$.

6. Given integers $a_1, a_2, \ldots, a_7$ and a permutation of them $a_{f(1)}, a_{f(2)}, \ldots, a_{f(7)}$, show that the product $(a_1 - a_{f(1)})(a_2 - a_{f(2)}) \ldots (a_7 - a_{f(7)})$ is always even.

7. How many games are there in a knock-out tournament amongst n people?

8. C is a fixed circle of radius r. L is a variable chord. D is one of the two areas bounded by C and L. A circle C' of maximal radius is inscribed in D. A is the area of D outside C'. Show that A is greatest when D is the larger of the two areas and the length of L is $16\pi r/(16 + \pi^2)$.

9. The altitudes of a triangle are 3, 4, 6. What are its sides?

10. The faces of the tetrahedron ABCD are all congruent. The angle between the edges AB and CD is x. Show that $\cos x = \sin(\angle ABC - \angle BAC)/\sin(\angle ABC + \angle BAC)$.

11. The sum of the reciprocals of n distinct positive integers is 1. Show that there is a unique set of such integers for $n = 3$. Given an example of such a set for every $n > 3$.

12. What is the largest number of points that can be placed on a spherical shell of radius 1 such that the distance between any two points is at least $\sqrt{2}$? What is the largest number such that the distance is $> \sqrt{2}$?

5th BMO 1969

1. Find the condition on the distinct real numbers a, b, c such that (x - a)(x - b)/(x - c) takes all real values. Sketch a graph where the condition is satisfied and another where it is not.

2. Find all real solutions to cos x + $\cos^5$x + cos 7x = 3.

3. For which positive integers n can we find distinct integers a, b, c, d, a', b', c', d' greater than 1 such that n^2 - 1 = aa' + bb' + cc' + dd'? Give the solution for the smallest n.

4. Find all integral solutions to a^2 - 3ab - a + b = 0.

5. A long corridor has unit width and a right-angle corner. You wish to move a pipe along the corridor and round the corner. The pipe may have any shape, but every point must remain in contact with the floor. What is the longest possible distance between the two ends of the pipe?

6. If a, b, c, d, e are positive integers, show that any divisor of both ae + b and ce + d also divides ad - bc.

7. (1) f is a real-valued function on the reals, not identically zero, and differentiable at x = 0. It satisfies f(x) f(y) = f(x+y) for all x, y. Show that f(x) is differentiable arbitrarily many times for all x and that if f(1) < 1, then f(0) + f(1) + f(2) + ... = 1/(1 - f(1)).

(2) Find the real-valued function f on the reals, not identically zero, and differentiable at x = 0 which satisfies f(x) f(y) = f(x-y) for all x, y.

(2) Find the real-valued function f on the reals, not identically zero, and differentiable at x = 0 which satisfies f(x) f(y) = f(x-y) for all x, y.

9. Let A_n be an n x n array of lattice points (n > 3). Is there a polygon with n^2 sides whose vertices are the points of A_n such that no two sides intersect except adjacent sides at a vertex? You should prove the result for n = 4 and 5, but merely state why it is plausible for n > 5.

10. Given a triangle, construct an equilateral triangle with the same area using ruler and compasses.

6th BMO 1970

1. (1) Find $1/\log_2 a + 1/\log_3 a + ... + 1/\log_n a$ as a quotient of two logs to base 2.

(2) Find the sum of the coefficients of $(1 + x - x^2)^3(1 - 3x + x^2)^2$ and the sum of the coefficients of its derivative.

2. Sketch the curve $x^2 + 3xy + 2y^2 + 6x + 12y + 4$. Where is the center of symmetry?

3. Morley's theorem is as follows. ABC is a triangle. C' is the point of intersection of the trisector of angle A closer to AB and the trisector of angle B closer to AB. A' and B' are defined similarly. Then A'B'C' is equilateral. What is the largest possible value of area A'B'C'/area ABC? Is there a minimum value?

4. Prove that any subset of a set of n positive integers has a non-empty subset whose sum is divisible by n.

5. What is the minimum number of planes required to divide a cube into at least 300 pieces?

6. y(x) is defined by $y' = f(x)$ in the region $|x| \le a$, where f is an even, continuous function. Show that (1) $y(-a) + y(a) = 2\, y(0)$ and (2) $\int_{-a}^{a} y(x)\, dx = 2a\, y(0)$. If you integrate numerically from (-a, 0) using 2N equal steps δ using $g(x_{n+1}) = g(x_n) + \delta \text{ x } g'(x_n)$, then the resulting solution does not satisfy (1). Suggest a modified method which ensures that (1) is satisfied.

7. ABC is a triangle with $\angle B = \angle C = 50^o$. D is a point on BC and E a point on AC such that $\angle BAD = 50^o$ and $\angle ABE = 30^o$. Find $\angle BED$.

8. 8 light bulbs can each be switched on or off by its own switch. State the total number of possible states for the 8 bulbs. What is the smallest number of switch changes required to cycle through all the states and return to the initial state?

9. Find rationals r and s such that $\sqrt{(2\sqrt{3} - 3)} = r^{1/4} - s^{1/4}$.

10. Find "some kind of 'formula' for" the number f(n) of incongruent right-angled triangles with shortest side n? Show that f(n) is unbounded. Does it tend to infinity?

7th BMO 1971

1. Factorise $(a + b)^7 - a^7 - b^7$. Show that $2n^3 + 2n^2 + 2n + 1$ is never a multiple of 3.

2. Let $a = 9^9$, $b = 9^a$, $c = 9^b$. Show that the last two digits of b and c are equal. What are they?

3. A and B are two vertices of a regular 2n-gon. The n longest diameters subtend angles a_1, a_2, ... , a_n and b_1, b_2, ... , b_n at A and B respectively. Show that $\tan^2 a_1 + \tan^2 a_2 + ... + \tan^2 a_n = \tan^2 b_1 + \tan^2 b_2 + ... + \tan^2 b_n$.

4. Given any n+1 distinct integers all less than 2n+1, show that there must be one which divides another.

5. The triangle ABC has circumradius R. $\angle A \geq \angle B \geq \angle C$. What is the upper limit for the radius of circles which intersect all three sides of the triangle?

6. (1) Let $I(x) = \int_c^x f(x, u)\, du$. Show that $I'(x) = f(x, x) + \int_c^x \partial f/\partial x\, du$.

(2) Find $\lim_{\theta\to 0} \cot\theta \sin(t \sin\theta)$.

(3) Let $G(t) = \int_0^t \cot\theta \sin(t \sin\theta)\, d\theta$. Prove that $G'(\pi/2) = 2/\pi$.

7. Find the probability that two points chosen at random on a segment of length h are a distance less than k apart.

8. A is a 3 x 2 real matrix, B is a 2 x 3 real matrix. AB = M where det M = 0 BA = det N where det N is non-zero, and $M^2 = kM$. Find det N in terms of k.

9. A solid spheres is fixed to a table. Another sphere of equal radius is placed on top of it at rest. The top sphere rolls off. Show that slipping occurs then the line of centers makes an angle θ to the vertical, where $2 \sin\theta = \mu(17 \cos\theta - 10)$. Assume that the top sphere has moment of inertia $2/5\ Mr^2$ about a diameter, where r is its radius.

8th BMO 1972

1. The relation R is defined on the set X. It has the following two properties: if aRb and bRc then cRa for distinct elements a, b, c; for distinct elements a, b either aRb or bRa but not both. What is the largest possible number of elements in X?

2. Show that there can be at most four lattice points on the hyperbola $(x + ay + c)(x + by + d) = 2$, where a, b, c, d are integers. Find necessary and sufficient conditions for there to be four lattice points.

3. C and C' are two unequal circles which intersect at A and B. P is an arbitrary point in the plane. What region must P lie in for there to exist a line L through P which contains chords of C and C' of equal length. Show how to construct such a line if it exists by considering distances from its point of intersection with AB or otherwise.

4. P is a point on a curve through A and B such that PA = a, PB = b, AB = c, and $\angle APB = \theta$. As usual, $c^2 = a^2 + b^2 - 2ab \cos \theta$. Show that $\sin^2\theta\, ds^2 = da^2 + db^2 - 2\, da\, db \cos \theta$, where s is distance along the curve. P moves so that for time t in the interval $T/2 < t < T$, $PA = h \cos(t/T)$, $PB = k \sin(t/T)$. Show that the speed of P varies as cosec θ.

5. A cube C has four of its vertices on the base and four of its vertices on the curved surface of a right circular cone R with semi-vertical angle x. Show that if x is varied the maximum value of vol C/vol R is at $\sin x = 1/3$.

6. Define the sequence a_n, by $a_1 = 0$, $a_2 = 1$, $a_3 = 2$, $a_4 = 3$, and $a_{2n} = a_{2n-5} + 2^n$, $a_{2n+1} = a_{2n} + 2^{n-1}$. Show that $a_{2n} = [17/7\ 2^{n-1}] - 1$, $a_{2n-1} = [12/7\ 2^{n-1}] - 1$.

7. Define sequences of integers by $p_1 = 2$, $q_1 = 1$, $r_1 = 5$, $s_1 = 3$, $p_{n+1} = p_n^2 + 3 q_n^2$, $q_{n+1} = 2 p_n q_n$, $r_n = p_n + 3 q_n$, $s_n = p_n + q_n$. Show that $p_n/q_n > \sqrt{3} > r_n/s_n$ and that p_n/q_n differs from $\sqrt{3}$ by less than $s_n/(2 r_n q_n^2)$.

8. Three children throw stones at each other every minute. A child who is hit is out of the game. The surviving player wins. At each throw each child chooses at random which of his two opponents to aim at. A has probability 3/4 of hitting the child he aims at, B has probability 2/3 and C has probability 1/2. No one ever hits a child he is not aiming at. What is the probability that A is eliminated in the first round and C wins.

9. A rocket, free of external forces, accelerates in a straight line. Its mass is M, the mass of its fuel is m exp(-kt) and its fuel is expelled at velociy v exp(-kt). If m is small compared to M, show that its terminal velocity is mv/(2M) times its initial velocity.

9th BMO 1973

1. A variable circle touches two fixed circles at P and Q. Show that the line PQ passes through one of two fixed points. State a generalisation to ellipses or conics.

2. Given any nine points in the interior of a unit square, show that we can choose 3 which form a triangle of area at most 1/8.

3. The curve C is the quarter circle $x^2 + y^2 = r^2$, x >= 0, y >= 0 and the line segment x = r, 0 >= y >= -h. C is rotated about the y-axis for form a surface of revolution which is a hemisphere capping a cylinder. An elastic string is stretched over the surface between (x, y, z) = (r sin θ, r cos θ, 0) and (-r, -h, 0). Show that if tan θ > r/h, then the string does not lie in the xy plane. You may assume spherical triangle formulae such as cos a = cos b cos c + sin b sin c cos A, or sin A cot B = sin c cot b - cos c cos A.

4. n equilateral triangles side 1 can be fitted together to form a convex equiangular hexagon. The three smallest possible values of n are 6, 10 or 13. Find all possible n.

5. Show that there is an infinite set of positive integers of the form 2^n - 7 no two of which have a common factor.

6. The probability that a teacher will answer a random question correctly is p. The probability that randomly chosen boy in the class will answer correctly is q and the probability that a randomly chosen girl in the class will answer correctly is r. The probability that a randomly chosen pupil's answer is the same as the teacher's answer is 1/2. Find the proportion of boys in the class.

7. From each 10000 live births, tables show that y will still be alive x years later. y(60) = 4820 and y(80) = 3205, and for some A, B the curve $Ax(100-x) + B/(x-40)^2$ fits the data well for 60 <= x <= 100. Anyone still alive at 100 is killed. Find the life expectancy in years to the nearest 0.1 year of someone aged 70.

8. T: z → (az + b)/(cz + d) is a map. M is the associated matrix

a b

c d

Show that if M is associated with T and M' with T' then the matrix MM' is associated with the map TT'. Find conditions on a, b, c, d for T^4 to be the identity map, but T^2 not to be the identity map.

9. Let L(θ) be the determinant:

x y 1

a + c cos θ b + c sin θ 1

l + n cos θ m + n sin θ 1

Show that the lines are concurrent and find their point of intersection.

10. Write a computer program to print out all positive integers up to 100 of the form $a^2 - b^2 - c^2$ where a, b, c are positive integers and $a \geq b + c$.

11. (1) A uniform rough cylinder with radius a, mass M, moment of inertia $Ma^2/2$ about its axis, lies on a rough horizontal table. Another rough cylinder radius b, mass m, moment of inertia $mb^2/2$ about its axis, rests on top of the first with its axis parallel. The cylinders start to roll. The plane containing the axes makes the angle θ with the vertical. Show the forces during the period when there is no slipping. Write down equations, which will give on elimination a differential equation for □, but you do not need to find the differential equation.

(2) Such a differential equation is $\theta_2(4 + 2 \cos \theta - 2 \cos^2\theta + 9k/2) + \theta_1^2 \sin \theta (2 \cos \theta - 1) = 3g(1 + k) (\sin \theta /(a + b)$, where $k = M/m$. Find θ_1 in terms of θ. Here θ_1 denotes $d\theta/dt$ and θ_2 denotes the second derivative.

10th BMO 1974

1. C is the curve $y = 4x^2/3$ for $x \geq 0$ and C' is the curve $y = 3x^2/8$ for $x \geq 0$. Find curve C'' which lies between them such that for each point P on C'' the area bounded by C, C'' and a horizontal line through P equals the area bounded by C'', C and a vertical line through P.

2. S is the set of all 15 dominoes (m, n) with $1 \leq m \leq n \leq 5$. Each domino (m, n) may be reversed to (n, m). How many ways can S be partitioned into three sets of 5 dominoes, so that the dominoes in each set can be arranged in a closed chain: (a, b), (b, c), (c, d), (d, e), (e, a)?

3. Show that there is no convex polyhedron with all faces hexagons.

4. A is the 16 x 16 matrix $(a_{i,j})$. $a_{1,1} = a_{2,2} = ... = a_{16,16} = a_{16,1} = a_{16,2} = ... = a_{16,15} = 1$ and all other entries are 1/2. Find A^{-1}.

5. In a standard pack of cards every card is different and there are 13 cards in each of 4 suits. If the cards are divided randomly between 4 players, so that each gets 13 cards, what is the probability that each player gets cards of only one suit?

6. ABC is a triangle. P is equidistant from the lines CA and BC. The feet of the perpendiculars from P to CA and BC are at X and Y. The perpendicular from P to the line AB meets the line XY at Z. Show that the line CZ passes through the midpoint of AB.

7. b and c are non-zero. $x^3 = bx + c$ has real roots α, β, γ. Find a condition which ensures that there are real p, q, r such that $\beta = p\alpha^2 + q\alpha + r$, $\gamma = p\beta^2\ q\beta + r$, $\alpha = p\gamma^2 + q\gamma + r$.

8. p is an odd prime. The product $(x + 1)(x + 2) ... (x + p - 1)$ is expanded to give $a_{p-1}x^{p-1} + ... + a_1x + a_0$. Show that $a_{p-1} = 1$, $a_{p-2} = p(p-1)/2!$, $2a_{p-3} = p(p-1)(p-2)/3! + a_{p-2}(p-1)(p-2)/2!$, ... , $(p-2)a_1 = p + a_{p-2}(p-1) + a_{p-3}(p-2) + ... + 3a_2$, $(p-1)a_0 = 1 + a_{p-2} + ... + a_1$. Show that $a_1, a_2, ... , a_{p-2}$ are divisible by p and $(a_0 + 1)$ is divisible by p. Show that for any integer x, $(x+1)(x+2) ... (x+p-1) - x^{p-1} + 1$ is divisible by p. Deduce Wilson's theorem that p divides $(p-1)! + 1$ and Fermat's theorem that p divides $x^{p-1} - 1$ for x not a multiple of p.

9. A uniform rod is attached by a frictionless joint to a horizontal table. At time zero it is almost vertical and starts to fall. How long does it take to reach the table? You may assume that $\int \text{cosec}\ x\ dx = \log |\tan x/2|$.

10. A long solid right circular cone has uniform density, semi-vertical angle x and vertex V. All points except those whose distance from V lie in the range a to b are removed. The resulting solid has mass M. Show that the gravitational attraction of the solid on a point of unit mass at V is $3/2\ GM(1 + \cos x)/(a^2 + ab + b^2)$.

11th BMO 1975

1. Find all positive integer solutions to $[1^{1/3}] + [2^{1/3}] + ... + [(n^3 - 1)^{1/3}] = 400$

2. The first k primes are divided into two groups. n is the product of the first group and n is the product of the second group. M is any positive integer divisible only by primes in the first group and N is any positive integer divisible only by primes in the second group. If d > 1 divides Mm - Nn, show that d exceeds the kth prime.

3. Show that if a disk radius 1 contains 7 points such that the distance between any two is at least 1, then one of the points must be at the center of the disk. [You may wish to use the pigeonhole principle.]

4. ABC is a triangle. Parallel lines are drawn through A, B, C meeting the lines BC, CA, AB at D, E, F respectively. Collinear points P, Q, R are taken on the segments AD, BE, CF respectively such that AP/PD = BQ/CE = CR/RF = k. Find k.

5. Let nCr represent the binomial coefficient n!/((n-r)! r!). Define f(x) = (2m)C0 + (2m)C1 cos x + (2m)C2 cos 2x + (2m)C3 cos 3x + ... + (2m)C(2m) cos 2mx. Let g(x) = (2m)C0 + (2m)C2 cos 2x + (2m)C4 cos 4x + ... + (2m)C(2m) cos 2mx. Find all x such that x/π is irrational and $\lim_{m\to\infty} g(x)/f(x) = 1/2$. You may use the identity: $f(x) = (2 \cos(x/2))^{2m} \cos mx$.

6. Show that for n > 1 and real numbers x > y > 1, $(x^{n+1} - 1)/(x^n - x) > (y^{n+1} - 1)/(y^n - y)$.

7. Show that for each n > 0 there is a unique set of real numbers $x_1, x_2, ... , x_n$ such that $(1 - x_1)^2 + (x_1 - x_2)^2 + ... + (x_{n-1} - x_n)^2 + x_n^2 = 1/(n + 1)$.

8. A wine glass has the shape of a right circular cone. It is partially filled with water so that when tilted the water just touches the lip at one end and extends halfway up at the other end. What proportion of the glass is filled with water?

12th BMO 1976

1. ABC is a triangle area k. Let d be the length of the shortest line segment which bisects the area of the triangle. Find d. Give an example of a curve which bisects the area and has length < d.

2. Prove that $x/(y + z) + y/(z + x) + z/(x + y) \geq 3/2$ for any positive reals x, y, z.

3. Given 50 distinct subsets of a finite set X, each containing more than $|X|/2$ elements, show that there is a subset of X with 5 elements which has at least one element in common with each of the 50 subsets.

4. Show that $8^n 19 + 17$ is not prime for any non-negative integer n.

5. aCb represents the binomial coefficient a!/((a - b)! b!). Show that for n a positive integer, $r \leq n$ and odd, $r' = (r - 1)/2$ and x, y reals we have: $\sum_0^{r'}$ nC(r-i) nCi $(x^{r-i}y^i + x^iy^{r-i}) = \sum_0^{r'}$ nC(r-i) (r-i)Ci $x^iy^i(x + y)^{r-2i}$.

6. A sphere has center O and radius r. A plane p, a distance r/2 from O, intersects the sphere in a circle C center O'. The part of the sphere on the opposite side of p to O is removed. V lies on the ray OO' a distance 2r from O'. A cone has vertex V and base C, so with the remaining part of the sphere it forms a surface S. XY is a diameter of C. Q is a point of the sphere in the plane through V, X and Y and in the plane through O parallel to p. P is a point on VY such that the shortest path from P to Q along the surface S cuts C at 45 deg. Show that $VP = r\sqrt{3} / \sqrt{(1 + 1/\sqrt{5})}$.

13th BMO 1977

1. f(n) is a function on the positive integers with non-negative integer values such that: (1) f(mn) = f(m) + f(n) for all m, n; (2) f(n) = 0 if the last digit of n is 3; (3) f(10) = 0. Show that f(n) = 0 for all n.

2. S is either the incircle or one of the excircles of the triangle ABC. It touches the line BC at X. M is the midpoint of BC and N is the midpoint of AX. Show that the center of S lies on the line MN.

3. (1) Show that $x(x - y)(x - z) + y(y - z)(y - x) + z(z - x)(z - y) \geq 0$ for any non-negative reals x, y, z.

(2) Hence or otherwise show that $x^6 + y^6 + z^6 + 3x^2y^2z^2 \geq 2(y^3z^3 + z^3x^3 + x^3y^3)$ for all real x, y, z.

4. $x^3 + qx + r = 0$, where r is non-zero, has roots u, v, w. Find the roots of $r^2x^3 + q^3x + q^3 = 0$ (*) in terms of u, v, w. Show that if u, v, w are all real, then (*) has no real root x satisfying $-1 < x < 3$.

5. Five spheres radius a all touch externally two spheres S and S' of radius a. We can find five points, one on each of the first five spheres, which form the vertices of a regular pentagon side 2a. Do the spheres S and S' intersect?

6. Find all n > 1 for which we can write $26(x + x^2 + x^3 + ... + x^n)$ as a sum of polynomials of degree n, each of which has coefficients which are a permutation of 1, 2, 3, ... , n.

14th BMO 1978

1. Find the point inside a triangle which has the largest product of the distances to the three sides.

2. Show that there is no rational number m/n with $0 < m < n < 101$ whose decimal expansion has the consecutive digits 1, 6, 7 (in that order).

3. Show that there is a unique sequence $a_1, a_2, a_3, \dots$ such that $a_1 = 1$, $a_2 > 1$, $a_{n+1}a_{n-1} = a_n^3 + 1$, and all terms are integral.

4. An altitude of a tetrahedron is a perpendicular from a vertex to the opposite face. Show that the four altitudes are concurrent iff each pair of opposite edges is perpendicular.

5. There are 11000 points inside a cube side 15. Show that there is a sphere radius 1 which contains at least 6 of the points.

6. Show that $2 \cos nx$ is a polynomial of degree n in $(2 \cos x)$. Hence or otherwise show that if k is rational then $\cos k\pi$ is $0, \pm 1/2, \pm 1$ or irrational.

15th BMO 1979

1. Find all triangles ABC such that AB + AC = 2 and AD + BD = $\sqrt{5}$, where AD is the altitude.

2. Three rays in space have endpoints at O. The angles between the pairs are α, β, γ, where $0 < \alpha < \beta < \gamma$. Show that there are unique points A, B, C, one on each ray, so that the triangles OAB, OBC, OCA all have perimeter 2s. Find their distances from O.

3. Show that the sum of any n distinct positive odd integers whose pairs all have different differences is at least $n(n^2 + 2)/3$.

4. f(x) is defined on the rationals and takes rational values. $f(x + f(y)) = f(x) f(y)$ for all x, y. Show that f must be constant.

5. Let p(n) be the number of partitions of n. For example, p(4) = 5: 1 + 1 + 1 + 1, 1 + 1 + 2, 2 + 2, 1 + 3, 4. Show that $p(n+1) \geq 2p(n) - p(n-1)$.

6. Show that the number $1 + 10^4 + 10^8 + ... + 10^{4n}$ is not prime for n > 0.

16th BMO 1980

1. Show that there are no solutions to $a^n + b^n = c^n$, with $n > 1$ is an integer, and a, b, c are positive integers with a and b not exceeding n.

2. Find a set of seven consecutive positive integers and a polynomial p(x) of degree 5 with integer coefficients such that p(n) = n for five numbers n in the set including the smallest and largest, and p(n) = 0 for another number in the set.

3. AB is a diameter of a circle. P, Q are points on the diameter and R, S are points on the same arc AB such that PQRS is a square. C is a point on the same arc such that the triangle ABC has the same area as the square. Show that the incenter I of the triangle ABC lies on one of the sides of the square and on the line joining A or B to R or S.

4. Find all real a_0 such that the sequence $a_0, a_1, a_2, \ldots$ defined by $a_{n+1} = 2^n - 3a_n$ has $a_{n+1} > a_n$ for all $n \geq 0$.

5. A graph has 10 points and no triangles. Show that there are 4 points with no edges between them.

17th BMO 1981 - Further International Selection Test

1. ABC is a triangle. Three lines divide the triangle into four triangles and three pentagons. One of the triangle has its three sides along the new lines, the others each have just two sides along the new lines. If all four triangles are congruent, find the area of each in terms of the area of ABC.

2. An *axis* of a solid is a straight line joining two points on its boundary such that a rotation about the line through an angle greater than 0 deg and less than 360 deg brings the solid into coincidence with itself. How many such axes does a cube have? For each axis indicate the minimum angle of rotation and how the vertices are permuted.

3. Find all real solutions to $x^2y^2 + x^2z^2 = axyz$, $y^2z^2 + y^2x^2 = bxyz$, $z^2x^2 + z^2y^2 = cxyz$, where a, b, c are fixed reals.

4. Find the remainder on dividing $x^{81} + x^{49} + x^{25} + x^9 + x$ by $x^3 - x$.

5. The sequence $u_0, u_1, u_2, \ldots$ is defined by $u_0 = 2$, $u_1 = 5$, $u_{n+1}u_{n-1} - u_n^2 = 6^{n-1}$. Show that all terms of the sequence are integral.

6. Show that for rational c, the equation $x^3 - 3cx^2 - 3x + c = 0$ has at most one rational root.

7. If x and y are non-negative integers, show that there are non-negative integers a, b, c, d such that $x = a + 2b + 3c + 7d$, $y = b + 2c + 5d$ iff $5x \geq 7y$.

18th BMO 1982 - Further International Selection Test

1. ABC is a triangle. The angle bisectors at A, B, C meet the circumcircle again at P, Q , R respectively. Show that AP + BQ + CR > AB + BC + CA.

2. The sequence $p_1, p_2, p_3, \ldots$ is defined as follows. $p_1 = 2$. p_{n+1} is the *largest* prime divisor of $p_1p_2 \ldots p_n + 1$. Show that 5 does not occur in the sequence.

3. a is a fixed odd positive integer. Find the largest positive integer n for which there are no positive integers x, y, z such that $ax + (a + 1)y + (a + 2)z = n$.

4. a and b are positive reals and n > 1 is an integer. $P_1 (x_1, y_1)$ and $P_2 (x_2, y_2)$ are two points on the curve $x^n - ay^n = b$ with positive real coordinates. If $y_1 < y_2$ and A is the area of the triangle OP_1P_2, show that $by_2 > 2ny_1^{n-1}a^{1-1/n}A$.

5. p(x) is a real polynomial such that $p(2x) = 2^{k-1}(p(x) + p(x + 1/2))$, where k is a non-negative integer. Show that $p(3x) = 3^{k-1}(p(x) + p(x + 1/3) + p(x + 2/3))$.

19th BMO 1983 - Further International Selection Test

1. Given points A and B and a line *l*, find the point P which minimises $PA^2 + PB^2 + PN^2$, where N is the foot of the perpendicular from P to *l*. State without proof a generalisation to three points.

2. Each pair of excircles of the triangle ABC has a common tangent which does not contain a side of the triangle. Show that one such tangent is perpendicular to OA, where O is the circumcenter of ABC.

3. *l*, *m*, and *n* are three lines in space such that neither *l* nor *m* is perpendicular to *n*. Variable points P on *l* and Q on *m* are such that PQ is perpendicular to *n*, The plane through P perpendicular to *m* meets *n* at R, and the plane through Q perpendicular to *l* meets *n* at S. Show that RS has constant length.

4. Show that for any positive reals a, b, c, d, e, f we have $ab/(a + b) + cd/(c + d) + ef/(e + f) \le (a + c + e)(b + d + f)/(a + b + c + d + e + f)$.

5. How many permutations a, b, c, d, e, f, g, h of 1, 2, 3, 4, 5, 6, 7, 8 satisfy $a < b$, $b > c$, $c < d$, $d > e$, $e < f$, $f > g$, $g < h$?

6. Find all positive integer solutions to $(n + 1)^m = n! + 1$.

7. Show that in a colony of mn + 1 mice, either there is a set of m + 1 mice, none of which is a parent of another, or there is an ordered set of n + 1 mice ($M_0, M_1, M_2, \ldots, M_n$) such that M_i is the parent of M_{i-1} for $i = 1, 2, \ldots, n$.

20th BMO 1984 - Further International Selection Test

1. In the triangle ABC, $\angle C = 90^o$. Find all points D such that AD·BC = AC·BD = AB·CD/√2.

2. ABCD is a tetrahedron such that DA = DB = DC = d and AB = BC = CA = e. M and N are the midpoints of AB and CD. A variable plane through MN meets AD at P and BC at Q. Show that AP/AD = BQ/BC. Find the value of this ratio in terms of d and e which minimises the area of MQNP.

3. Find the maximum and minimum values of cos x + cos y + cos z, where x, y, z are non-negative reals with sum 4π/3.

4. Let b_n be the number of partitions of n into non-negative powers of 2. For example $b^4 = 4$: 1 + 1 + 1 + 1, 1 + 1 + 2, 2 + 2, 4. Let c_n be the number of partitions which include at least one of every power of 2 from 1 up to the highest in the partition. For example, $c_4 = 2$: 1 + 1 + 1 + 1, 1 + 1 + 2. Show that $b_{n+1} = 2c_n$.

5. Show that for any positive integers m, n we can find a polynomial p(x) with integer coefficients such that $| p(x) - m/n | \leq 1/n^2$ for all x in some interval of length 1/n.

21st BMO 1985

1. Prove that $\sum_1^n \sum_1^n | x_i - x_j | \le n^2$ for all real x_i such that $0 \le x_i \le 2$. When does equality hold?

2. (1) The incircle of the triangle ABC touches BC at L. LM is a diameter of the incircle. The ray AM meets BC at N. Show that | NL | = | AB - AC |.

(2) A variable circle touches the segment BC at the fixed point T. The other tangents from B and C to the circle (apart from BC) meet at P. Find the locus of P.

3. Let { x } denote the nearest integer to x, so that x - 1/2 ≤ { x } < x + 1/2. Define the sequence $u_1, u_2, u_3, \ldots$ by $u_1 = 1$. $u_{n+1} = u_n + \{ u_n\sqrt{2} \}$. So, for example, $u_2 = 2$, $u_3 = 5$, $u_4 = 12$. Find the units digit of u_{1985}.

4. A, B, C, D are points on a sphere of radius 1 such that the product of the six distances between the points is 512/27. Prove that ABCD is a regular tetrahedron.

5. Let b_n be the number of ways of partitioning the set {1, 2, ... , n} into non-empty subsets. For example, $b_3 = 5$: 123; 12, 3; 13, 2; 23, 1; 1, 2, 3. Let c_n be the number of partitions where each part has at least two elements. For example, $c_4 = 4$: 1234; 12, 34; 13, 24; 14, 23. Show that $c_n = b_{n-1} - b_{n-2} + \ldots + (-1)^n b_1$.

6. Find all non-negative integer solutions to $5^a 7^b + 4 = 3^c$.

22nd BMO 1986 - Further International Selection Test

1. A *rational point* is a point both of whose coordinates are rationals. Let A, B, C, D be rational points such that AB and CD are not both equal and parallel. Show that there is just one point P such that the triangle PCD can be obtained from the triangle PAB by enlargement and rotation about P. Show also that P is rational.

2. Find the maximum value of $x^2y + y^2z + z^2x$ for reals x, y, z with sum zero and sum of squares 6.

3. $P_1, P_2, \ldots, P_n$ are distinct subsets of {1, 2, ... , n} with two elements. Distinct subsets P_i and P_j have an element in common iff {i, j} is one of the P_k. Show that each member of {1, 2, ... , n} belongs to just two of the subsets.

4. $m \le n$ are positive integers. nCm denotes the binomial coefficient n!/(m! (n-m)!). Show that nCm nC(m-1) is divisible by n. Find the smallest positive integer k such that k nCm nC(m-1) nC(m-2) is divisible by n^2 for all m, n such that $1 < m \le n$. For this value of k and fixed n, find the greatest common divisor of the n - 1 integers (k nCm nC(m-1) nC(m-2))/n^2 where $1 < m \le n$.

5. C and C' are fixed circles. A is a fixed point on C, and A' is a fixed point on C'. B is a variable point on C. B' is the point on C' such that A'B' is parallel to AB. Find the locus of the midpoint of BB'.

24th BMO 1988 - Further International Selection Test

1. ABC is an equilateral triangle. S is the circle diameter AB. P is a point on AC such that the circle center P radius PC touches S at T. Show that AP/AC = 4/5. Find AT/AC.

2. Show that the number of ways of dividing {1, 2, ... , 2n} into n sets of 2 elements is 1.3.5 ... (2n-1). There are 5 married couples at a party. How many ways may the 10 people be divided into 5 pairs if no married couple may be paired together? For example, for 2 couples a, A, b, B the answer is 2: ab, AB; aB, bA.

3. The real numbers a, b, c, x, y, z satisfy: $x^2 - y^2 - z^2 = 2ayz$, $-x^2 + y^2 - z^2 = 2bzx$, $-x^2 - y^2 + z^2 = 2cxy$, and $xyz \neq 0$. Show that $x^2(1 - b^2) = y^2(1 - a^2) = xy(ab - c)$ and hence find $a^2 + b^2 + c^2 - 2abc$ (independently of x, y, z).

4. Find all positive integer solutions to 1/a + 2/b - 3/c = 1.

5. L and M are skew lines in space. A, B are points on L, M respectively such that AB is perpendicular to L and M. P, Q are variable points on L, M respectively such that PQ is of constant length. P does not coincide with A and Q does not coincide with B. Show that the center of the sphere through A, B, P, Q lies on a fixed circle whose center is the midpoint of AB.

6. Show that if there are triangles with sides a, b, c, and A, B, C, then there is also a triangle with sides $\sqrt{(a^2 + A^2)}$, $\sqrt{(b^2 + B^2)}$, $\sqrt{(c^2 + C^2)}$.

25th BMO 1989 - Further International Selection Test

1. Find the smallest positive integer a such that $ax^2 - bx + c = 0$ has two distinct roots in the interval $0 < x < 1$ for some integers b, c.

2. Find the number of different ways of arranging five As, five Bs and five Cs in a row so that each letter is adjacent to an identical letter. Generalise to n letters each appearing five times.

3. f(x) is a polynomial of degree n such that f(0) = 0, f(1) = 1/2, f(2) = 2/3, f(3) = 3/4, ... , f(n) = n/(n+1). Find f(n+1).

4. D is a point on the side AC of the triangle ABC such that the incircles of BAD and BCD have equal radii. Express | BD | in terms of the lengths a = | BC |, b = | CA |, c = | AB |.

26th BMO 1990 - Further International Selection Test

1. Show that if a polynomial with integer coefficients takes the value 1990 at four different integers, then it cannot take the value 1997 at any integer.

2. The fractional part { x } of a real number is defined as x - [x]. Find a positive real x such that { x } + { 1/x } = 1 (*). Is there a rational x satisfying (*)?

3. Show that $\sqrt{(x^2 + y^2 - xy)} + \sqrt{(y^2 + z^2 - yz)} \geq \sqrt{(z^2 + x^2 + zx)}$ for any positive real numbers x, y, z.

4. A rectangle is *inscribed* in a triangle if its vertices all lie on the boundary of the triangle. Given a triangle T, let d be the shortest diagonal for any rectangle inscribed in T. Find the maximum value of d^2/area T for all triangles T.

5. ABC is a triangle with incenter I. X is the center of the excircle opposite A. Show that AI·AX = AB·AC and AI·BX·CX = AX·BI·CI.

27th BMO 1991 - Further International Selection Test

1. ABC is a triangle with $\angle B = 90^o$ and M the midpoint of AB. Show that sin ACM $\leq$ 1/3.

2. Twelve dwarfs live in a forest. Some pairs of dwarfs are friends. Each has a black hat and a white hat. Each dwarf consistently wears one of his hats. However, they agree that on the nth day of the New Year, the nth dwarf modulo 12 will visit each of his friends. (For example, the 2nd dwarf visits on days 2, 14, 26 and so on.) If he finds that a majority of his friends are wearing a different color of hat, then he will immediately change color. No other hat changes are made. Show that after a while no one changes hat.

3. A triangle has sides a, b, c with sum 2. Show that $a^2 + b^2 + c^2 + 2abc < 2$.

4. Let N be the smallest positive integer such that at least one of the numbers x, 2x, 3x, ... , Nx has a digit 2 for every real number x. Find N. Failing that, find upper and lower bounds and show that the upper bound does not exceed 20.

28th BMO 1992 - Round 2

1. p is an odd prime. Show that there are unique positive integers m, n such that $m^2 = n(n + p)$. Find m and n in terms of p.

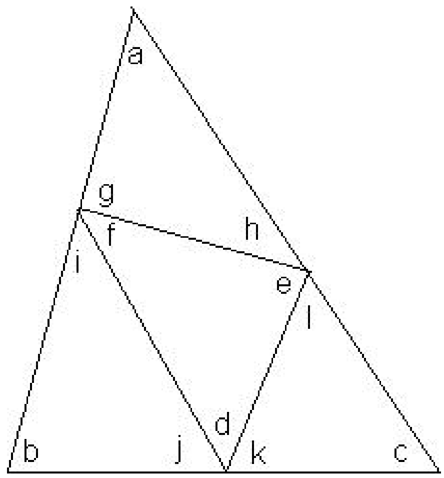
2. Show that $12/(w + x + y + z) \le 1/(w + x) + 1/(w + y) + 1/(w + z) + 1/(x + y) + 1/(x + z) + 1/(y + z) \le 3(1/w + 1/x + 1/y + 1/z)/4$ for any positive reals w, x, y, z.

3. The circumradius R of a triangle with sides a, b, c satisfies $a^2 + b^2 = c^2 - R^2$. Find the angles of the triangle.

4. Each edge of a connected graph with n points is colored red, blue or green. Each point has exactly three edges, one red, one blue and one green. Show that n must be even and that such a colored graph is possible for any even $n > 2$. X is a subset of $1 < k < n$ points. In order to isolate X from the other points (so that there is no edge between a point in X and a point not in X) it is necessary and sufficient to delete R red edges, B blue edges and G green edges. Show that R, B, G are all even or all odd.

29th BMO 1993 - Round 2

1. The angles in the diagram below are measured in some unknown unit, so that a, b, ... , k, l are all distinct positive integers. Find the smallest possible value of a + b + c and give the corresponding values of a, b, ... , k, l.



2. p > 3 is prime. $m = (4^p - 1)/3$. Show that $2^{m-1} = 1 \bmod m$.

3. P is a point inside the triangle ABC. x = ∠BPC - ∠A, y = ∠CPA - ∠B, z = ∠APB - ∠C. Show that PA sin A/sin x = PB sin B/sin y = PC sin C/sin z.

4. For 0 < m < 10, let S(m, n) is the set of all positive integers with n 1s, n 2s, n 3s, ... , n ms. For a positive integer N let d(N) be the sum of the absolute differences between all pairs of adjacent digits. For example, d(122313) = 1 + 0 + 1 + 2 + 2 = 6. Find the mean value of d(N) for N in S(m, n).

30th BMO 1994 - Round 2

1. Find the smallest integer $n > 1$ such that $(1^2 + 2^2 + 3^2 + ... + n^2)/n$ is a square.

2. How many incongruent triangles have integer sides and perimeter 1994?

3. A, P, Q, R, S are distinct points on a circle such that $\angle PAQ = \angle QAR = \angle RAS$. Show that $AR(AP + AR) = AQ(AQ + AS)$.

4. How many perfect squares are there mod 2^n?

31st BMO 1995 - Round 2

1. Find all positive integers $a \ge b \ge c$ such that $(1 + 1/a)(1 + 1/b)(1 + 1/c) = 2$.

2. ABC is a triangle. D, E, F are the midpoints of BC, CA, AB. Show that $\angle DAC = \angle ABE$ iff $\angle AFC = \angle ADB$.

3. x, y, z are real numbers such that $x < y < z$, $x + y + z = 6$ and $xy + yz + zx = 9$. Show that $0 < x < 1 < y < 3 < z < 4$.

4. (1) How many ways can 2n people be grouped into n teams of 2?

(2) Show that $(mn)!\,(mn)!$ is divisible by $m!^{n+1}\, n!^{m+1}$ for all positive integers m, n.

32nd BMO 1996 - Round 2

1. Find all non-negative integer solutions to $2^m + 3^n = k^2$.

2. The triangle ABC has sides a, b, c, and the triangle UVW has sides u, v, w such that $a^2 = u(v + w - u)$, $b^2 = v(w + u - v)$, $c^2 = w(u + v - w)$. Show that ABC must be acute angled and express the angles U, V, W in terms of the angles A, B, C.

3. The circles C and C' lie inside the circle S. C and C' touch each other externally at K and touch S at A and A' respectively. The common tangent to C and C' at K meets S at P. The line PA meets C again at B, and the line PA' meets C' again at B'. Show that BB' is a common tangent to C and C'.

4. Find all positive real solutions to $w + x + y + z = 12$, $wxyz = wx + wy + wz + xy + xz + yz + 27$.

33rd BMO 1997 - Round 2

1. M and N are 9-digit numbers. If any digit of M is replaced by the corresponding digit of N (eg the 10s digit of M replaced by the 10s digit of N), then the resulting integer is a multiple of 7. Show that if any digit of N is replaced by the corresponding digit of M, then the resulting integer must be a multiple of 7. Find d > 9, such that the result remains true when M and N are d-digit numbers.

2. ABC is an acute-angled triangle. The median BM and the altitude CF have equal length, and ∠MBC = ∠FCA. Show that ABC must be equilateral.

3. Find the number of polynomials of degree 5 with distinct coefficients from the set {1, 2, ... , 9} which are divisible by $x^2 - x + 1$.

4. Let S be the set {1/1, 1/2, 1/3, 1/4, ... }. The subset {1/20, 1/8, 1/5} is an arithmetic progression of length 3 and is maximal, because it cannot be extended (within S) to a longer arithmetic progression. Find a maximal arithmetic progression in S of length 1996. Is there a maximal arithmetic progression in S of length 1997?

34th BMO 1998

1. A station issues 3800 tickets covering 200 destinations. Show that there are at least 6 destinations for which the number of tickets sold is the same. Show that this is not necessarily true for 7.

2. The triangle ABC has $\angle A > \angle C$. P lies inside the triangle so that $\angle PAC = \angle C$. Q is taken outside the triangle so that BQ parallel to AC and PQ is parallel to AB. R is taken on AC (on the same side of the line AP as C) so that $\angle PRQ = \angle C$. Show that the circles ABC and PQR touch.

3. a, b, c are positive integers satisfying 1/a - 1/b = 1/c and d is their greatest common divisor. Prove that abcd and d(b - a) are squares.

4. Show that:

$$xy + yz + zx = 12$$

$$xyz - x - y - z = 2$$

have a unique solution in the positive reals. Show that there is a solution with x, y, z distinct reals.

35th BMO 1999 - Round 2

1. Let $X_n = \{1, 2, 3, \dots, n\}$. For which n can we partition X_n into two parts with the same sum? For which n can we partition X_n into three parts with the same sum?

2. A circle is inscribed in a hexagon ABCDEF. It touches AB, CD and EF at their midpoints (L, M, N respectively) and touches BC, DE, FA at the points P, Q, R. Prove that LQ, MR, NP are concurrent.

3. Show that $xy + yz + zx \le 2/7 + 9xyz/7$ for non-negative reals x, y, z with sum 1.

4. Find the smallest possible sum of digits for a number of the form $3n^2 + n + 1$ (where n is a positive integer). Does there exist a number of this form with sum of digits 1999?

36th BMO 2000

1. Two circles meet at A and B and touch a common tangent at C and D. Show that triangles ABC and ABD have the same area.

2. Find the smallest value of $x^2 + 4xy + 4y^2 + 2z^2$ for positive reals x, y, z with product 32.

3. Find positive integers m, n such that $(m^{1/3} + n^{1/3} - 1)^2 = 49 + 20 (6^{1/3})$.

4. Find a set of 10 distinct positive integers such that no 6 members of the set have a sum divisible by 6. Is it possible to find such a set with 11 members?

37th BMO 2001

1. A has a marbles and B has b < a marbles. Starting with A each gives the other enough marbles to double the number he has. After 2n such transfers A has b marbles. Find a/b in terms of n.

2. Find all integer solutions to $m^2n + 1 = m^2 + 2mn + 2m + n$.

3. ABC is a triangle with AB greater than AC. AD is the angle bisector. E is the point on AB such that ED is perpendicular to BC. F is the point on AC such that DE bisects angle BEF. Show that $\angle FDC = \angle BAD$.

4. n dwarfs with heights 1, 2, 3, ... , n stand in a circle. S is the sum of the (non-negative) differences between each adjacent pair of dwarfs. What are the maximum and minimum possible values of S?

38th BMO 2002

1. From the foot of an altitude in an acute-angled triangle perpendiculars are drawn to the other two sides. Show that the distance between their feet is independent of the choice of altitude.

2. n people wish to sit at a round table which has n chairs. The first person takes an seat. The second person sits one place to the right of the first person, the third person sits two places to the right of the second person, the fourth person sits three places to the right of the third person and so on. For which n is this possible?

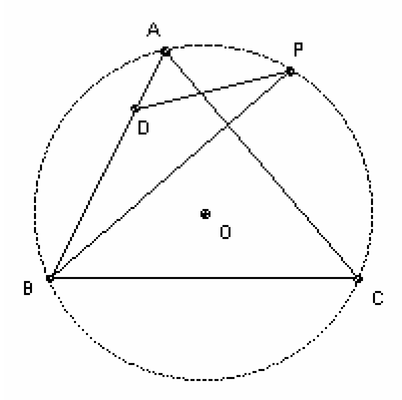
3. The real sequence x_1, x_1, x_2, ... is defined by $x_0 = 1$, $x_{n+1} = (3x_n + \sqrt{(5x_n^2 - 4)})/2$. Show that all the terms are integers.

4. S_1, S_2, ... , S_n are spheres of radius 1 arranged so that each touches exactly two others. P is a point outside all the spheres. Let x_1, x_2, ... , x_n be the distances from P to the n points of contact between two spheres and y_1, y_2, ... , y_n be the lengths of the tangents from P to the spheres. Show that $x_1x_2 \ldots x_n \geq y_1y_2 \ldots y_n$.

39th BMO 2003

1. Find all integers $0 < a < b < c$ such that $b - a = c - b$ and none of a, b, c have a prime factor greater than 3.

2. D is a point on the side AB of the triangle ABC such that AB = 4·AD. P is a point on the circumcircle such that angle ADP = angle C. Show that PB = 2·PD.



3. f is a bijection on the positive integers. Show that there are three positive integers $a_0 < a_1 < a_2$ in arithmetic progression such that $f(a_0) < f(a_1) < f(a_2)$. Is there necessarily an arithmetic progression $a_1 < a_2 < ... < a_{2003}$ such that $f(a_0) < f(a_1) < ... < f(a_{2003})$?

4. Let X be the set of non-negative integers and $f : X \to X$ a map such that $(f(2n+1))^2 - (f(2n))^2 = 6 f(n) + 1$ and $f(2n) \geq f(n)$ for all n in X. How many numbers in f(X) are less than 2003?

40th BMO 2004

1. ABC is an equilateral triangle. D is a point on the side BC (not at the endpoints). A circle touches BC at D and meets the side AB at M and N, and the side AC at P and Q. Show that BD + AM + AN = CD + AP + AQ.

2. Show that there is a multiple of 2004 whose binary expression has exactly 2004 0s and 2004 1s.

3. a, b, c are reals with sum zero. Show that $a^3 + b^3 + c^3 > 0$ iff $a^5 + b^5 + c^5 > 0$. Prove the same result for 4 reals.

4. The decimal $0.a_1a_2a_3a_4...$ hs the property that there are at most 2004 distinct blocks $a_ka_{k+1}...a_{k+2003}$ in the expansion. Show that the decimal must be rational.

Brasil (1979 – 2003)

1st Brasil 1979

1. Show that if $a < b$ are in the interval $[0, \pi/2]$ then $a - \sin a < b - \sin b$. Is this true for $a < b$ in the interval $[\pi, 3\pi/2]$?

2. The remainder on dividing the polynomial $p(x)$ by $x^2 - (a+b)x + ab$ (where a and b are unequal) is $mx + n$. Find the coefficients m, n in terms of a, b. Find m, n for the case $p(x) = x^{200}$ divided by $x^2 - x - 2$ and show that they are integral.

3. The vertex C of the triangle ABC is allowed to vary along a line parallel to AB. Find the locus of the orthocenter.

4. Show that the number of positive integer solutions to $x_1 + 2^3x_2 + 3^3x_3 + ... + 10^3x_{10} = 3025$ (*) equals the number of non-negative integer solutions to the equation $y_1 + 2^3y_2 + 3^3y_3 + ... + 10^3y_{10} = 0$. Hence show that (*) has a unique solution in positive integers and find it.

5.(i) ABCD is a square with side 1. M is the midpoint of AB, and N is the midpoint of BC. The lines CM and DN meet at I. Find the area of the triangle CIN.

(ii) The midpoints of the sides AB, BC, CD, DA of the parallelogram ABCD are M, N, P, Q respectively. Each midpoint is joined to the two vertices not on its side. Show that the area outside the resulting 8-pointed star is 2/5 the area of the parallelogram.

(iii) ABC is a triangle with CA = CB and centroid G. Show that the area of AGB is 1/3 of the area of ABC.

(iv) Is (ii) true for all convex quadrilaterals ABCD?

2nd Brasil 1980

1. Box A contains black balls and box B contains white balls. Take a certain number of balls from A and place them in B. Then take the same number of balls from B and place them in A. Is the number of white balls in A then greater, equal to, or less than the number of black balls in B?

2. Show that for any positive integer $n > 2$ we can find n distinct positive integers such that the sum of their reciprocals is 1.

3. Given a triangle ABC and a point P_0 on the side AB. Construct points P_i, Q_i, R_i as follows. Q_i is the foot of the perpendicular from P_i to BC, R_i is the foot of the perpendicular from Q_i to AC and P_i is the foot of the perpendicular from R_{i-1} to AB. Show that the points P_i converge to a point P on AB and show how to construct P.

4. Given 5 points of a sphere radius r, show that two of the points are a distance $\leq r\sqrt{2}$ apart.

3rd Brasil 1981

1. For which k does the system $x^2 - y^2 = 0$, $(x-k)^2 + y^2 = 1$ have exactly (1) two, (2) three real solutions?

2. Show that there are at least 3 and at most 4 powers of 2 with m digits. For which m are there 4?

3. Given a sheet of paper and the use of a rule, compass and pencil, show how to draw a straight line that passes through two given points, if the length of the ruler and the maximum opening of the compass are both less than half the distance between the two points. You may not fold the paper.

4. A graph has 100 points. Given any four points, there is one joined to the other three. Show that one point must be joined to all 99 other points. What is the smallest number possible of such points (that are joined to all the others)?

5. Two thieves stole a container of 8 liters of wine. How can they divide it into two parts of 4 liters each if all they have is a 3 liter container and a 5 liter container? Consider the general case of dividing m+n liters into two equal amounts, given a container of m liters and a container of n liters (where m and n are positive integers). Show that it is possible iff m+n is even and (m+n)/2 is divisible by gcd(m,n).

6. The centers of the faces of a cube form a regular octahedron of volume V. Through each vertex of the cube we may take the plane perpendicular to the long diagonal from the vertex. These planes also form a regular octahedron. Show that its volume is 27V.

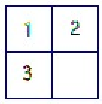
4th Brasil 1982

1. The angles of the triangle ABC satisfy $\angle A/\angle C = \angle B/\angle A = 2$. The incenter is O. K, L are the excenters of the excircles opposite B and A respectively. Show that triangles ABC and OKL are similar.

2. Any positive integer n can be written in the form $n = 2^b(2c+1)$. We call 2c+1 the *odd part* of n. Given an odd integer n > 0, define the sequence $a_0, a_1, a_2, \ldots$ as follows: $a_0 = 2^n-1$, a_{k+1} is the odd part of $3a_k+1$. Find a_n.

3. S is a (k+1) x (k+1) array of lattice points. How many squares have their vertices in S?

4. Three numbered tiles are arranged in a tray as shown: Show that we cannot interchange the 1 and the 3 by a sequence of moves where we slide a tile to the adjacent vacant space.



5. Show how to construct a line segment length $(a^4 + b^4)^{1/4}$ given segments length a and b.

6. Five spheres of radius r are inside a right circular cone. Four of the spheres lie on the base of the cone. Each touches two of the others and the sloping sides of the cone. The fifth sphere touches each of the other four and also the sloping sides of the cone. Find the volume of the cone.

5th Brasil 1983

1. Show that there are only finitely many solutions to $1/a + 1/b + 1/c = 1/1983$ in positive integers.

2. An equilateral triangle ABC has side a. A square is constructed on the outside of each side of the triangle. A right regular pyramid with sloping side a is placed on each square. These pyramids are rotated about the sides of the triangle so that the apex of each pyramid comes to a common point above the triangle. Show that when this has been done the other vertices of the bases of the pyramids (apart from the vertices of the triangle) form a regular hexagon.

3. Show that $1 + 1/2 + 1/3 + ... + 1/n$ is not an integer for $n > 1$.

4. Show that it is possible to color each point of a circle red or blue so that no right-angled triangle inscribed in the circle has its vertices all the same color.

5. Show that $1 \le n^{1/n} \le 2$ for all positive integers n. Find the smallest k such that $1 \le n^{1/n} \le k$ for all positive integers n.

6. Show that the maximum number of spheres of radius 1 that can be placed touching a fixed sphere of radius 1 so that no pair of spheres has an interior point in common is between 12 and 14.

6th Brasil 1984

1. Find all solutions in positive integers to $(n+1)^k - 1 = n!$.

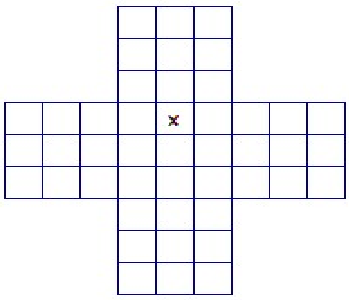
2. Each day 289 students are divided into 17 groups of 17. No two students are ever in the same group more than once. What is the largest number of days that this can be done?

3. Given a regular dodecahedron of side a. Take two pairs of opposite faces: E, E' and F, F'. For the pair E, E' take the line joining the centers of the faces and take points A and C on the line each a distance m outside one of the faces. Similarly, take B and D on the line joining the centers of F, F' each a distance m outside one of the faces. Show that ABCD is a rectangle and find the ratio of its side lengths.

4. ABC is a triangle with $\angle A = 90^o$. For a point D on the side BC, the feet of the perpendiculars to AB and AC are E and F. For which point D is EF a minimum?

5. ABCD is any convex quadrilateral. Squares center E, F, G, H are constructed on the outside of the edges AB, BC, CD and DA respectively. Show that EG and FH are equal and perpendicular.

6. There is a piece on each square of the solitaire board shown except for the central square. A move can be made when there are three adjacent squares in a horizontal or vertical line with two adjacent squares occupied and the third square vacant. The move is to remove the two pieces from the occupied squares and to place a piece on the third square. (One can regard one of the pieces as hopping over the other and taking it.) Is it possible to end up with a single piece on the board, on the square marked X?



7th Brasil 1985

1. a, b, c, d are integers with $ad \neq bc$. Show that $1/((ax+b)(cx+d))$ can be written in the form $r/(ax+b) + s/(cx+d)$. Find the sum $1/1\cdot4 + 1/4\cdot7 + 1/7\cdot10 + ... + 1/2998\cdot3001$.

2. Given n points in the plane, show that we can always find three which give an angle $\leq \pi/n$.

3. A convex quadrilateral is inscribed in a circle of radius 1. Show that the its perimeter less the sum of its two diagonals lies between 0 and 2.

4. a, b, c, d are integers. Show that $x^2 + ax + b = y^2 + cy + d$ has infinitely many integer solutions iff $a^2 - 4b = c^2 - 4d$.

5. A, B are reals. Find a necessary and sufficient condition for $Ax + B[x] = Ay + B[y]$ to have no solutions except $x = y$.

8th Brasil 1986

1. A ball moves endlessly on a circular billiard table. When it hits the edge it is reflected. Show that if it passes through a point on the table three times, then it passes through it infinitely many times.

2. Find the number of ways that a positive integer n can be represented as a sum of one or more consecutive positive integers.

3. The Poincare plane is a half-plane bounded by a line R. The lines are taken to be (1) the half-lines perpendicular to R, and (2) the semicircles with center on R. Show that given any line L and any point P not on L, there are infinitely many lines through P which do not intersect L. Show that if ABC is a triangle, then the sum of its angles lies in the interval $(0, \pi)$.

4. Find all 10 digit numbers $a_0a_1...a_9$ such that for each k, a_k is the number of times that the digit k appears in the number.

5. A number is written in each square of a chessboard, so that each number not on the border is the mean of the 4 neighboring numbers. Show that if the largest number is N, then there is a number equal to N in the border squares.

9th Brasil 1987

1. $p(x_1, x_2, ..., x_n)$ is a polynomial with integer coefficients. For each positive integer r, k(r) is the number of n-tuples $(a_1, a_2, ..., a_n)$ such that $0 \le a_i \le r-1$ and $p(a_1, a_2, ..., a_n)$ is prime to r. Show that if u and v are coprime then $k(u \cdot v) = k(u) \cdot k(v)$, and if p is prime then $k(p^s) = p^{n(s-1)} k(p)$.

2. Given a point p inside a convex polyhedron P. Show that there is a face F of P such that the foot of the perpendicular from p to F lies in the interior of F.

3. Two players play alternately. The first player is given a pair of positive integers (x_1, y_1). Each player must replace the pair (x_n, y_n) that he is given by a pair of non-negative integers (x_{n+1}, y_{n+1}) such that $x_{n+1} = \min(x_n, y_n)$ and $y_{n+1} = \max(x_n, y_n) - k \cdot x_{n+1}$ for some positive integer k. The first player to pass on a pair with $y_{n+1} = 0$ wins. Find for which values of x_1/y_1 the first player has a winning strategy.

4. Given points $A_1 (x_1, y_1, z_1)$, $A_2 (x_2, y_2, z_2)$, ... , $A_n (x_n, y_n, z_n)$ let P (x, y, z) be the point which minimizes $\sum (|x - x_i| + |y - y_i| + |z - z_i|)$. Give an example (for each $n > 4$) of points A_i for which the point P lies outside the convex hull of the points A_i.

5. A and B wish to divide a cake into two pieces. Each wants the largest piece he can get. The cake is a triangular prism with the triangular faces horizontal. A chooses a point P on the top face. B then chooses a vertical plane through the point P to divide the cake. B chooses which piece to take. Which point P should A choose in order to secure as large a slice as possible?

10th Brasil 1988

1. Find all primes which can be written both as a sum of two primes and as a difference of two primes.

2. P is a fixed point in the plane. A, B, C are points such that PA = 3, PB = 5, PC = 7 and the area ABC is as large as possible. Show that P must be the orthocenter of ABC.

3. Let N be the natural numbers and N' = N □ {0}. Find all functions f:N→N' such that f(xy) = f(x) + f(y), f(30) = 0 and f(x) = 0 for all x = 7 mod 10.

4. Two triangles have the same incircle. Show that if a circle passes through five of the six vertices of the two triangles, then it also passes through the sixth.

5. A figure on a computer screen shows n points on a sphere, no four coplanar. Some pairs of points are joined by segments. Each segment is colored red or blue. For each point there is a key that switches the colors of all segments with that point as endpoint. For every three points there is a sequence of key presses that makes the three segments between them red. Show that it is possible to make all the segments on the screen red. Find the smallest number of key presses that can turn all the segments red, starting from the worst case.

11th Brasil 1989

1. The triangle vertices (0,0), (0,1), (2,0) is repeatedly reflected in the three lines AB, BC, CA where A is (0,0), B is (3,0), C is (0,3). Show that one of the images has vertices (24,36), (24,37) and (26,36).

2. n is a positive integer such that n(n+1)/3 is a square. Show that n is a multiple of 3, and n+1 and n/3 are squares.

3. Let Z be the integers. $f : Z \to Z$ is defined by $f(n) = n - 10$ for $n > 100$ and $f(n) = f(f(n+11))$ for $n \le 100$. Find the set of possible values of f.

4. A and B play a game. Each has 10 tokens numbered from 1 to 10. The board is two rows of squares. The first row is numbered 1 to 1492 and the second row is numbered 1 to 1989. On the nth turn, A places his token number n on any empty square in either row and B places his token on any empty square in the other row. B wins if the order of the tokens is the same in the two rows, otherwise A wins. Which player has a winning strategy? Suppose each player has k tokens, numbered from 1 to k. Who has the winning strategy? Suppose that both rows are all the integers? Or both all the rationals?

5. The circumcenter of a tetrahedron lies inside the tetrahedron. Show that at least one of its edges is at least as long as the edge of a regular tetrahedron with the same circumsphere.

12th Brasil 1990

1. Show that a convex polyhedron with an odd number of faces has at least one face with an even number of edges.

2. Show that there are infinitely many positive integer solutions to $a^3 + 1990b^3 = c^4$.

3. Each face of a tetrahedron is a triangle with sides a, b, c and the tetrahedon has circumradius 1. Find $a^2 + b^2 + c^2$.

4. ABCD is a convex quadrilateral. E, F, G, H are the midpoints of sides AB, BC, CD, DA respectively. Find the point P such that area PHAE = area PEBF = area PFCG = area PGDH.

5. Given that f(x) = (ax+b)/(cx+d), f(0) ≠ 0, f(f(0)) ≠ 0. Put F(x) = f(...(f(x) ...) (where there are n fs). If F(0) = 0, show that F(x) = x for all x where the expression is defined.

13th Brasil 1991

1. At a party every woman dances with at least one man, and no man dances with every woman. Show that there are men M and M' and women W and W' such that M dances with W, M' dances with W', but M does not dance with W', and M' does not dance with W.

2. P is a point inside the triangle ABC. The line through P parallel to AB meets AC at AC at A_0 and BC at B_0. Similarly, the line through P parallel to CA meets AB at A_1 and BC at C_1, and the line through P parallel to BC meets AB at B_2 and AC at C_2. Find the point P such that $A_0B_0 = A_1B_1 = A_2C_2$.

3. Given k > 0, the sequence a_1, a_2, a_3, ... is defined by its first two members and $a_{n+2} = a_{n+1} + (k/n)a_n$. For which k can we write a_n as a polynomial in n? For which k can we write an+1/an = p(n)/q(n)?

4. Show that there is a number of the form 199...91 (with n 9s) with n > 2 which is divisible by 1991.

5. $P_0 = (1,0)$, $P_1 = (1,1)$, $P_2 = (0,1)$, $P_3 = (0,0)$. P_{n+4} is the midpoint of P_nP_{n+1}. Q_n is the quadrilateral $P_nP_{n+1}P_{n+2}P_{n+3}$. A_n is the interior of Q_n. Find $\cap_{n\geq 0}A_n$.

14th Brasil 1992

1. The polynomial $x^3 + px + q$ has three distinct real roots. Show that $p < 0$.

2. Show that there is a positive integer n such that the first 1992 digits of n^{1992} are 1.

3. Given positive real numbers $x_1, x_2, \ldots, x_n$ find the polygon $A_0A_1..A_n$ with $A_0A_1 = x_1$, $A_1A_2 = x_2, \ldots, A_{n-1}A_n = x_n$ which has greatest area.

4. ABC is a triangle. Find D on AC and E on AB such that area ADE = area DEBC and DE has minimum possible length.

5. Let d(n) be the number of positive divisors of n. Show that $n(1/2 + 1/3 + \ldots + 1/n) \le d(1) + d(2) + \ldots + d(n) \le n(1 + 1/2 + 1/3 + \ldots + 1/n)$.

6. Given a set of n elements, find the largest number of subsets such that no subset is contained in any other.

7. Find all solutions in positive integers to $n^a + n^b = n^c$.

8. In a chess tournament each player plays every other player once. A player gets 1 point for a win, ½ point for a draw and 0 for a loss. Both men and women played in the tournament and each player scored the same total of points against women as against men. Show that the total number of players must be a square.

9. Show that for each $n > 5$ it is possible to find a convex polyhedron with all faces congruent such that each face has another face parallel to it.

15th Brasil 1993

1. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 8$, $a_2 = 18$, $a_{n+2} = a_{n+1}a_n$. Find all terms which are perfect squares.

2. A real number with absolute value less than 1 is written in each cell of an n x n array, so that the sum of the numbers in each 2 x 2 square is zero. Show that for n odd the sum of all the numbers is less than n.

3. Given a circle and its center O, a point A inside the circle and a distance h, construct a triangle BAC with $\angle A = 90^o$, B and C on the circle and the altitude from A length h.

4. ABCD is a convex quadrilateral with $\angle BAC = 30^o$, $\angle CAD = 20^o$, $\angle ABD = 50^o$, $\angle DBC = 30^o$. If the diagonals intersect at P, show that PC = PD.

5. Find a real-valued function f(x) on the non-negative reals such that f(0) = 0, and f(2x+1) = 3f(x) + 5 for all x.

16th Brasil 1994

1. The edges of a cube are labeled from 1 to 12 in an arbitrary manner. Show that it is not possible to get the sum of the edges at each vertex the same. Show that we can get eight vertices with the same sum if one of the labels is changed to 13.

2. Given any convex polygon, show that there are three consecutive vertices such that the polygon lies inside the circle through them.

3. We are given n objects of identical appearance, but different mass, and a balance which can be used to compare any two objects (but only one object can be placed in each pan at a time). How many times must we use the balance to find the heaviest object and the lightest object?

4. Show that if the positive real numbers a, b satisfy $a^3 = a+1$ and $b^6 = b+3a$, then $a > b$.

5. Call a *super-integer* an infinite sequence of decimal digits: $...d_n...d_2d_1$. Given two such super-integers $...c_n...c_2c_1$ and $...d_n...d_2d_1$, their product$...p_n...p_2p_1$ is formed by taking $p_n...p_2p_1$ to be the last n digits of the product $c_n...c_2c_1$ and $d_n...d_2d_1$. Can we find two non-zero super-integers with zero product (a zero super-integer has all its digits zero).

6. A triangle has semi-perimeter s, circumradius R and inradius r. Show that it is right-angled iff $2R = s - r$.

17th Brasil 1995

A1. ABCD is a quadrilateral with a circumcircle center O and an inscribed circle center I. The diagonals intersect at S. Show that if two of O, I, S coincide, then it must be a square.

A2. Find all real-valued functions on the positive integers such that f(x + 1019) = f(x) for all x, and f(xy) = f(x) f(y) for all xy.

A3. Let p(n) be the largest prime which divides n. Show that there are infinitely many positive integers n such that p(n) < p(n+1) < p(n+2).

B1. A regular tetrahedron has side L. What is the smallest x such that the tetrahedron can be passed through a loop of twine of length x?

B2. Show that the nth root of a rational (for n a positive integer) cannot be a root of the polynomial $x^5 - x^4 - 4x^3 + 4x^2 + 2$.

B3. X has n elements. F is a family of subsets of X each with three elements, such that any two of the subsets have at most one element in common. Show that there is a subset of X with at least √(2n) members which does not contain any members of F.

18th Brasil 1996

A1. Show that the equation $x^2 + y^2 + z^2 = 3xyz$ has infinitely many solutions in positive integers.

A2. Does there exist a set of $n > 2$ points in the plane such that no three are collinear and the circumcenter of any three points of the set is also in the set?

A3. Let f(n) be the smallest number of 1s needed to represent the positive integer n using only 1s, + signs, x signs and brackets. For example, you could represent 80 with 13 1s as follows: (1+1+1+1+1)x(1+1+1+1)x(1+1+1+1). Show that $3 \log_3 n \le f(n) \le 5 \log_3 n$ for $n > 1$.

B1. ABC is acute-angled. D s a variable point on the side BC. O_1 is the circumcenter of ABD, O_2 is the circumcenter of ACD, and O is the circumcenter of AO_1O_2. Find the locus of O.

B2. There are n boys $B_1, B_2, \ldots, B_n$ and n girls $G_1, G_2, \ldots, G_n$. Each boy ranks the girls in order of preference, and each girl ranks the boys in order of preference. Show that we can arrange the boys and girls into n pairs so that we cannot find a boy and a girl who prefer each other to their partners. For example if (B_1, G_3) and (B_4, G_7) are two of the pairs, then it must not be the case that B_4 prefers G_3 to G_7 and G_3 prefers B_4 to B_1.

B3. Let p(x) be the polynomial $x^3 + 14x^2 - 2x + 1$. Let $p^n(x)$ denote $p(p^{n-1}(x))$. Show that there is an integer N such that $p^N(x) - x$ is divisible by 101 for all integers x.

19th Brasil 1997

A1. Given R, $r > 0$. Two circles are drawn radius R, r which meet in two points. The line joining the two points is a distance D from the center of one circle and a distance d from the center of the other. What is the smallest possible value for D+d?

A2. A is a set of n non-negative integers. We say it has *property P* if the set $\{x + y: x, y \in A\}$ has $n(n+1)/2$ elements. We call the largest element of A minus the smallest element, the *diameter* of A. Let f(n) be the smallest diameter of any set A with property P. Show that $n^2/4 \le f(n) < n^3$.

A3. Let R be the reals, show that there are no functions f, g: R → R such that $g(f(x)) = x^3$ and $f(g(x)) = x^2$ for all x. Let S be the set of all real numbers > 1. Show that there are functions f, g : S → S satsfying the condition above.

B1. Let F_n be the Fibonacci sequence $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$. Put $V_n = \sqrt{(F_n^2 + F_{n+2}^2)}$. Show that V_n, V_{n+1}, V_{n+2} are the sides of a triangle of area ½.

B2. c is a rational. Define $f^0(x) = x$, $f^{n+1}(x) = f(f^n(x))$. Show that there are only finitely many x such that the sequence $f^0(x), f^1(x), f^2(x), \ldots$ takes only finitely many values.

B3. f is a map on the plane such that two points a distance 1 apart are always taken to two points a distance 1 apart. Show that for any d, f takes two points a distance d apart to two points a distance d apart.

20th Brasil 1998

A1. 15 positive integers < 1998 are relatively prime (no pair has a common factor > 1). Show that at least one of them must be prime.

A2. ABC is a triangle. D is the midpoint of AB, E is a point on the side BC such that BE = 2 EC and $\angle$ADC = $\angle$BAE. Find $\angle$BAC.

A3. Two players play a game as follows. There $n > 1$ rounds and $d \geq 1$ is fixed. In the first round A picks a positive integer m_1, then B picks a positive integer $n_1 \neq m_1$. In round k (for k = 2, ... , n), A picks an integer m_k such that $m_{k-1} < m_k \leq m_{k-1} + d$. Then B picks an integer n_k such that $n_{k-1} < n_k \leq n_{k-1} + d$. A gets $\gcd(m_k,n_{k-1})$ points and B gets $\gcd(m_k,n_k)$ points. After n rounds, A wins if he has at least as many points as B, otherwise he loses. For each n, d which player has a winning strategy?

B1. Two players play a game as follows. The first player chooses two non-zero integers A and B. The second player forms a quadratic with A, B and 1998 as coefficients (in any order). The first player wins iff the equation has two distinct rational roots. Show that the first player can always win.

B2. Let N = {0, 1, 2, 3, ... }. Find all functions $f : N \to N$ which satisfy $f(2f(n)) = n + 1998$ for all n.

B3. Two mathematicians, lost in Berlin, arrived on the corner of Barbarossa street with Martin Luther street and need to arrive on the corner of Meininger street with Martin Luther street. Unfortunately they don't know which direction to go along Martin Luther Street to reach Meininger Street nor how far it is, so they must go fowards and backwards along Martin Luther street until they arrive on the desired corner. What is the smallest value for a positive integer K so that they can be sure that if there are N blocks between Barbarossa street and Meininger street then they can arrive at their destination by walking no more than KN blocks (no matter what N turns out to be)?

21st Brasil 1999

A1. ABCDE is a regular pentagon. The star ACEBD has area 1. AC and BE meet at P, BD and CE meet at Q. Find the area of APQD.

A2. Let d_n be the nth decimal digit of √2. Show that d_n cannot be zero for all of n = 1000001, 1000002, 1000003, ... , 3000000.

A3. How many pieces can be placed on a 10 x 10 board (each at the center of its square, at most one per square) so that no four pieces form a rectangle with sides parallel to the sides of the board?

B1. A spherical planet has finitely many towns. If there is a town at X, then there is also a town at X', the antipodal point. Some pairs of towns are connected by direct roads. No such roads cross (except at endpoints). If there is a direct road from A to B, then there is also a direct road from A' to B'. It is possible to get from any town to any other town by some sequence of roads. The populations of two towns linked by a direct road differ by at most 100. Show that there must be two antipodal towns whose populations differ by at most 100.

B2. n teams wish to play n(n-1)/2 games so that each team plays every other team just once. No team may play more than once per day. What is the minimum number of days required for the tournament?

B3. Given any triangle ABC, show how to construct A' on the side AB, B' on the side BC, C' on the side CA, so that ABC and A'B'C' are similar (with ∠A = ∠A', ∠B = ∠B' and ∠C = ∠C').

22nd Brasil 2000

A1. A piece of paper has top edge AD. A line L from A to the bottom edge makes an angle x with the line AD. We want to trisect x. We take B and C on the vertical ege through A such that AB = BC. We then fold the paper so that C goes to a point C' on the line L and A goes to a point A' on the horizontal line through B. The fold takes B to B'. Show that AA' and AB' are the required trisectors.

A2. Let s(n) be the sum of all positive divisors of n, so s(6) = 12. We say n is *almost perfect* if s(n) = 2n - 1. Let mod(n, k) denote the residue of n modulo k (in other words, the remainder of dividing n by k). Put t(n) = mod(n, 1) + mod(n, 2) + ... + mod(n, n). Show that n is almost perfect iff t(n) = t(n-1).

A3. Define f on the positive integers by $f(n) = k^2 + k + 1$, where 2^k is the highest power of 2 dividing n. Find the smallest n such that f(1) + f(2) + ... + f(n) ≥ 123456.

B1. An infinite road has traffic lights at intervals of 1500m. The lights are all synchronised and are alternately green for 3/2 minutes and red for 1 minute. For which v can a car travel at a constant speed of v m/s without ever going through a red light?

B2. X is the set of all sequences $a_1, a_2, \dots, a_{2000}$ such that each of the first 1000 terms is 0, 1 or 2, and each of the remaining terms is 0 or 1. The distance between two members a and b of X is defined as the number of i for which a_i and b_i are unequal. Find the number of functions f : X → X which preserve distance.

B3. C is a wooden cube. We cut along every plane which is perpendicular to the segment joining two distinct vertices and bisects it. How many pieces do we get?

23rd Brasil 2001

A1. Prove that $(a + b)(a + c) \geq 2(abc(a + b + c))^{1/2}$ for all positive reals.

A2. Given $a_0 > 1$, the sequence $a_0, a_1, a_2, \ldots$ is such that for all $k > 0$, a_k is the smallest integer greater than a_{k-1} which is relatively prime to all the earlier terms in the sequence. Find all a_0 for which all terms of the sequence are primes or prime powers.

A3. ABC is a triangle. The points E and F divide AB into thirds, so that AE = EF = FB. D is the foot of the perpendicular from E to the line BC, and the lines AD and CF are perpendicular. $\angle ACF = 3 \angle BDF$. Find DB/DC.

B1. A calculator treats angles as radians. It initially displays 1. What is the largest value that can be achieved by pressing the buttons cos or sin a total of 2001 times? (So you might press cos five times, then sin six times and so on with a total of 2001 presses.)

B2. An *altitude* of a convex quadrilateral is a line through the midpoint of a side perpendicular to the opposite side. Show that the four altitudes are concurrent iff the quadrilateral is cyclic.

B3. A one-player game is played as follows. There is bowl at each integer on the x-axis. All the bowls are initially empty, except for that at the origin, which contains n stones. A move is either (A) to remove two stones from a bowl and place one in each of the two adjacent bowls, or (B) to remove a stone from each of two adjacent bowls and to add one stone to the bowl immediately to their left. Show that only a finite number of moves can be made and that the final position (when no more moves are possible) is independent of the moves made (for given n).

24th Brasil 2002

A1. Show that there is a set of 2002 distinct positive integers such that the sum of one or more elements of the set is never a square, cube, or higher power.

A2. ABCD is a cyclic quadrilateral and M a point on the side CD such that ADM and ABCM have the same area and the same perimeter. Show that two sides of ABCD have the same length.

A3. The squares of an m x n board are labeled from 1 to mn so that the squares labeled i and i+1 always have a side in common. Show that for some k the squares k and k+3 have a side in common.

B1. For any non-empty subset A of {1, 2, ... , n} define f(A) as the largest element of A minus the smallest element of A. Find ∑ f(A) where the sum is taken over all non-empty subsets of {1, 2, ... , n}.

B2. A finite collection of squares has total area 4. Show that they can be arranged to cover a square of side 1.

B3. Show that we cannot form more than 4096 binary sequences of length 24 so that any two differ in at least 8 positions.

25th Brasil 2003

A1. Find the smallest positive prime that divides $n^2 + 5n + 23$ for some integer n.

A2. Let S be a set with n elements. Take a positive integer k. Let A_1, A_2, ... A_k be any distinct subsets of S. For each i take $B_i = A_i$ or $S - A_i$. Find the smallest k such that we can always choose B_i so that □ $B_i = S$.

A3. ABCD is a parallelogram with perpendicular diagonals. Take points E, F, G, H on sides AB, BC, CD, DA respectively so that EF and GH are tangent to the incircle of ABCD. Show that EH and FG are parallel.

B1. Given a circle and a point A inside the circle, but not at its center. Find points B, C, D on the circle which maximise the area of the quadrilateral ABCD.

B2. f(x) is a real-valued function defined on the positive reals such that (1) if $x < y$, then $f(x) < f(y)$, (2) $f(2xy/(x+y)) \geq (f(x) + f(y))/2$ for all x. Show that $f(x) < 0$ for some value of x.

B3. A graph G with n vertices is called *great* if we can label each vertex with a different positive integer $\leq [n^2/4]$ and find a set of non-negative integers D so that there is an edge between two vertices iff the difference between their labels is in D. Show that if n is sufficiently large we can always find a graph with n vertices which is not great.

CanMO (1969 – 2003)

1st CanMO 1969

1. a, b, c, d, e, f are reals such that a/b = c/d = e/f; p, q, r are reals, not all zero; and n is a positive integer. Show that $(a/b)^n = (p\,a^n + q\,c^n + r\,e^n)/(p\,b^n + q\,d^n + r\,f^n)$.

2. If x is a real number not less than 1, which is larger: $\sqrt{x+1} - \sqrt{x}$ or $\sqrt{x} - \sqrt{x-1}$?

3. A right-angled triangle has longest side c and other side lengths a and b. Show that $a + b \le c\sqrt{2}$. When do we have equality?

4. The sum of the distances from a point inside an equilateral triangle of perimeter length p to the sides of the triangle is s. Show that $s\sqrt{12} = p$.

5. ABC is a triangle with |BC| = a, |CA| = b. Show that the length of the angle bisector of C is (2ab cos C/2)/(a + b).

6. Find 1.1! + 2.2! + ... + n.n! .

7. Show that there are no integer solutions to $a^2 + b^2 = 8c + 6$.

8. f is a function defined on the positive integers with integer values. Given that (1) f(2) = 2, (2) f(mn) = f(m) f(n) for all m,n, and (3) f(m) > f(n) for all m, n such that m > n, show that f(n) = n for all n.

9. Show that the shortest side of a cyclic quadrilateral with circumradius 1 is at most $\sqrt{2}$.

10. P is a point on the hypoteneuse of an isosceles, right-angled triangle. Lines are drawn through P parallel to the other two sides, dividing the triangle into two smaller triangles and a rectangle. Show that the area of one of these component figures is at least 4/9 of the area of the original triangle.

2nd CanMO 1970

1. Find all triples of real numbers such that the product of any two of the numbers plus the third is 2.

2. The triangle ABC has angle A > 90^o. The altitude from A is AD and the altitude from B is BE. Show that BC + AD ≥ AC + BE. When do we have equality?

3. Every ball in a collection is one of two colors and one of two weights. There is at least one of each color and at least one of each weight. Show that there are two balls with different color and different weight.

4. Find all positive integers whose first digit is 6 and such that the effect of deleting the first digit is to divide the number by 25. Show that there is no positive integer such that the deletion of its first digit divides it by 35.

5. A quadrilateral has one vertex on each side of a square side 1. Show that the sum of the squares of its sides is at least 2 and at most 4.

6. Given three non-collinear points O, A, B show how to construct a circle center O such that the tangents from A and B are parallel.

7. Given any sequence of five integers, show that three terms have sum divisible by 3.

8. P lies on the line y = x and Q lies on the line y = 2x. Find the locus for the midpoint of PQ, if |PQ| = 4.

9. Let $a_1 = 0$, $a_{2n+1} = a_{2n} = n$. Let $s(n) = a_1 + a_2 + ... + a_n$. Find a formula for s(n) and show that s(m + n) = mn + s(m - n) for m > n.

10. A monic polynomial p(x) with integer coefficients takes the value 5 at four distinct integer values of x. Show that it does not take the value 8 at any integer value of x.

3rd CanMO 1971

1. A diameter and a chord of a circle intersect at a point inside the circle. The two parts of the chord are length 3 and 5 and one part of the diameter is length 1. What is the radius of the circle?

2. If two positive real numbers x and y have sum 1, show that $(1 + 1/x)(1 + 1/y) \geq 9$.

3. ABCD is a quadrilateral with AB = CD and $\angle ABC > \angle BCD$. Show that AC > BD.

4. Find all real a such that $x^2 + ax + 1 = x^2 + x + a = 0$ for some real x.

5. A polynomial with integral coefficients has odd integer values at 0 and 1. Show that it has no integral roots.

6. Show that $n^2 + 2n + 12$ is not a multiple of 121 for any integer n.

7. Find all five digit numbers such that the number formed by deleting the middle digit divides the original number.

8. Show that the sum of the lengths of the perpendiculars from a point inside a regular pentagon to the sides (or their extensions) is constant. Find an expression for it in terms of the circumradius.

9. Find the locus of all points in the plane from which two flagpoles appear equally tall. The poles are heights h and k and are a distance 2a apart.

10. n people each have exactly one unique secret. How many phone calls are needed so that each person knows all n secrets? You should assume that in each phone call the caller tells the other person every secret he knows, but learns nothing from the person he calls.

4th CanMO 1972

1. Three unit circles are arranged so that each touches the other two. Find the radii of the two circles which touch all three.

2. $x_1, x_2, \ldots, x_n$ are non-negative reals. Let $s = \sum_{i<j} x_i x_j$. Show that at least one of the x_i has square not exceeding $2s/(n^2 - n)$.

3. Show that 10201 is composite in base $n > 2$. Show that 10101 is composite in any base.

4. Show how to construct a convex quadrilateral ABCD given the lengths of each side and the fact that AB is parallel to CD.

5. Show that there are no positive integers m, n such that $m^3 + 11^3 = n^3$.

6. Given any distinct real numbers x, y, show that we can find integers m, n such that $mx + ny > 0$ and $nx + my < 0$.

7. Show that the roots of $x^2 - 198x + 1$ lie between 1/198 and 197.9949494949... . Hence show that $\sqrt{2} < 1.41$*421356* (where the digits 421356 repeat). Is it true that $\sqrt{2} < 1.41421356$?

8. X is a set with n elements. Show that we cannot find more than 2^{n-1} subsets of X such that every pair of subsets has non-empty intersection.

9. Given two pairs of parallel lines, find the locus of the point the sum of whose distances from the four lines is constant.

10. Find the longest possible geometric progression in {100, 101, 102, ... , 1000}.

5th CaMO 1973

1. (1) For what x do we have x < 0 and x < 1/(4x) ? (2) What is the greatest integer n such that 4n + 13 < 0 and n(n+3) > 16? (3) Give an example of a rational number between 11/24 and 6/13. (4) Express 100000 as a product of two integers which are not divisible by 10. (5) Find $1/\log_2 36 + 1/\log_3 36$.

2. Find all real numbers x such that x + 1 = |x + 3| - |x - 1|.

3. Show that if p and p+2 are primes then p = 3 or 6 divides p+1.

4. Let $P_0, P_1, \ldots, P_8$ be a convex 9-gon. Draw the diagonals P_0P_3, P_0P_6, P_0P_7, P_1P_3, P_4P_6, thus dividing the 9-gon into seven triangles. How many ways can we label these triangles from 1 to 7, so that P_n belongs to triangle n for n = 1, 2, ... , 7.

5. Let s(n) = 1 + 1/2 + 1/3 + ... + 1/n. Show that s(1) + s(2) + ... + s(n-1) = n s(n) - n.

6. C is a circle with chord AB (not a diameter). XY is any diameter. Find the locus of the intersection of the lines AX and BY.

7. Let $a_n = 1/(n(n+1))$. (1) Show that $1/n = 1/(n+1) + a_n$. (2) Show that for any integer n > 1 there are positive integers r < s such that $1/n = a_r + a_{r+1} + \ldots + a_s$.

6th CanMO 1974

1. (1) given $x = (1 + 1/n)^n$, $y = (1 + 1/n)^{n+1}$, show that $x^y = y^x$. (2) Show that $1^2 - 2^2 + 3^2 - 4^2 + ... + (-1)^{n+1}n^2 = (-1)^{n+1}(1 + 2 + ... + n)$.

2. Given the points A (0, 1), B (0, 0), C (1, 0), D (2, 0), E (3, 0), F (3, 1). Show that angle FBE + angle FCE = angle FDE.

3. All coefficients of the polynomial p(x) are non-negative and none exceed p(0). If p(x) has degree n, show that the coefficient of x^{n+1} in $p(x)^2$ is at most $p(1)^2/2$.

4. What is the maximum possible value for the sum of the absolute values of the differences between each pair of n non-negative real numbers which do not exceed 1?

5. AB is a diameter of a circle. X is a point on the circle other than the midpoint of the arc AB. BX meets the tangent at A at P, and AX meets the tangent at B at Q. Show that the line PQ, the tangent at X and the line AB are concurrent.

6. What is the largest integer n which cannot be represented as 8a + 15b with a and b non-negative integers?

7. Bus A leaves the terminus every 20 minutes, it travels a distance 1 mile to a circular road of length 10 miles and goes clockwise around the road, and then back along the same road to to the terminus (a total distance of 12 miles). The journey takes 20 minutes and the bus travels at constant speed. Having reached the terminus it immediately repeats the journey. Bus B does the same except that it leaves the terminus 10 minutes after Bus A and travels the opposite way round the circular road. The time taken to pick up or set down passengers is negligible. A man wants to catch a bus a distance $0 < x < 12$ miles from the terminus (along the route of Bus A). Let f(x) the maximum time his journey can take (waiting time plus journey time to the terminus). Find f(2) and f(4). Find the value of x for which f(x) is a maximum. Sketch f(x).

7th CanMO 1975

1. Evaluate $(1\cdot2\cdot4 + 2\cdot4\cdot8 + 3\cdot6\cdot12 + 4\cdot8\cdot16 + ... + n\cdot2n\cdot4n)^{1/3}/(1\cdot3\cdot9 + 2\cdot6\cdot18 + 3\cdot9\cdot27 + 4\cdot12\cdot36 + ... + n\cdot3n\cdot9n)^{1/3}$.

2. Define the real sequence $a_1, a_2, a_3, ...$ by $a_1 = 1/2$, $n^2a_n = a_1 + a_2 + ... + a_n$. Evaluate a_n.

3. Sketch the points in the x, y plane for which $[x]^2 + [y]^2 = 4$.

4. Find all positive real x such that x - [x], [x], x form a geometric progression.

5. Four points on a circle divide it into four arcs. The four midpoints form a quadrilateral. Show that its diagonals are perpendicular.

6. 15 guests with different names sit down at a circular table, not realizing that there is a name card at each place. Everyone is in the wrong place. Show that the table can be rotated so that at least two guests match their name cards. Give an example of an arrangement where just one guest is correct, but rotating the table does not improve the situation.

7. Is $\sin(x^2)$ periodic?

8. Find all real polynomials p(x) such that $p(p(x)) = p(x)^n$ for some positive integer n.

8th CanMO 1976

1. Given four unequal weights in geometric progression, show how to find the heaviest weight using a balance twice.

2. The real sequence $x_0, x_1, x_2, \ldots$ is defined by $x_0 = 1$, $x_1 = 2$, $n(n+1) x_{n+1} = n(n-1) x_n - (n-2) x_{n-1}$. Find $x_0/x_1 + x_1x_2 + \ldots + x_{50}/x_{51}$.

3. n+2 students played a tournament. Each pair played each other once. A player scored 1 for a win, 1/2 for a draw and nil for a loss. Two students scored a total of 8 and the other players all had equal total scores. Find n.

4. C lies on the segment AB. P is a variable point on the circle with diameter AB. Q lies on the line CP on the opposite side of C to P such that PC/CQ = AC/CB. Find the locus of Q.

5. Show that a positive integer is a sum of two or more consecutive positive integers iff it is not a power of 2.

6. The four points A, B, C, D in space are such that the angles ABC, BCD, CDA, DAB are all right angles. Show that the points are coplanar.

7. $p(x, y)$ is a symmetric polynomial with the factor $(x - y)$. Show that $(x - y)^2$ is a factor.

8. A graph has 9 points and 36 edges. Each edge is colored red or blue. Every triangle has a red edge. Show that there are four points with all edges between them red.

9th CanMO 1977

1. Show that there are no positive integers m, n such that $4m(m+1) = n(n+1)$.

2. X is a point inside a circle center O other than O. Which points P on the circle maximise $\angle OPX$?

3. Find the smallest positive integer b for which $7 + 7b + 7b^2$ is a fourth power.

4. The product of two polynomials with integer coefficients has all its coefficients even, but at least one not divisible by 4. Show that one of the two polynomials has all its coefficients even and that the other has at least one odd coefficient.

5. A right circular cone has base radius 1. The vertex is K. P is a point on the circumference of the base. The distance KP is 3. A particle travels from P around the cone and back by the shortest route. What is its minimum distance from K?

6. The real sequence $x_1, x_2, x_3, \ldots$ is defined by $x_1 = 1 + k$, $x_{n+1} = 1/x_n + k$, where $0 < k < 1$. Show that every term exceeds 1.

7. Given m+1 equally spaced horizontal lines and n+1 equally spaced vertical lines forming a rectangular grid with (m+1)(n+1) nodes. Let f(m, n) be the number of paths from one corner to the opposite corner along the grid lines such that the path does not visit any node twice. Show that $f(m, n) \le 2^{mn}$.

10th CanMO 1978

1. A square has tens digit 7. What is the units digit?

2. Find all positive integers m, n such that $2m^2 = 3n^3$. **3.** Find the real solution x, y, z to x + y + z = 5, xy + yz + zx = 3 with the largest z.

4. ABCD is a convex quadrilateral with area 1. The lines AD, BC meet at X. The midpoints of the diagonals AC and BD are Y and Z. Find the area of the triangle XYZ.

5. Two players play a game on an initially empty 3 x 3 board. Each player in turn places a black or white piece on an unoccupied square of the board. Each player may play either color. When the board is full player A gets one point for every row, column or main diagonal with 0 or 2 black pieces on it. Player B gets one point for every row, column or main diagonal with 1 or 3 black pieces on it. Can the game end in a draw? Which player has a winning strategy if player A plays first? If player B plays first?

6. Sketch the graph of $x^3 + xy + y^3 = 3$.

11th CanMO 1979

1. If $a > b > c > d$ is an arithmetic progression of positive reals and $a > h > k > d$ is a geometric progression of positive reals, show that $bc \geq hk$.

2. Show that two tetrahedra do not necessarily have the same sum for their dihedral angles.

3. Given five distinct integers greater than one, show that the sum of the inverses of the four lowest common multiples of the adjacent pairs is at most 15/16. [Two of the numbers are adjacent if none of the others lies between them.]

4. A dog is standing at the center of a circular yard. A rabbit is at the edge. The rabbit runs round the edge at constant speed v. The dog runs towards the rabbit at the same speed v, so that it always remains on the line between the center and the rabbit. Show that it reaches the rabbit when the rabbit has run one quarter of the way round.

5. The lattice is the set of points (x, y) in the plane with at least one coordinate integral. Let f(n) be the number of walks of n steps along the lattice starting at the origin. Each step is of unit length from one intersection point to another. We only count walks which do not visit any point more than once. Find f(n) for n =1, 2, 3, 4 and show that $2^n < f(n) \leq 4\cdot 3^{n-1}$.

12th CanMO 1980

1. If the 5-digit decimal number a679b is a multiple of 72 find a and b.

2. The numbers 1 to 50 are arranged in an arbitrary manner into 5 rows of 10 numbers each. Then each row is rearranged so that it is in increasing order. Then each column is arranged so that it is in increasing order. Are the rows necessarily still in increasing order?

3. Find the triangle with given angle A and given inradius r with the smallest perimeter.

4. A fair coin is tossed repeatedly. At each toss 1 is scored for a head and 2 for a tail. Show that the probability that at some point the score is n is $(2 + (-1/2)^n)/3$.

5. Do any polyhedra other than parallelepipeds have the property that all cross sections parallel to any given face have the same perimeter?

13th CanMO 1981

1. Show that there are no real solutions to [x] + [2x] + [4x] + [8x] + [16x] + [32x] = 12345.

2. The circle C has radius 1 and touches the line L at P. The point X lies on C and Y is the foot of the perpendicular from X to L. Find the maximum possible value of area PXY (as X varies).

3. Given a finite set of lines in the plane, show that an arbitrarily large circle can be drawn which does not meet any of them. Show that there is a countable set of lines in the plane such that any circle with positive radius meets at least one of them.

4. p(x) and q(x) are real polynomials such that p(q(x)) = q(p(x)) and p(x) = q(x) has no real solutions. Show that p(p(x)) = q(q(x)) has no real solutions.

5. 11 groups perform at a festival. Each day any groups not performing watch the others (but groups performing that day do not watch the others). What is the smallest number of days for which the festival can last if every group watches every other group at least once during the festival?

14th CanMO 1982

1. Given a quadrilateral ABCD and a point P, take A' so that PA' is parallel to AB and of equal length. Similarly take PB', PC', PD' equal and parallel to BC, CD, DA respectively. Show that the area of A'B'C'D' is twice that of ABCD.

2. Show that the roots of $x^3 - x^2 - x - 1$ are all distinct. If the roots are a, b, c show that $(a^{1982} - b^{1982})/(a - b) + (b^{1982} - c^{1982})/(b - c) + (c^{1982} - a^{1982})/(c - a)$ is an integer.

3. What is the smallest number of points in n-dimensional space R^n such that every point of R^n is an irrational distance from at least one of the points.

4. Show that the number of permutations of 1, 2, ... , n with no fixed points is one different from the number with exactly one fixed point.

5. Let the altitudes of a tetrahedron ABCD be AA', BB', CC', DD' (so that A' lies in the plane BCD and similarly for B', C', D'). Take points A", B", C", D" on the rays AA', BB', CC', DD' respectively so that AA'·AA" = BB'·BB" = CC'·CC" = DD'·DD". Show that the centroid of A"B"C"D" is the same as the centroid of ABCD.

15th CanMO 1983

1. Find all solutions to n! = a! + b! + c! .

2. Find all real-valued functions f on the reals whose graphs remain unchanged under all transformations $(x, y) \to (2^k x, 2^k(kx + y))$, where k is real.

3. Is the volume of a tetrahedron determined by the areas of its faces?

4. Show that we can find infinitely many positive integers n such that $2^n - n$ is a multiple of any given prime p.

5. Show that the geometric mean of a set S of positive numbers equals the geometric mean of the geometric means of all non-empty subsets of S.

16th CanMO 1984

1. Show that the sum of 1984 consecutive positive integers cannot be a square.

2. You have keyring with n identical keys. You wish to color code the keys so that you can distinguish them. What is the smallest number of colors you need? [For example, you could use three colors for eight keys: R R R R G B R R. Starting with the blue key and moving away from the green key uniquely distinguishes each of the red keys.]

3. Show that there are infinitely many integers which have no zeros and which are divisible by the sum of their digits.

4. An acute-angled triangle has unit area. Show that there is a point inside the triangle which is at least $2/(3^{3/4})$ from any vertex.

5. Given any seven real numbers show we can select two, x and y, such that $0 \le (x - y)/(1 + xy) \le 1/\sqrt{3}$.

17th CanMO 1985

1. A triangle has sides 6, 8, 10. Show that there is a unique line which bisects the area and the perimeter.

2. Is there an integer which is doubled by moving its first digit to the end? [For example, 241 does not work because 412 is not 2 x 241.]

3. A regular 1985-gon is inscribed in a circle (so each vertex lies on the circle). Another regular 1985-gon is circumcribed about the same circle (so that each side touches the circle). Show that the sum of the perimeters of the two polygons is at least twice the circumference of the circle. [Assume tan x >= x for 0 <= x < 90 deg.]

4. Show that n! is divisible by 2^{n-1} iff n is a power of 2.

5. Define the real sequence $x_1, x_2, x_3, \ldots$ by $x_1 = k$, where $1 < k < 2$, and $x_{n+1} = x_n - x_n^2/2 + 1$. Show that $|x_n - \sqrt{2}| < 1/2^n$ for $n > 2$.

18th CanMO 1986

1. The triangle ABC has angle B = 90^o. The point D is taken on the ray AC, the other side of C from A, such that CD = AB. $\angle$CBD = 30^o. Find AC/CD.

2. Three competitors A, B, C compete in a number of sporting events. In each event a points is awarded for a win, b points for second place and c points for third place. There are no ties. The final score was A 22, B 9, C 9. B won the 100 meters. How many events were there and who came second in the high jump?

3. A chord AB of constant length slides around the curved part of a semicircle. M is the midpoint of AB, and C is the foot of the perpendicular from A onto the diameter. Show that angle ACM does not change.

4. Show that (1 + 2 + ... + n) divides ($1^k + 2^k + ... + n^k$) for k odd.

5. The integer sequence $a_1, a_2, a_3, ...$ is defined by $a_1 = 39$, $a_2 = 45$, $a_{n+2} = a_{n+1}^2 - a_n$. Show that infinitely many terms of the sequence are divisible by 1986.

19th CanMO 1987

1. Find all positive integer solutions to $n! = a^2 + b^2$ for $n < 14$.

2. Find all the ways in which the number 1987 can be written in another base as a three digit number with the digits having the same sum 25.

3. ABCD is a parallelogram. X is a point on the side BC such that ACD, ACX and ABX are all isosceles. Find the angles of the parallelogram.

4. n stationary people each fire a water pistol at the nearest person. They are arranged so that the nearest person is always unique. If n is odd, show that at least one person is not hit. Does one person always escape if n is even?

5. Show that $[\sqrt{(4n + 1)}] = [\sqrt{(4n + 2)}] = [\sqrt{(4n + 3)}] = [\sqrt{n} + \sqrt{(n + 1)}]$ for all positive integers n.

20th CanMO 1988

1. For what real values of k do $1988x^2 + kx + 8891$ and $8891x^2 + kx + 1988$ have a common zero?

2. Given a triangle area A and perimeter p, let S be the set of all points a distance 5 or less from a point of the triangle. Find the area of S.

3. Given $n > 4$ points in the plane, some of which are colored red and the rest black. No three points of the same color are collinear. Show that we can find three points of the same color, such that two of the points do not have a point of the opposite color on the segment joining them.

4. Define two integer sequences $a_0, a_1, a_2, \dots$ and $b_0, b_1, b_2, \dots$ as follows. $a_0 = 0$, $a_1 = 1$, $a_{n+2} = 4a_{n+1} - a_n$, $b_0 = 1$, $b_1 = 2$, $b_{n+2} = 4b_{n+1} - b_n$. Show that $b_n^2 = 3a_n^2 + 1$.

5. If S is a sequence of positive integers let p(S) be the product of the members of S. Let m(S) be the arithmetic mean of p(T) for all non-empty subsets T of S. S' is formed from S by appending an additional positive integer. If m(S) = 13 and m(S') = 49, find S'.

21st CanMO 1989

1. How many permutations of 1, 2, 3, ... , n have each number larger than all the preceding numbers or smaller than all the preceding numbers?

2. Each vertex of a right angle triangle of area 1 is reflected in the opposite side. What is the area of the triangle formed by the three reflected points?

3. Tranform a number by taking the sum of its digits. Start with 1989^{1989} and make four transformations. What is the result?

4. There are five ladders. There are also some ropes. Each rope attaches a rung of one ladder to a rung of another ladder. No ladder has two ropes attached to the same rung. A monkey starts at the bottom of each ladder and climbs. Each time it reaches a rope, it traverses the rope to the other ladder and continues climbing up the other ladder. Show that each monkey eventually reaches the top of a different ladder.

5. For every permutation $a_1, a_2, ... , a_n$ of 1, 2, 4, 8, ... , 2^{n-1} form the product of all n partial sums $a_1 + a_2 + ... + a_k$. Find the sum of the inverses of all these products.

22nd CanMO 1990

1. A competition is played amongst $n > 1$ players over d days. Each day one player gets a score of 1, another a score of 2, and so on up to n. At the end of the competition each player has a total score of 26. Find all possible values for (n, d).

2. $n(n + 1)/2$ distinct numbers are arranged at random into n rows. The first row has 1 number, the second has 2 numbers, the third has 3 numbers and so on. Find the probability that the largest number in each row is smaller than the largest number in each row with more numbers.

3. The feet of the perpendiculars from the intersection point of the diagonals of a convex cyclic quadrilateral to the sides form a quadrilateral q. Show that the sum of the lengths of each pair of opposite sides of q is equal.

4. A particle can travel at a speed of 2 meters/sec along the x-axis and 1 meter/sec elsewhere. Starting at the origin, which regions of the plane can the particle reach within 1 second.

5. N is the positive integers, R is the reals. The function $f : N \to R$ satisfies $f(1) = 1$, $f(2) = 2$ and $f(n+2) = f(n+2 - f(n+1)) + f(n+1 - f(n))$. Show that $0 \leq f(n+1) - f(n) \leq 1$. Find all n for which $f(n) = 1025$.

23rd CanMO 1991

1. Show that there are infinitely many solutions in positive integers to $a^2 + b^5 = c^3$.

2. Find the sum of all positive integers which have n 1s and n 0s when written in base 2.

3. Show that the midpoints of all chords of a circle which pass through a fixed point lie on another circle.

4. Can ten distinct numbers $a_1, a_2, b_1, b_2, b_3, c_1, c_2, d_1, d_2, d_3$ be chosen from {0, 1, 2, ... , 14}, so that the 14 differences $|a_1 - b_1|, |a_1 - b_2|, |a_1 - b_3|, |a_2 - b_1|, |a_2 - b_2|, |a_2 - b_3|, |c_1 - d_1|, |c_1 - d_2|, |c_1 - d_3|, |c_2 - d_1|, |c_2 - d_2|, |c_2 - d_3|, |a_1 - c_1|, |a_2 - c_2|$ are all distinct?

5. An equilateral triangle side n is divided into n^2 equilateral triangles side 1 by lines parallel to its sides. How many parallelograms can be formed from the small triangles? [For example, if n = 3, there are 15, nine composed of two small triangles and six of four.]

24th CanMO 1992

1. Show that n! is divisible by (1 + 2 + ... + n) iff n+1 is not an odd prime.

2. Show that $x(x - z)^2 + y(y - z)^2 \geq (x - z)(y - z)(x + y - z)$ for all non-negative reals x, y, z. When does equality hold?

3. ABCD is a square. X is a point on the side AB, and Y is a point on the side CD. AY meets DX at R, and BY meets CX at S. How should X and Y be chosen to maximise the area of the quadrilateral XRYS?

4. Find all real solutions to $x^2(x + 1)^2 + x^2 = 3(x + 1)^2$.

5. There are 2n+1 cards. There are two cards with each integer from 1 to n and a joker. The cards are arranged in a line with the joker in the center position (with n cards each side of it). For which n < 11 can we arrange the cards so that the two cards with the number k have just k-1 cards between them (for k = 1, 2, ... , n)?

25th CanMO 1993

1. Show that there is a unique triangle such that (1) the sides and an altitude have lengths with are 4 consecutive integers, and (2) the foot of the altitude is an integral distance from each vertex.

2. Show that the real number k is rational iff the sequence k, k + 1, k + 2, k + 3, ... contains three (distinct) terms which form a geometric progression.

3. The medians from two vertices of a triangle are perpendicular, show that the sum of the cotangent of the angles at those vertices is at least 2/3.

4. Several schools took part in a tournament. Each player played one match against each player from a different school and did not play anyone from the same school. The total number of boys taking part differed from the total number of girls by 1. The total number of matches with both players of the same sex differed by at most one from the total number of matches with players of opposite sex. What is the largest number of schools that could have sent an odd number of players to the tournament?

5. A sequence of positive integers a_1, a_2, a_3, ... is defined as follows. $a_1 = 1$, $a_2 = 3$, $a_3 = 2$, $a_{4n} = 2a_{2n}$, $a_{4n+1} = 2a_{2n} + 1$, $a_{4n+2} = 2a_{2n+1} + 1$, $a_{4n+3} = 2a_{2n+1}$. Show that the sequence is a permutation of the positive integers.

26th CanMO 1994

1. Find -3/1! + 7/2! - 13/3! + 21/4! - 31/5! + ... + (1994^2 + 1994 + 1)/1994!

2. Show that every power of (√2 - 1) can be written in the form √(k+1) - √k.

3. 25 people sit in circle. They vote for or against an issue every hour. Each person changes his vote iff his vote was different from both his neighbours on the previous vote. Show that after a while no one's vote changes.

4. AB is the diameter of a circle. C is a point not on the line AB. The line AC cuts the circle again at X and the line BC cuts the circle again at Y. Find cos ACB in terms of CX/CA and CY/CB.

5. ABC is an acute-angled triangle. K is a point inside the triangle on the altitude AD. The line BK meets AC at Y, and the line CK meets AB at Z. Show that ∠ ADY = ∠ ADZ.

27th CanMO 1995

1. Find g(1/1996) + g(2/1996) + g(3/1996) + ... + g(1995/1996) where $g(x) = 9^x/(3 + 9^x)$.

2. Show that $x^x y^y z^z >= (xyz)^{(x+y+z)/3}$ for positive reals x, y, z.

3. A convex n-gon is divided into m quadrilaterals. Show that at most m - n/2 + 1 of the quadrilaterals have an angle exceeding 180^o.

4. Show that for any n > 0 and k ≥ 0 we can find infinitely many solutions in positive integers to $x_1^3 + x_2^3 + ... + x_n^3 = y^{3k+2}$.

5. 0 < k < 1 is a real number. Define f: [0, 1] → [0, 1] by f(x) = 0 for x ≤ k, $1 - (\sqrt{(kx)} + \sqrt{((1-k)(1-x))})^2$ for x > k. Show that the sequence 1, f(1), f(f(1)), f(f(f(1))), ... eventually becomes zero.

28th CanMO 1996

1. The roots of $x^3 - x - 1 = 0$ are r, s, t. Find $(1 + r)/(1 - r) + (1 + s)/(1 - s) + (1 + t)/(1 - t)$.

2. Find all real solutions to the equations $x = 4z^2/(1 + 4z^2)$, $y = 4x^2/(1 + 4x^2)$, $z = 4y^2/(1 + 4y^2)$.

3. Let N be the number of permutations of 1, 2, 3, ... , 1996 in which 1 is fixed and each number differs from its neighbours by at most 2. Is N divisible by 3?

4. In the triangle ABC, AB = AC and the bisector of angle B meets AC at E. If BC = BE + EA find angle A.

5. Let $x_1, x_2, \ldots, x_m$ be positive rationals with sum 1. What is the maximum and minimum value of $n - [n x_1] - [n x_2] - \ldots - [n x_m]$ for positive integers n?

29th CanMO 1997

1. How many pairs of positive integers have greatest common divisor 5! and least common multiple 50! ?

2. A finite number of closed intervals of length 1 cover the interval [0, 50]. Show that we can find a subset of at least 25 intervals with every pair disjoint.

3. Show that 1/44 > (1/2)(3/4)(5/6) ... (1997/1998) > 1/1999.

4. Two opposite sides of a parallelogram subtend supplementary angles at a point inside the parallelogram. Show that the line joining the point to a vertex subtends equal angles at the two adjacent vertices.

5. Find $\sum_{0\le k\le n} (-1)^k$ nCk /($k^3 + 9k^2 + 26k + 24$), where nCk is the binomial coefficient n!/(k! (n-k)!).

30th CanMO 1998

1. How many real x satisfy x = [x/2] + [x/3] + [x/5]?

2. Find all real x equal to √(x - 1/x) + √(1 - 1/x).

3. Show that if n > 1 is an integer then (1 + 1/3 + 1/5 + ... + 1/(2n-1))/(n+1) > (1/2 + 1/4 + ... + 1/2n)/n.

4. The triangle ABC has ∠A = 40^o and ∠B = 60^o. X is a point inside the triangle such that ∠XBA = 20^o and ∠XCA = 10^o. Show that AX is perpendicular to BC.

5. Show that non-negative integers a <= b satisfy $(a^2 + b^2) = n^2(ab + 1)$, where n is a positive integer, iff they are consecutive terms in the sequence a_k defined by $a_0 = 0$, $a_1 = n$, $a_{k+1} = n^2a_k - a_{k-1}$.

31st CanMO 1999

1. Find all real solutions to the equation $4x^2 - 40[x] + 51 = 0$.

2. ABC is equilateral. A circle with center on the line through A parallel to BC touches the segment BC. Show that the length of arc of the circle inside ABC is independent of the position of the circle.

3. Find all positive integers which equal the square of their number of positive divisors.

4. X is a subset of eight elements of {1, 2, 3, ... , 17}. Show that there are three pairs of (distinct) elements with the same difference.

5. x, y, z are non-negative reals with sum 1, show that $x^2y + y^2z + z^2x \le 4/27$. When do we have equality?

32nd CanMO 2000

1. Three runners start together and run around a track length 3L at different constant speeds, not necessarily in the same direction (so, for example, they may all run clockwise, or one may run clockwise). Show that there is a moment when any given runner is a distance L or more from both the other runners (where distance is measured around the track in the shorter direction).

2. How many permutations of 1901, 1902, 1903, ... , 2000 are such that none of the sums of the first n permuted numbers is divisible by 3 (for n = 1, 2, 3, ... , 2000)?

3. Show that in any sequence of 2000 integers each with absolute value not exceeding 1000 such that the sequence has sum 1, we can find a subsequence of one or more terms with zero sum.

4. ABCD is a convex quadrilateral with AB = BC, $\angle$CBD = 2 $\angle$ADB, and $\angle$ABD = 2 $\angle$CDB. Show that AD = DC.

5. A non-increasing sequence of 100 non-negative reals has the sum of the first two terms at most 100 and the sum of the remaining terms at most 100. What is the largest possible value for the sum of the squares of the terms?

33rd CanMO 2001

1. A quadratic with integral coefficients has two distinct positive integers as roots, the sum of its coefficients is prime and it takes the value -55 for some integer. Show that one root is 2 and find the other root.

2. The numbers -10, -9, -8, ... , 9, 10 are arranged in a line. A player places a token on the 0 and throws a fair coin 10 times. For each head the token is moved one place to the left and for each tail it is moved one place to the left. If we color one or more numbers black and the remainder white, we find that the chance of the token ending up on a black number is m/n with m + n = 2001. What is the largest possible total for the black numbers?

3. The triangle ABC has AB and AC unequal. The angle bisector of A meets the perpendicular bisector of BC at X. The line joining the feet of the perpendiculars from X to AB and AC meets BC at D. Find BD/DC.

4. A rectangular table has every entry a positive integer. n is a fixed positive integer. A move consists of either subtracting n from every element in a column or multiplying every element in a row by n. Find all n such that we can always end up with all zeros whatever the size or content of the starting table.

5. A_0, A_1, A_2 lie on a circle radius 1 and A_1A_2 is not a diameter. The sequence A_n is defined by the statement that A_n is the circumcenter of $A_{n-1}A_{n-2}A_{n-3}$. Show that A_1, A_5, A_9, A_{13}, ... are collinear. Find all A_1A_2 for which $A_1A_{1001}/A_{1001}A_{2001}$ is the 500th power of an integer.

34th CanMO 2002

1. What is the largest possible number of elements in a subset of {1, 2, 3, ... , 9} such that the sum of every pair (of distinct elements) in the subset is different?

2. We say that the positive integer m satisfies condition X if every positive integer less than m is a sum of distinct divisors of m. Show that if m and n satisfy condition X, then so does mn.

3. Show that $x^3/(yz) + y^3/(zx) + z^3/(xy) \geq x + y + z$ for any positive reals x, y, z. When do we have equality?

4. ABC is an equilateral triangle. C lies inside a circle center O through A and B. X and Y are points on the circle such that AB = BX and C lies on the chord XY. Show that CY = AO.

5. Let X be the set of non-negative integers. Find all functions f: X → X such that $x f(y) + y f(x) = (x + y) f(x^2 + y^2)$ for all x, y.

35th CanMO 2003

1. The angle between the hour and minute hands of a standard 12-hour clock is exactly 1^o. The time is an integral number n of minutes after noon (where $0 < n < 720$). Find the possible values of n.

2. What are the last three digits of 2003^N, where $N = 2002^{2001}$.

3. Find all positive real solutions to $x^3 + y^3 + z^3 = x + y + z$, $x^2 + y^2 + z^2 = xyz$.

4. Three fixed circles pass through the points A and B. X is a variable point on the first circle different from A and B. The line AX meets the other two circles at Y and Z (with Y between X and Z). Show that XY/YZ is independent of the position of X.

5. S is any set of n distinct points in the plane. The shortest distance between two points of S is d. Show that there is a subset of at least n/7 points such that each pair is at least a distance $d\sqrt{3}$ apart.

Eötvös Competition (1894 – 2004)

1st Eötvös Competition 1894

1. Show that { (m, n): 17 divides 2m + 3n} = { (m, n): 17 divides 9m + 5n}.

2. Given a circle C, and two points A, B inside it, construct a right-angled triangle PQR with vertices on C and hypoteneuse QR such that A lies on the side PQ and B lies on the side PR. For which A, B is this not possible?

3. A triangle has sides length a, a + d, a + 2d and area S. Find its sides and angles in terms of d and S. Give numerical answers for d = 1, S = 6.

2nd Eötvös Competition 1895

1. n cards are dealt to two players. Each player has at least one card. How many possible hands can the first player have?

2. ABC is a right-angled triangle. Construct a point P inside ABC so that the angles PAB, PBC, PCA are equal.

3. A triangle ABC has sides BC = a, CA = b, AB = c. Given (1) the radius R of the circumcircle, (2) a, (3) t = b/c, determine b, c and the angles A, B, C.

3rd Eötvös Competition 1896

1. For a positive integer n, let p(n) be the number of prime factors of n. Show that $\ln n \geq p(n) \ln 2$.

2. Show that if (a, b) satisfies $a^2 - 3ab + 2b^2 + a - b = a^2 - 2ab + b^2 - 5a + 7b = 0$, then it also satisfies $ab - 12a + 15b = 0$.

3. Given three points P, Q, R in the plane, find points A, B, C such that P is the foot of the perpendicular from A to BC, Q is the foot of the perpendicular from B to CA, and R is the foot of the perpendicular from C to AB. Find the lengths AB, BC, CA in terms of PQ, QR and RP.

4th Eötvös Competition 1897

1. ABC is a right-angled triangle. Show that $\sin A \sin B \sin(A - B) + \sin B \sin C \sin(B - C) + \sin C \sin A \sin(C - A) + \sin(A - B) \sin(B - C) \sin(C - A) = 0$.

2. ABC is an arbitrary triangle. Show that $\sin(A/2) \sin(B/2) \sin(C/2) < 1/4$.

3. The line L contains the distinct points A, B, C, D in that order. Construct a rectangle whose sides (or their extensions) intersect L at A, B, C, D and such that the side which intersects L at C has length k. How many such rectangles are there?

5th Eötvös Competition 1898

1. For which positive integers n does 3 divide $2^n + 1$?

2. Triangles ABC, PQR satisfy (1) ∠A = ∠P, (2) |∠B - ∠C| < |∠Q - ∠R|. Show that sin A + sin B + sin C > sin P + sin Q + sin R. What angles A, B, C maximise sin A + sin B + sin C?

3. The line L contains the distinct points A, B, C, D in that order. Construct a square such that two opposite sides (or their extensions) intersect L at A, B, and the other two sides (or their extensions) intersect L at C, D.

6th Eötvös Competition 1899

1. ABCDE is a regular pentagon (with vertices in that order) inscribed in a circle of radius 1. Show that AB·AC = √5.

2. The roots of the quadratic $x^2 - (a + d) x + ad - bc = 0$ are α and β. Show that α^3 and β^3 are the roots of $x^2 - (a^3 + d^3 + 3abc + 3bcd) x + (ad - bc)^3 = 0$.

3. Show that $2903^n - 803^n - 464^n + 261^n$ is divisible by 1897.

7th Eötvös Competition 1900

1. d is not divisible by 5. For some integer n, $a n^3 + b n^2 + c n + d$ is divisible by 5. Show that for some integer m, $a + b m + c m^2 + d m^3$ is divisible by 5.

2. Construct the triangle ABC given c, r and r', where c = |AB|, r is the radius of the inscribed circle, and r' is the radius of the other circle tangent to the segment AB and the lines BC and CA.

3. Two particles fall from rest 300 m under the influence of gravity alone. One particle leaves when the other has already fallen 1 μm. How far apart are they when the first particle reaches the end point (to the nearest 100 μm)?

8th Eötvös Competition 1901

1. Show that 5 divides $1^n + 2^n + 3^n + 4^n$ iff 4 does not divide n.

2. Let α = cot π/8, β = cosec π/8. Show that α satisfies a quadratic and β a quartic, both with integral coefficients and leading coefficient 1.

3. Let d be the greatest common divisor of a and b. Show that exactly d elements of {a, 2a, 3a, ... , ba} are divisible by b.

9th Eötvös Competition 1902

1. Let $p(x) = ax^2 + bx + c$ be a quadratic with real coefficients. Show that we can find reals d, e, f so that $p(x) = d/2\ x(x - 1) + ex + f$, and that p(n) is always integral for integral n iff d, e, f are integers.

2. P is a variable point outside the fixed sphere S with center O. Show that the surface area of the sphere center P radius PO which lies inside S is independent of P.

3. The triangle ABC has area k and angle $A = \theta$, and is such that BC is as small as possible. Find AB and AC.

10th Eötvös Competition 1903

1. Prove that $2^{p-1}(2^p - 1)$ is perfect when $2^p - 1$ is prime. [A perfect number equals the sum of its (positive) divisors, excluding the number itself.]

2. α and β are real and $a = \sin \alpha$, $b = \sin \beta$, $c = \sin(\alpha+\beta)$. Find a polynomial p(x, y, z) with integer coefficients, such that p(a, b, c) = 0. Find all values of (a, b) for which there are less than four distinct values of c.

3. ABCD is a rhombus. C_A is the circle through B, C, D; C_B is the circle through A, C, D; C_C is the circle through A, B, D; and C_D is the circle through A, B, C. Show that the angle between the tangents to C_A and C_C at B equals the angle between the tangents to C_B and C_D at A.

11th Eötvös Competition 1904

1. A pentagon inscribed in a circle has equal angles. Show that it has equal sides.

2. Let a be an integer, and let $p(x_1, x_2, \ldots, x_n) = \sum_1^n k\, x_k$. Show that the number of integral solutions $(x_1, x_2, \ldots, x_n)$ to $p(x_1, x_2, \ldots, x_n) = a$, with all $x_i > 0$ equals the number of integral solutions $(x_1, x_2, \ldots, x_n)$ to $p(x_1, x_2, \ldots, x_n) = a - n(n + 1)/2$, with all $x_i \geq 0$.

3. R is a rectangle. Find the set of all points which are closer to the center of the rectangle than to any vertex.

12th Eötvös Competition 1905

1. For what positive integers m, n can we find positive integers a, b, c such that $a + mb = n$ and $a + b = m^c$. Show that there is at most one such solution for each m, n.

2. Divide the unit square into 9 equal squares (forming a 3 x 3 array) and color the central square red. Now subdivide each of the 8 uncolored squares into 9 equal squares and color each central square red. Repeat n times, so that the side length of the smallest squares is $1/3^n$. How many squares are uncolored? What is the total red area as $n \to \infty$?

3. ABC is a triangle and R any point on the segment AB. Let P be the intersection of the line BC and the line through A parallel to CR. Let Q be the intersection of the line AC and the line through B parallel to CR. Show that $1/AP + 1/BQ = 1/CR$.

13th Eötvös Competition 1906

1. Let α be a real number, not an odd multiple of π. Prove that tan α/2 is rational iff cos α and sin α are rational.

2. Show that the centers of the squares on the outside of the four sides of a rhombus form a square.

3. $(a_1, a_2, ..., a_n)$ is a permutation of (1, 2, ... , n). Show that $\prod (a_i - i)$ is even if n is odd.

14th Eötvös Competition 1907

1. Show that the quadratic $x^2 + 2mx + 2n$ has no rational roots for odd integers m, n.

2. Let r be the radius of a circle through three points of a parallelogram. Show that any point inside the parallelogram is a distance ≤ r from at least one of its vertices.

3. Show that the decimal expansion of a rational number must repeat from some point on. [In other words, if the fractional part of the number is $0.a_1a_2a_3 ...$, then $a_{n+k} = a_n$ for some k > 0 and all n > some n_0.]

15th Eötvös Competition 1908

1. m and n are odd. Show that 2^k divides $m^3 - n^3$ iff it divides m - n.

2. Let a right angled triangle have side lengths a > b > c. Show that for n > 2, $a^n > b^n + c^n$.

3. Let the vertices of a regular 10-gon be $A_1, A_2, ..., A_{10}$ in that order. Show that $A_1A_4 - A_1A_2$ is the radius of the circumcircle.

16th Eötvös Competition 1909

1. Prove that $(n + 1)^3 \neq n^3 + (n - 1)^3$ for any positive integer n.

2. α is acute. Show that α < (sin α + tan α)/2.

3. ABC is a triangle. The feet of the altitudes from A, B, C are P, Q, R respectively, and P, Q, R are distinct points. The altitudes meet at O. Show that if ABC is acute, then O is the center of the circle inscribed in the triangle PQR, and that A, B, C are the centers of the other three circles that touch all three sides of PQR (extended if necessary). What happens if ABC is not acute?

17th Eötvös Competition 1910

1. α, β, γ are real and satisfy $\alpha^2 + \beta^2 + \gamma^2 = 1$. Show that $-1/2 \le \alpha\beta + \beta\gamma + \gamma\alpha \le 1$.

2. If ac, bc + ad, bd = 0 (mod n) show that bc, ad = 0 (mod n).

3. ABC is a triangle with angle C = 120^o. Find the length of the angle bisector of angle C in terms of BC and CA.

18th Eötvös Competition 1911

1. Real numbers a, b, c, A, B, C satisfy $b^2 < ac$ and $aC - 2bB + cA = 0$. Show that $B^2 \ge AC$.

2. L_1, L_2, L_3, L_4 are diameters of a circle C radius 1 and the angle between any two is either $\pi/2$ or $\pi/4$. If P is a point on the circle, show that the sum of the fourth powers of the distances from P to the four diameters is 3/2.

3. Prove that $3^n + 1$ is not divisible by 2^n for n > 1.

19th Eötvös Competition 1912

1. How many n-digit decimal integers have all digits 1, 2 or 3. How many also contain each of 1, 2, 3 at least once?

2. Prove that $5^n + 2\,3^{n-1} + 1 = 0$ (mod 8).

3. ABCD is a quadrilateral with vertices in that order. Prove that AC is perpendicular to BD iff $AB^2 + CD^2 = BC^2 + DA^2$.

20th Eötvös Competition 1913

1. Prove that $n!\, n! > n^n$ for n > 2.

2. Let A and B be diagonally opposite vertices of a cube. Prove that the midpoints of the 6 edges not containing A or B form a regular (planar) hexagon.

3. If d is the greatest common divisor of a and b, and D is the greatest common divisor of A and B, show that dD is the greatest common divisor of aA, aB, bA and bB.

21st Eötvös Competition 1914

1. Circles C and C' meet at A and B. The arc AB of C' divides the area inside C into two equal parts. Show that its length is greater than the diameter of C.

2. a, b, c are reals such that $|ax^2 + bx + c| \le 1$ for all $x \le |1|$. Show that $|2ax + b| \le 4$ for all $|x| \le 1$.

3. A circle meets the side BC of the triangle ABC at A_1, A_2. Similarly, it meets CA at B_1, B_2, and it meets AB at C_1, C_2. The perpendicular to BC at A_1, the perpendicular to CA at B_1 and the perpendicular to AB at C_1 meet at a point. Show that the perpendiculars at A_2, B_2, C_2 also meet at a point.

22nd Eötvös Competition 1915

1. Given any reals A, B, C, show that $An^2 + Bn + C < n!$ for all sufficiently large integers n.

2. A triangle lies entirely inside a polygon. Show that its perimeter does not exceed the perimeter of the polygon.

3. Show that a triangle inscribed in a parallelogram has area at most half that of the parallelogram.

23rd Eötvös Competition 1916

1. a, b are positive reals. Show that $1/x + 1/(x-a) + 1/(x+b) = 0$ has two real roots one in [a/3, 2a/3] and the other in [-2b/3, -b/3].

2. ABC is a triangle. The bisector of $\angle C$ meets AB at D. Show that $CD^2 < CA \cdot CB$.

3. The set {1, 2, 3, 4, 5} is divided into two parts. Show that one part must contain two numbers and their difference.

24th Eötvös Competition 1917

1. a, b are integers. The solutions of $y - 2x = a$, $y^2 - xy + x^2 = b$ are rational. Show that they must be integers.

2. A square has 10s digit 7. What is its units digit?

3. A, B are two points inside a given circle C. Show that there are infinitely many circles through A, B which lie entirely inside C.

25th Eötvös Competition 1918

1. AC is the long diagonal of a parallelogram ABCD. The perpendiculars from C meet the lines AB and AD at P and Q respectively. Show that $AC^2 = AB\cdot AP + AD\cdot AQ$.

2. Find three distinct positive integers a, b, c such that 1/a + 1/b + 1/c is an integer.

3. The real quadratics $ax^2 + 2bx + c$ and $Ax^2 + 2Bx + C$ are non-negative for all real x. Show that $aAx^2 + 2bBx + cC$ is also non-negative for all real x.

26th Eötvös Competition 1922

1. Show that given four non-coplanar points A, B, P, Q there is a plane with A, B on one side and P, Q on the other, and with all the points the same distance from the plane.

2. Show that we cannot factorise $x^4 + 2x^2 + 2x + 2$ as the product of two quadratics with integer coefficients.

3. Let S be any finite set of distinct positive integers which are not divisible by any prime greater than 3. Prove that the sum of their reciprocals is less than 3.

27th Eötvös Competition 1923

1. The circles OAB, OBC, OCA have equal radius r. Show that the circle ABC also has radius r.

2. Let x be a real number and put y = (x + 1)/2. Put $a_n = 1 + x + x^2 + ... + x^n$, and $b_n = 1 + y + y^2 + ... + y^n$. Show that $\sum_0^n a_m\ (n+1)C(m+1) = 2^n\ b_n$, where aCb is the binomial coefficient a!/(b! (a-b)!).

3. Show that an infinite arithmetic progression of unequal terms cannot consist entirely of primes.

28th Eötvös Competition 1924

1. The positive integers a, b, c are such that there are triangles with sides a^n, b^n, c^n for all positive integers n. Show that at least two of a, b, c must be equal.

2. What is the locus of the point (in the plane), the sum of whose distances to a given point and line is fixed?

3. Given three points in the plane, how does one construct three distinct circles which touch in pairs at the three points?

29th Eötvös Competition 1925

1. Given four integers, show that the product of the six differences is divisible by 12.

2. How many zeros does the the decimal representation of 1000! end with?

3. Show that the inradius of a right-angled triangle is less than 1/4 of the length of the hypoteneuse and less than 1/2 the length of the shortest side.

30th Eötvös Competition 1926

1. Show that for any integers m, n the equations w + x + 2y + 2z = m, 2w - 2x + y - z = n have a solution in integers.

2. Show that the product of four consecutive integers cannot be a square.

3. A circle or radius R rolls around the inside of a circle of radius 2R, what is the path traced out by a point on its circumference?

31st Eötvös Competition 1927

1. a, b, c, d are each relatively prime to n = ad - bc, and r and s are integers. Show ar + bs is a multiple of n iff cr + ds is a multiple of n.

2. Find the sum of all four digit numbers (written in base 10) which contain only the digits 1 - 5 and contain no digit twice.

3. r is the inradius of the triangle ABC and r' is the exradius for the circle touching AB. Show that $4r\,r' \le c^2$, where c is the length of the side AB.

32nd Eötvös Competition 1928

1. Show that for any real x, at least one of x, 2x, 3x, ... , (n-1)x differs from an integer by no more than 1/n.

2. The numbers 1, 2, ... , n are arranged around a circle so that the difference between any two adjacent numbers does not exceed 2. Show that this can be done in only one way (treating rotations and reflections of an arrangement as the same arrangement).

3. Given two points A, B and a line L in the plane, find the point P on the line for which max(AP, BP) is as short as possible.

33rd Eötvös Competition 1929

1. Coins denomination 1, 2, 10, 20 and 50 are available. How many ways are there of paying 100?

2. Show that $\sum_{i=0 \text{ to } k}$ nCi $(-x)^i$ is positive for all $0 \le x < 1/n$ and all $k \le n$, where nCi is the binomial coefficient.

3. L, M, N are three lines through a point such that the angle between any pair is 60^o. Show that the set of points P in the plane of ABC whose distances from the lines L, M, N are less than a, b, c respectively is the interior of hexagon iff there is a triangle with sides a, b, c. Find the perimeter of this hexagon.

34th Eötvös Competition 1930

1. How many integers (1) have 5 decimal digits, (2) have last digit 6, and (3) are divisible by 3?

2. A straight line is drawn on an 8 x 8 chessboard. What is the largest possible number of the unit squares with interior points on the line?

3. An acute-angled triangle has circumradius R. Show that any interior point of the triangle other than the circumcenter is a distance > R from at least one vertex and a distance < R from at least one vertex.

35th Eötvös Competition 1931

1. Prove that there is just one solution in integers $m > n$ to $2/p = 1/m + 1/n$ for p an odd prime.

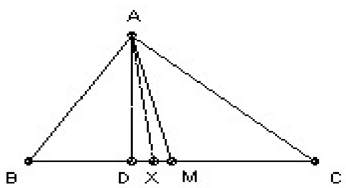
2. Show that an odd square cannot be expressed as the sum of five odd squares.

3. Find the point P on the line AB which maximizes $1/(AP + AB) + 1/(BP + AB)$.

36th Eötvös Competition 1932

1. Show that if m is a multiple of a^n, then $(a + 1)^m - 1$ is a multiple of a^{n+1}.

2. ABC is a triangle with AB and AC unequal. AM, AX, AD are the median, angle bisector and altitude. Show that X always lies between D and M, and that if the triangle is acute-angled, then angle MAX < angle DAX.



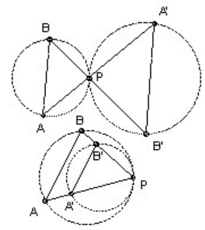
3. An acute-angled triangle has angles A < B < C. Show that sin 2A > sin 2B > sin 2C.

37th Eötvös Competition 1933

1. If $x^2 + y^2 = u^2 + v^2 = 1$ and $xu + yv = 0$ for real x, y, u, v, find $xy + uv$.

2. S is a set of 16 squares on an 8 x 8 chessboard such that there are just 2 squares of S in each row and column. Show that 8 black pawns and 8 white pawns can be placed on these squares so that there is just one white pawn and one black pawn in each row and column.

3. A and B are points on the circle C, which touches a second circle at a third point P. The lines AP and BP meet the second circle again at A' and B' respectively. Show that triangles ABP and A'B'P are similar.



38th Eötvös Competition 1934

1. E is the product 2·4·6 ... 2n, and D is the product 1·3·5 ... (2n-1). Show that, for some m, $D \cdot 2^m$ is a multiple of E.

2. Given a circle, find the inscribed polygon with the largest sum of the squares of its sides.

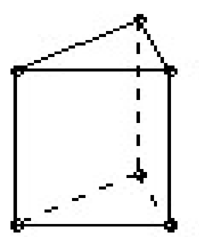
3. For i and j positive integers, let R_{ij} be the rectangle with vertices at (0, 0), (i, 0), (0, j), (i, j). Show that any infinite set of R_{ij} must have two rectangles one of which covers the other.

39th Eötvös Competition 1935

1. $x_1, x_2, \dots, x_n$ is any sequence of positive reals and y_i is any permutation of x_i. Show that $\sum x_i/y_i \geq n$.

2. S is a finite set of points in the plane. Show that there is at most one point P in the plane such that if A is any point of S, then there is a point A' in S with P the midpoint of AA'.

3. Each vertex of a triangular prism is labeled with a real number. If each number is the arithmetic mean of the three numbers on the adjacent vertices, show that the numbers are all equal.



40th Eötvös Competition 1936

1. Show that 1/(1·2) + 1/(3·4) + 1/(5·6) + ... + 1/((2n-1)·2n) = 1/(n+1) + 1/(n+2) + ... + 1/(2n).

2. ABC is a triangle. Show that, if the point P inside the triangle is such that the triangles PAB, PBC, PCA have equal area, then P must be the centroid.

3. Given any positive integer N, show that there is just one solution to m + ½(m + n - 1)(m + n - 2) = N in positive integers.

41st Eötvös Competition 1937

1. $a_1, a_2, \ldots, a_n$ is any finite sequence of positive integers. Show that $a_1!\ a_2!\ \ldots\ a_n! < (S + 1)!$ where $S = a_1 + a_2 + \ldots + a_n$.

2. P, Q, R are three points in space. The circle C_P passes through Q and R, the circle C_Q passes through R and P, and the circle C_R passes through P and Q. The tangents to C_Q and C_R at P coincide. Similarly, the tangents to C_R and C_P at Q coincide, and the tangents to C_P and C_Q at R coincide. Show that the circles are either coplanar or lie on the surface of the same sphere.

3. $A_1, A_2, \ldots, A_n$ are points in the plane, no three collinear. The distinct points P and Q in the plane do not coincide with any of the A_i and are such that $PA_1 + \ldots + PA_n = QA_1 + \ldots + QA_n$. Show that there is a point R in the plane such that $RA_1 + \ldots + RA_n < PA_1 + \ldots + PA_n$.

42nd Eötvös Competition 1938

1. Show that a positive integer n is the sum of two squares iff 2n is the sum of two squares.

2. Show that $1/n + 1/(n+1) + 1/(n+2) + \ldots + 1/n^2 > 1$ for integers $n > 1$.

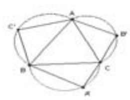
3. Show that for every acute-angled triangle ABC there is a point in space P such that (1) if Q is any point on the line BC, then AQ subtends an angle 90^o at P, (2) if Q is any point on the line CA, then BQ subtends an angle 90^o at P, and (3) if Q is any point on the line AB, then CQ subtends an angle 90^o at P.

43rd Eötvös Competition 1939

1. Show that $(a + a')(c + c') \geq (b + b')^2$ for all real numbers a, a', b, b', c, c' such that $aa' > 0$, $ac \geq b^2$, $a'c' \geq b'^2$.

2. Find the highest power of 2 dividing $2^n!$

3. ABC is acute-angled. A' is a point on the semicircle diameter BC (lying on the opposite side of BC to A). B' and C' are similar. Show how to construct such points so that AB' = AC', BC' = BA' and CA' = CB'.



44th Eötvös Competition 1940

1. Each button in a box is big or small, and white or black. There is at least one big button, at least one small button, at least one white button, and at least one black button. Show that there are two buttons with different size and color.

2. m and n are different positive integers. Show that $2^{2m} + 1$ and $2^{2n} + 1$ are coprime.

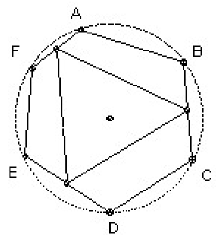
3. T is a triangle. Show that there is a triangle T' whose sides are equal to the medians of T, and that T'' is similar to T.

45th Eötvös Competition 1941

1. Prove that $(1+x)(1+x^2)(1+x^4) \dots (1+x^{2n-1}) = 1 + x + x^2 + x^3 + \dots + x^{2n-1}$.

2. The a parallelogram has its vertices at lattice points and there is at least one other lattice point inside the parallelogram or on its sides. Show that its area is greater than 1.

3. ABCDEF is a hexagon with vertices on a circle radius R (in that order). The three sides AB, CD, EF have length R. Show that the midpoints of BC, DE, FA form an equilateral triangle.

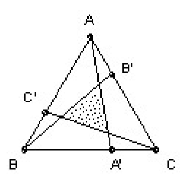


46th Eötvös Competition 1942

1. Show that no triangle has two sides each shorter than its corresponding altitude (from the opposite vertex).

2. a, b, c, d are integers. For all integers m, n we can find integers h, k such that ah + bk = m and ch + dk = n. Show that |ad - bc| = 1.

3. ABC is an equilateral triangle with area 1. A' is the point on the side BC such that BA' = 2·A'C. Define B' and C' similarly. Show that the lines AA', BB' and CC' enclose a triangle with area 1/7.



47th Eötvös Competition 1943

1. Show that a graph has an even number of points of odd degree.

2. P is any point inside an acute-angled triangle. D is the maximum and d is the minimum distance PX for X on the perimeter. Show that $D \geq 2d$, and find when $D = 2d$.

3. $x_1 < x_2 < x_3 < x_4$ are real. y_1, y_2, y_3, y_4 is any permutation of x_1, x_2, x_3, x_4. What are the smallest and largest possible values of $(y_1 - y_2)^2 + (y_2 - y_3)^2 + (y_3 - y_4)^2 + (y_4 - y_1)^2$.

48th Kürschák Competition 1947

1. Prove that $46^{2n+1} + 296 \cdot 13^{2n+1}$ is divisible by 1947.

2. Show that any graph with 6 points has a triangle or three points which are not joined to each other.

3. What is the smallest number of disks radius ½ that can cover a disk radius 1?

49th Kürschák Competition 1948

1. Knowing that 23 October 1948 was a Saturday, which is more frequent for New Year's Day, Sunday or Monday?

2. A convex polyhedron has no diagonals (every pair of vertices are connected by an edge). Prove that it is a tetrahedron.

3. Prove that among any n positive integers one can always find some (at least one) whose sum is divisible by n.

50th Kürschák Competition 1949

1. Prove that $\sin x + \frac{1}{2} \sin 2x + (1/3) \sin 3x > 0$ for $0 < x < 180^o$.

2. P is a point on the base of an isosceles triangle. Lines parallel to the sides through P meet the sides at Q and R. Show that the reflection of P in the line QR lies on the circumcircle of the triangle.

3. Which positive integers cannot be represented as a sum of (two or more) consecutive integers?

51st Kürschák Competition 1950

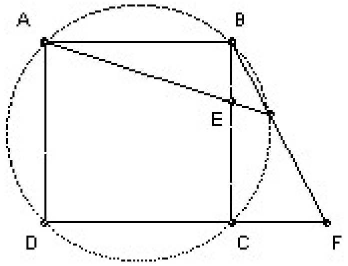
1. Several people visited a library yesterday. Each one visited the library just once (in the course of yesterday). Amongst any three of them, there were two who met in the library. Prove that there were two moments T and T' yesterday such that everyone who visited the library yesterday was in the library at T or T' (or both).

2. Three circles C_1, C_2, C_3 in the plane touch each other (in three different points). Connect the common point of C_1 and C_2 with the other two common points by straight lines. Show that these lines meet C_3 in diametrically opposite points.

3. (x_1, y_1, z_1) and (x_2, y_2, z_2) are triples of real numbers such that for every pair of integers (m, n) at least one of $x_1m + y_1n + z_1$, $x_2m + y_2n + z_2$ is an even integer. Prove that one of the triples consists of three integers.

52nd Kürschák Competition 1951

1. ABCD is a square. E is a point on the side BC such that BE = BC/3, and F is a point on the ray DC such that CF = DC/2. Prove that the lines AE and BF intersect on the circumcircle of the square.



2. For which m > 1 is (m-1)! divisible by m?

3. An open half-plane is the set of all points lying to one side of a line, but excluding the points on the line itself. If four open half-planes cover the plane, show that one can select three of them which still cover the plane.

53rd Kürschák Competition 1952

1. A circle C touches three pairwise disjoint circles whose centers are collinear and none of which contains any of the others. Show that its radius must be larger than the radius of the middle of the three circles.

2. Show that if we choose any n+2 distinct numbers from the set {1, 2, 3, ... , 3n} there will be two whose difference is greater than n and smaller than 2n.

3. ABC is a triangle. The point A' lies on the side opposite to A and BA'/BC = k, where 1/2 < k < 1. Similarly, B' lies on the side opposite to B with CB'/CA = k, and C' lies on the side opposite to C with AC'/AB = k. Show that the perimeter of A'B'C' is less than k times the perimeter of ABC.

54th Kürschák Competition 1953

1. A and B are any two subsets of {1, 2, ... , n-1} such that |A| + |B| > n-1. Prove that one can find a in A and b in B such that a + b = n.

2. n and d are positive integers such that d divides $2n^2$. Prove that $n^2 + d$ cannot be a square.

3. ABCDEF is a convex hexagon with all its sides equal. Also A + C + E = B + D + F. Show that A = D, B = E and C = F.

55th Kürschák Competition 1954

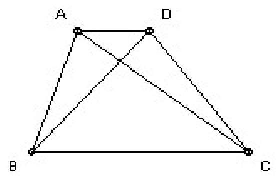
1. ABCD is a convex quadrilateral with AB + BD ≤ AC + CD. Prove that AB < AC.

2. Every planar section of a three-dimensional body B is a disk. Show that B must be a ball.

3. A tournament is arranged amongst a finite number of people. Every person plays every other person just once and each game results in a win to one of the players (there are no draws). Show that there must a person X such that, given any other person Y in the tournament, either X beat Y, or X beat Z and Z beat Y for some Z.

56th Kürschák Competition 1955

1. Prove that if the two angles on the base of a trapezoid are different, then the diagonal starting from the smaller angle is longer than the other diagonal.



2. How many five digit numbers are divisible by 3 and contain the digit 6?

3. The vertices of a triangle are lattice points (they have integer coordinates). There are no other lattice points on the boundary of the triangle, but there is exactly one lattice point inside the triangle. Show that it must be the centroid.

57th Kürschák Competition 1957

1. ABC is an acute-angled triangle. D is a variable point in space such that all faces of the tetrahedron ABCD are acute-angled. P is the foot of the perpendicular from D to the plane ABC. Find the locus of P as D varies.

2. A factory produces several types of mug, each with two colors, chosen from a set of six. Every color occurs in at least three different types of mug. Show that we can find three mugs which together contain all six colors.

3. What is the largest possible value of $|a_1 - 1| + |a_2 - 2| + ... + |a_n - n|$, where $a_1, a_2, ... , a_n$ is a permutation of 1, 2, ... , n?

58th Kürschák Competition 1958

1. Given any six points in the plane, no three collinear, show that we can always find three which form an obtuse-angled triangle with one angle at least 120^o.

2. Show that if m and n are integers such that $m^2 + mn + n^2$ is divisible by 9, then they must both be divisible by 3.

3. The hexagon ABCDEF is convex and opposite sides are parallel. Show that the triangles ACE and BDF have equal area.

59th Kürschák Competition 1959

1. a, b, c are three distinct integers and n is a positive integer. Show that $a^n/((a - b)(a - c)) + b^n/((b - a)(b - c)) + c^n/((c - a)(c - b))$ is an integer.

2. The angles subtended by a tower at distances 100, 200 and 300 from its foot sum to 90^o. What is its height?

3. Three brothers and their wives visited a friend in hospital. Each person made just one visit, so that there were six visits in all. Some of the visits overlapped, so that each of the three brothers met the two other brothers' wives during a visit. Show that one brother must have met his own wife during a visit.

60th Kürschák Competition 1960

1. Among any four people at a party there is one who has met the three others before the party. Show that among any four people at the party there must be one who has met everyone at the party before the party.

2. Let $a_1 = 1, a_2, a_3, \ldots$ be a sequence of positive integers such that $a_k < 1 + a_1 + a_2 + \ldots + a_{k-1}$ for all $k > 1$. Prove that every positive integer can be expressed as a sum of a_is.

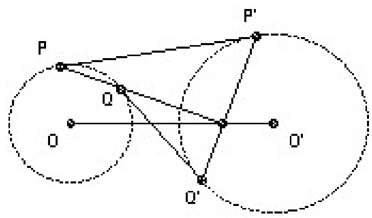
3. E is the midpoint of the side AB of the square ABCD, and F, G are any points on the sides BC, CD such that EF is parallel to AG. Show that FG touches the inscribed circle of the square.

61st Kürschák Competition 1961

1. Given any four distinct points in the plane, show that the ratio of the largest to the smallest distance between two of them is at least $\sqrt{2}$.

2. x, y, z are positive reals less than 1. Show that at least one of (1 - x)y, (1 - y)z and (1 - z)x does not exceed 1/4.

3. Two circles centers O and O' are disjoint. PP' is an outer tangent (with P on the circle center O, and P' on the circle center O'). Similarly, QQ' is an inner tangent (with Q on the circle center O, and Q' on the circle center O'). Show that the lines PQ and P'Q' meet on the line OO'.



62nd Kürschák Competition 1962

1. Show that the number of ordered pairs (a, b) of positive integers with lowest common multiple n is the same as the number of positive divisors of n^2.

2. Show that given any n+1 diagonals of a convex n-gon, one can always find two which have no common point.

3. P is any point of the tetrahedron ABCD except D. Show that at least one of the three distances DA, DB, DC exceeds at least one of the distances PA, PB and PC.

63rd Kürschák Competition 1963

1. mn students all have different heights. They are arranged in m > 1 rows of n > 1. In each row select the shortest student and let A be the height of the tallest such. In each column select the tallest student and let B be the height of the shortest such. Which of the following are possible: A < B, A = B, A > B? If a relation is possible, can it always be realized by a suitable arrangement of the students?

2. A is an acute angle. Show that (1 + 1/sin A)(1 + 1/cos A) > 5.

3. A triangle has no angle greater than 90^o. Show that the sum of the medians is greater than four times the circumradius.

64th Kürschák Competition 1964

1. ABC is an equilateral triangle. D and D' are points on opposite sides of the plane ABC such that the two tetrahedra ABCD and ABCD' are congruent (but not necessarily with the vertices in that order). If the polyhedron with the five vertices A, B, C, D, D' is such that the angle between any two adjacent faces is the same, find DD'/AB.

2. At a party every girl danced with at least one boy, but not with all of them. Similarly, every boy danced with at least one girl, but not with all of them. Show that there were two girls G and G' and two boys B and B', such that each of B and G danced, B' and G' danced, but B and G' did not dance, and B' and G did not dance.

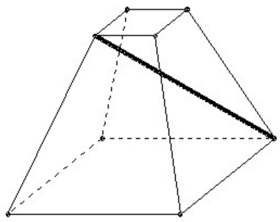
3. Show that for any positive reals w, x, y, z we have $\left((w^2 + x^2 + y^2 + z^2)/4\right)^{1/2} \geq \left((wxy + wxz + wyz + xyz)/4\right)^{1/3}$.

65th Kürschák Competition 1965

1. What integers a, b, c satisfy $a^2 + b^2 + c^2 + 3 < ab + 3b + 2c$?

2. D is a closed disk radius R. Show that among any 8 points of D one can always find two whose distance apart is less than R.

3. A pyramid has square base and equal sides. It is cut into two parts by a plane parallel to the base. The lower part (which has square top and square base) is such that the circumcircle of the base is smaller than the circumcircles of the lateral faces. Show that the shortest path on the surface joining the two endpoints of a spatial diagonal lies entirely on the lateral faces.



66th Kürschák Competition 1966

1. Can we arrange 5 points in space to form a pentagon with equal sides such that the angle between each pair of adjacent edges is 90^o?

2. Show that the n digits after the decimal point in $(5 + \sqrt{26})^n$ are all equal.

3. Do there exist two infinite sets of non-negative integers such that every non-negative integer can be uniquely represented in the form a + b with a in A and b in B?

67th Kürschák Competition 1967

1. A is a set of integers which is closed under addition and contains both positive and negative numbers. Show that the difference of any two elements of A also belongs to A.

2. A convex n-gon is divided into triangles by diagonals which do not intersect except at vertices of the n-gon. Each vertex belongs to an odd number of triangles. Show that n must be a multiple of 3.

3. For a vertex X of a quadrilateral, let h(X) be the sum of the distances from X to the two sides not containing X. Show that if a convex quadrilateral ABCD satisfies h(A) = h(B) = h(C) = h(D), then it must be a parallelogram.

68th Kürschák Competition 1968

1. In an infinite sequence of positive integers every element (starting with the second) is the harmonic mean of its neighbors. Show that all the numbers must be equal.

2. There are 4n segments of unit length inside a circle radius n. Show that given any line L there is a chord of the circle parallel or perpendicular to L which intersects at least two of the 4n segments.

3. For each arrangement X of n white and n black balls in a row, let f(X) be the number of times the color changes as one moves from one end of the row to the other. For each k such that 0 < k < n, show that the number of arrangements X with f(X) = n - k is the same as the number with f(X) = n + k.

69th Kürschák Competition 1969

1. Show that if $2 + 2\sqrt{(28n^2 + 1)}$ is an integer, then it is a square (for n an integer).

2. A triangle has side lengths a, b, c and angles A, B, C as usual (with b opposite B etc). Show that if a(1 - 2 cos A) + b(1 - 2 cos B) + c(1 - 2 cos C) = 0, then the triangle is equilateral.

3. We are given 64 cubes, each with five white faces and one black face. One cube is placed on each square of a chessboard, with its edges parallel to the sides of the board. We are allowed to rotate a complete row of cubes about the axis of symmetry running through the cubes or to rotate a complete column of cubes about the axis of symmetry running through the cubes. Show that by a sequence of such rotations we can always arrange that each cube has its black face uppermost.

70th Kürschák Competition 1970

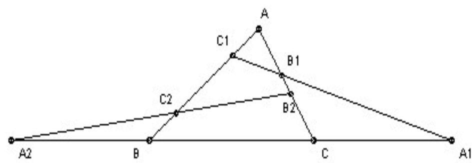
1. What is the largest possible number of acute angles in an n-gon which is not self-intersecting (no two non-adjacent edges interesect)?

2. A valid lottery ticket is formed by choosing 5 distinct numbers from 1, 2, 3, ... , 90. What is the probability that the winning ticket contains at least two consecutive numbers?

3. n points are taken in the plane, no three collinear. Some of the line segments between the points are painted red and some are painted blue, so that between any two points there is a unique path along colored edges. Show that the uncolored edges can be painted (each edge either red or blue) so that all triangles have an odd number of red sides.

71st Kürschák Competition 1971

1. A straight line cuts the side AB of the triangle ABC at C_1, the side AC at B_1 and the line BC at A_1. C_2 is the reflection of C_1 in the midpoint of AB, and B_2 is the reflection of B_1 in the midpoint of AC. The lines B_2C_2 and BC intersect at A_2. Prove that $\sin B_1A_1C/\sin C_2A_2B = B_2C_2/B_1C_1$.



2. Given any 22 points in the plane, no three collinear. Show that the points can be divided into 11 pairs, so that the 11 line segments defined by the pairs have at least five different intersections.

3. There are 30 boxes each with a unique key. The keys are randomly arranged in the boxes, so that each box contains just one key and the boxes are locked. Two boxes are broken open, thus releasing two keys. What is the probability that the remaining boxes can be opened without forcing them?

72nd Kürschák Competition 1972

1. A triangle has side lengths a, b, c. Prove that $a(b - c)^2 + b(c - a)^2 + c(a - b)^2 + 4abc > a^3 + b^3 + c^3$.

2. A class has n > 1 boys and n girls. For each arrangement X of the class in a line let f(X) be the number of ways of dividing the line into two non-empty segments, so that in each segment the number of boys and girls is equal. Let the number of arrangements with f(X) = 0 be A, and the number of arrangements with f(X) = 1 be B. Show that B = 2A.

3. ABCD is a square side 10. There are four points P_1, P_2, P_3, P_4 inside the square. Show that we can always construct line segments parallel to the sides of the square of total length 25 or less, so that each P_i is linked by the segments to both of the sides AB and CD. Show that for some points P_i it is not possible with a total length less than 25.

73rd Kürschák Competition 1973

1. For what positive integers n, k (with k < n) are the binomial coefficients nC(k-1), nCk and nC(k+1) three successive terms of an arithmetic progression?

2. For any positive real r, let d(r) be the distance of the nearest lattice point from the circle center the origin and radius r. Show that d(r) tends to zero as r tends to infinity.

3. n > 4 planes are in general position (so every 3 planes have just one common point, and no point belongs to more than 3 planes). Show that there are at least (2n-3)/4 tetrahedra among the regions formed by the planes.

74th Kürschák Competition 1974

1. A library has one exit and one entrance and a blackboard at each. Only one person enters or leaves at a time. As he does so he records the number of people found/remaining in the library on the blackboard. Prove that at the end of the day exactly the same numbers will be found on the two blackboards (possibly in a different order).

2. S_n is a square side 1/n. Find the smallest k such that the squares S_1, S_2, S_3, ... can be put into a square side k without overlapping.

3. Let $p_k(x) = 1 - x + x^2/2! - x^3/3! + ... + (-x)^{2k}/(2k)!$ Show that it is non-negative for all real x and all positive integers k.

75th Kürschák Competition 1975

1. Transform the equation $ab^2(1/(a + c)^2 + 1/(a - c)^2) = (a - b)$ into a simpler form, given that $a > c \geq 0$, $b > 0$.

2. Prove or disprove: given any quadrilateral inscribed in a convex polygon, we can find a rhombus inscribed in the polygon with side not less than the shortest side of the quadrilateral.

3. Let $x_0 = 5$, $x_{n+1} = x_n + 1/x_n$. Prove that $45 < x_{1000} < 45.1$.

76th Kürschák Competition 1976

1. ABCD is a parallelogram. P is a point outside the parallelogram such that angles PAB and PCB have the same value but opposite orientation. Show that angle APB = angle DPC.

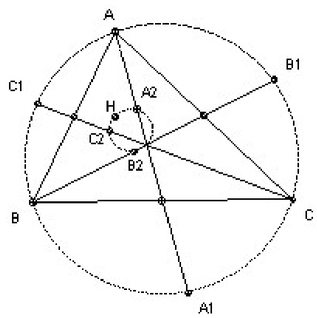
2. A lottery ticket is a choice of 5 distinct numbers from 1, 2, 3, ... , 90. Suppose that 5^5 distinct lottery tickets are such that any two of them have a common number. Prove that one can find four numbers such that every ticket contains at least one of the four.

3. Prove that if the quadratic $x^2 + ax + b$ is always positive (for all real x) then it can be written as the quotient of two polynomials whose coefficients are all positive.

77th Kürschák Competition 1977

1. Show that there are no integers n such that $n^4 + 4^n$ is a prime greater than 5.

2. ABC is a triangle with orthocenter H. The median from A meets the circumcircle again at A_1, and A_2 is the reflection of A_1 in the midpoint of BC. The points B_2 and C_2 are defined similarly. Show that H, A_2, B_2 and C_2 lie on a circle.



3. Three schools each have n students. Each student knows a total of n+1 students at the other two schools. Show that there must be three students, one from each school, who know each other.

78th Kürschák Competition 1978

1. a and b are rationals. Show that if $ax^2 + by^2 = 1$ has a rational solution (in x and y), then it must have infinitely many.

2. The vertices of a convex n-gon are colored so that adjacent vertices have different colors. Prove that if n is odd, then the polygon can be divided into triangles with non-intersecting diagonals such that no diagonal has its endpoints the same color.

3. A triangle has inradius r and circumradius R. Its longest altitude has length H. Show that if the triangle does not have an obtuse angle, then $H \geq r + R$. When does equality hold?

79th Kürschák Competition 1979

1. The base of a convex pyramid has an odd number of edges. The lateral edges of the pyramid are all equal, and the angles between neighbouring faces are all equal. Show that the base must be a regular polygon.

2. f is a real-valued function defined on the reals such that $f(x) \le x$ and $f(x+y) \le f(x) + f(y)$ for all x, y. Prove that $f(x) = x$ for all x.

3. An n x n array of letters is such that no two rows are the same. Show that it must be possible to omit a column, so that the remaining table has no two rows the same.

80th Kürschák Competition 1980

1. Every point in space is colored with one of 5 colors. Prove that there are four coplanar points with different colors.

2. $n > 1$ is an odd integer. Show that there are positive integers a and b such that $4/n = 1/a + 1/b$ iff n has a prime divisor of the form 4k-1.

3. There are two groups of tennis players, one of 1000 players and the other of 1001 players. The players can ranked according to their ability. A higher ranking player always beats a lower ranking player (and the ranking never changes). We know the ranking within each group. Show how it is possible in 11 games to find the player who is 1001st out of 2001.

81st Kürschák Competition 1981

1. Given any 5 points A, B, P, Q, R (in the plane) show that $AB + PQ + QR + RP \le AP + AQ + AR + BP + BQ + BR$.

2. $n > 2$ is even. The squares of an n x n chessboard are painted with $n^2/2$ colors so that there are exactly two squares of each color. Prove that one can always place n rooks on squares of different colors so that no two are in the same row or column.

3. Divide the positive integer n by the numbers 1, 2, 3, ... , n and denote the sum of the remainders by r(n). Prove that for infinitely many n we have $r(n) = r(n+1)$.

82nd Kürschák Competition 1982

1. A cube has all 4 vertices of one face at lattice points and integral side-length. Prove that the other vertices are also lattice points.

2. Show that for any integer $k > 2$, there are infinitely many positive integers n such that the lowest common multiple of n, n+1, ... , n+k-1 is greater than the lowest common multiple of n+1, n+2, ... , n+k.

3. The integers are colored with 100 colors, so that all the colors are used and given any integers $a < b$ and $A < B$ such that $b - a = B - A$, with a and A the same color and b and B the same color, we have that the whole intervals [a, b] and [A, B] are identically colored. Prove that -1982 and 1982 are different colors.

83rd Kürschák Competition 1983

1. Show that the only rational solution to $x^3 + 3y^3 + 9z^3 - 9xyz = 0$ is $x = y = z = 0$.

2. The polynomial $x^n + a_1x^{n-1} + ... + a_{n-1}x + 1$ has non-negative coefficients and n real roots. Show that its value at 2 is at least 3^n.

3. The n+1 points $P_1, P_2, ... , P_n, Q$ lie in the plane and no 3 are collinear. Given any two distinct points P_i and P_j, there is a third point P_k such that Q lies inside the triangle $P_iP_jP_k$. Prove that n must be odd.

84th Kürschák Competition 1984

1. If we write out the first four rows of the Pascal triangle and add up the columns we get:

1

1 1

1 2 1

1 3 3 1

1 1 4 3 4 1 1

If we write out the first 1024 rows of the triangle and add up the columns, how many of the resulting 2047 totals will be odd?

2. $A_1B_1A_2$, $B_1A_2B_2$, $A_2B_2A_3$, $B_2A_3B_3$, ... , $A_{13}B_{13}A_{14}$, $B_{13}A_{14}B_{14}$, $A_{14}B_{14}A_1$, $B_{14}A_1B_1$ are thin triangular plates with all their edges equal length, joined along their common edges. Can the network of plates be folded (along the edges A_iB_i) so that all 28 plates lie in the same plane? (They are allowed to overlap).

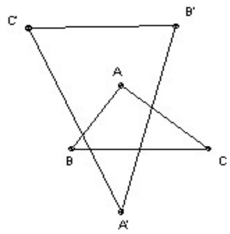
3. A and B are positive integers. We are given a collection of n integers, not all of which are different. We wish to derive a collection of n distinct integers. The allowed move is to take any two integers in the collection which are the same (m and m) and to replace them by m + A and m - B. Show that we can always derive a collection of n distinct integers by a finite sequence of moves.

85th Kürschák Competition 1985

1. The convex polygon P_0P_1 ... P_n is divided into triangles by drawing non-intersecting diagonals. Show that the triangles can be labeled with the numbers 1, 2, ... , n-1 so that the triangle labeled i contains the vertex P_i (for each i).

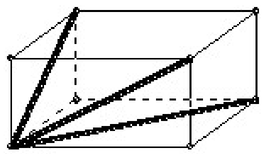
2. For each prime dividing a positive integer n, take the largest power of the prime not exceeding n and form the sum of these prime powers. For example, if n = 100, the sum is $2^6 + 5^2 = 89$. Show that there are infinitely many n for which the sum exceeds n.

3. Vertex A of the triangle ABC is reflected in the opposite side to give A'. The points B' and C' are defined similarly. Show that the area of A'B'C' is less than 5 times the area of ABC.



86th Kürschák Competition 1986

1. Prove that three half-lines from a given point contain three face diagonals of a cuboid iff the half-lines make with each other three acute angles whose sum is 180°.



2. Given n > 2, find the largest h and the smallest H such that $h < x_1/(x_1 + x_2) + x_2/(x_2 + x_3) + ... + x_n/(x_n + x_1) < H$ for all positive real $x_1, x_2, ... , x_n$.

3. k numbers are chosen at random from the set {1, 2, ... , 100}. For what values of k is the probability ½ that the sum of the chosen numbers is even?

87th Kürschák Competition 1987

1. Find all quadruples (a, b, c, d) of distinct positive integers satisfying a + b = cd and c + d = ab.

2. Does there exist an infinite set of points in space such that at least one, but only finitely many, points of the set belong to each plane?

3. A club has 3n+1 members. Every two members play just one of tennis, chess and table-tennis with each other. Each member has n tennis partners, n chess partners and n table-tennis partners. Show that there must be three members of the club, A, B and C such that A and B play chess together, B and C play tennis together and C and A play table-tennis together.

88th Kürschák Competition 1988

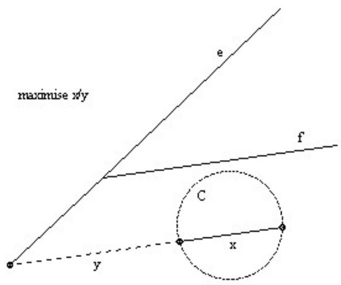
1. P is a point inside a convex quadrilateral ABCD such that the areas of the triangles PAB, PBC, PCD and PDA are all equal. Show that one of its diagonals must bisect the area of the quadrilateral.

2. What is the largest possible number of triples $a < b < c$ that can be chosen from 1, 2, 3, ... , n such that for any two triples $a < b < c$ and $a' < b' < c'$ at most one of the equations $a = a'$, $b = b'$, $c = c'$ holds?

3. PQRS is a convex quadrilateral whose vertices are lattice points. The diagonals of the quadrilateral intersect at E. Prove that if the sum of the angles at P and Q is less than 180^o then the triangle PQE contains a lattice point apart from P and Q either on its boundary or in its interior.

89th Kürschák Competition 1989

1. Given two non-parallel lines e and f and a circle C which does not meet either line. Construct the line parallel to f such that the length of its segment inside C divided by the length of its segment from C to e (and outside C) is as large as possible.



2. Let S(n) denote the sum of the decimal digits of the positive integer n. Find the set of all positive integers m such that s(km) = s(m) for k = 1, 2, ... , m.

3. Walking in the plane, we are allowed to move from (x, y) to one of the four points $(x, y \pm 2x)$, $(x \pm 2y, y)$. Prove that if we start at $(1, \sqrt{2})$, then we cannot return there after finitely many moves.

90th Kürschák Competition 1990

1. Show that for p an odd prime and n a positive integer, there is at most one divisor d of n^2p such that $d + n^2$ is a square.

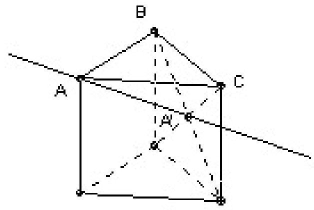
2. I is the incenter of the triangle ABC and A' is the center of the excircle opposite A. The bisector of angle BIC meets the side BC at A". The points B', C', B", C" are defined similarly. Show that the lines A'A", B'B", C'C" are concurrent.

3. A coin has probability p of heads and probability 1-p of tails. The outcome of each toss is independent of the others. Show that it is possible to choose p and k, so that if we toss the coin k times we can assign the 2^k possible outcomes amongst 100 children, so that each has the same 1/100 chance of winning. [A child wins if one of its outcomes occurs.]

91st Kürschák Competition 1991

1. a >= 1, b >= 1 and c > 0 are reals and n is a positive integer. Show that ($(ab + c)^n$ - c) <= a^n ($(b + c)^n$ - c).

2. ABC is a face of a convex irregular triangular prism (the triangular faces are not necessarily congruent or parallel). The diagonals of the quadrilateral face opposite A meet at A'. The points B' and C' are defined similarly. Show that the lines AA', BB' and CC' are concurrent.



3. There are 998 red points in the plane, no three collinear. What is the smallest k for which we can always choose k blue points such that each triangle with red vertices has a blue point inside?

92nd Kürschák Competition 1992

1. Given n positive integers a_i, define $S_k = \sum a_i^k$, $A = S_2/S_1$, and $C = (S_3/n)^{1/3}$. For each of n = 2, 3 which of the following is true: (1) A >= C; (2) A <= C; or (3) A may be > C or < C, depending on the choice of a_i?

2. Let $f_1(k)$ be the sum of the (base 10) digits of k. Define $f_n(k) = f_1(f_{n-1}(k))$. Find $f_{1992}(2^{1991})$.

3. A finite number of points are given in the plane, no three collinear. Show that it is possible to color the points with two colors so that it is impossible to draw a line in the plane with exactly three points of the same color on one side of the line.

93rd Kürschák Competition 1993

1. a and b are positive integers. Show that there are at most a finite number of integers n such that $an^2 + b$ and $a(n + 1)^2 + b$ are both squares.

2. The triangle ABC is not isosceles. The incircle touches BC, CA, AB at K, L, M respectively. N is the midpoint of LM. The line through B parallel to LM meets the line KL at D, and the line through C parallel to LM meets the line MK at E. Show that D, E and N are collinear.

3. Find the minimum value of $x^{2n} + 2x^{2n-1} + 3x^{2n-2} + ... + 2n x + (2n+1)$ for real x.

94th Kürschák Competition 1994

1. Let r > 1 denote the ratio of two adjacent sides of a parallelogram. Determine how the largest possible value of the acute angle included by the diagonals depends on r.

2. Prove that after removing any n-3 diagonals of a convex n-gon, it is always possible to choose n-3 non-intersecting diagonals amongst those remaining, but that n-2 diagonals can be removed so that it is not possible to find n-3 non-intersecting diagonals amongst those remaining.

3. For k = 1, 2, ... , n, H_k is a disjoint union of k intervals of the real line. Show that one can find [(n + 1)/2] disjoint intervals which belong to different H_k.

95th Kürschák Competition 1995

1. A rectangle has its vertices at lattice points and its sides parallel to the axes. Its smaller side has length k. It is divided into triangles whose vertices are all lattice points, such that each triangle has area ½. Prove that the number of the triangles which are right-angled is at least 2k.

2. A polynomial in n variables has the property that if each variable is given one of the values 1 and -1, then the result is positive whenever the number of variables set to -1 is even and negative when it is odd. Prove that the degree of the polynomial is at least n.

3. A, B, C, D are points in the plane, no three collinear. The lines AB and CD meet at E, and the lines BC and DA meet at F. Prove that the three circles with diameters AC, BD and EF either have a common point or are pairwise disjoint.

96th Kürschák Competition 1996

1. The diagonals of a trapezium are perpendicular. Prove that the product of the two lateral sides is not less than the product of the two parallel sides.

2. Two delegations A and B, with the same number of delegates, arrived at a conference. Some of the delegates knew each other already. Prove that there is a non-empty subset A' of A such that either each member in B knew an odd number of members from A', or each member of B knew an even number of members from A'.

3. 2kn+1 diagonals are drawn in a convex n-gon. Prove that among them there is a broken line having 2k+1 segments which does not go through any point more than once. Moreover, this is not necessarily true if kn diagonals are drawn.

97th Kürschák Competition 1997

1. Let S be the set of points with coordinates (m, n), where $0 <= m, n < p$. Show that we can find p points in S with no three collinear.

2. A triangle ABC has incenter I and circumcenter O. The orthocenter of the three points at which the incircle touches its sides is X. Show that I, O and X are collinear.

3. Show that the edges of a planar graph can be colored with three colors so that there is no monochromatic circuit.

99th Kürschák Competition 1999

1. Let e(k) be the number of positive even divisors of k, and let o(k) be the number of positive odd divisors of k. Show that the difference between e(1) + e(2) + ... + e(n) and o(1) + o(2) + ... + o(n) does not exceed n.

2. ABC is an arbitrary triangle. Construct an interior point P such that if A' is the foot of the perpendicular from P to BC, and similarly for B' and C', then the centroid of A'B'C' is P.

3. Prove that every set of integers with more than 2^k members has a subset B with k+2 members such that any two non-empty subsets of B with the same number of members have different sums.

100th Kürschák Competition 2000

1. The square $0 \le x \le n$, $0 \le y \le n$ has $(n+1)^2$ lattice points. How many ways can each of these points be colored red or blue, so that each unit square has exactly two red vertices?

2. ABC is any non-equilateral triangle. P is any point in the plane different from the vertices. Let the line PA meet the circumcircle again at A'. Define B' and C' similarly. Show that there are exactly two points P for which the triangle A'B'C' is equilateral and that the line joining them passes through the circumcenter.

3. k is a non-negative integer and the integers $a_1, a_2, \ldots, a_n$ give at least 2k different remainders on division by n+k. Prove that among the a_i there are some whose sum is divisible by n+k.

101st Kürschák Competition 2001

1. Given any 3n-1 points in the plane, no three collinear, show that it is possible to find 2n whose convex hull is not a triangle.

2. k > 2 is an integer and n > kC3 (where aCb is the usual binomial coefficient a!/(b! (a-b)!)). Show that given 3n distinct real numbers a_i, b_i, c_i (where i = 1, 2, ... , n), there must be at least k+1 distinct numbers in the set $\{a_i + b_i, b_i + c_i, c_i + a_i \mid i = 1, 2, ... , n\}$. Show that the statement is not always true for n = kC3.

3. The vertices of the triangle ABC are lattice points and there is no smaller triangle similar to ABC with its vertices at lattice points. Show that the circumcenter of ABC is not a lattice point.

102th Kürschák Competition 2002

1. ABC is an acute-angled non-isosceles triangle. H is the orthocenter, I is the incenter and O is the circumcenter. Show that if one of the vertices lies on the circle through H, I and O, then at least two vertices lie on it.

2. The Fibonacci numbers are defined by $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$. Suppose that a rational a/b belongs to the open interval $(F_n/F_{n-1}, F_{n+1}/F_n)$. Prove that $b \geq F_{n+1}$.

3. S is a convex 3^n gon. Show that we can choose a set of triangles, such that the edges of each triangle are sides or diagonals of S, and every side or diagonal of S belongs to just one triangle.

INMO (1995 – 2004)

INMO 1995

1. ABC is an acute-angled triangle with $\angle A = 30^o$. H is the orthocenter and M is the midpoint of BC. T is a point on HM such that HM = MT. Show that AT = 2 BC.

2. Show that there are infinitely many pairs (a,b) of coprime integers (which may be negative, but not zero) such that $x^2 + ax + b = 0$ and $x^2 + 2ax + b$ have integral roots.

3. Show that more 3 element subsets of {1, 2, 3, ... , 63} have sum greater than 95 than have sum less than 95.

4. ABC is a triangle with incircle K, radius r. A circle K', radius r', lies inside ABC and touches AB and AC and touches K externally. Show that $r'/r = \tan^2((\pi-A)/4)$.

5. $x_1, x_2, ... , x_n$ are reals > 1 such that $|x_i - x_{i+1}| < 1$ for i < n. Show that $x_1/x_2 + x_2/x_3 + ... + x_{n-1}/x_n + x_n/x_1 < 2n-1$.

6. Find all primes p for which $(2^{p-1} - 1)/p$ is a square.

INMO 1996

1. Given any positive integer n, show that there are distinct positive integers a, b such that a + k divides b + k for k = 1, 2, ... , n. If a, b are positive integers such that a + k divides b + k for all positive integers k, show that a = b.

2. C, C' are concentric circles with radii R, 3R respectively. Show that the orthocenter of any triangle inscribed in C must lie inside the circle C'. Conversely, show that any point inside C' is the orthocenter of some circle inscribed in C.

3. Find reals a, b, c, d, e such that $3a = (b + c + d)^3$, $3b = (c + d + e)^3$, $3c = (d + e + a)^3$, $3d = (e + a + b)^3$, $3e = (a + b + c)^3$.

4. X is a set with n elements. Find the number of triples (A, B, C) such that A, B, C are subsets of X, A is a subset of B, and B is a proper subset of C.

5. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$, $a_2 = 2$, $a_{n+2} = 2a_{n+1} - a_n + 2$. Show that for any m, $a_m a_{m+1}$ is also a term of the sequence.

6. A 2n x 2n array has each entry 0 or 1. There are just 3n 0s. Show that it is possible to remove all the 0s by deleting n rows and n columns.

INMO 1997

1. ABCD is a parallelogram. A line through C does not pass through the interior of ABCD and meets the lines AB, AD at E, F respectively. Show that $AC^2 + CE \cdot CF = AB \cdot AE + AD \cdot AF$.

2. Show that there do not exist positive integers m, n such that $m/n + (n+1)/m = 4$.

3. a, b, c are distinct reals such that $a + 1/b = b + 1/c = c + 1/a = t$ for some real t. Show that $t = -abc$.

4. In a unit square, 100 segments are drawn from the center to the perimeter, dividing the square into 100 parts. If all parts have equal perimeter p, show that $1.4 < p < 1.5$.

5. Find the number of 4 x 4 arrays with entries from {0, 1, 2, 3} such that the sum of each row is divisible by 4, and the sum of each column is divisible by 4.

6. a, b are positive reals such that the cubic $x^3 - ax + b = 0$ has all its roots real. α is the root with smallest absolute value. Show that $b/a < \alpha \leq 3b/2a$.

INMO 1998

1. C is a circle with center O. AB is a chord not passing through O. M is the midpoint of AB. C' is the circle diameter OM. T is a point on C'. The tangent to C' at T meets C at P. Show that $PA^2 + PB^2 = 4 PT^2$.

2. a, b are positive rationals such that $a^{1/3} + b^{1/3}$ is also a rational. Show that $a^{1/3}$ and $b^{1/3}$ are rational.

3. p, q, r, s are integers and s is not a multiple of 5. If there is an integer a such that $pa^3 + qa^2 + ra + s$ is a multiple of 5, show that there is an integer b such that $sb^3 + rb^2 + qb + p$ is a multiple of 5.

4. ABCD is a cyclic quadrilateral inscribed in a circle radius 1. If $AB \cdot BC \cdot CD \cdot DA \geq 4$, show that ABCD is a square.

5. The quadratic $x^2 - (a+b+c)x + (ab+bc+ca) = 0$ has non-real roots. Show that a, b, c, are all positive and that there is a triangle with sides $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$.

6. $a_1, a_2, \ldots, a_{2n}$ is a sequence with two copies each of 0, 1, 2, ... , n-1. A subsequence of n elements is chosen so that its arithmetic mean is integral and as small as possible. Find this minimum value.

INMO 1999

1. ABC is an acute-angled triangle. AD is an altitude, BE a median, and CF an angle bisector. CF meets AD at M, and DE at N. FM = 2, MN = 1, NC = 3. Find the perimeter of ABC.

2. A rectangular field with integer sides and perimeter 3996 is divided into 1998 equal parts, each with integral area. Find the dimensions of the field.

3. Show that $x^5 + 2x + 1$ cannot be factorised into two polynomials with integer coefficients (and degree ≥ 1).

4. X, X' are concentric circles. ABC, A'B'C' are equilateral triangles inscribed in X, X' respectively. P, P' are points on the perimeters of X, X' respectively. Show that $P'A^2 + P'B^2 + P'C^2 = A'P^2 + B'P^2 + C'P^2$.

5. Given any four distinct reals, show that we can always choose three A, b, C, such that the equations $ax^2 + x + b = 0$, $bx^2 + x + c = 0$, $cx^2 + x + a = 0$ either all have real roots, or all have non-real roots.

6. For which n can {1, 2, 3, ... , 4n} be divided into n disjoint 4-element subsets such that for each subset one element is the arithmetic mean of the other three?

INMO 2000

1. The incircle of ABC touches BC, CA, AB at K, L, M respectively. The line through A parallel to LK meets MK at P, and the line through A parallel to MK meets LK at Q. Show that the line PQ bisects AB and bisects AC.

2. Find the integer solutions to $a + b = 1 - c$, $a^3 + b^3 = 1 - c^2$.

3. a, b, c are non-zero reals, and x is real and satisfies $[bx + c(1-x)]/a = [cx + a(1-x)]/b = [ax + b(1-x)]/b$. Show that $a = b = c$.

4. In a convex quadrilateral PQRS, PQ = RS, $SP = (\sqrt{3} + 1)QR$, and $\angle RSP - \angle SQP = 30^o$. Show that $\angle PQR - \angle QRS = 90^o$.

5. a, b, c are reals such that $0 \le c \le b \le a \le 1$. Show that if α is a root of $z^3 + az^2 + bz + c = 0$, then $|\alpha| \le 1$.

6. Let f(n) be the number of incongruent triangles with integral sides and perimeter n, eg f(3) = 1, f(4) = 0, f(7) = 2. Show that f(1999) > f(1996) and f(2000) = f(1997).

INMO 2001

1. ABC is a triangle which is not right-angled. P is a point in the plane. A', B', C' are the reflections of P in BC, CA, AB. Show that [incomplete].

2. Show that $a^2 + b^2 + c^2 = (a-b)(b-c)(c-a)$ has infinitely many integral solutions.

3. a, b, c are positive reals with product 1. Show that $a^{b+c}b^{c+a}c^{a+b} \le 1$.

4. Show that given any nine integers, we can find four, a, b, c, d such that a + b - c - d is divisible by 20. Show that this is not always true for eight integers.

5. ABC is a triangle. M is the midpoint of BC. $\angle MAB = \angle C$, and $\angle MAC = 15^o$. Show that $\angle AMC$ is obtuse. If O is the circumcenter of ADC, show that AOD is equilateral.

6. Find all real-valued functions f on the reals such that f(x+y) = f(x) f(y) f(xy) for all x, y.

INMO 2002

1. ABCDEF is a convex hexagon. Consider the following statements. (1) AB is parallel to DE, (2) BC is parallel to EF, (3) CD is parallel to FA, (4) AE = BD, (5) BF = CE, (6) CA = DF. Show that if any five of these statements are true then the hexagon is cyclic.

2. Find the smallest positive value taken by $a^3 + b^3 + c^3 - 3abc$ for positive integers a, b, c. Find all a, b, c which give the smallest value.

3. x, y are positive reals such that $x + y = 2$. Show that $x^3y^3(x^3 + y^3) \le 2$.

4. Do there exist 100 lines in the plane, no three concurrent, such that they intersect in exactly 2002 points?

4. Do there exist 100 lines in the plane, no three concurrent, such that they intersect in exactly 2002 points?

5. Do there exist distinct positive integers a, b, c such that a, b, c, -a+b+c, a-b+c, a+b-c, a+b+c form an arithmetic progression (in some order).

6. The numbers 1, 2, 3, ... , n^2 are arranged in an n x n array, so that the numbers in each row increase from left to right, and the numbers in each column increase from top to bottom. Let a_{ij} be the number in position i, j. Let b_j be the number of possible value for a_{jj}. Show that $b_1 + b_2 + ... + b_n = n(n^2-3n+5)/3$.

INMO 2003

1. ABC is acute-angled. P is an interior point. The line BP meets AC at E, and the line CP meets AB at F. AP meets EF at D. K is the foot of the perpendicular from D to BC. Show that KD bisects $\angle$EKF.

2. Find all primes p, q and even n > 2 such that $p^n + p^{n-1} + ... + p + 1 = q^2 + q + 1$.

3. Show that $8x^4 - 16x^3 + 16x^2 - 8x + k = 0$ has at least one real root for all real k. Find the sum of the non-real roots.

4. Find all 7-digit numbers which use only the digits 5 and 7 and are divisible by 35.

5. ABC has sides a, b, c. The triangle A'B'C' has sides a + b/2, b + c/2, c + a/2. Show that its area is at least (9/4) area ABC.

6. Each lottery ticket has a 9-digit numbers, which uses only the digits 1, 2, 3. Each ticket is colored red, blue or green. If two tickets have numbers which differ in all nine places, then the tickets have different colors. Ticket 122222222 is red, and ticket 222222222 is green. What color is ticket 123123123?

INMO 2004

1. ABCD is a convex quadrilateral. K, L, M, N are the midpoints of the sides AB, BC, CD, DA. BD bisects KM at Q. QA = QB = QC = QD, and LK/LM = CD/CB. Prove that ABCD is a square.

2. $p > 3$ is a prime. Find all integers a, b, such that $a^2 + 3ab + 2p(a+b) + p^2 = 0$.

3. If α is a real root of $x^5 - x^3 + x - 2 = 0$, show that $[\alpha^6] = 3$.

4. ABC is a triangle, with sides a, b, c (as usual), circumradius R, and exradii r_a, r_b, r_c. If $2R \le r_a$, show that $a > b$, $a > c$, $2R > r_b$, and $2R > r_c$.

5. S is the set of all (a, b, c, d, e, f) where a, b, c, d, e, f are integers such that $a^2 + b^2 + c^2 + d^2 + e^2 = f^2$. Find the largest k which divides abcdef for all members of S.

6. Show that the number of 5-tuples (a, b, c, d, e) such that abcde = 5(bcde + acde + abde + abce + abcd) is odd.

Irish (1988 – 2003)

1st Irish 1988

1. One face of a pyramid with square base and all edges 2 is glued to a face of a regular tetrahedron with edge length 2 to form a polyhedron. What is the total edge length of the polyhedron?

2. P is a point on the circumcircle of the square ABCD between C and D. Show that $PA^2 - PB^2 = PB \cdot PD - PA \cdot PC$.

3. E is the midpoint of the arc BC of the circumcircle of the triangle ABC (on the opposite side of the line BC to A). DE is a diameter. Show that $\angle DEA$ is half the difference between the $\angle B$ and $\angle C$.

4. The triangle ABC (with sidelengths a, b, c as usual) satisfies $\log(a^2) = \log(b^2) + \log(c^2) - \log(2bc \cos A)$. What can we say about the triangle?

5. Let $X = \{1, 2, 3, 4, 5, 6, 7\}$. How many 7-tuples $(X_1, X_2, X_3, X_4, X_5, X_6, X_7)$ are there such that each X_i is a different subset of X with three elements and the union of the X_i is X?

6. Each member of the sequence $a_1, a_2, \dots, a_n$ belongs to the set $\{1, 2, \dots, n-1\}$ and $a_1 + a_2 + \dots + a_n < 2n$. Show that we can find a subsequence with sum n.

7. Put $f(x) = x^3 - x$. Show that the set of positive real A such that for some real x we have $f(x + A) = f(x)$ is the interval $(0, 2]$.

8. The sequence of nonzero reals $x_1, x_2, x_3, \dots$ satisfies $x_n = x_{n-2}x_{n-1}/(2x_{n-2} - x_{n-1})$ for all $n > 2$. For which (x_1, x_2) does the sequence contain infinitely many integral terms?

9. The year 1978 had the property that 19 + 78 = 97. In other words the sum of the number formed by the first two digits and the number formed by the last two digits equals the number formed by the middle two digits. Find the closest years either side of 1978 with the same property.

10. Show that $(1 + x)^n \geq (1 - x)^n + 2nx(1 - x^2)^{(n-1)/2}$ for all $0 \leq x \leq 1$ and all positive integers n.

11. Given a positive real k, for which real x_0 does the sequence $x_0, x_1, x_2, \dots$ defined by $x_{n+1} = x_n(2 - k x_n)$ converge to $1/k$?

12. Show that for n a positive integer we have $\cos^4 k + \cos^4 2k + \dots \cos^4 nk = 3n/8 - 5/16$ where $k = \pi/(2n+1)$.

13. ABC is a triangle with $AB = 2 \cdot AC$ and E is the midpoint of AB. The point F lies on the line EC and the point G lies on the line BC such that A, F, G are collinear and FG = AC. Show that $AG^3 = AB \cdot CE^2$.

14. $a_1, a_2, \dots, a_n$ are integers and $m < n$ is a positive integer. Put $S_i = a_i + a_{i+1} + \dots + a_{i+m}$, and $T_i = a_{m+i} + a_{m+i+1} + \dots + a_{n-1+i}$, for where we use the usual cyclic subscript convention, whereby subscripts are reduced to the range 1, 2, ... , n by subtracting multiples of n as necessary. Let m(h, k) be the number of elements i in $\{1, 2, \dots, n\}$ for which $S_i = h \bmod 4$ and $T_i = b \bmod 4$. Show that $m(1, 3) = m(3, 1) \bmod 4$ iff $m(2, 2)$ is even.

15. X is a finite set. $X_1, X_2, \dots, X_n$ are distinct subsets of X ($n > 1$), each with 11 elements, such that the intersection of any two subsets has just one element and given any two elements of X, there is an X_i containing them both. Find n.

2nd Irish 1989

A1. S is a square side 1. The points A, B, C, D lie on the sides of S in that order, and each side of S contains at least one of A, B, C, D. Show that $2 \le AB^2 + BC^2 + CD^2 + DA^2 \le 4$.

A2. A *sumsquare* is a 3 x 3 array of positive integers such that each row, each column and each of the two main diagonals has sum m. Show that m must be a multiple of 3 and that the largest entry in the array is at most 2m/3 - 1.

A3. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$, $a_{2n} = a_n$, $a_{2n+1} = a_{2n} + 1$. Find the largest value in $a_1, a_2, \ldots, a_{1989}$ and the number of times it occurs.

A4. n^2 ends with m equal non-zero digits (in base 10). What is the largest possible value of m?

A5. An n-digit number has the property that if you cyclically permute its digits it is always divisible by 1989. What is the smallest possible value of n? What is the smallest such number? [If we cyclically permute the digits of 3701 we get 7013, 137, 1370, and 3701.]

B1. L is a fixed line, A a fixed point, k > 0 a fixed real. P is a variable point on L, Q is the point on the ray AP such that $AP \cdot AQ = k^2$. Find the locus of Q.

B2. Each of n people has a unique piece of information. They wish to share the information. A person may pass another person a message containing all the pieces of information that he has. What is the smallest number of messages that must be passed so that each person ends up with all n pieces of information? For example, if A, B, C start by knowing a, b, c respectively. Then four messages suffice: A passes a to B; B passes a and b to C; C passes a, b and c to A; C passes a, b, c to B.

B3. Let k be the product of the distances from P to the sides of the triangle ABC. Show that if P is inside ABC, then $AB \cdot BC \cdot CA \ge 8k$ with equality iff ABC is equilateral.

B4. Show that $(n + \sqrt{(n^2 + 1)})^{1/3} + (n - \sqrt{(n^2 + 1)})^{1/3}$ is a positive integer iff $n = m(m^2 + 3)/2$ for some positive integer m.

B5.(a) Show that $2nCn < 2^{2n}$ and that it is divisible by all primes p such that $n < p < 2n$ (where $2nCn = (2n)!/(n!\,n!)$).

(b) Let $\pi(x)$ denote the number of primes $\le x$. Show that for $n > 2$ we have $\pi(2n) < \pi(n) + 2n/\log_2 n$ and $\pi(2^n) < (1/n)\, 2^{n+1} \log_2(n-1)$. Deduce that for $x \ge 8$, $\pi(x) < (4x/\log_2 x) \log_2(\log_2 x)$.

3rd Irish 1990

1. Find the number of rectangles with sides parallel to the axes whose vertices are all of the form (a, b) with a and b integers such that $0 \le a, b \le n$.

2. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 2$, a_n is the largest prime divisor of $a_1a_2 \ldots a_{n-1} + 1$. Show that 5 does not occur in the sequence.

3. Does there exist a function f(n) on the positive integers which takes positive integer values and satisfies $f(n) = f(f(n-1)) + f(f(n+1))$ for all $n > 1$?

4. Find the largest n for which we can find a real number x satisfying:

$$2^1 < x^1 + x^2 < 2^2$$
$$2^2 < x^2 + x^3 < 2^3$$
$$\ldots$$
$$2^n < x^n + x^{n+1} < 2^{n+1}.$$

5. In the triangle ABC, $\angle A = 90^o$. X is the foot of the perpendicular from A, and D is the reflection of A in B. Y is the midpoint of XC. Show that DX is perpendicular to AY.

6. If all $a_n = \pm 1$ and $a_1a_2 + a_2a_3 + \ldots a_{n-1}a_n + a_na_1 = 0$, show that n is a multiple of 4.

7. Show that $1/3^3 + 1/4^3 + \ldots + 1/n^3 < 1/12$.

8. $p_1 < p_2 < \ldots < p_{15}$ are primes forming an arithmetic progression, show that the difference must be a multiple of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$.

9. Let $a_n = 2\cos(t/2^n) - 1$. Simplify $a_1a_2 \ldots a_n$ and deduce that it tends to $(2\cos t + 1)/3$.

10. Let T be the set of all (2k-1)-tuples whose entries are all 0 or 1. There is a subset S of T with 2^k elements such that given any element x of T, there is a unique element of S which disagrees with x in at most 3 positions. If $k > 5$, show that it must be 12.

4th Irish 1991

A1. Given three points X, Y, Z, show how to construct a triangle ABC which has circumcenter X, Y the midpoint of BC and BZ an altitude.

A2. Find all polynomials $p(x)$ of degree $\le n$ which satisfy $p(x^2) = p(x)^2$ for all real x.

A3. For any positive integer n, define $f(n) = 10n$, $g(n) = 10n+4$, and for any even positive integer n, define $h(n) = n/2$. Show that starting from 4 we can reach any positive integer by some finite sequence of the operations f, g, h.

A4. 8 people decide to hold daily meetings subject to the following rules. At least one person must attend each day. A different set of people must attend on different days. On day N for each $1 \le k < N$, at least one person must attend who was present on day k. How many days can the meetings be held?

A5. Find all polynomials $x^n + a_1x^{n-1} + ... + a_n$ such that each of the coefficients $a_1, a_2, ... , a_n$ is ± 1 and all the roots are real.

B1. Prove that the sum of m consecutive squares cannot be a square for m = 3, 4, 5, 6. Give an example of 11 consecutive squares whose sum is a square.

B2. Define $a_n = (n^2 + 1)/\sqrt{(n^4 + 4)}$ for n = 1, 2, 3, ... , and let $b_n = a_1a_2 ... a_n$. Show that $b_n = (\sqrt{2} \sqrt{(n^2+1)})/\sqrt{(n^2+2n+2)}$, and hence that $1/(n+1)^3 < b_n/\sqrt{2} - n/(n+1) < 1/n^3$.

B3. ABC is a triangle and L is the line through C parallel to AB. The angle bisector of A meets BC at D and L at E. The angle bisector of B meets AC at F and L at G. If DE = FG show that CA = CB.

B4. P is the set of positive rationals. Find all functions $f:P \to P$ such that $f(x) + f(1/x) = 1$ and $f(2x) = 2 f(f(x))$ for all x.

B5. A non-empty subset S of the rationals satisfies: (1) 0 □ S; (2) if $a, b \in S$, then $a/b \in S$; (3) there is a non-zero rational q not in S such that if s is a non-zero rational not in S, then $s = qt$ for some $t \in S$. Show that every element of S is a sum of two elements of S.

5th Irish 1992

A1. Give a geometric description for the set of points (x, y) such that $t^2 + yt + x \geq 0$ for all real t satisfying $|t| \leq 1$.

A2. How many (x, y, z) satisfy $x^2 + y^2 + z^2 = 9$, $x^4 + y^4 + z^4 = 33$, $xyz = -4$?

A3. A has n elements. How many (B, C) are such that $\square \neq B \square C \square A$?

A4. ABC is a triangle with circumradius R. A', B', C' are points on BC, CA, AB such that AA', BB', CC' are concurrent. Show that the AB'·BC'·CA'/area A'B'C' = 2R.

A5. A triangle has two vertices with rational coordinates. Show that the third vertex has rational coordinates iff each angle X of the triangle has $X = 90^o$ or tan X rational.

B1. Let $m = \sum k^3$, where the sum is taken over $1 \leq k < n$ such that k is relatively prime to n. Show that m is a multiple of n.

B2. The digital root of a positive integer is obtained by repeatedly taking the product of the digits until we get a single-digit number. For example 24378 → 1344 → 48 → 32 → 6. Show that if n has digital root 1, then all its digits are 1.

B3. All three roots of $az^3 + bz^2 + cz + d$ have negative real part. Show that $ab > 0$, $bc > ad > 0$.

B4. Each diagonal of a convex pentagon divides the pentagon into a quadrilateral and a triangle of unit area. Find the area of the pentagon.

B5. Show that for any positive reals a_i, b_i, we have $(a_1a_2 \ldots a_n)^{1/n} + (b_1b_2 \ldots b_n)^{1/n} \leq ((a_1+b_1)(a_2+b_2) \ldots (a_n+b_n))^{1/n}$ with equality iff $a_1/b_1 = a_2/b_2 = \ldots = a_n/b_n$.

6th Irish 1993

A1. The real numbers x, y satisfy $x^3 - 3x^2 + 5x - 17 = 0$, $y^3 - 3y^2 + 5y + 11 = 0$. Find x + y.

A2. Find which positive integers can be written as the sum and product of the same sequence of two or more positive integers. (For example 10 = 5+2+1+1+1 = 5·2·1·1·1).

A3. A triangle ABC has fixed incircle. BC touches the incircle at the fixed point P. B and C are varied so that PB·PC is constant. Find the locus of A.

A4. The polynomial $x^n + a_{n-1}x^{n-1} + ... + a_0$ has real coefficients. All its roots are real and lie in the interval (0, 1). Also f(1) = |f(0)|. Show that the product of the roots does not exceed $1/2^n$.

A5. The points z_1, z_2, z_3, z_4, z_5 form a convex pentagon P in the complex plane. The origin and the points $\alpha z_1, ... , \alpha z_5$ all lie inside the pentagon. Show that $|\alpha| \le 1$ and $Re(\alpha) + Im(\alpha) \tan(\pi/5) \le 1$.

B1. Given 5 lattice points in the plane, show that at least one pair of points has a distinct lattice point on the segment joining them.

B2. $a_1, a_2, ... , a_n$ are distinct reals. $b_1, b_2, ... , b_n$ are reals. There is a real number α such that $\prod_{1\le k\le n} (a_i + b_k) = \alpha$ for i = 1, 2, ... , n. Show that there is a real β such that $\prod_{1\le k\le n} (a_k + b_j) = \beta$ for j = 1, 2, ... , n.

B3. Given positive integers $r \le n$, show that $\sum_d$ (n-r+1)Cd (r-1)C(d-1) = nCr, where nCr denotes the usual binomial coefficient and the sum is taken over all positive d ≤ n-r+1 and ≤ r.

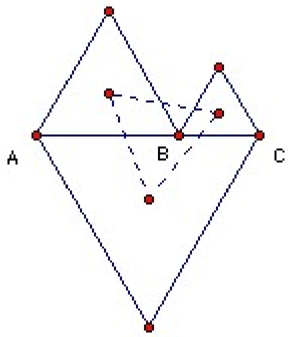
B4. Show that sin x + (sin 3x)/3 + (sin 5x)/5 + ... + (sin(2n-1)x)/(2n-1) > 0 for all x in (0, π).

B5. An m x n rectangle is divided into unit squares. Show that a diagonal of the rectangle intersects m + n - gcd(m,n) of the squares. An a x b x c box is divided into unit cubes. How many cubes does a long diagonal of the box intersect?

7th Irish 1994

A1. m, n are positive integers with $n > 3$ and $m^2 + n^4 = 2(m-6)^2 + 2(n+1)^2$. Prove that $m^2 + n^4 = 1994$.

A2. B is an arbitrary point on the segment AC. Equilateral triangles are drawn as shown. Show that their centers form an equilateral triangle whose center lies on AC.



A3. Find all real polynomials p(x) satisfying $p(x^2) = p(x)p(x-1)$ for all x.

A4. An n x n array of integers has each entry 0 or 1. Find the number of arrays with an even number of 1s in every row and column.

A5. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 2$, $a_{n+1} = a_n^2 - a_n + 1$. Show that $1/a_1 + 1/a_2 + \ldots + 1/a_n$ lies in the interval $(1-½^N, 1-½^{2N})$, where $N = 2^{n-1}$.

B1. The sequence $x_1, x_2, x_3, \ldots$ is defined by $x_1 = 2$, $nx_n = 2(2n-1)x_{n-1}$. Show that every term is integral.

B2. p, q, r are distinct reals such that $q = p(4-p)$, $r = q(4-q)$, $p = r(4-r)$. Find all possible values of p+q+r.

B3. Prove that $n((n+1)^{2/n} - 1) < \sum_1^n (2i+1)/i^2 < n(1 - 1/n^{2/(n-1)}) + 4$.

B4. w, a, b, c are distinct real numbers such that the equations:

$x + y + z = 1$

$xa^2 + yb^2 + zc^2 = w^2$

$xa^3 + yb^3 + zc^3 = w^3$

$xa^4 + yb^4 + zc^4 = w^4$

have a real solution x, y, z. Express w in terms of a, b, c.

B5. A square is partitioned into n convex polygons. Find the maximum number of edges in the resulting figure. You may assume Euler's formula for a polyhedron: V + F = E + 2, where V is the no. of vertices, F the no. of faces and E the no. of edges.

8th Irish 1995

A1. There are n^2 students in a class. Each week they are arranged into n teams of n players. No two students can be in the same team in more than one week. Show that the arrangement can last for at most n+1 weeks.

A2. Find all integers n for which $x^2 + nxy + y^2 = 1$ has infinitely many distinct integer solutions x, y.

A3. X lies on the line segment AD. B is a point in the plane such that $\angle ABX > 120^o$. C is a point on the line segment BX, show that $(AB + BC + CD) \le 2AD/\sqrt{3}$.

A4. X_k is the point (k, 0). There are initially 2n+1 disks, all at X_0. A move is to take two disks from X_k and to move one to X_{k-1} and the other to X_{k+1}. Show that whatever moves are chosen, after n(n+1)(2n+1)/6 moves there is one disk at X_k for $|k| \le n$.

A5. Find all real-valued functions f(x) such that xf(x) - yf(y) = (x-y) f(x+y) for all real x, y.

B1. Show that for every positive integer n, $n^n \le (n!)^2 \le ((n+1)(n+2)/6)^n$.

B2. a, b, c are complex numbers. All roots of $z^3 + az^2 + bz + c = 0$ satisfy |z| = 1. Show that all roots of $z^3 + |a|z^2 + |b|z + |c| = 0$ also satisfy |z| = 1.

B3. S is the square $\{(x,y) : 0 \le x, y \le 1\}$. For each $0 < t < 1$, C_t is the set of points (x, y) in S such that $x/t + y/(1-t) \ge 1$. Show that the set $\cap C_t$ is the points (x, y) in S such that $\sqrt{x} + \sqrt{y} \ge 1$.

B4. Given points P, Q, R show how to construct a triangle ABC such that P, Q, R are on BC, CA, AB respectively and P is the midpoint of BC, CQ/QA = AR/RB = 2. You may assume that P, Q, R are positioned so that such a triangle exists.

B5. n < 1995 and n = abcd, where a, b, c, d are distinct primes. The positive divisors of n are $1 = d_1 < d_2 < ... < d_{16} = n$. Show that $d_9 - d_8 \ne 22$.

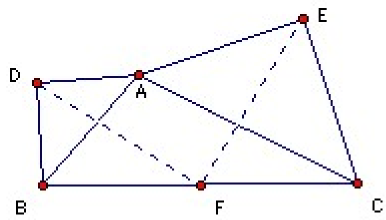
9th Irish 1996

A1. Find gcd(n!+1, (n+1)!).

A2. Let s(n) denote the sum of the digits of n. Show that $s(2n) \leq 2s(n) \leq 10s(2n)$ and that there is a k such that s(k) = 1996 s(3k).

A3. R denotes the reals. $f : [0,1] \to R$ satisfies f(1) = 1, $f(x) \geq 0$ for all $x \in [0,1]$, and if x, y, x+y all $\in [0,1]$, then $f(x+y) \geq f(x) + f(y)$. Show that $f(x) \leq 2x$ for all $x \in [0,1]$.

A4. ABC is any triangle. D, E are constructed as shown so that ABD and ACE are right-angled isosceles triangles, and F is the midpoint of BC. Show that DEF is a right-angled isosceles triangle.



A5. Show how to dissect a square into at most 5 pieces so that the pieces can be reassembled to form three squares all of different size.

B1. The Fibonacci sequence $F_0, F_1, F_2, \ldots$ is defined by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$. Show that $F_{n+60} - F_n$ is divisible by 10 for all n, but for any $1 \leq k < 60$ there is some n such that $F_{n+k} - F_n$ is not divisible by 10. Similarly, show that $F_{n+300} - F_n$ is divisible by 100 for all n, but for any $1 \leq k < 300$ there is some n such that $F_{n+k} - F_n$ is not divisible by 100.

B2. Show that $2^{1/2}4^{1/4}8^{1/8} \ldots (2^n)^{1/2n} < 4$.

B3. If p is a prime, show that $2^p + 3^p$ cannot be an nth power (for n > 1). **B4.** ABC is an acute-angled triangle. The altitudes are AD, BE, CF. The feet of the perpendiculars from A, B, C to EF, FD, DE respectively are P, Q, R. Show that AP, BQ, CR are concurrent.

B5. 33 disks are placed on a 5 x 9 board, at most one disk per square. At each step every disk is moved once so that after the step there is at most one disk per square. Each disk is moved alternately one square up/down and one square left/right. So a particular disk might be moved L,U,L,D,L,D,R,U ... in successive steps. Prove that only finitely many steps are possible. Show that with 32 disks it is possible to have infinitely many steps.

10th Irish 1997

A1. Find all integer solutions to $1 + 1996m + 1998n = mn$.

A2. ABC is an equilateral triangle. M is a point inside the triangle. D, E, F are the feet of the perpendiculars from M to BC, CA, AB. Find the locus of M such that $\angle FDE = 90^o$.

A3. Find all polynomials p(x) such that $(x-16)\, p(2x) = (16x-16)\, p(x)$.

A4. a, b, c are non-negative reals such that $a + b + c \ge abc$. Show that $a^2 + b^2 + c^2 \ge abc$.

A5. Let S denote the set of odd integers > 1. For $x \in S$, define f(x) to be the largest integer such that $2^{f(x)} < x$. For a, b $\in$ S define $a * b = a + 2^{f(a)-1}(b-3)$. For example, f(5) = 2, so $5 * 7 = 5 + 2(7-3) = 13$. Similarly, f(7) = 2, so $7 * 5 = 7 + 2(5-3) = 11$. Show that a * b is always an odd integer > 1 and that the operation * is associative.

B1. Let σ(n) denote the sum of the positive divisors of n. Show that if σ(n) > 2n, then σ(mn) > 2mn for any m.

B2. The quadrilateral ABCD has an inscribed circle. $\angle A = \angle B = 120^o$, $\angle C = 30^o$ and BC = 1. Find AD.

B3. A subset of {0, 1, 2, ... , 1997} has more than 1000 elements. Show that it must contain a power of 2 or two distinct elements whose sum is a power of 2.

B4. How many 1000-digit positive integers have all digits odd, and are such that any two adjacent digits differ by 2?

B5. p is an odd prime. We say n satisfies K_p if the set {1, 2, ... , n} can be partitioned into p disjoint parts, such that the sum of the elements in each part is the same. For example, 5 satisfies K_3 because {1, 2, 3, 4, 5} = {1, 4} &cup {2, 3} □ {5}. Show that if n satisfies K_p, then n or n+1 is a multiple of p. Show that if n is a multiple of 2p, then n satisfies K_p.

11th Irish 1998

A1. Show that $x^8 - x^5 - 1/x + 1/x^4 \geq 0$ for all $x \neq 0$.

A2. P is a point inside an equilateral triangle. Its distances from the vertices are 3, 4, 5. Find the area of the triangle.

A3. Show that the 4 digit number mnmn cannot be a cube in base 10. Find the smallest base $b > 1$ for which it can be a cube.

A4. Show that 7 disks radius 1 can be arranged to cover a disk radius 2.

A5. x is real and $x^n - x$ is an integer for $n = 2$ and some $n > 2$. Show that x must be an integer.

B1. Find all positive integers n with exactly 16 positive divisors $1 = d_1 < d_2 < ... < d_{16} = n$ such that $d_6 = 18$ and $d_9 - d_8 = 17$.

B2. Show that for positive reals a, b, c we have $9/(a+b+c) \leq 2/(a+b) + 2/(b+c) + 2/(c+a) \leq 1/a + 1/b + 1/c$.

B3. Let N be the set of positive integers. Show that we can partition N into three disjoint parts such that if $|m-n| = 2$ or 5, then m and n are in different parts. Show that we can partition N into four disjoint parts such that if $|m-n| = 2$, 3 or 5, then m and n are in different parts, but that this is not possible with only three disjoint parts.

B4. The sequence $x_0, x_1, x_2, ...$ is defined by $x_0 = a$, $x_1 = b$, $x_{n+2} = (1+x_{n+1})/x_n$. Find x_{1998}.

B5. Find the smallest possible perimeter for a triangle ABC with integer sides such that $\angle A = 2\angle B$ and $\angle C > 90^o$.

12th Irish 1999

A1. Find all real solutions to $x^2/(x+1-\sqrt{(x+1)})^2 < (x^2+3x+18)/(x+1)^2$.

A2. The Fibonacci sequence is defined by $F_0 = 1$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$. Show that some Fibonacci number is divisible by 1000.

A3. In the triangle ABC, AD is an altitude, BE is an angle bisector and CF is a median. Show that they are concurrent iff $a^2(a-c) = (b^2-c^2)(a+c)$.

A4. Show that a 10000 x 10000 board can be tiled by 1 x 3 tiles and a 2 x 2 tile placed centrally, but not if the 2 x 2 tile is placed in a corner.

A5. The sequence $u_0, u_1, u_2, \ldots$ is defined as follows. $u_0 = 0$, $u_1 = 1$, and u_{n+1} is the smallest integer $> u_n$ such that there is no arithmetic progression u_i, u_j, u_{n+1} with $i < j < n+1$. Find u_{100}.

B1. Solve: $y^2 = (x+8)(x^2+2)$ and $y^2 - (8+4x)y + (16+16x-5x^2) = 0$.

B2. f(n) is a function defined on the positive integers with positive integer values such that $f(ab) = f(a)f(b)$ when a, b are relatively prime and $f(p+q) = f(p)+f(q)$ for all primes p, q. Show that $f(2) = 2$, $f(3) = 3$ and $f(1999) = 1999$.

B3. Show that $a^2/(a+b) + b^2/(b+c) + c^2/(c+d) + d^2/(d+a) \geq 1/2$ for positive reals a, b, c, d such that $a + b + c + d = 1$, and that we have equality iff $a = b = c = d$.

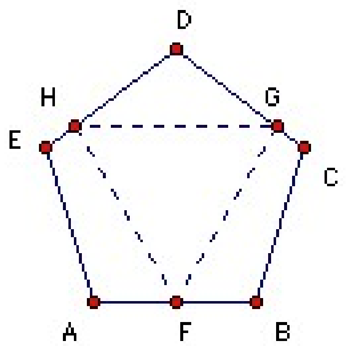
B4. Let d(n) be the number of positive divisors of n. Find all n such that $n = d(n)^4$.

B5. ABCDEF is a convex hexagon such that AB = BC, CD = DE, EF = FA and $\angle B + \angle D + \angle F = 360^\circ$. Show that the perpendiculars from A to FB, C to BD, and E to DF are concurrent.

13th Irish 2000

A1. Let S be the set of all numbers of the form n^2+n+1. Show that the product of n^2+n+1 and $(n+1)^2+(n+1)+1$ is in S, but give an example of a, b $\in$ S with $ab \notin S$.

A2. ABCDE is a regular pentagon side 1. F is the midpoint of AB. G, H are points on DC, DE respectively such that $\angle DFG = \angle DFH = 30o$. Show that FGH is equilateral and GH = 2 cos 18o cos236o/cos 6o. A square is inscribed in FGH with one side on GH. Show that its side has length $GH\sqrt{3}/(2+\sqrt{3})$.



A3. Let $f(n) = 5n^{13} + 13n^5 + 9an$. Find the smallest positive integer a such that f(n) is divisible by 65 for every integer n.

A4. A strictly increasing sequence $a_1 < a_2 < ... < a_M$ is called a *weak AP* if we can find an arithmetic progression $x_0, x_1, ... , x_M$ such that $x_{n-1} \le a_n < x_n$ for n = 1, 2, ... , M. Show that any strictly increasing sequence of length 3 is a weak AP. Show that any subset of {0, 1, 2, ... , 999} with 730 members has a weak AP of length 10.

A5. Let $y = x^2 + 2px + q$ be a parabola which meets the x- and y-axes in three distinct points. Let C_{pq} be the circle through these points. Show that all circles C_{pq} pass through a common point.

B1. Show that $x^2y^2(x^2+y^2) \le 2$ for positive reals x, y such that x+y = 2.

B2. ABCD is a cyclic quadrilateral with circumradius R, side lengths a, b, c, d and area S. Show that $16R^2S^2 = (ab+cd)(ac+bd)(ad+bc)$. Deduce that $RS\sqrt{2} \ge (abcd)^{3/4}$ with equality iff ABCD is a square.

B3. For each positive integer n, find all positive integers m which can be written as $1/a_1 + 2/a_2 + ... + n/a_n$ for some positive integers $a_1 < a_2 < ... < a_n$.

B4. Show that in any set of 10 consecutive integers there is one which is relatively prime to each of the others.

B5. p(x) is a plynomial with non-negative real coefficients such that p(4) = 2, p(16) = 8. Show that $p(8) \le 4$ and find all polynomials where equality holds.

14th Irish 2001

A1. Find all solutions to $a! + b! + c! = 2^n$.

A2. ABC is a triangle. Show that the medians BD and CE are perpendicular iff $b^2 + c^2 = 5a^2$.

A3. p is an odd prime which can be written as a difference of two fifth powers. Show that $\sqrt{(4p+1)/5} = (n^2+1)/2$ for some odd integer n.

A4. Show that $2n/(3n+1) \le \sum_{n<k\le 2n} 1/k \le (3n+1)/(4n+4)$.

A5. Show that $(a^2b^2(a+b)^2/4)^{1/3} \le (a^2+10ab+b^2)/12$ for all reals a, b such that ab > 0. When do we have equality? Find all real numbers a, b for which $(a^2b^2(a+b)^2/4)^{1/3} \le (a^2+ab+b^2)/3$.

B1. Find the smallest positive integer m for which $55^n + m32^n$ is a multiple of 2001 for some odd n.

B2. Three circles each have 10 black beads and 10 white beads randomly arranged on them. Show that we can always rotate the beads around the circles so that in 5 corresponding positions the beads have the same color.

B3. P is a point on the altitude AD of the triangle ABC. The lines BP, CP meet CA, AB at E, F respectively. Show that AD bisects $\angle$ EDF.

B4. Find all non-negative reals for which $(13 + \sqrt{x})^{1/3} + (13 - \sqrt{x})^{1/3}$ is an integer.

B5. Let N be the set of positive integers. Find all functions f : N → N such that f(m + f(n)) = f(m) + n for all m, n.

15th Irish 2002

A1. The triangle ABC has a, b, c = 29, 21, 20 respectively. The points D, E lie on the segment BC with BD = 8, DE = 12, EC = 9. Find ∠ DAE.

A2. A graph has n points. Each point has degree at most 3. If there is no edge between two points, then there is a third point joined to them both. What is the maximum possible value of n? What is the maximum if the graph contains a triangle?

A3. Find all positive integer solutions to p(p+3) + q(q+3) = n(n+3), where p and q are primes.

A4. Define the sequence a_1, a_2, a_3, ... by $a_1 = a_2 = a_3 = 1$, $a_{n+3} = (a_{n+2}a_{n+1} + 2)/a_n$. Show that all terms are integers.

A5. Show that $x/(1-x) + y/(1-y) + z/(1-z) \geq 3(xyz)^{1/3}/(1 - (xyz)^{1/3})$ for positive reals x, y, z all < 1.

B1. For which n can we find a cyclic shift $a_1, a_2, \ldots , a_n$ of 1, 2, 3, ... , n (ie i, i+1, i+2, ... , n, 1, 2, ... , i-1 for some i) and a permutation $b_1, b_2, \ldots , b_n$ of 1, 2, 3, ... , n such that $1 + a_1 + b_1 = 2 + a_2 + b_2 = \ldots = n + a_n + b_n$?

B2. n = p·q·r·s, where p, q, r, s are distinct primes such that s = p + r, p(p + q + r + s) = r(s - q) and qs = 1 + qr + s. Find n.

B3. Let Q be the rationals. Find all functions f : Q → Q such that f(x + f(y)) = f(x) + y for all x, y.

B4. Show that $k^n - [k^n] = 1 - 1/k^n$, where $k = 2 + \sqrt{3}$.

B5. The incircle of the triangle ABC touches BC at D and AC at E. The sides have integral lengths and $|AD^2 - BE^2| \leq 2$. Show that AC = BC.

16th Irish 2003

A1. Find all integral solutions to $(m^2 + n)(m + n^2) = (m + n)^3$.

A2. QB is a chord of the circle parallel to the diameter PA. The lines PB and QA meet at R. S is taken so that PORS is a parallelogram (where O is the center of the circle). Show that SP = SQ.

A3. Find $([1^{1/2}] - [1_{1/3}]) + ([2_{1/2}] - [2_{1/3}]) + ... + ([2003^{1/2} - [2003^{1/3}])$.

A4. A, B, C, D, E, F, G, H compete in a chess tournament. Each pair plays at most once and no five players all play each other. Write a possible arrangment of 24 games which satisfies the conditions and show that no arrangement of 25 games works.

A5. Let R be the reals and R^+ the positive reals. Show that there is no function $f : R^+ \rightarrow R$ such that $f(y) > (y - x) f(x)^2$ for all x, y such that $y > x$.

B1. A triangle has side lengths a, b, c with sum 2. Show that $1 \leq ab + bc + ca - abc \leq 1 + 1/27$.

B2. ABCD is a quadrilateral. The feet of the perpendiculars from D to AB, BC are P, Q respectively, and the feet of the perpendiculars from B to AD, CD are R, S respectively. Show that if $\angle PSR = \angle SPQ$, then PR = QS.

B3. Find all integer solutions to $m^2 + 2m = n^4 + 20n^3 + 104n^2 + 40n + 2003$.

B4. Given real positive a, b, find the largest real c such that $c \leq \max(ax + 1/(ax), bx + 1/bx)$ for all positive real x.

B5. N distinct integers are to be chosen from {1, 2, ... , 2003} so that no two of the chosen integers differ by 10. How many ways can this be done for N = 1003? Show that it can be done in $(3 \cdot 5151 + 7 \cdot 1700)\ 101^7$ ways for N = 1002.

Mexican (1987 – 2003)

1st Mexican 1987

A1. a/b and c/d are positive fractions in their lowest terms such that a/b + c/d = 1. Show that b = d.

A2. How many positive integers divide 20! ?

A3. L and L' are parallel lines and P is a point midway between them. The variable point A lies L, and A' lies on L' so that ∠APA' = 90^o. X is the foot of the perpendicular from P to the line AA'. Find the locus of X as A varies.

A4. Let N be the product of all positive integers ≤ 100 which have exactly three positive divisors. Find N and show that it is a square.

B1. ABC is a triangle with ∠A = 90^o. M is a variable point on the side BC. P, Q are the feet of the perpendiculars from M to AB, AC. Show that the areas of BPM, MQC, AQMP cannot all be equal.

B2. Prove that $(n^3 - n)(5^{8n+4} + 3^{4n+2})$ is a multiple of 3804 for all positive integers n.

B3. Show that $n^2 + n - 1$ and $n^2 + 2n$ have no common factor.

B4. ABCD is a tetrahedron. The plane ABC is perpendicular to the line BD. ∠ADB = ∠CDB = 45^o and ∠ABC = 90^o. Find ∠ADC. A plane through A perpendicular to DA meets the line BD at Q and the line CD at R. If AD = 1, find AQ, AR, and QR.

2nd Mexican 1988

A1. In how many ways can we arrange 7 white balls and 5 black balls in a line so that there is at least one white ball between any two black balls?

A2. If m and n are positive integers, show that 19 divides 11m + 2n iff it divides 18m + 5n.

A3. Two circles of different radius R and r touch externally. The three common tangents form a triangle. Find the area of the triangle in terms of R and r.

A4. How many ways can we find 8 integers $a_1, a_2, \ldots , a_8$ such that $1 \le a_1 \le a_2 \le \ldots \le a_8 \le 8$?

B1. a and b are relatively prime positive integers, and n is an integer. Show that the greatest common divisor of a^2+b^2-nab and a+b must divide n+2.

B2. B and C are fixed points on a circle. A is a variable point on the circle. Find the locus of the incenter of ABC as A varies.

B3. [unclear]

B4. Calculate the volume of an octahedron which has an inscribed sphere of radius 1.

3rd Mexican 1989

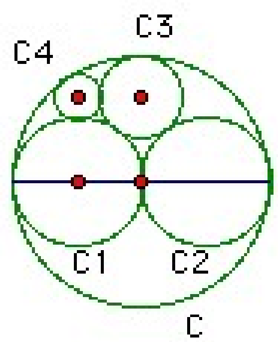
A1. The triangle ABC has AB = 5, the medians from A and B are perpendicular and the area is 18. Find the lengths of the other two sides.

A2. Find integers m and n such that n^2 is a multiple of m, m^3 is a multiple of n^2, n^4 is a multiple of m^3, m^5 is a multiple of n^4, but n^6 is not a multiple of m^5.

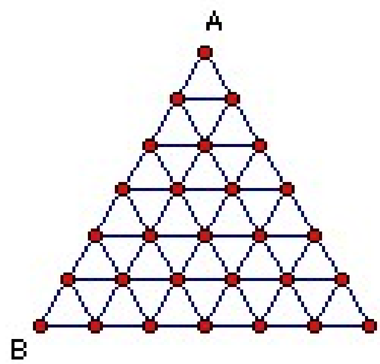
A3. Show that there is no positive integer of 1989 digits, at least three of them 5, such that the sum of the digits is the same as the product of the digits.

B1. Find a positive integer n with decimal expansion $a_m a_{m-1} \dots a_0$ such that $a_1 a_0 a_m a_{m-1} \dots a_2 0$ = 2n.

B2. C_1 and C_2 are two circles of radius 1 which touch at the center of a circle C of radius 2. C_3 is a circle inside C which touches C, C_1 and C_2. C_4 is a circle inside C which touches C, C_1 and C_3. Show that the centers of C, C_1, C_3 and C_4 form a rectangle.

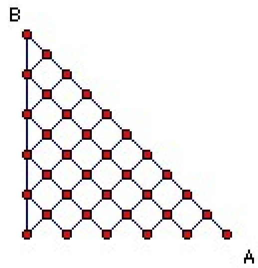


B3. How many paths are there from A to B which do not pass through any vertex twice and which move only downwards or sideways, never up?



4th Mexican 1990

A1. How many paths are there from A to the line BC if the path does not go through any vertex twice and always moves to the left?



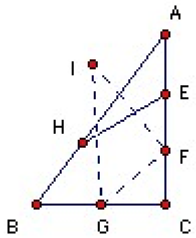
A2. ABC is a triangle with $\angle B = 90^o$ and altitude BH. The inradii of ABC, ABH, CBH are r, r_1, r_2. Find a relation between them.

A3. Show that $n^{n-1} - 1$ is divisible by $(n-1)^2$ for $n > 2$.

B1. Find 0/1 + 1/1 + 0/2 + 1/2 + 2/2 + 0/3 + 1/3 + 2/3 + 3/3 + 0/4 + 1/4 + 2/4 + 3/4 + 4/4 + 0/5 + 1/5 + 2/5 + 3/5 + 4/5 + 5/5 + 0/6 + 1/6 + 2/6 + 3/6 + 4/6 + 5/6 + 6/6.

B2. Given 19 points in the plane with integer coordinates, no three collinear, show that we can always find three points whose centroid has integer coordinates.

B3. ABC is a triangle with $\angle C = 90^o$. E is a point on AC, and F is the midpoint of EC. CH is an altitude. I is the circumcenter of AHE, and G is the midpoint of BC. Show that ABC and IGF are similar.



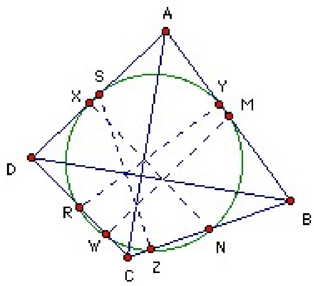
5th Mexican 1991

A1. Find the sum of all positive irreducible fractions less than 1 whose denominator is 1991.

A2. n is palindromic (so it reads the same backwards as forwards, eg 15651) and n = 2 mod 3, n = 3 mod 4, n = 0 mod 5. Find the smallest such positive integer. Show that there are infinitely many such positive integers.

A3. 4 spheres of radius 1 are placed so that each touches the other three. What is the radius of the smallest sphere that contains all 4 spheres?

B1. ABCD is a convex quadrilateral with AC perpendicular to BD. M, N, R, S are the midpoints of AB, BC, CD, DA. The feet of the perpendiculars from M, N, R, S to CD, DA, AB, BC are W, X, Y, Z. Show that M, N, R, S, W, X, Y, Z lie on the same circle.



B2. The sum of the squares of two consecutive positive integers can be a square, for example $3^2 + 4^2 = 5^2$. Show that the sum of the squares of 3 or 6 consecutive positive integers cannot be a square. Give an example of the sum of the squares of 11 consecutive positive integers which is a square.

B3. Let T be a set of triangles whose vertices are all vertices of an n-gon. Any two triangles in T have either 0 or 2 common vertices. Show that T has at most n members.

6th Mexican 1992

A1. The tetrahedron OPQR has the $\angle POQ = \angle POR = \angle QOR = 90^o$. X, Y, Z are the midpoints of PQ, QR and RP. Show that the four faces of the tetrahedron OXYZ have equal area.

A2. Given a prime number p, how many 4-tuples (a, b, c, d) of positive integers with $0 < a, b, c, d < p-1$ satisfy $ad = bc \bmod p$?

A3. Given 7 points inside or on a regular hexagon, show that three of them form a triangle with area $\leq 1/6$ the area of the hexagon.

B1. Show that $1 + 11^{11} + 111^{111} + 1111^{1111} + ... + 1111111111^{1111111111}$ is divisible by 100.

B2. x, y, z are positive reals with sum 3. Show that $6 < \sqrt{(2x+3)} + \sqrt{(2y+3)} + \sqrt{(2z+3)} < 3\sqrt{5}$.

B3. ABCD is a rectangle. I is the midpoint of CD. BI meets AC at M. Show that the line DM passes through the midpoint of BC. E is a point outside the rectangle such that AE = BE and $\angle AEB = 90^o$. If BE = BC = x, show that EM bisects $\angle AMB$. Find the area of AEBM in terms of x.

7th Mexican 1993

A1. ABC is a triangle with $\angle A = 90^o$. Take E such that the triangle AEC is outside ABC and AE = CE and $\angle AEC = 90^o$. Similarly, take D so that ADB is outside ABC and similar to AEC. O is the midpoint of BC. Let the lines OD and EC meet at D', and the lines OE and BD meet at E'. Find area DED'E' in terms of the sides of ABC.

A2. Find all numbers between 100 and 999 which equal the sum of the cubes of their digits.

A3. Given a pentagon of area 1993 and 995 points inside the pentagon, let S be the set containing the vertices of the pentagon and the 995 points. Show that we can find three points of S which form a triangle of area ≤ 1.

B1. f(n,k) is defined by (1) f(n,0) = f(n,n) = 1 and (2) f(n,k) = f(n-1,k-1) + f(n-1,k) for 0 < k < n. How many times do we need to use (2) to find f(3991,1993)?

B2. OA, OB, OC are three chords of a circle. The circles with diameters OA, OB meet again at Z, the circles with diameters OB, OC meet again at X, and the circles with diameters OC, OA meet again at Y. Show that X, Y, Z are collinear.

B3. p is an odd prime. Show that p divides n(n+1)(n+2)(n+3) + 1 for some integer n iff p divides m^2 - 5 for some integer m.

8th Mexican 1994

A1. The sequence 1, 2, 4, 5, 7, 9 ,10, 12, 14, 16, 17, ... is formed as follows. First we take one odd number, then two even numbers, then three odd numbers, then four even numbers, and so on. Find the number in the sequence which is closest to 1994.

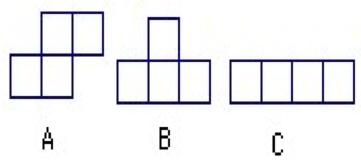
A2. The 12 numbers on a clock face are rearranged. Show that we can still find three adjacent numbers whose sum is 21 or more.

A3. ABCD is a parallelogram. Take E on the line AB so that BE = BC and B lies between A and E. Let the line through C perpendicular to BD and the line through E perpendicular to AB meet at F. Show that $\angle$DAF = $\angle$BAF.

B1. A capricious mathematician writes a book with pages numbered from 2 to 400. The pages are to be read in the following order. Take the last unread page (400), then read (in the usual order) all pages which are not relatively prime to it and which have not been read before. Repeat until all pages are read. So, the order would be 2, 4, 5, ... , 400, 3, 7, 9, ... , 399, ... What is the last page to be read?

B2. ABCD is a convex quadrilateral. Take the 12 points which are the feet of the altitudes in the triangles ABC, BCD, CDA, DAB. Show that at least one of these points must lie on the sides of ABCD.

B3. Show that we cannot tile a 10 x 10 board with 25 pieces of type A, or with 25 pieces of type B, or with 25 pieces of type C.



9th Mexican 1995

A1. N students are seated at desks in an m x n array, where m, n ≥ 3. Each student shakes hands with the students who are adjacent horizontally, vertically or diagonally. If there are 1020 handshakes, what is N?

A2. 6 points in the plane have the property that 8 of the distances between them are 1. Show that three of the points form an equilateral triangle with side 1.

A3. A, B, C, D are consecutive vertices of a regular 7-gon. AL and AM are tangents to the circle center C radius CB. N is the point of intersection of AC and BD. Show that L, M, N are collinear.

B1. Find 26 elements of {1, 2, 3, ... , 40} such that the product of two of them is never a square. Show that one cannot find 27 such elements.

B2. ABCDE is a convex pentagon such that the triangles ABC, BCD, CDE, DEA and EAB have equal area. Show that (1/4) area ABCDE < area ABC < (1/3) area ABCDE.

B3. A 1 or 0 is placed on each square of a 4 x 4 board. One is allowed to change each symbol in a row, or change each symbol in a column, or change each symbol in a diagonal (there are 14 diagonals of lengths 1 to 4). For which arrangements can one make changes which end up with all 0s?

10th Mexican 1996

A1. ABCD is a quadrilateral. P and Q are points on the diagonal BD such that the points are in the order B, P, Q, D and BP = PQ = QD. The line AP meets BC at E, and the line Q meets CD at F. Show that ABCD is a parallelogram iff E and F are the midpoints of their sides.

A2. 64 tokens are numbered 1, 2, ... , 64. The tokens are arranged in a circle around 1996 lamps which are all turned off. Each minute the tokens are all moved. Token number n is moved n places clockwise. More than one token is allowed to occupy the same place. After each move we count the number of tokens which occupy the same place as token 1 and turn on that number of lamps. Where is token 1 when the last lamp is turned on?

A3. Show that it is not possible to cover a 6 x 6 board with 1 x 2 dominos so that each of the 10 lines of length 6 that form the board (but do not lie along its border) bisects at least one domino. But show that we can cover a 5 x 6 board with 1 x 2 dominos so that each of the 9 lines of length 5 or 6 that form the board (but do not lie along its border) bisects at least one domino.

B1. For which n can we arrange the numbers 1, 2, 3, ... , 16 in a 4 x 4 array so that the eight row and column sums are all distinct and all multiples of n?

B2. Arrange the numbers 1, 2, 3, ... , n^2 in order in a n x n array (so that the first row is 1, 2, 3, ... , n, the second row is n+1, n+2, ... , 2n, and so on). For each path from 1 to n^2 which consists entirely of steps to the right and steps downwards, find the sum of the numbers in the path. Let M be the largest such sum and m the smallest. Show that M - m is a cube and that we cannot get the sum 1996 for a square of any size.

B3. ABC is an acute-angled triangle with AB < BC < AC. The points A', B', C' are such that AA' is perpendicular to BC and has the same length. Similarly, BB' is perpendicular to AC and has the same length, and CC' is perpendicular to AB and has the same length. The orthocenter H of ABC and A' are on the same side of A. Similarly, H and B' are on the same side of B, and H and C' are on the same side of C. Also $\angle AC'B = 90^o$. Show that A', B', C' are collinear.

11th Mexican 1997

A1. Find all primes p such that $8p^4$ - 3003 is a (positive) prime.

A2. ABC is a triangle with centroid G. P, P' are points on the side BC, Q is a point on the side AC, R is a point on the side AB, such that AR/RB = BP/PC = CQ/QA = CP'/P'B. The lines AP' and QR meet at K. Show that P, G and K are collinear.

A3. Show that it is possible to place the numbers 1, 2, ... , 16 on the squares of a 4 x 4 board (one per square), so that the numbers on two squares which share a side differ by at most 4. Show that it is not possible to place them so that the difference is at most 3.

B1. 3 non-collinear points in space determine a unique plane, which contains the points. What is the smallest number of planes determined by 6 points in space if no three points are collinear and the points do not all lie in the same plane?

B2. ABC is a triangle. P, Q, R are points on the sides BC, CA, AB such that BQ, CR meet at A', CR, AP meet at B', AP, BQ meet at C' and we have AB' = B'C', BC' = C'A', CA' = A'B'. Find area PQR/area ABC.

B3. Show that we can represent 1 as $1/5 + 1/a_1 + 1/a_2 + ... + 1/a_n$ (for positive integers a_i) in infinitely many different ways.

12th Mexican 1998

A1. Given a positive integer we can take the sum of the squares of its digits. If repeating this operation a finite number of times gives 1 we call the number *tame*. Show that there are infinitely many pairs (n, n+1) of consecutive tame integers.

A2. The lines L and L' meet at A. P is a fixed point on L. A variable circle touches L at P and meets L' at Q and R. The bisector of $\angle$QPR meets the circle again at T. Find the locus of T as the circle varies.

A3. Each side and diagonal of an octagon is colored red or black. Show that there are at least 7 triangles whose vertices are vertices of the octagon and whose sides are the same color.

B1. Find all positive integers that can be written as $1/a_1 + 2/a_2 + ... + 9/a_9$, where a_i are positive integers.

B2. AB, AC are the tangents from A to a circle. Q is a point on the segment AC. The line BQ meets the circle again at P. The line through Q parallel to AB meets BC at J. Show that PJ is parallel to AC iff $BC^2 = AC \cdot QC$.

B3. Given 5 points, no 4 in the same plane, how many planes can be equidistant from the points? (A plane is equidistant from the points if the perpendicular distance from each point to the plane is the same.)

13th Mexican 1999

A1. 1999 cards are lying on a table. Each card has a red side and a black side and can be either side up. Two players play alternately. Each player can remove any number of cards showing the same color from the table or turn over any number of cards of the same color. The winner is the player who removes the last card. Does the first or second player have a winning strategy?

A2. Show that there is no arithmetic progression of 1999 distinct positive primes all less than 12345.

A3. P is a point inside the triangle ABC. D, E, F are the midpoints of AP, BP, CP. The lines BF, CE meet at L; the lines CD, AF meet at M; and the lines AE, BD meet at N. Show that area DNELFM = (1/3) area ABC. Show that DL, EM, FN are concurrent.

B1. 10 squares of a chessboard are chosen arbitrarily and the center of each chosen square is marked. The side of a square of the board is 1. Show that either two of the marked points are a distance $\leq \sqrt{2}$ apart or that one of the marked points is a distance 1/2 from the edge of the board.

B2. ABCD has AB parallel to CD. The exterior bisectors of $\angle$B and $\angle$C meet at P, and the exterior bisectors of $\angle$A and $\angle$D meet at Q. Show that PQ is half the perimeter of ABCD.

B3. A polygon has each side integral and each pair of adjacent sides perpendicular (it is not necessarily convex). Show that if it can be covered by non-overlapping 2 x 1 dominos, then at least one of its sides has even length.

14th Mexican 2000

A1. A, B, C, D are circles such that A and B touch externally at P, B and C touch externally at Q, C and D touch externally at R, and D and A touch externally at S. A does not intersect C, and B does not intersect D. Show that PQRS is cyclic. If A and C have radius 2, B and D have radius 3, and the distance between the centers of A and C is 6, find area PQRS.

A2. A triangle is constructed like that below, but with 1, 2, 3, ... , 2000 as the first row. Each number is the sum of the two numbers immediately above. Find the number at the bottom of the triangle.

1 2 3 4 5

3 5 7 9

8 12 16

20 28

48

A3. If A is a set of positive integers, take the set A' to be all elements which can be written as $\pm a_1 \pm a_2 \dots \pm a_n$, where a_i are distinct elements of A. Similarly, form A" from A'. What is the smallest set A such that A" contains all of 1, 2, 3, ... , 40?

B1. Given positive integers a, b (neither a multiple of 5) we construct a sequence as follows: $a_1 = 5$, $a_{n+1} = a\, a_n + b$. What is the largest number of primes that can be obtained before the first composite member of the sequence?

B2. Given an n x n board with squares colored alternately black and white like a chessboard. An allowed move is to take a rectangle of squares (with one side greater than one square, and both sides odd or both sides even) and change the color of each square in the rectangle. For which n is it possible to end up with all the squares the same color by a sequence of allowed moves?

B3. ABC is a triangle with $\angle B > 90^o$. H is a point on the side AC such that AH = BH and BH is perpendicular to BC. D, E are the midpoints of AB, BC. The line through H parallel to AB meets DE at F. Show that $\angle BCF = \angle ACD$.

15th Mexican 2001

A1. Find all 7-digit numbers which are multiples of 21 and which have each digit 3 or 7.

A2. Given some colored balls (at least three different colors) and at least three boxes. The balls are put into the boxes so that no box is empty and we cannot find three balls of different colors which are in three different boxes. Show that there is a box such that all the balls in all the other boxes have the same color.

A3. ABCD is a cyclic quadrilateral. M is the midpoint of CD. The diagonals meet at P. The circle through P which touches CD at M meets AC again at R and BD again at Q. The point S on BD is such that BS = DQ. The line through S parallel to AB meets AC at T. Show that AT = RC.

B1. For positive integers n, m define f(n,m) as follows. Write a list of 2001 numbers a_i, where $a_1 = m$, and a_{k+1} is the residue of a_k^2 mod n (for k = 1, 2, ... , 2000). Then put $f(n,m) = a_1 - a_2 + a_3 - a_4 + a_5 - ... + a_{2001}$. For which $n \geq 5$ can we find m such that $2 \leq m \leq n/2$ and $f(m,n) > 0$?

B2. ABC is a triangle with AB < AC and $\angle A = 2 \angle C$. D is the point on AC such that CD = AB. Let L be the line through B parallel to AC. Let L meet the external bisector of $\angle A$ at M and the line through C parallel to AB at N. Show that MD = ND.

B3. A collector of rare coins has coins of denominations 1, 2, ... , n (several coins for each denomination). He wishes to put the coins into 5 boxes so that: (1) in each box there is at most one coin of each denomination; (2) each box has the same number of coins and the same denomination total; (3) any two boxes contain all the denominations; (4) no denomination is in all 5 boxes. For which n is this possible?

16th Mexican 2002

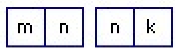
A1. The numbers 1 to 1024 are written one per square on a 32 x 32 board, so that the first row is 1, 2, ... , 32, the second row is 33, 34, ... , 64 and so on. Then the board is divided into four 16 x 16 boards and the position of these boards is moved round clockwise, so that AB goes to DA, DC goes to CB

then each of the 16 x 16 boards is divided into four equal 8 x 8 parts and each of these is moved around in the same way (within the 16 x 16 board). Then each of the 8 x 8 boards is divided into four 4 x 4 parts and these are moved around, then each 4 x 4 board is divided into 2 x 2 parts which are moved around, and finally the squares of each 2 x 2 part are moved around. What numbers end up on the main diagonal (from the top left to bottom right)?

A2. ABCD is a parallelogram. K is the circumcircle of ABD. The lines BC and CD meet K again at E and F. Show that the circumcenter of CEF lies on K.

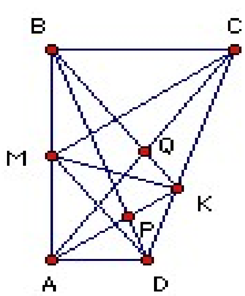
A3. Does n^2 have more divisors = 1 mod 4 or = 3 mod 4?

B1. A domino has two numbers (which may be equal) between 0 and 6, one at each end. The domino may be turned around. There is one domino of each type, so 28 in all. We want to form a chain in the usual way, so that adjacent dominos have the same number at the adjacent ends. Dominos can be added to the chain at either end. We want to form the chain so that after each domino has been added the total of all the numbers is odd. For example, we could place first the domino (3,4), total 3 + 4 = 7. Then (1,3), total 1 + 3 + 3 + 4 = 11, then (4,4), total 11 + 4 + 4 = 19. What is the largest number of dominos that can be placed in this way? How many maximum-length chains are there?



B2. A *trio* is a set of three distinct integers such that two of the numbers are divisors or multiples of the third. Which trio contained in {1, 2, ... , 2002} has the largest possible sum? Find all trios with the maximum sum.

B3. ABCD is a quadrilateral with ∠A = ∠B = 90^o. M is the midpoint of AB and ∠CMD = 90^o. K is the foot of the perpendicular from M to CD. AK meets BD at P, and BK meets AC at Q. Show that ∠AKB = 90^o and KP/PA + KQ/QB = 1.



17th Mexican 2003

A1. Find all positive integers with two or more digits such that if we insert a 0 between the units and tens digits we get a multiple of the original number.

A2. A, B, C are collinear with B betweeen A and C. K_1 is the circle with diameter AB, and K_2 is the circle with diameter BC. Another circle touches AC at B and meets K_1 again at P and K_2 again at Q. The line PQ meets K_1 again at R and K_2 again at S. Show that the lines AR and CS meet on the perpendicular to AC at B.

A3. At a party there are n women and n men. Each woman likes r of the men, and each man likes r of then women. For which r and s must there be a man and a woman who like each other?

B1. The quadrilateral ABCD has AB parallel to CD. P is on the side AB and Q on the side CD such that AP/PB = DQ/CQ. M is the intersection of AQ and DP, and N is the intersection of PC and QB. Find MN in terms of AB and CD.

B2. Some cards each have a pair of numbers written on them. There is just one card for each pair (a,b) with $1 \le a < b \le 2003$. Two players play the following game. Each removes a card in turn and writes the product ab of its numbers on the blackboard. The first player who causes the greatest common divisor of the numbers on the blackboard to fall to 1 loses. Which player has a winning strategy?

B3. Given a positive integer n, an allowed move is to form 2n+1 or 3n+2. The set S_n is the set of all numbers that can be obtained by a sequence of allowed moves starting with n. For example, we can form $5 \rightarrow 11 \rightarrow 35$ so 5, 11 and 35 belong to S_5. We call m and n *compatible* if $S_m \cap S_n$ is non-empty. Which members of {1, 2, 3, ... , 2002} are compatible with 2003?

Polish (1983 – 2003)

34th Polish 1983

A1. The angle bisectors of the angles A, B, C in the triangle ABC meet the circumcircle again at K, L, M. Show that |AK| + |BL| + |CM| > |AB| + |BC| + |CA|.

A2. For given n, we choose k and m at random subject to $0 \le k \le m \le 2^n$. Let p_n be the probability that the binomial coefficient mCk is even. Find $\lim_{n\to\infty} p_n$.

A3. Q is a point inside the n-gon $P_1P_2...P_n$ which does not lie on any of the diagonals. Show that if n is even, then Q must lie inside an even number of triangles $P_iP_jP_k$.

B1. Given a real numbers $x \in (0,1)$ and a positive integer N, prove that there exist positive integers a, b, c, d such that (1) a/b < x < c/d, (2) c/d - a/b < 1/n, and (3) qr - ps = 1.

B2. There is a piece in each square of an m x n rectangle on an infinite chessboard. An allowed move is to remove two pieces which are adjacent horizontally or vertically and to place a piece in an empty square adjacent to the two removed and in line with them (as shown below)

X X . to . . X, or . to X

X .

X .

Show that if mn is a multiple of 3, then it is not possible to end up with only one piece after a sequence of moves.

B3. Show that if the positive integers a, b, c, d satisfy ab = cd, then we have gcd(a,c) gcd(a,d) = a gcd(a,b,c,d).

35th Polish 1984

A1. X is a set with $n > 2$ elements. Is there a function $f : X \to X$ such that the composition f^{n-1} is constant, but f^{n-2} is not constant?

A2. Given n we define $a_{i,j}$ as follows. For i, j = 1, 2, ... , n, $a_{i,j} = 1$ for j = i, and 0 for $j \neq i$. For i = 1, 2, ... , n, j = n+1, ... , 2n, $a_{i,j} = -1/n$. Show that for any permutation p of (1, 2, ... , 2n) we have $\sum_{i=1}^n |\sum_{k=1}^n a_{i,p(k)}| \geq n/2$.

A3. W is a regular octahedron with center O. P is a plane through the center O. K(O, r_1) and K(O, r_2) are circles center O and radii r_1, r_2 such that K(O, r_1) □ P∩W □ K(O, r_2). Show that $r_1/r_2 \leq (\sqrt{3})/2$.

B1. We throw a coin n times and record the results as the sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$, using 1 for head, 2 for tail. Let $\beta_j = \alpha_1 + \alpha_2 + \ldots + \alpha_j$ and let p(n) be the probability that the sequence $\beta_1, \beta_2, \ldots, \beta_n$ includes the value n. Find p(n) in terms of p(n-1) and p(n-2).

B2. Six disks with diameter 1 are placed so that they cover the edges of a regular hexagon with side 1. Show that no vertex of the hexagon is covered by two or more disks.

B3. There are 1025 cities, $P_1, \ldots, P_{1025}$ and ten airlines $A_1, \ldots, A_{10}$, which connect some of the cities. Given any two cities there is at least one airline which has a direct flight between them. Show that there is an airline which can offer a round trip with an odd number of flights.

36th Polish 1985

A1. Find the largest k such that for every positive integer n we can find at least k numbers in the set {n+1, n+2, ... , n+16} which are coprime with n(n+17).

A2. Given a square side 1 and 2n positive reals $a_1, b_1, \ldots, a_n, b_n$ each ≤ 1 and satisfying $\sum a_ib_i \geq 100$. Show that the square can be covered with rectangles R_i with sides length (a_i, b_i) parallel to the square sides.

A3. The function f : R → R satisfies $f(3x) = 3f(x) - 4f(x)^3$ for all real x and is continuous at x = 0. Show that $|f(x)| \leq 1$ for all x.

B1. P is a point inside the triangle ABC is a triangle. The distance of P from the lines BC, CA, AB is d_a, d_b, d_c respectively. Show that $2/(1/d_a + 1/d_b + 1/d_c) < r < (d_a + d_b + d_c)/2$, where r is the inradius.

B2. p(x,y) is a polynomial such that p(cos t, sin t) = 0 for all real t. Show that there is a polynomial q(x,y) such that $p(x,y) = (x^2 + y^2 - 1)\, q(x,y)$.

B3. There is a convex polyhedron with k faces. Show that if k/2 of the faces are such that no two have a common edge, then the polyhedron cannot have an inscribed sphere.

37th Polish 1986

A1. A square side 1 is covered with m^2 rectangles. Show that there is a rectangle with perimeter at least 4/m.

A2. Find the maximum possible volume of a tetrahedron which has three faces with area 1.

A3. p is a prime and m is a non-negative integer < p-1. Show that $\sum_{j=1}^{p} j^m$ is divisible by p.

B1. Find all n such that there is a real polynomial f(x) of degree n such that f(x) ≥ f '(x) for all real x.

B2. There is a chess tournament with 2n players (n > 1). There is at most one match between each pair of players. If it is not possible to find three players who all play each other, show that there are at most n^2 matches. Conversely, show that if there are at most n^2 matches, then it is possible to arrange them so that we cannot find three players who all play each other.

B3. ABC is a triangle. The feet of the perpendiculars from B and C to the angle bisector at A are K, L respectively. N is the midpoint of BC, and AM is an altitude. Show that K,L,N,M are concyclic.

38th Polish 1987

A1. There are $n \geq 2$ points in a square side 1. Show that one can label the points $P_1, P_2, \ldots, P_n$ such that $\sum_{i=1}^{n} |P_{i-1} - P_i|^2 \leq 4$, where we use cyclic subscripts, so that P_0 means P_n.

A2. A regular n-gon is inscribed in a circle radius 1. Let X be the set of all arcs PQ, where P, Q are distinct vertices of the n-gon. 5 elements $L_1, L_2, \ldots, L_5$ of X are chosen at random (so two or more of the L_i can be the same). Show that the expected length of $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5$ is independent of n.

A3. w(x) is a polynomial with integral coefficients. Let p_n be the sum of the digits of the number w(n). Show that some value must occur infinitely often in the sequence $p_1, p_2, p_3, \ldots$.

B1. Let S be the set of all tetrahedra which satisfy (1) the base has area 1, (2) the total face area is 4, and (3) the angles between the base and the other three faces are all equal. Find the element of S which has the largest volume.

B2. Find the smallest n such that n^2-n+11 is the product of four primes (not necessarily distinct).

B3. A plane is tiled with regular hexagons of side 1. A is a fixed hexagon vertex. Find the number of paths P such that (1) one endpoint of P is A, (2) the other endpoint of P is a hexagon vertex, (3) P lies along hexagon edges, (4) P has length 60, and (5) there is no shorter path along hexagon edges from A to the other endpoint of P.

39th Polish 1988

A1. The real numbers $x_1, x_2, ... , x_n$ belong to the interval (0,1) and satisfy $x_1 + x_2 + ... + x_n = m + r$, where m is an integer and $r \in [0,1)$. Show that $x_1^2 + x_2^2 + ... + x_n^2 \leq m + r^2$.

A2. For a permutation $P = (p_1, p_2, ... , p_n)$ of (1, 2, ... , n) define X(P) as the number of j such that $p_i < p_j$ for every $i < j$. What is the expected value of X(P) if each permutation is equally likely?

A3. W is a polygon. W has a center of symmetry S such that if P belongs to W, then so does P', where S is the midpoint of PP'. Show that there is a parallelogram V containing W such that the midpoint of each side of V lies on the border of W.

B1. d is a positive integer and $f : [0,d] \to R$ is a continuous function with f(0) = f(d). Show that there exists $x \in [0,d-1]$ such that f(x) = f(x+1).

B2. The sequence $a_1, a_2, a_3, ...$ is defined by $a_1 = a_2 = a_3 = 1$, $a_{n+3} = a_{n+2}a_{n+1} + a_n$. Show that for any positive integer r we can find s such that a_s is a multiple of r.

B3. Find the largest possible volume for a tetrahedron which lies inside a hemisphere of radius 1.

40th Polish 1989

A1. An even number of politicians are sitting at a round table. After a break, they come back and sit down again in arbitrary places. Show that there must be two people with the same number of people sitting between them as before the break.

A2. k_1, k_2, k_3 are three circles. k_2 and k_3 touch externally at P, k_3 and k_1 touch externally at Q, and k_1 and k_2 touch externally at R. The line PQ meets k_1 again at S, the line PR meets k_1 again at T. The line RS meets k_2 again at U, and the line QT meets k_3 again at V. Show that P, U, V are collinear.

A3. The edges of a cube are labeled from 1 to 12. Show that there must exist at least eight triples (i, j, k) with $1 \le i < j < k \le 12$ so that the edges i, j, k are consecutive edges of a path. But show that the labeling can be done so that we cannot find nine such triples.

B1. n, k are positive integers. A_0 is the set {1, 2, ... , n}. A_i is a randomly chosen subset of A_{i-1} (with each subset having equal probability). Show that the expected number of elements of A_k is $n/2^k$.

B2. Three circles of radius a are drawn on the surface of a sphere of radius r. Each pair of circles touches externally and the three circles all lie in one hemisphere. Find the radius of a circle on the surface of the sphere which touches all three circles.

B3. Show that for positive reals a, b, c, d we have $((ab + ac + ad + bc + bd + cd)/6)^{1/2} \ge ((abc + abd + acd + bcd)/4)^{1/3}$.

41st Polish 1990

A1. Find all real-valued functions f on the reals such that $(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2-y^2)$ for all x, y.

A2. For $n > 1$ and positive reals $x_1, x_2, \dots, x_n$, show that $x_1^2/(x_1^2+x_2x_3) + x_2^2/(x_2^2+x_3x_4) + \dots + x_n^2/(x_n^2+x_1x_2) \le n-1$.

A3. In a tournament there are n players. Each pair of players play each other just once. There are no draws. Show that either (1) one can divide the players into two groups A and B, such that every player in A beat every player in B, or (2) we can label the players $P_1, P_2, \dots, P_n$ such that P_i beat P_{i+1} for i = 1, 2, ... n (where we use cyclic subscripts, so that P_{n+1} means P_1).

B1. A triangle with each side length at least 1 lies inside a square side 1. Show that the center of the square lies inside the triangle.

B2. $a_1, a_2, a_3, \dots$ is a sequence of positive integers such that $\lim_{n\to\infty} n/a_n = 0$. Show that we can find k such that there are at least 1990 squares between $a_1 + a_2 + \dots + a_k$ and $a_1 + a_2 + \dots + a_{k+1}$.

B3. Show that $\sum_{k=0}^{[n/3]} (-1)^k$ nC3k is a multiple of 3 for $n > 2$. (nCm is the binomial coefficient).

42nd Polish 1991

A1. Do there exist tetrahedra T_1, T_2 such that (1) vol T_1 > vol T_2, and (2) every face of T_2 has larger area than any face of T_1?

A2. Let F(n) be the number of paths P_0, P_1, ... , P_n of length n that go from P_0 = (0,0) to a lattice point P_n on the line y = 0, such that each P_i is a lattice point and for each i < n, P_i and P_{i+1} are adjacent lattice points a distance 1 apart. Show that F(n) = (2n)Cn.

A3. N is a number of the form $\sum_{k=1}^{60} a_k k^{kk}$, where each a_k = 1 or -1. Show that N cannot be a 5th power.

B1. Let V be the set of all vectors (x,y) with integral coordinates. Find all real-valued functions f on V such that (a) f(**v**) = 1 for all **v** of length 1; (b) f(**v** + **w**) = f(**v**) + f(**w**) for all perpendicular **v**, **w** $\in$ V. (The vector (0,0) is considered to be perpendicular to any vector.)

B2. k_1, k_2 are circles with different radii and centers K_1, K_2. Neither lies inside the other, and they do not touch or intersect. One pair of common tangents meet at A on K_1K_2, the other pair meet at B on K_1K_2. P is any point on k_1. Show that there is a diameter of K_2 with one endpoint on the line PA and the other on the line PB.

B3. The real numbers x, y, z satisfy $x^2 + y^2 + z^2 = 2$. Show that $x + y + z \le 2 + xyz$. When do we have equality?

43rd Polish 1992

A1. Segments AC and BD meet at P, and |PA| = |PD|, |PB| = |PC|. O is the circumcenter of the triangle PAB. Show that OP and CD are perpendicular.

A2. Find all functions $f : Q^+ \to Q^+$, where Q^+ is the positive rationals, such that $f(x+1) = f(x) + 1$ and $f(x^3) = f(x)^3$ for all x.

A3. Show that for real numbers $x_1, x_2, \ldots , x_n$ we have $\sum_{i=1}^m (\sum_{j=1}^n x_i x_j/(i+j)) \geq 0$. When do we have equality?

B1. The functions $f_0, f_1, f_2, \ldots$ are defined on the reals by $f_0(x) = 8$ for all x, $f_{n+1}(x) = \sqrt{(x^2 + 6f_n(x))}$. For all n solve the equation $f_n(x) = 2x$.

B2. The base of a regular pyramid is a regular 2n-gon $A_1A_2 \ldots A_{2n}$. A sphere passes through the apex S of the pyramid and cuts the edge SA_i at B_i (for i = 1, 2, ... , 2n). Show that $\sum SB_{2i-1} = \sum SB_{2i}$.

B3. Show that $k^3!$ is divisible by $(k!)^{k2+k+1}$.

44th Polish 1993

A1. Find all rational solutions to:

$t^2 - w^2 + z^2 = 2xy$

$t^2 - y^2 + w^2 = 2xz$

$t^2 - w^2 + x^2 = 2yz$.

A2. A circle center O is inscribed in the quadrilateral ABCD. AB is parallel to and longer than CD and has midpoint M. The line OM meets CD at F. CD touches the circle at E. Show that DE = CF iff AB = 2CD.

A3. g(k) is the greatest odd divisor of k. Put f(k) = k/2 + k/g(k) for k even, and $2^{(k+1)/2}$ for k odd. Define the sequence $x_1, x_2, x_3, \ldots$ by $x_1 = 1$, $x_{n+1} = f(x_n)$. Find n such that $x_n = 800$.

B1. P is a convex polyhedron with all faces triangular. The vertices of P are each colored with one of three colors. Show that the number of faces with three vertices of different colors is even.

B2. Find all real-valued functions f on the reals such that f(-x) = -f(x), f(x+1) = f(x) + 1 for all x, and $f(1/x) = f(x)/x^2$ for $x \neq 0$.

B3. Is the volume of a tetrahedron determined by the areas of its faces and its circumradius?

45th Polish 1994

A1. Find all triples (x,y,z) of positive rationals such that x + y + z, 1/x + 1/y + 1/z and xyz are all integers.

A2. L, L' are parallel lines. C is a circle that does not intersect L. A is a variable point on L. The two tangents to C from A meet L' in two points with midpoint M. Show that the line AM passes through a fixed point (as A varies).

A3. k is a fixed positive integer. Let a_n be the number of maps f from the subsets of {1, 2, ... , n} to {1, 2, ... , k} such that for all subsets A, B of {1, 2, ... , n} we have f(A ∩ B) = min(f(A), f(B)). Find $\lim_{n\to\infty} a_n^{1/n}$.

B1. m, n are relatively prime. We have three jugs which contain m, n and m+n liters. Initially the largest jug is full of water. Show that for any k in {1, 2, ... , m+n} we can get exactly k liters into one of the jugs.

B2. A parallelepiped has vertices $A_1, A_2, \dots, A_8$ and center O. Show that $4 \sum |OA_i|^2 \le (\sum|OA_i|)^2$.

B3. The distinct reals $x_1, x_2, \dots, x_n$ (n > 3) satisfy $\sum x_i = 0$, &sum $x_i^2 = 1$. Show that four of the numbers a, b, c, d must satisfy $a + b + c + nabc \le \sum x_i^3 \le a + b + d + nabd$.

46th Polish 1995

A1. How many subsets of {1, 2, ... , 2n} do not contain two numbers with sum 2n+1?

A2. The diagonals of a convex pentagon divide it into a small pentagon and ten triangles. What is the largest number of the triangles that can have the same area?

A3. $p \geq 5$ is prime. The sequence $a_0, a_1, a_2, \ldots$ is defined by $a_0 = 1, a_1 = 1, \ldots, a_{p-1} = p-1$ and $a_n = a_{n-1} + a_{n-p}$ for $n \geq p$. Find a_p^3 mod p.

B1. The positive reals $x_1, x_2, \ldots, x_n$ have harmonic mean 1. Find the smallest possible value of $x_1 + x_2^2/2 + x_3^3/3 + \ldots + x_n^n/n$.

B2. An urn contains n balls labeled 1, 2, ... , n. We draw the balls out one by one (without replacing them) until we obtain a ball whose number is divisible by k. Find all k such that the expected number of balls removed is k.

B3. PA, PB, PC are three rays in space. Show that there is just one pair of points B', C' with B' on the ray PB and C' on the ray PC such that PC' + B'C' = PA + AB' and PB' + B'C' = PA + AC'.

47th Polish 1996

A1. Find all pairs (n,r) with n a positive integer and r a real such that $2x^2+2x+1$ divides $(x+1)^n - r$.

A2. P is a point inside the triangle ABC such that ∠PBC = ∠PCA < ∠PAB. The line PB meets the circumcircle of ABC again at E. The line CE meets the circumcircle of APE again at F. Show that area APEF/area ABP does not depend on P.

A3. a_i, x_i are positive reals such that $a_1 + a_2 + ... + a_n = x_1 + x_2 + ... + x_n = 1$. Show that $2 \sum_{i<j} x_i x_j \le (n-2)/(n-1) + \sum a_i x_i^2/(1-a_i)$. When do we have equality?

B1. ABCD is a tetrahedron with ∠BAC = ∠ACD and ∠ABD = ∠BDC. Show that AB = CD.

B2. Let p(k) be the smallest prime not dividing k. Put q(k) = 1 if p(k) = 2, or the product of all primes < p(k) if p(k) > 2. Define the sequence $x_0, x_1, x_2, ...$ by $x_0 = 1$, $x_{n+1} = x_n p(x_n)/q(x_n)$. Find all n such that $x_n = 111111$.

B3. Let S be the set of permutations $a_1 a_2 ... a_n$ of 123...n such that $a_i \ge i$. An element of S is chosen at random. Find all n such that the probability that the chosen permutation satisfies $a_i \le i+1$ exceeds 1/3.

48th Polish 1997

A1. The positive integers $x_1, x_2, \ldots, x_7$ satisfy $x_6 = 144$, $x_{n+3} = x_{n+2}(x_{n+1}+x_n)$ for n = 1, 2, 3, 4. Find x_7.

A2. Find all real solutions to $3(x^2 + y^2 + z^2) = 1$, $x^2y^2 + y^2z^2 + z^2x^2 = xyz(x + y + z)^3$.

A3. ABCD is a tetrahedron. DE, DF, DG are medians of triangles DBC, DCA, DAB. The angles between DE and BC, between DF and CA, and between DG and AB are equal. Show that area DBC ≤ area DCA + area DAB.

B1. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 0$, $a_n = a_{[n/2]} + (-1)^{n(n+1)/2}$. Show that for any positive integer k we can find n in the range $2^k \le n < 2^{k+1}$ such that $a_n = 0$.

B2. ABCDE is a convex pentagon such that DC = DE and $\angle C = \angle E = 90^o$. F is a point on the side AB such that AF/BF = AE/BC. Show that $\angle FCE = \angle FDE$ and $\angle FEC = \angle BDC$.

B3. Given any n points on a unit circle show that at most $n^2/3$ of the segments joining two points have length $> \sqrt{2}$.

49th Polish 1998

A1. Find all solutions in positive integers to $a + b + c = xyz$, $x + y + z = abc$.

A2. Fn is the Fibonacci sequence $F_0 = F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$. Find all pairs $m > k \geq 0$ such that the sequence $x_0, x_1, x_2, \ldots$ defined by $x_0 = F_k/F_m$ and $x_{n+1} = (2x_n - 1)/(1 - x_n)$ for $x_n \neq 1$, or 1 if $x_n = 1$, contains the number 1.

A3. PABCDE is a pyramid with ABCDE a convex pentagon. A plane meets the edges PA, PB, PC, PD, PE in points A', B', C', D', E' distinct from A, B, C, D, E and P. For each of the quadrilaterals ABB'A', BCC'B, CDD'C', DEE'D', EAA'E' take the intersection of the diagonals. Show that the five intersections are coplanar.

B1. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 1$, $a_n = a_{n-1} + a_{[n/2]}$. Does the sequence contain infinitely many multiples of 7?

B2. The points D, E on the side AB of the triangle ABC are such that $(AD/DB)(AE/EB) = (AC/CB)^2$. Show that $\angle ACD = \angle BCE$.

B3. S is a board containing all unit squares in the xy plane whose vertices have integer coordinates and which lie entirely inside the circle $x^2 + y^2 = 1998^2$. +1 is written in each square of S. An allowed move is to change the sign of every square in S in a given row, column or diagonal. Can we end up with all -1s by a sequence of allowed moves?

50th Polish 1999

A1. D is a point on the side BC of the triangle ABC such that AD > BC. E is a point on the side AC such that AE/EC = BD/(AD-BC). Show that AD > BE.

A2. Given 101 distinct non-negative integers less than 5050 show that one choose four a, b, c, d such that a + b - c - d is a multiple of 5050.

A3. Show that one can find 50 distinct positive integers such that the sum of each number and its digits is the same.

B1. For which n do the equations have a solution in integers:

$x_1^2 + x_2^2 + 50 = 16x_1 + 12x_2$

$x_2^2 + x_3^2 + 50 = 16x_2 + 12x_3$

...

$x_{n-1}^2 + x_n^2 + 50 = 16x_{n-1} + 12x_n$

$x_n^2 + x_1^2 + 50 = 16x_n + 12x_1$

B2. Show that $\sum_{1\le i<j\le n} (|a_i-a_j| + |b_i-b_j|) \le \sum_{1\le i<j\le n} |a_i-b_j|$ for all integers a_i, b_i.

B3. The convex hexagon ABCDEF satisfies $\angle A + \angle C + \angle E = 360^o$ and AB·CD·EF = BC·DE·FA. Show that AB·FD·EC = BF·DE·CA.

51st Polish 2000

A1. How many solutions in non-negative reals are there to the equations:

$x_1 + x_n^2 = 4x_n$

$x_2 + x_1^2 = 4x_1$

...

$x_n + x_{n-1}^2 = 4x_{n-1}$?

A2. The triangle ABC has AC = BC. P is a point inside the triangle such that $\angle PAB = \angle PBC$. M is the midpoint of AB. Show that $\angle APM + \angle BPC = 180^o$.

A3. The sequence $a_1, a_2, a_3, \ldots$ is defined as follows. a_1 and a_2 are primes. a_n is the greatest prime divisor of $a_{n-1} + a_{n-2} + 2000$. Show that the sequence is bounded.

B1. $PA_1A_2...A_n$ is a pyramid. The base $A_1A_2...A_n$ is a regular n-gon. The apex P is placed so that the lines PA_i all make an angle 60^o with the plane of the base. For which n is it possible to find B_i on PA_i for i = 2, 3, ... , n such that $A_1B_2 + B_2B_3 + B_3B_4 + \ldots + B_{n-1}B_n + B_nA_1 < 2A_1P$?

B2. For each $n \geq 2$ find the smallest k such that given any subset S of k squares on an n x n chessboard we can find a subset T of S such that every row and column of the board has an even number of squares in T.

B3. p(x) is a polynomial of odd degree which satisfies $p(x^2-1) = p(x)^2 - 1$ for all x. Show that p(x) = x.

52nd Polish 2001

A1. Show that $x_1 + 2x_2 + 3x_3 + ... + nx_n \le ½n(n-1) + x_1 + x_2^2 + x_3^3 + ... + x_n^n$ for all non-negative reals x_i.

A2. P is a point inside a regular tetrahedron with edge 1. Show that the sum of the distances from P to the vertices is at most 3.

A3. The sequence $x_1, x_2, x_3, ...$ is defined by $x_1 = a$, $x_2 = b$, $x_{n+2} = x_{n+1} + x_n$, where a and b are reals. A number c is a *repeated value* if it occurs in the sequence more than once. Show that we can choose a, b so that the sequence has more than 2000 repeated values, but not so that it has infinitely many repeated values.

B1. a and b are integers such that $2^n a + b$ is a square for all non-negative integers n. Show that a = 0.

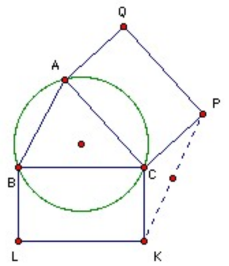
B2. ABCD is a parallelogram. K is a point on the side BC and L is a point on the side CD such that BK·AD = DL·AB. DK and BL meet at P. Show that ∠DAP = ∠BAC.

B3. Given a set of 2000 distinct positive integers under 10^{100}, show that one can find two non-empty disjoint subsets which have the same number of elements, the same sum and the same sum of squares.

53rd Polish 2002

A1. Find all triples of positive integers (a, b, c) such that $a^2 + 1$ and $b^2 + 1$ are prime and $(a^2 + 1)(b^2 + 1) = c^2 + 1$.

A2. ABC is an acute-angled triangle. BCKL, ACPQ are rectangles on the outside of two of the sides and have equal area. Show that the midpoint of PK lies on the line through C and the circumcenter.



A3. Three non-negative integers are written on a blackboard. A move is to replace two of the integers by their sum and (non-negative) difference. Can we always get two zeros by a sequence of moves?

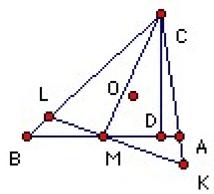
B1. Given any finite sequence x1, x2, ... , xn of at least 3 positive integers, show that either $\sum_1^n x_i/(x_{i+1} + x_{i+2}) \geq n/2$ or $\sum_1^n x_i/(x_{i-1} + x_{i-2}) \geq n/2$. (We use the cyclic subscript convention, so that x_{n+1} means x_1 and x_{-1} means x_{n-1} etc).

B2. ABC is a triangle. A sphere does not intersect the plane of ABC. There are 4 points K, L, M, P on the sphere such that AK, BL, CM are tangent to the sphere and AK/AP = BL/BP = CM/CP. Show that the sphere touches the circumsphere of ABCP.

B3. k is a positive integer. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = k+1$, $a_{n+1} = a_n^2 - ka_n + k$. Show that a_m and a_n are coprime (for $m \neq n$).

54th Polish 2003

A1. ABC is acute-angled. M is the midpoint of AB. A line through M meets the lines CA, CB at K, L with CK = CL. O is the circumcenter of CKL and CD is an altitude of ABC. Show that OD = OM.



A2. $0 \le k_1 < k_2 < ... < k_n$ are integers. $0 < a < 1$ is a real. Show that $(1-a)(a^{k_1} + a^{k_2} + ... + a^{k_n})^2 < (1+a)(a^{2k_1} + a^{2k_2} + ... + a^{2k_n})$.

A3. Find all polynomials p(x) with integer coefficients such that p(n) divides $2^n - 1$ for n = 1, 2, 3, ...

B1. p is a prime and a, b, c, are distinct positive integers less than p such that $a^3 = b^3 = c^3$ mod p. Show that $a^2 + b^2 + c^2$ is divisible by a + b + c.

B2. ABCD is a tetrahedron. The insphere touches the face ABC at H. The exsphere opposite D (which also touches the face ABC and the three planes containing the other faces) touches the face ABC at O. If O is the circumcenter of ABC, show that H is the orthocenter of ABC.

B3. n is even. Show that there is a permutation $a_1a_2...a_n$ of 12...n such that $a_{i+1} \in \{2a_i, 2a_i-1, 2a_i-n, 2a_i-n-1\}$ for i = 1, 2, ... , n (and we use the cyclic subscript convention, so that a_{n+1} means a_1).

Spanish (1990 – 2003)

26th Spanish 1990

A1. Show that $\sqrt{x} + \sqrt{y} + \sqrt{(xy)} = \sqrt{x} + \sqrt{(y + xy + 2y\sqrt{x})}$. Hence show that $\sqrt{3} + \sqrt{(10 + 2\sqrt{3})} = \sqrt{(5 + \sqrt{22})} + \sqrt{(8 - \sqrt{22} + 2\sqrt{(15 - 3\sqrt{22})})}$.

A2. Every point of the plane is painted with one of three colors. Can we always find two points a distance 1 apart which are the same color?

A3. Show that $[(4 + \sqrt{11})^n]$ is odd for any positive integer n.

B1. Show that $((a+1)/2 + ((a+3)/6)\sqrt{((4a+3)/3)}\ )^{1/3} + ((a+1)/2 - ((a+3)/6)\sqrt{((4a+3)/3)}\ )^{1/3}$ is independent of a for $a \geq 3/4$ and find it.

B2. ABC is a triangle with area S. Points A', B', C' are taken on the sides BC, CA, AB, so that AC'/AB = BA'/BC = CB'/CA = k, where $0 < k < 1$. Find the area of A'B'C' in terms of S and k. Find the value of k which minimises the area. The line through A' parallel to AB and the line through C' parallel to AC meet at P. Find the locus of P as k varies.

B3. There are n points in the plane so that no two pairs are the same distance apart. Each point is connected to the nearest point by a line. Show that no point is connected to more than 5 points.

27th Spanish 1991

A1. Let S be the set of all points in the plane with integer coordinates. Let T be the set of all segments AB, where A, B $\in$ S and AB has integer length. Prove that we cannot find two elements of T making an angle 45^o. Is the same true in three dimensions?

A2. a, b are distinct elements of {0,1,-1}. A is the matrix:

$$\begin{matrix} a+b & a+b^2 & a+b^3 & \dots & a+b^m \\ a^2+b & a^2+b^2 & a^2+b^3 & \dots & a^2+b^m \\ a^3+b & a^3+b^2 & a^3+b^3 & \dots & a^3+b^m \\ & & \dots & & \\ a^n+b & a^n+b^2 & a^n+b^3 & \dots & a^n+b^m \end{matrix}$$

Find the smallest possible number of columns of A such that any other column is a linear combination of these columns with integer coefficients.

A3. What condition must be satisfied by the coefficients u, v, w if the roots of the polynomial $x^3 - ux^2 + vx - w$ can be the sides of a triangle?

B1. The incircle of ABC touches BC, CA, AB at A', B', C' respectively. The line A'C' meets the angle bisector of A at D. Find $\angle$ ADC.

B2. Let s(n) be the sum of the binary digits of n. Find s(1) + s(2) + s(3) + ... + s(2^k) for each positive integer k.

B3. Find the integral part of $1/\sqrt{1} + 1/\sqrt{2} + 1/\sqrt{3} + \dots + 1/\sqrt{1000}$.

28th Spanish 1992

A1. Find the smallest positive integer N which is a multiple of 83 and is such that N^2 has exactly 63 positive divisors.

A2. Given two circles (neither inside the other) with different radii, a line L, and k > 0, show how to construct a line L' parallel to L so that L intersects the two circles in chords with total length k.

A3. a, b, c, d are positive integers such that $(a+b)^2 + 2a + b = (c+d)^2 + 2c + d$. Show that a = c and b = d. Show that the same is true if a, b, c, d satisfy $(a+b)^2 + 3a + b = (c+d)^2 + 3c + d$. But show that there exist a, b, c, d such that $(a+b)^2 + 4a + b = (c+d)^2 + 4c + d$, but $a \neq c$ and $b \neq d$.

B1. Show that there are infinitely many primes in the arithmetic progression 3, 7, 11, 15, ...

B2. Given the triangle ABC, show how to find geometrically the point P such that $\angle PAB = \angle PBC = \angle PCA$. Express this angle in terms of $\angle A$, $\angle B$, $\angle C$ using trigonometric functions.

B3. For each positive integer n let S(n) be the set of complex numbers z such that |z| = 1 and $(z + 1/z)^n = 2^{n-1}(z^n + 1/z^n)$. Find S(2), S(3), S(4). Find an upper bound for |S(n)| for $n \geq 5$.

29th Spanish 1993

A1. There is a reunion of 201 people from 5 different countries. In each group of 6 people, at least two have the same age. Show that there must be at least 5 people with the same country, age and sex.

A2. In the triangle of numbers below, each number is the sum of the two immediately above:

0 1 2 3 4 ... 1991 1992 1993

1 3 5 7 ... 3983 3985

4 8 12 ... 7968

...

Show that the bottom number is a multiple of 1993.

A3. Show that for any triangle $2r \le R$ (where r is the inradius and R is the circumradius).

B1. Show that for any prime $p \neq 2, 5$, infinitely many numbers of the form 11...1 are multiples of p.

B2. Given a 4 x 4 grid of points as shown below. The points at two opposite corners are marked A and D as shown. How many ways can we choose a set of two further points {B,C} so that the six distances between A, B, C, D are all distinct?

How many of the sets of 4 points are geometrically distinct (so that one cannot be obtained from another by a reflection, rotation etc)? Give the points coordinates (x,y) from (1,1) to (4,4). Take the *grid-distance* between (x,y) and (u,v) to be $|x-u| + |y-v|$. Show that the sum of the six grid-distances between the points is always the same.

B3. A casino game uses the diagram shown. At the start a ball appears at S. Each time the player presses a button, the ball moves to one of the adjacent letters (joined by a line segment) (with equal probability). If the ball returns to S the player loses. If the ball reaches G, then the player wins. Find the probability that the player wins and the expected number of button presses.

30th Spanish 1994

A1. Show that if an (infinite) arithmetic progression includes a square, then it must include infinitely many squares.

A2. Take three-dimensional coordinates with origin O. C is the point (0,0,c). P is a point on the x-axis, and Q is a point on the y-axis such that OP + OQ = k, where k is fixed. Let W be the center of the sphere through O, C, P, Q. Let W' be the projection of W on the xy-plane. Find the locus of W' as P and Q vary. Find also the locus of W as P and Q vary.

A3. The tourism office is collecting figures on the number of sunny days and the number of rainy days in the regions A, B, C, D, E, F.

	sunny/rainy	unclassifiable
A	336	29
B	321	44
C	335	30
D	343	22
E	329	36
F	330	35

If one region is excluded then the total number of rainy days in the other regions is one-third of the total number of sunny days in those regions. Which region is excluded?

B1. The triangle ABC has $\angle A = 36^o$, $\angle B = 72^o$, $\angle C = 72^o$. The bisector of $\angle C$ meets AB at D. Find the angles of BCD. Express the length BC in terms of AC, without using any trigonometric functions.

B2. 21 counters are arranged in a 3 x 7 grid. Some of the counters are black and some white. Show that one can always find 4 counters of the same color at the vertices of a rectangle.

B3. A convex n-gon is divided into m triangles, so that no two triangles have interior points in common, and each side of a triangle is either a side of the polygon or a side of another triangle. Show that m + n must be even. Given m, n, find the number of triangle sides in the interior of the polygon and the number of vertices in the interior of the polygon.

31st Spanish 1995

A1. X is a set of 100 distinct positive integers such that if a, b, c $\in$ X (not necessarily distinct), then there is a triangle with sides a, b, c which is not obtuse. Let S(X) be the sum of the perimeters of all the possible triangles. Find the minimum possible value of S(X).

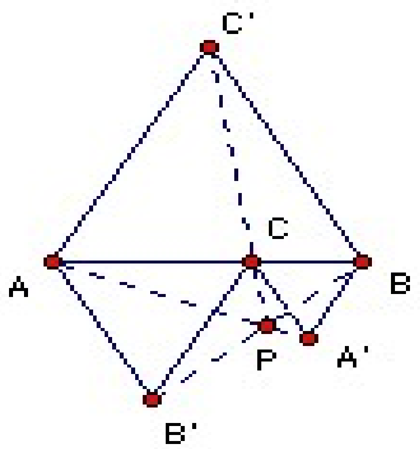
A2. A finite number of paper disks are arranged so that no disk lies inside another, but there is some overlapping. Show that if we cut out the parts which do not overlap we cannot rearrange them to form disks.

A3. ABC is a triangle with centroid G. A line through G meets the side AB at P and the side AC at Q. Show that $(PB/PA)(QC/QA) \le 1/4$.

B1. p is a prime number. Find all integral solutions to $p(m+n) = mn$.

B2. Given that the equations $x^3 + mx - n = 0$, $nx^3 - 2m^2x^2 - 5mnx - 2m^3 - n^2 = 0$ (where $n \neq 0$) have a common root, show that the first must have two equal roots and find the roots of the two equations in terms of n.

B3. C is a variable point on the segment AB. Equilateral triangles AB'C and BA'C are constructed on the same side of AB, and the equilateral triangle ABC' is constructed on the opposite side of AB. Show that AA', BB', CC' meet at some point P. Find the locus of P as C varies. Show that the centers of the three equilateral triangles form an equilateral triangle and lie on a fixed circle (as C varies).



32nd Spanish 1996

A1. The integers m, n are such that (m+1)/n + (n+1)/m is an integer. Show that gcd(m,n) ≤ √(m+n).

A2. G is the centroid of ABC. Show that if AB + GC = AC + GB, then the triangle is isosceles.

A3. $p(x) = ax^2+bx+c$, $q(x) = cx^2+bx+a$, and $|p(-1)| \le 1$, $|p(0)| \le 1$, $|p(1)| \le 1$. Show that $|p(x)| \le 5/4$ and $|q(x)| \le 2$ for $x \in [-1,1]$.

B1. Discuss the existence of solutions to the equation $\sqrt{(x^2-p)} + 2\sqrt{(x^2-1)} = x$ for varying values of the real parameter p.

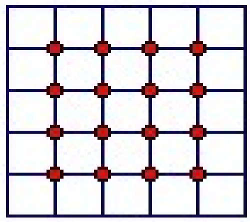
B2. In Port Aventura there are 16 secret agents. Each of the agents watches some of his rivals. It is known that if agent A watches agent B, then agent B does not watch agent A. It is possible to find 10 agents such that the first watches the second, the second watches the third, ... , and the tenth watches the first. Show that it is possible to find a cycle of 11 such agents.

B3. Take a cup made of 6 regular pentagons, so that two such cups could be put together to form a regular dodecahedron. The edge length is 1. What volume of liquid will the cup hold?

33rd Spanish 1997

A1. An arithmetic progression has 100 terms. The sum of the terms is -1, and the sum of the even-numbered terms is 1. Find the sum of the squares of the terms.

A2. X is the set of 16 points shown. What is the largest number of elements of X that we can choose so that no three of the chosen points form an isosceles triangle?



A3. Let S be the set of all parabolas $y = x^2 + px + q$ whose graphs cut the coordinate axes in three distinct points. Let C(p,q) be the circle through the three points. Show that all circles C(p,q) have a common point.

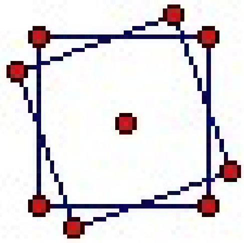
B1. Given a prime p, find all integers k such that $\sqrt{(k^2-kp)}$ is integral.

B2. Q is a convex quadrilateral with area 1. Show that the sum of the sides and diagonals is at least $2(2+\sqrt{2})$.

B3. A car wishes to make a circuit of a circular road. There are some tanks along the road that contain between them just enough gasoline for the car to make the trip. The car has a tank large enough to hold all the gasoline for a complete circuit, but the tank is initially empty. Show that irrespective of the number of tanks, their positions, and the amount each contains, it is possible to find a starting point on the road which will allow the car to make a complete circuit (refueling when it reaches a tank).

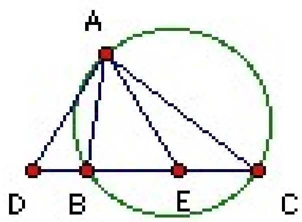
34th Spanish 1998

A1. A square side 1 is rotated through an angle θ about its center. Find the area common to the original and rotated squares.



A2. Find all 4-digit numbers which equal the cube of the sum of their digits.

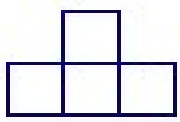
A3. ABC is a triangle. D, E are points on the line BC such that AD, AE are parallel to the tangents to the circumcircle at C, B. Show that $BE/CD = (AB/AC)^2$.



B1. A triangle has angles A, B, C such that tan A, tan B, tan C are all positive integers. Find A, B, C.

B2. Let N be the set of positive integers. Find all functions $f : N \rightarrow N$ which are strictly increasing and which satisfy $f(n + f(n)) = 2 f(n)$ for all n.

B3. For which values of n is it possible to tile an n x n square with tiles of the type:

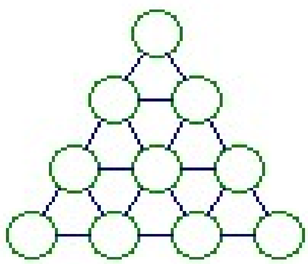


35th Spanish 1999

A1. A and B are points of the parabola $y = x^2$. The tangents at A and B meet at C. The median of the triangle ABC from C has length m. Find area ABC in terms of m.

A2. Show that there is an infinite sequence $a_1, a_2, a_3, \ldots$ of positive integers such that $a_1^2 + a_2^2 + \ldots + a_n^2$ is a square for all n.

A3. A game is played on the board shown. A token is placed on each circle. Each token has a black side and a white side. Initially the topmost token has the black face showing, the others have the white face showing. A move is to remove token showing its black face and to turn over the tokens on the adjacent circles (joined by a line). Is it possible to remove all the tokens by a sequence of moves?



B1. A box contains 900 cards, labeled from 100 to 999. Cards are removed one at a time without replacement. What is the smallest number of cards that must be removed to guarantee that at least three of the digit sums of the cards removed are equal?

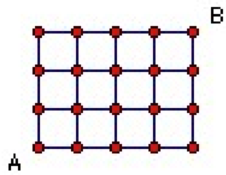
B2. G is the centroid of the triangle ABC. The distances of G from the three sides are g_a, g_b, g_c. Show that $g_a \geq 2r/3$, and $(g_a + g_b + g_c) \geq 3r$, where r is the inradius.

B3. Three families of parallel lines divide the plane into N regions. No three lines pass through the same point. What is the smallest number of lines needed to get N > 1999?

36th Spanish 2000

A1. Let $p(x) = x^4 + ax^3 + bx^2 + cx + 1$, $q(x) = x^4 + cx^3 + bx^2 + ax + 1$. Find conditions on a, b, c (assuming $a \neq c$) so that p(x) and q(x) have two common roots. In this case solve p(x) = q(x) = 0.

A2. The diagram shows a network of roads. The distance from one node to an adjacent node is 1. P goes from A to B by a path length 7, and Q goes from B to A by a path length 7. Each goes at the same constant speed. At each junction with two possible directions to take, each has probability 1/2. Find the probability that P and Q meet.



A3. Circles C and C' meet at A and B. P, P' are variable points on C, C' such that P, B, P' are collinear. Show that the perpendicular bisector of PP' passes through a fixed point M (which depends only on C and C').

B1. Find the largest integer N such that [N/3] has three digits, all equal, and [N/3] = 1 + 2 + 3 + ... + n for some n.

B2. Given 4 points inside or on the perimeter of a square side 1, show that two of them must be distance ≤ 1 apart.

B3. Let N be the set of positive integers. Show that there is no function f : N → N such that f(f(n)) = n+1 for all n.

37th Spanish 2001

A1. Show that the graph of the polynomial p(x) is symmetric about the point (a,b) iff there is a polynomial q(x) such that $p(x) = b + (x-1)\, q((x-a)^2)$.

A2. P is a point inside the triangle ABC equidistant from A and B. Exterior triangles BQC and CRA are constructed similar to APB. Show that P, Q, C, R are collinear or form a parallelogram.

A3. Five segments are such that any three of them can be used to form a triangle. Show that at least one of these triangles is acute-angled.

B1. Can we arrange the digits 0 to 9 into a 3 x 3 array so that the six numbers (the three rows left to right, and the three columns top to bottom) add up to 2001?

B2. ABCD is a quadrilateral inscribed in a circle radius 1 with AB a diameter. It has an inscribed circle. Show that $CD \le 2\sqrt{5} - 4$.

B3. Let N be the set of positive integers. Find a function $f : N \to N$ such that $f(1) = f(2^n) = 1$ for all $n \in N$, and $f(2^n + m) = f(n) + 1$ for $m < 2^n$. Find the maximum value of f(n) for $n \le 2001$. Find the smallest n such that f(n) = 2001.

38th Spanish 2002

A1. Find all polynomials p(x) such that $p(x^2-y^2) \equiv p(x+y)\, p(x-y)$.

A2. AD is an altitude of the triangle ABC and H is the orthocenter. Find a relation between $\angle B$ and $\angle C$ in terms of k = AD/HD. Given B, C and k, find the locus of A.

A3. The function f is defined on the positive integers and satisfies f(2) = 1, f(2n) = f(n), f(2n+1) = f(2n) + 1. Find the maximum value M of f(n) for $1 \le n \le 2002$ and find how many n satisfy f(n) = M.

B1. r(n) is the number obtained by writing the digits of n in reverse order, and s(n) is the sum of the digits of n. Find all 3-digit numbers n such that 2r(n) + s(n) = n.

B2. Given 2002 line segments in the plane with total length 1. Show that there is a line L such that the projections of the segments onto L have total length < 2/3.

B3. r vertices of a regular (6n+1)-gon are colored red and the remaining vertices are colored blue. Show that the number of isosceles triangles with all vertices the same color depends only on n and r.

39th Spanish 2003

A1. For which primes $p \neq 2$ or 5 is there a multiple of p whose digits are all 9? For example, $999999 = 13 \cdot 76923$.

A2. Does there exist a finite set M of at least two real numbers such that if $a, b \in M$, then $2a - b^2 \in M$?

A3. H is the orthocenter of ABC and AB = CH. Find $\angle C$.

B1. α is a real root of $x^3 + 2x^2 + 10x - 20 = 0$. Show that α^2 is irrational.

B2. A hexagon has all its angles equal and sides 1, 2, 3, 4, 5, 6 in that order. What is its area?

B3. 2n white balls and 2n black balls are arranged in a line. Show that however they are arranged it is possible to find 2n consecutive balls, just n of which are white.

Swedish (1961 – 2003)

1st Swedish 1961

1. Let S be the system of equations (1) $y(x^4 - y^2 + x^2) = x$, (2) $x(x^4 - y^2 + x^2) = 1$. Take S' to be the system of equations (1) and $x\cdot(1) - y\cdot(2)$ (or $y = x^2$). Show that S and S' do not have the same set of solutions and explain why.

2. Show that $x_1/x_n + x_2/x_{n-1} + x_3/x_{n-2} + ... + x_n/x_1 \geq n$ for any positive reals $x_1, x_2, ... , x_n$.

3. For which n is it possible to put n identical candles in a candlestick and to light them as follows. For i = 1, 2, ... , n, exactly i candles are lit on day i and burn for exactly one hour. At the end of day n, all n candles must be burnt out. State a possible rule for deciding which candles to light on day i.

4. 288 points are placed inside a square ABCD of side 1. Show that one can draw a set S of lines length 1 parallel to AB joining AD and BC, and additional lines parallel to AD joining each of the 288 point to a line in S, so that the total length of all the lines is less than 24. Is there a stronger result?

5. n is a positive integer. Show that $x^6/6 + x^2 - nx$ has exactly one minimum a_n. Show that for some k, $\lim_{n\to\infty} a_n/n^k$ exists and is non-zero. Find k and the limit.

2nd Swedish 1962

1. Find all polynomials f(x) such that f(2x) = f '(x) f ''(x).

2. ABCD is a square side 1. P and Q lie on the side AB and R lies on the side CD. What are the possible values for the circumradius of PQR?

3. Find all pairs (m, n) of integers such that n^2 - 3mn + m - n = 0.

4. Which of the following statements are true?

(A) X implies Y, or Y implies X, where X is the statement, the lines L_1, L_2, L_3 lie in a plane, and Y is the statement, each pair of the lines L_1, L_2, L_3 intersect.

(B) Every sufficiently large integer n satisfies n = $a^4 + b^4$ for some integers a, b.

(C) There are real numbers $a_1, a_2, ... , a_n$ such that $a_1 \cos x + a_2 \cos 2x + ... + a_n \cos nx > 0$ for all real x.

5. Find the largest cube which can be placed inside a regular tetrahedron with side 1 so that one of its faces lies on the base of the tetrahedron.

3rd Swedish 1963

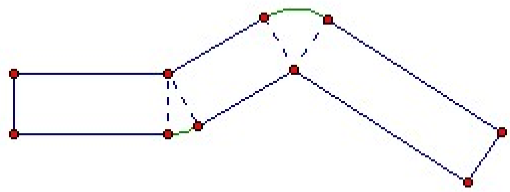
1. How many positive integers have square less than 10^7?

2. The squares of a chessboard have side 4. What is the circumference of the largest circle that can be drawn entirely on the black squares of the board?

3. What is the remainder on dividing $1234^{567} + 89^{1011}$ by 12?

4. Given the real number k, find all differentiable real-valued functions f(x) defined on the reals such that f(x+y) = f(x) + f(y) + f(kxy) for all x, y.

5. A road has constant width. It is made up of finitely many straight segments joined by corners, where the inner corner is a point and the outer side is a circular arc. The direction of the straight sections is always between NE (45°) and SSE (157½°). A person wishes to walk along the side of the road from point A to point B on the same side. He may only cross the street perpendicularly. What is the shortest route?



6. The real-valued function f(x) is defined on the reals. It satisfies |f(x)| ≤ A, |f ''(x)| ≤ B for some positive A, B (and all x). Show that |f '(x)| ≤ C, for some fixed C, which depends only on A and B. What is the smallest possible value of C?

4th Swedish 1964

1. Find the side lengths of the triangle ABC with area S and $\angle BAC = x$ such that the side BC is as short as possible.

2. Find all positive integers m, n such that $n + (n+1) + (n+2) + ... + (n+m) = 1000$.

3. Find a polynomial with integer coefficients which has $\sqrt{2} + \sqrt{3}$ and $\sqrt{2} + 3^{1/3}$ as roots.

4. Points $H_1, H_2, ... , H_n$ are arranged in the plane so that each distance $H_iH_j \le 1$. The point P is chosen to minimise $\max(PH_i)$. Find the largest possible value of $\max(PH_i)$ for $n = 3$. Find the best upper bound you can for $n = 4$.

5. $a_1, a_2, ... , a_n$ are constants such that $f(x) = 1 + a_1 \cos x + a_2 \cos 2x + ... + a_n \cos nx \ge 0$ for all x. We seek estimates of a_1. If $n = 2$, find the smallest and largest possible values of a_1. Find corresponding estimates for other values of n.

5th Swedish 1965

1. The feet of the altitudes in the triangle ABC are A', B', C'. Find the angles of A'B'C' in terms of the angles A, B, C. Show that the largest angle in A'B'C' is at least as big as the largest angle in ABC. When is it equal?

2. Find all positive integers m, n such that $m^3 - n^3 = 999$.

3. Show that for every real $x \geq ½$ there is an integer n such that $|x - n^2| \leq \sqrt{(x - ¼)}$.

4. Find constants A > B such that f(1/(1+2x))/f(x) is independent of x, where f(x) = (1 + Ax)/(1 + Bx) for all real x ≠ -1/B. Put $a_0 = 1$, $a_{n+1} = 1/(1 + 2a_n)$. Find an expression for a_n by considering $f(a_0)$, $f(a_1)$, ...

5. Let S be the set of all real polynomials $f(x) = ax^3 + bx^2 + cx + d$ such that $|f(x)| \leq 1$ for all $-1 \leq x \leq 1$. Show that the set of possible |a| for f in S is bounded above and find the smallest upper bound.

6th Swedish 1966

1. Let {x} denote the fractional part of x = x - [x]. The sequences x_1, x_2, x_3, ... and y_1, y_2, y_3, ... are such that lim $\{x_n\}$ = lim $\{y_n\}$ = 0. Is it true that lim $\{x_n + y_n\} = 0$? lim $\{x_n - y_n\} = 0$?

2. $a_1 + a_2 + ... + a_n = 0$, for some k we have $a_j \le 0$ for $j \le k$ and $a_j \ge 0$ for $j > k$. If a_i are not all 0, show that $a_1 + 2a_2 + 3a_3 + ... + na_n > 0$.

3. Show that an integer = 7 mod 8 cannot be sum of three squares.

4. Let f(x) = 1 + 2/x. Put $f_1(x) = f(x)$, $f_2(x) = f(f_1(x))$, $f_3(x) = f(f_2(x))$, ... Find the solutions to $x = f_n(x)$ for n > 0.

5. Let f(r) be the number of lattice points inside the circle radius r, center the origin. Show that $\lim_{r\to\infty} f(r)/r^2$ exists and find it. If the limit is k, put $g(r) = f(r) - kr^2$. Is it true that $\lim_{r\to\infty} g(r)/r^h = 0$ for any h < 2?

7th Swedish 1967

1. p parallel lines are drawn in the plane and q lines perpendicular to them are also drawn. How many rectangles are bounded by the lines?

2. You are given a ruler with two parallel straight edges a distance d apart. It may be used (1) to draw the line through two points, (2) given two points a distance $\geq$ d apart, to draw two parallel lines, one through each point, (3) to draw a line parallel to a given line, a distance d away. One can also (4) choose an arbitrary point in the plane, and (5) choose an arbitrary point on a line. Show how to construct (A) the bisector of a given angle, and (B) the perpendicular to the midpoint of a given line segment.

3. Show that there are only finitely many triples (a, b, c) of positive integers such that 1/a + 1/b + 1/c = 1/1000.

4. The sequence $a_1, a_2, a_3, \ldots$ of positive reals is such that $\sum a_i$ diverges. Show that there is a sequence $b_1, b_2, b_3, \ldots$ of positive reals such that $\lim b_n = 0$ and $\sum a_ib_i$ diverges.

5. $a_1, a_2, a_3, \ldots$ are positive reals such that $a_n^2 \geq a_1 + a_2 + \ldots + a_{n-1}$. Show that for some $C > 0$ we have $a_n \geq Cn$ for all n.

6. The vertices of a triangle are lattice points. There are no lattice points on the sides (apart from the vertices) and n lattice points inside the triangle. Show that its area is n + ½. Find the formula for the general case where there are also m lattice points on the sides (apart from the vertices).

8th Swedish 1968

1. Find the maximum and minimum values of $x^2 + 2y^2 + 3z^2$ for real x, y, z satisfying $x^2 + y^2 + z^2 = 1$.

2. How many different ways (up to rotation) are there of labeling the faces of a cube with the numbers 1, 2, ... , 6?

3. Show that the sum of the squares of the sides of a quadrilateral is at least the sum of the squares of the diagonals. When does equality hold?

4. For $n \neq 0$, let f(n) be the largest k such that 3^k divides n. If M is a set of $n > 1$ integers, show that the number of possible values for f(m-n), where m, n belong to M cannot exceed n-1.

5. Let a, b be non-zero integers. Let m(a, b) be the smallest value of cos ax + cos bx (for real x). Show that for some r, $m(a, b) \leq r < 0$ for all a, b.

9th Swedish 1969

1. Find all integers m, n such that $m^3 = n^3 + n$.

2. Show that tan $\pi/3n$ is irrational for all positive integers n.

3. $a_1 \geq a_2 \geq \ldots \geq a_n$ is a sequence of reals. $b_1, b_2, b_3, \ldots b_n$ is any rearrangement of the sequence $B_1 \geq B_2 \geq \ldots \geq B_n$. Show that $\sum a_i b_i \leq$ &sum $a_i B_i$.

4. Define g(x) as the largest value of $|y^2 - xy|$ for y in [0, 1]. Find the minimum value of g (for real x).

5. Let $N = a_1a_2 \ldots a_n$ in binary. Show that if $a_1 - a_2 + a_3 - \ldots + (-1)^{n-1}a_n = 0 \bmod 3$, then $N = 0 \bmod 3$.

6. Given 3n points in the plane, no three collinear, is it always possible to form n triangles (with vertices at the points), so that no point in the plane lies in more than one triangle?

10th Swedish 1970

1. Show that infinitely many positive integers cannot be written as a sum of three fourth powers of integers.

2. 6 open disks in the plane are such that the center of no disk lies inside another. Show that no point lies inside all 6 disks.

3. A polynomial with integer coefficients takes the value 5 at five distinct integers. Show that it does not take the value 9 at any integer.

4. Let $p(x) = (x - x_1)(x - x_2)(x - x_3)$, where x_1, x_2 and x_3 are real. Show that $p(x)\, p''(x) \le p'(x)^2$ for all x.

5. A 3 x 1 paper rectangle is folded twice to give a square side 1. The square is folded along a diagonal to give a right-angled triangle. A needle is driven through an interior point of the triangle, making 6 holes in the paper. The paper is then unfolded. Where should the point be in order to maximise the smallest distance between any two holes?

6. Show that $(n - m)!/m! \le (n/2 + 1/2)^{n-2m}$ for positive integers m, n with $2m \le n$.

11th Swedish 1971

1. Show that $(1 + a + a^2)^2 < 3(1 + a^2 + a^4)$ for real $a \neq 1$.

2. An arbitrary number of lines divide the plane into regions. Show that the regions can be colored red and blue so that neighboring regions have different colors.

3. A table is covered by 15 pieces of paper. Show that we can remove 7 pieces so that the remaining 8 cover at least 8/15 of the table.

4. Find $(65533^3 + 65534^3 + 65535^3 + 65536^3 + 65537^3 + 65538^3 + 65539^3)/(32765\cdot 32766 + 32767\cdot 32768 + 32768\cdot 32769 + 32770\cdot 32771)$.

5. Show that $\max_{|x|\leq t} |1 - a \cos x| \geq \tan^2(t/2)$ for a positive and $t \in (0, \pi/2)$.

6. 99 cards each have a label chosen from 1, 2, ... , 99, such that no (non-empty) subset of the cards has labels with total divisible by 100. Show that the labels must all be equal.

12th Swedish 1972

1. Find the largest real number a such that x - 4y = 1, ax + 3y = 1 has an integer solution.

2. A rectangular grid of streets has m north-south streets and n east-west streets. For which m, n > 1 is it possible to start at an intersection and drive through each of the other intersections just once before returning to the start?

3. A steak temperature 5^o is put into an oven. After 15 minutes, it has temperature 45^o. After another 15 minutes it has temperature 77^o. The oven is at a constant temperature. The steak changes temperature at a rate proportional to the difference between its temperature and that of the oven. Find the oven temperature.

4. Put $x = \log_{10}2$, $y = \log_{10}3$. Then 15 < 16 implies 1 - x + y < 4x, so 1 + y < 5x. Derive similar inequalities from 80 < 81 and 243 < 250. Hence show that $0.47 < \log_{10}3 < 0.482$.

5. Show that $\int_0^1 (1/(1 + x^n))\, dx > 1 - 1/n$ for all positive integers n.

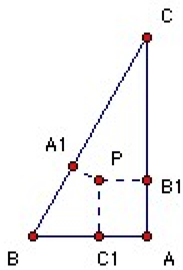
6. $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are sequences of positive integers. Show that we can find m < n such that $a_m \le a_n$ and $b_m \le b_n$.

13th Swedish 1973

1. $\log_8 2 = 0.2525$ in base 8 (to 4 places of decimals). Find $\log_8 4$ in base 8 (to 4 places of decimals).

2. The Fibonacci sequence $f_1, f_2, f_3, \ldots$ is defined by $f_1 = f_2 = 1$, $f_{n+2} = f_{n+1} + f_n$. Find all n such that $f_n = n^2$.

3. ABC is a triangle with $\angle A = 90^o$, $\angle B = 60^o$. The points A_1, B_1, C_1 on BC, CA, AB respectively are such that $A_1B_1C_1$ is equilateral and the perpendiculars (to BC at A_1, to CA at B_1 and to AB at C_1) meet at a point P inside the triangle. Find the ratios $PA_1:PB_1:PC_1$.



4. p is a prime. Find all relatively prime positive integers m, n such that $m/n + 1/p^2 = (m + p)/(n + p)$.

5. f(x) is a polynomial of degree 2n. Show that all polynomials p(x), q(x) of degree at most n such that f(x)q(x) - p(x) has the form $\sum_{2n<k\le 3n} (a^k + x^k)$, have the same p(x)/q(x).

6. f(x) is a real valued function defined for $x \ge 0$ such that $f(0) = 0$, $f(x+1) = f(x) + \sqrt{x}$ for all x, and $f(x) < \frac{1}{2} f(x - \frac{1}{2}) + \frac{1}{2} f(x + \frac{1}{2})$ for all $x \ge \frac{1}{2}$. Show that $f(\frac{1}{2})$ is uniquely determined.

14th Swedish 1974

1. Let $a_n = 2^{n-1}$ for $n > 0$. Let $b_n = \sum_{r+s\leq n} a_r a_s$. Find $b_n - b_{n-1}$, $b_n - 2b_{n-1}$ and b_n.

2. Show that $1 - 1/k \leq n(k^{1/n} - 1) \leq k - 1$ for all positive integers n and positive reals k.

3. Let $a_1 = 1$, $a_2 = 2^{a_1}$, $a_3 = 3^{a_2}$, $a_4 = 4^{a_3}$, ... , $a_9 = 9^{a_8}$. Find the last two digits of a_9.

4. Find all polynomials $p(x)$ such that $p(x^2) = p(x)^2$ for all x. Hence find all polynomials $q(x)$ such that $q(x^2 - 2x) = q(x-2)^2$.

5. Find the smallest positive real t such that $x_1 + x_3 = 2t\, x_2$, $x_2 + x_4 = 2t\, x_3$, $x_3 + x_5 = 2t\, x_4$ has a solution x_1, x_2, x_3, x_4, x_5 in non-negative reals, not all zero.

6. For which n can we find positive integers $a_1, a_2, \ldots, a_n$ such that $a_1^2 + a_2^2 + \ldots + a_n^2$ is a square?

15th Swedish 1975

1. A is the point (1, 0), L is the line $y = kx$ (where $k > 0$). For which points P (t, 0) can we find a point Q on L such that AQ and QP are perpendicular?

2. Is there a positive integer n such that the fractional part of $(3 + \sqrt{5})^n > 0.99$?

3. Show that $a^n + b^n + c^n \geq ab^{n-1} + bc^{n-1} + ca^{n-1}$ for real $a, b, c \geq 0$ and n a positive integer.

4. P_1, P_2, P_3, Q_1, Q_2, Q_3 are distinct points in the plane. The distances P_1Q_1, P_2Q_2, P_3Q_3 are equal. P_1P_2 and Q_2Q_1 are parallel (not antiparallel), similarly P_1P_3 and Q_3Q_1, and P_2P_3 and Q_3Q_2. Show that P_1Q_1, P_2Q_2 and P_3Q_3 intersect in a point.

5. Show that n divides $2^n + 1$ for infinitely many positive integers n.

6. $f(x)$ is defined for $0 \leq x \leq 1$ and has a continuous derivative satisfying $|f'(x)| \leq C |f(x)|$ for some positive constant C. Show that if $f(0) = 0$, then $f(x) = 0$ for the entire interval.

16th Swedish 1976

1. In a tournament every team plays every other team just once. Each game is won by one of the teams (there are no draws). Each team loses at least once. Show that there must be three teams A, B, C such that A beat B, B beat C and C beat A.

2. For which real a are there distinct reals x, y such that $x = a - y^2$ and $y = a - x^2$?

3. If a, b, c are rational, show that $1/(b-c)^2 + 1/(c-a)^2 + 1/(a-b)^2$ is the square of a rational.

4. A number is placed in each cell of an n x n board so that the following holds: (A) the cells on the boundary all contain 0; (B) other cells on the main diagonal are each 1 greater than the mean of the numbers to the left and right; (C) other cells are the mean of the numbers to the left and right. Show that (B) and (C) remain true if "left and right" is replaced by "above and below".

5. f(x) is defined for $x \geq 0$ and has a continuous derivative. It satisfies $f(0) = 1$, $f'(0) = 0$ and $(1 + f(x))\, f''(x) = 1 + x$. Show that f is increasing and that $f(1) \leq 4/3$.

6. Show that there are only finitely many integral solutions to $3^m - 1 = 2^n$ and find them.

17th Swedish 1977

1. p is a prime. Find the largest integer d such that p^d divides p^4!

2. There is a point inside an equilateral triangle side d whose distance from the vertices is 3, 4, 5. Find d.

3. Show that the only integral solution to $xy + yz + zx = 3n^2 - 1$, $x + y + z = 3n$ with $x \geq y \geq z$ is $x = n+1$, $y = n$, $z = n-1$.

4. Show that if $\cos x/\cos y + \sin x/\sin y = -1$, then $\cos^3 y/\cos x + \sin^3 y/\sin x = 1$.

5. The numbers 1, 2, 3, ... , 64 are written in the cells of an 8 x 8 board (in some order, one per cell). Show that at least four 2 x 2 squares have sum > 100.

6. Show that there are positive reals a, b, c such that $a^2 + b^2 + c^2 > 2$, $a^3 + b^3 + c^3 < 2$, and $a^4 + b^4 + c^4 > 2$.

18th Swedish 1978

1. $a > b > c > d \geq 0$ are reals such that $a + d = b + c$. Show that $x^a + x^d \geq x^b + x^c$ for $x > 0$.

2. Let s_m be the number 66 ... 6 with m 6s. Find $s_1 + s_2 + ... + s_n$.

3. Two satellites are orbiting the earth in the equatorial plane at an altitude h above the surface. The distance between the satellites is always d, the diameter of the earth. For which h is there always a point on the equator at which the two satellites subtend an angle of 90^o?

4. $b_0, b_1, b_2, ...$ is a sequence of positive reals such that the sequence $b_0, c\, b_1, c^2b_2, c^3b_3, ...$ is convex for all $c > 0$. (A sequence is convex if each term is at most the arithmetic mean of its two neighbors.) Show that $\ln b_0, \ln b_1, \ln b_2, ...$ is convex.

5. $k > 1$ is fixed. Show that for n sufficiently large for every partition of $\{1, 2, ... , n\}$ into k disjoint subsets we can find $a \neq b$ such that a and b are in the same subset and a+1 and b+1 are in the same subset. What is the smallest n for which this is true?

6. p(x) is a polynomial of degree n with leading coefficient c, and q(x) is a polynomial of degree m with leading coefficient c, such that $p(x)^2 = (x^2 - 1)q(x)^2 + 1$. Show that $p'(x) = nq(x)$.

19th Swedish 1979

1. Solve the equations:

$x_1 + 2x_2 + 3x_3 + ... + (n-1)x_{n-1} + nx_n = n$

$2x_1 + 3x_2 + 4x_3 + ... + nx_{n-1} + x_n = n-1$

$3x_1 + 4x_2 + 5x_3 + ... + x_{n-1} + 2x_n = n-2$

...

$(n-1)x_1 + nx_2 + x_3 + ... + (n-3)x_{n-1} + (n-2)x_n = 2$

$nx_1 + x_2 + 2x_3 + ... + (n-2)x_{n-1} + (n-1)x_n = 1.$

2. Find rational x in (3, 4) such that $\sqrt{(x-3)}$ and $\sqrt{(x+1)}$ are rational.

3. Express $x^{13} + 1/x^{13}$ as a polynomial in $y = x + 1/x$.

4. f(x) is continuous on the interval $[0, \pi]$ and satisfies $\int_0^\pi f(x)\,dx = 0$, $\int_0^\pi f(x)\cos x\,dx = 0$. Show that f(x) has at least two zeros in the interval $(0, \pi)$.

5. Find the smallest positive integer a such that for some integers b, c the polynomial $ax^2 - bx + c$ has two distinct zeros in the interval (0, 1).

6. Find the sharpest inequalities of the form $a\cdot AB < AG < b\cdot AB$ and $c\cdot AB < BG < d\cdot AB$ for all triangles ABC with centroid G such that $GA > GB > GC$.

20th Swedish 1980

1. Show that $\log_{10}2$ is irrational.

2. $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ and $b_1, b_2, b_3, b_4, b_5, b_6, b_7$ are two permutations of 1, 2, 3, 4, 5, 6, 7. Show that $|a_1 - b_1|, |a_2 - b_2|, |a_3 - b_3|, |a_4 - b_4|, |a_5 - b_5|, |a_6 - b_6|, |a_7 - b_7|$ are not all different.

3. Let T(n) be the number of dissimilar (non-degenerate) triangles with all side lengths integral and $\le$ n. Find T(n+1) - T(n).

4. The functions f and g are positive and continuous. f is increasing and g is decreasing. Show that $\int_0^1 f(x)\, g(x)\, dx \le \int_0^1 f(x)\, g(1-x)\, dx$.

5. A *word* is a string of the symbols a, b which can be formed by repeated application of the following: (1) ab is a word; (2) if X and Y are words, then so is XY; (3) if X is a word, then so is aXb. How many words have 12 letters?

6. Find the smallest constant c such that for every 4 points in a unit square there are two a distance $\le$ c apart.

21st Swedish 1981

1. Let N = 11 ... 122 ... 25, where there are n 1s and n+1 2s. Show that N is a perfect square.

2. Does $x^y = z$, $y^z = x$, $z^x = y$ have any solutions in positive reals apart from x = y = z = 1?

3. Find all polynomials p(x) of degree 5 such that p(x) + 1 is divisible by $(x-1)^3$ and p(x) - 1 is divisible by $(x+1)^3$.

4. A cube side 5 is divided into 125 unit cubes. N of the small cubes are black and the rest white. Find the smallest N such that there must be a row of 5 black cubes parallel to one of the edges of the large cube.

5. ABC is a triangle. X, Y, Z lie on BC, CA, AB respectively. Show that area XYZ cannot be smaller than each of area AYZ, area BZX, area CXY.

6. Show that there are infinitely many triangles with side lengths a, b, c, where a is a prime, b is a power of 2 and c is the square of an odd integer.

22nd Swedish 1982

1. How many solutions does $x^2 - [x^2] = (x - [x])^2$ have satisfying $1 \le x \le n$?

2. Show that $abc \ge (a+b-c)(b+c-a)(c+a-b)$ for positive reals a, b, c.

3. Show that there is a point P inside the quadrilateral ABCD such that the triangles PAB, PBC, PCD, PDA have equal area. Show that P must lie on one of the diagonals.

4. ABC is a triangle with AB = 33, AC = 21 and BC = m, an integer. There are points D, E on the sides AB, AC respectively such that AD = DE = EC = n, an integer. Find m.

5. Each point in a 12 x 12 array is colored red, white or blue. Show that it is always possible to find 4 points of the same color forming a rectangle with sides parallel to the sides of the array.

6. Show that $(2a-1) \sin x + (1-a) \sin(1-a)x \ge 0$ for $0 \le a \le 1$ and $0 \le x \le \pi$

23rd Swedish 1983

1. The positive integers are grouped as follows: 1, 2+3, 4+5+6, 7+8+9+10, ... Find the value of the nth sum.

2. Show that $\cos x^2 + \cos y^2 - \cos xy < 3$ for reals x, y.

3. The equations $2x_1 - x_2 = 1$, $-x_1 + 2x_2 - x_3 = 1$, $-x_2 + 2x_3 - x_4 = 1$, $-x_3 + 3x_4 - x_5 = 1$, ... , $-x_{n-2} + 2x_{n-1} - x_n = 1$, $-x_{n-1} + 2x_n = 1$ have a solution in positive integers x_i. Show that n must be even.

4. C, C' are concentric circles with radii R, R'. A rectangle has two adjacent vertices on C and the other two vertices on C'. Find its sides if its area is as large as possible.

5. Show that a unit square can be covered with three equal disks with radius $< 1/\sqrt{2}$. What is the smallest possible radius?

6. Show that the only real solution to $x(x+y)^2 = 9$, $x(y^3 - x^3) = 7$ is $x = 1$, $y = 2$.

24th Swedish 1984

1. A, B are points inside a circle C. Show that there is a circle through A and B which lies entirely inside C.

2. Each point in a 3 x 7 array is colored yellow or blue. Show that it is always possible to find 4 points of the same color forming a rectangle with sides parallel to the side of the array.

3. Show that $((a+1)/(b+1))^{b+1} \geq (a/b)^b$ for positive reals a, b.

4. Find all positive integers m, n such that all roots of $(x^2 - mx + n)(x^2 - nx + m)$ are positive integers.

5. Find all positive integer solutions to $a^3 - b^3 - c^3 = 3abc$, $a^2 = 2(a + b + c)$.

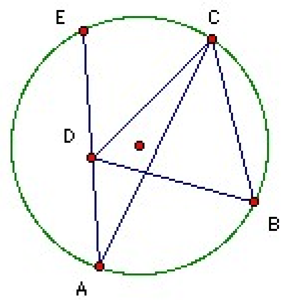
6. The positive integers $a_1, a_2, \dots, a_{14}$ satisfy $\sum 3^{a_i} = 6558$. Show that they must be two copies of each of 1, 2, ... , 7.

25th Swedish 1985

1. Show that $(a-b)^2/8a < (a+b)/2 - \sqrt{ab} < (a-b)^2/8b$ for reals $a > b > 0$.

2. Find the smallest positive integer n such that if the first digit is moved to become the last digit, then the new number is 7n/2.

3. BA = BC. D is inside the circle through A, B, C, such that BCD is equilateral. AD meets the circle again at E. Show that DE equals the radius of the circle.



5. A (a, 0), B (0, b), C (c, d) is a triangle with a, b, c, d, > 0. Show that its perimeter is at least 2 CO (where O is the origin).

6. A town has several clubs. Given any two residents there is exactly one club that both belong to. Given any two clubs, there is exactly one resident who belongs to both. Each club has at least 3 members. At least one club has 17 members. How many residents are there?

26th Swedish 1986

1. Show that $x^6 - x^5 + x^4 - x^3 + x^2 - x + 3/4$ has no real zeros.

2. ABCD is a quadrilateral area S. Its diagonals meet at X. Area ABX = X_1, area CDX = X_2. Show that $\sqrt{X_1} + \sqrt{X_2} \le \sqrt{X}$, with equality iff AB is parallel to CD.

3. N is a positive integer > 2. Show that there are the same number of pairs of positive integers $a < b \le N$ such that $b/a > 2$ and such that $b/a < 2$.

4. Show that the only solution to $x + y^2 + z^3 = 3$, $y + z^2 + x^3 = 3$, $z + x^2 + y^3 = 3$ in positive reals is $x = y = z = 1$.

5. In an m x n array of reals the difference between the smallest and largest number in each row is at most $d > 0$. We now rearrange each column in decreasing order. Show that after the rearrangement the difference between the smallest and largest number in each row is still at most d.

6. A finite number of intervals cover [0, 1]. Show that one can find a subset of pairwise disjoint intervals with total length at least 1/2.

27th Swedish 1987

1. In a 4 x 4 array of real numbers the sum of each row, column and main diagonal is k. Show that the sum of the four corner numbers is also k.

2. A circular disk radius R is divided into two equal parts by a circle. Show that the arc of the circle inside the disk has length > 2R.

3. 10 closed intervals of length 1 lie in the interval [0,4]. Show that there is a point belonging to at least four of them.

4. f(x) is differentiable on [0,1] and f(0) = f(1) = 0. Show that there is a point y in [0,1] such that $|f'(y)| = 4 \int_0^1 |f(x)|\, dx$.

5. Show that for some t > 0, we have 1/(1+a) + 1/(1+b) + 1/(1+c) + 1/(1+d) > t for all positive a, b, c, d such that abcd = 1. Find the smallest such t.

6. A baker uses n different spices. He bakes 10 loaves. Each loaf has a different combination of spices and uses more than n/2 spices. Show that there are three spices such that every loaf has at least one.

28th Swedish 1988

1. A triangle has sides $a > b > c$ and corresponding altitudes h_a, h_b, h_c. Show that $a + h_a > b + h_b > c + h_c$.

2. 6 ducks are swimming on a pond radius 5. Show that any moment there are two ducks a distance at most 5 apart.

3. x_i are reals. Show that for $n = 3$, $x_1 + x_2 + ... + x_n = 0$ implies $x_1x_2 + x_2x_3 + ... + x_{n-1}x_n + x_nx_1 \le 0$. For which $n > 3$ is this true?

4. $p(x)$ is a polynomial of degree 3 with 3 distinct real zeros. How many real zeros does $p'(x)^2 - 2p(x)p''(x)$ have?

5. Show that there is a constant $c > 1$ such that if the positive integers m, n satisfy $m/n < \sqrt{7}$, then $7 - m^2/n^2 \ge c/n^2$. What is the largest such c?

6. The sequence $a_1, a_2, a_3, ...$ is defined by $a_1 = 1$, $a_{n+1} = \sqrt{(a_n^2 + 1/a_n)}$. Show that $1/2 \le a_n/n^c \le 2$ for some c (and all n).

29th Swedish 1989

1. Show that in base n^2+1 the numbers $n^2(n^2+2)^2$ and $n^4(n^2+2)^2$ have the same digits but in opposite order.

2. Find all continuous functions f(x) such that $f(x) + f(x^2) = 2$ for all real x.

3. For which positive integers n is $n^3 - 18n^2 + 115n - 391$ a cube?

4. ABCD is a regular tetrahedron. Find a point P on the edge BD such that the sphere diameter AP touches the edge CD.

5. The positive reals x_1, x_2, x_3, x_4, x_5 satisfy $x_1 < x_2$ and $x_2 \le$ each of x_3, x_4, x_5. Also $a > 0$. Show that $1/(x_1+x_3)^a + 1/(x_2+x_4)^a + 1/(x_2+x_5)^a < 1/(x_1+x_2)^a + 1/(x_2+x_3)^a + 1/(x_4+x_5)^a$.

6. 4n points are arranged around a circle. The points are colored alternately yellow and blue. The yellow points are divided into pairs and each pair is joined by a yellow line segment. Similarly for the blue points. At most two segments meet at any point inside the circle. Show that there are at least n points of intersection between a yellow segment and a blue segment.

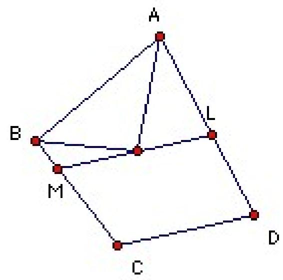
30th Swedish 1990

1. Let $d_1, d_2, \ldots, d_k$ be the positive divisors of n = 1990!. Show that $\sum d_i/\sqrt{n} = \sum \sqrt{n}/d_i$.

2. The points $A_1, A_2, \ldots, A_{2n}$ are equally spaced in that order along a straight line with $A_1A_2 = k$. P is chosen to minimise $\sum PA_i$. Find the minimum.

3. Find all a, b such that sin x + sin a ≥ b cos x for all x.

4. ABCD is a quadrilateral. The bisectors of ∠A and ∠B meet at E. The line through E parallel to CD meets AD at L and BC at M. Show that LM = AL + BM.



5. Find all monotonic positive functions f(x) defined on the positive reals such that f(xy) f(f(y)/x) = 1 for all x, y.

6. Find all positive integers m, n such that 117/158 > m/n > 97/131 and n ≤ 500.

31st Swedish 1991

1. Find all positive integers m, n such that $1/m + 1/n - 1/(mn) = 2/5$.

2. x, y are positive reals such that $x - \sqrt{x} \le y - 1/4 \le x + \sqrt{x}$. Show that $y - \sqrt{y} \le x - 1/4 \le y + \sqrt{y}$.

3. The sequence $x_0, x_1, x_2, \dots$ is defined by $x_0 = 0$, $x_{k+1} = [(n - \sum_0^k x_i)/2]$. Show that $x_k = 0$ for all sufficiently large k and that the sum of the non-zero terms x_k is n-1.

4. $x_1, x_2, \dots, x_8$ is a permutation of 1, 2, ... , 8. A *move* is to take x_3 or x_8 and place it at the start to from a new sequence. Show that by a sequence of moves we can always arrive at 1, 2, ... , 8.

5. Show that there are infinitely many odd positive integers n such that in binary n has more 1s than n^2.

6. Given any triangle, show that we can always pick a point on each side so that the three points form an equilateral triangle with area at most one quarter of the original triangle.

32nd Swedish 1992

1. Is $(19^{92} - 91^{29})/90$ an integer?

2. The squares in a 9 x 9 grid are numbered from 11 to 99, where the first digit is the row and the second the column. Each square is colored black or white. Squares 44 and 49 are black. Every black square shares an edge with at most one other black square, and each white square shares an edge with at most one other white square. What color is square 99?

3. Solve:

$2x_1 - 5x_2 + 3x_3 \geq 0$

$2x_2 - 5x_3 + 3x_4 \geq 0$

...

$2x_{23} - 5x_{24} + 3x_{25} \geq 0$

$2x_{24} - 5x_{25} + 3x_1 \geq 0$

$2x_{25} - 5x_1 + 3x_2 \geq 0$

4. Find all positive integers a, b, c such that $a < b$, $a < 4c$, and $b c^3 \leq a c^3 + b$.

5. A triangle has sides a, b, c with longest side c, and circumradius R. Show that if $a^2 + b^2 = 2cR$, then the triangle is right-angled.

6. (x_1, y_1), (x_2, y_2), (x_3, y_3) lie on a straight line and on the curve $y^2 = x^3$. Show that $x_1/y_1 + x_2/y_2 + x_3/y_3 = 0$.

33rd Swedish 1993

1. n and 3n have the same digit sum. Show that n is a multiple of 9.

2. A railway line passes through (in order) the 11 stations A, B, ... , K. The distance from A to K is 56. The distances AC, BD, CE, ... , IK are each ≤ 12. The distances AD, BE, CF, ... , HK are each ≥ 17. Find the distance between B and G.

3. Show that if ab are integers with ab even, then $a^2 + b^2 + x^2 = y^2$ has an integral solution x, y.

4. * is a real-valued operation on the non-zero reals which satisfies (1) a * a = 1, (2) a * (b * c) = (a * b) c (in other words, the ordinary product of (a * b) and c), for all a, b, c. Solve x * 36 = 216.

5. Given a triangle with sides a, b, c, we form, if we can, a triangle with sides s-a, s-b, s-c, where s = (a+b+c)/2. For which triangles can this be repeated indefinitely?

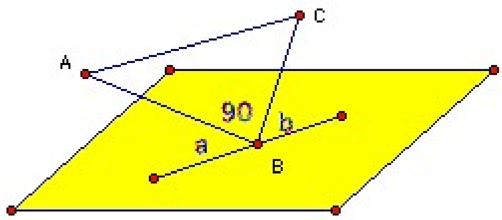
6. For reals a, b define the function f(x) = 1/(ax+b). For which a, b are there distinct reals x_1, x_2, x_3 such that $f(x_1) = x_2$, $f(x_2) = x_3$, $f(x_3) = x_1$.

34th Swedish 1994

1. $x\sqrt{8} + 1/(x\sqrt{8}) = \sqrt{8}$ has two real solutions x_1, x_2. The decimal expansion of x_1 has the digit 6 in place 1994. What digit does x_2 have in place 1994?

2. In the triangle ABC the medians from B and C are perpendicular. Show that cot B + cot C $\geq 2/3$.

3. The vertex B of the triangle ABC lies in the plane P. The plane of the triangle meets the plane in a line L. The angle between L and AB is a, and the angle between L and BC is b. The angle between the two planes is c. Angle ABC is 90^o. Show that $\sin^2 c = \sin^2 a + \sin^2 b$.



4. Find all integers m, n such that $2n^3 - m^3 = m n^2 + 11$.

5. The polynomial $x^k + a_1x^{k-1} + a_2x^{k-2} + ... + a_k$ has k distinct real roots. Show that $a_1^2 > 2ka_2/(k-1)$.

6. Let N be the set of non-negative integers. The function f:N→N satisfies f(a+b) = f(f(a)+b) for all a, b and f(a+b) = f(a)+f(b) for a+b < 10. Also f(10) = 1. How many three digit numbers n satisfy f(n) = f(N), where N is the "tower" 2, 3, 4, 5, in other words, it is 2^a, where $a = 3^b$, where $b = 4^5$?

35th Swedish 1995

1. The pages of a booklet are numbered from 1 to 2n. A single sheet (of 2 pages) is removed. The numbers of the remaining pages sum to 963. How many pages did the booklet have originally and which pages were removed?

2. X left home between 4 and 5 and returned between 5 and 6 to find that the hands of his clock had changed places. What time did he leave?

3. a, b, x, y are positive reals such that $a + b + x + y < 2$. If $a + b^2 = x + y^2$ and $a^2 + b = x^2 + y$, show that $a = x$ and $b = y$.

4. $a \geq b \geq c > 0$ are reals such that $abc = 1$ and $a + b + c > 1/a + 1/b + 1/c$. Show that $a > 1 > b$.

5. A, B, C, D are points on a circle radius r in that order. $AB = BC = CD = s < r$ and $AD = s+r$. Find the angles of the quadrilateral.

6. What is the largest number of binary sequences of length 10 such that each pair are different in at least 6 places?

36th Swedish 1996

1. A triangle area T is divided into six regions by lines drawn through a point inside the triangle parallel to the sides. The three triangular regions have areas T_1, T_2, T_3. Show that $\sqrt{T} = \sqrt{T_1} + \sqrt{T_2} + \sqrt{T_3}$.

2. Find $n > 1$ so that with stamp denominations n and n+2 it is possible to obtain any value $\geq 2n+2$.

3. For $x \geq 1$, define $p_n(x) = \frac{1}{2}(x + \sqrt{(x^2-1)})^n + \frac{1}{2}(x - \sqrt{(x^2-1)})^n$. Show that $p_n(x) \geq 1$ and $p_{mn}(x) = p_m(p_n(x))$.

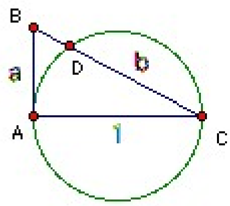
4. The pentagon ABCDE is incribed in a circle. $\angle A \leq \angle B \leq \angle C \leq \angle D \leq \angle E$. Show that $\angle C > \pi/2$ and that this is the best possible lower bound.

5. Show that we can divide $\{1, 2, 3, \ldots, 2^n\}$ into two disjoint parts S, T such that $\sum_{k \in S} k^m = \sum_{k \in T} k^m$ for $m = 0, 1, 2, \ldots, n-1$.

6. A rectangle is constructed from 1 x 6 rectangles. Show that one of its sides is a multiple of 6.

37th Swedish 1997

1. AC is a diameter of a circle. AB is a tangent. BC meets the circle again at D. AC = 1, AB = a, CD = b. Show that $1/(a^2 + ½) < b/a < 1/a^2$.



2. ABC is a triangle. BD is the angle bisector. The point E on AB is such that $\angle$ ACE = (2/5) $\angle$ ACB. BD and CE meet at P. ED = DC = CP. Find the angles of ABC.

3. a, b are integers with odd sum. Show that every integer can be written as $x^2 - y^2 + ax + by$ for some integers x, y.

4. A and B play a game as follows. Each throws a dice. Suppose A gets x and B gets y. If x and y have the same parity, then A wins. If not, then they make a list of all two digit numbers $ab \le xy$ with $1 \le a, b \le 6$. Then they take turns (starting with A) replacing two numbers on the list by their (non-negative) difference. When just one number remains, it is compared to x. If it has the same parity A wins, otherwise B wins. Find the probability that A wins.

5. Let s(n) be the sum of the digits of n. Show that for n > 1 and ≠ 10, there is a unique integer f(n) ≥ 2 such that s(k) + s(f(n) - k) = n for each 0 < k < f(n).

6. M is a union of finitely many disjoint intervals with total length > 1. Show that there are two distinct points in M whose difference is an integer.

38th Swedish 1998

1. Find all positive integers a, b, c, such that $(8a-5b)^2 + (3b-2c)^2 + (3c-7a)^2 = 2$.

2. ABC is a triangle. Show that $c \geq (a+b) \sin(C/2)$.

3. A cube side 5 is made up of unit cubes. Two small cubes are *adjacent* if they have a common face. Can we start at a cube adjacent to a corner cube and move through all the cubes just once? (The path must always move from a cube to an adjacent cube).

4. ABCD is a quadrilateral with $\angle A = 90^o$, AD = a, BC = b, AB = h, and area (a+b)h/2. What can we say about $\angle B$?

5. Show that for any n > 5 we can find positive integers $x_1, x_2, \ldots, x_n$ such that $1/x_1 + 1/x_2 + \ldots + 1/x_n = 1997/1998$. Show that in any such equation there must be two of the n numbers with a common divisor (> 1).

6. Show that for some c > 0, we have $|2^{1/3} - m/n| > c/n^3$ for all integers m, n with $n \geq 1$.

39th Swedish 1999

1. Solve $|||||x^2-x-1| - 2| - 3| - 4| - 5| = x^2 + x - 30$.

2. Circle C center O touches externally circle C' center O'. A line touches C at A and C' at B. P is the midpoint of AB. Show that $\angle OPO' = 90^o$.

3. Find non-negative integers a, b, c, d such that $5^a + 6^b + 7^c + 11^d = 1999$.

4. An equilateral triangle side x has its vertices on the sides of a square side 1. What are the possible values of x?

5. x_i are non-negative reals. $x_1 + x_2 + ... + x_n = s$. Show that $x_1x_2 + x_2x_3 + ... + x_{n-1}x_n \le s^2/4$.

6. S is any sequence of at least 3 positive integers. A *move* is to take any a, b in the sequence such that neither divides the other and replace them by gcd(a,b) and lcm(a,b). Show that only finitely many moves are possible and that the final result is independent of the moves made, except possibly for order.

40th Swedish 2000

1. Each of the numbers 1, 2, ... , 10 is colored red or blue. 5 is red and at least one number is blue. If m, n are different colors and $m+n \leq 10$, then $m+n$ is blue. If m, n are different colors and $mn \leq 10$, then mn is red. Find all the colors.

2. p(x) is a polynomial such that $p(y^2+1) = 6y^4 - y^2 + 5$. Find $p(y^2-1)$.

3. Are there any integral solutions to $n^2 + (n+1)^2 + (n+2)^2 = m^2$?

4. The vertices of a triangle are three-dimensional lattice points. Show that its area is at least ½.

5. Let f(n) be defined on the positive integers and satisfy: f(prime) = 1; f(ab) = a f(b) + f(a) b. Show that f is unique and find all n such that n = f(n).

6. Solve $y(x+y)^2 = 9$, $y(x^3-y^3) = 7$.

41st Swedish 2001

1. Show that if we take any six numbers from the following array, one from each row and column, then the product is always the same:

4	6	10	14	22	26
6	9	15	21	33	39
10	15	25	35	55	65
16	24	40	56	88	104
18	27	45	63	99	117
20	30	50	70	110	130

2. Show that $(\sqrt{52} + 5)^{1/3} - (\sqrt{52} - 5)^{1/3}$ is rational.

3. Show that if $b = (a+c)/2$ in the triangle ABC, then $\cos(A-C) + 4 \cos B = 3$.

4. ABC is a triangle. A circle through A touches the side BC at D and intersects the sides AB and AC again at E, F respectively. EF bisects $\angle AFD$ and $\angle ADC = 80^o$. Find $\angle ABC$.

5. Find all polynomials $p(x)$ such that $p'(x)^2 = c\, p(x)\, p''(x)$ for some constant c.

6. A chessboard is covered with 32 dominos. Each domino covers two adjacent squares. Show that the number of horizontal dominos with a white square on the left equals the number with a white square on the right.

42nd Swedish 2002

1. 268 numbers are written around a circle. The 17th number is 3, the 83rd is 4 and the 144th is 9, The sum of every 20 consecutive numbers is 72. Find the 210th number.

2. A, B, C can walk at 5km/hr. They have a car that can accomodate any two of them whch travels at 50km/hr. Can they reach a point 62km away in less than 3 hrs?

3. C is the circle center (0,1), radius 1. P is the parabola $y = ax^2$. They meet at (0, 0). For what values of a do they meet at another point or points?

4. For which integers $n \geq 8$ is $n^{1/(n-7)}$ an integer?

5. The reals a, b satisfy $a^3 - 3a^2 + 5a - 17 = 0$, $b^3 - 3b^2 + 5b + 11 = 0$. Find a+b.

6. A tetrahedron has five edges length 3 and circumradius 2. What is the length of the sixth edge?

43rd Swedish 2003

1. The real numbers w, x, y, z are all non-negative and satisfy: $y = x - 2003$, $z = 2y - 2003$, $w = 3z - 2003$. Find the solution with the smallest x.

2. A lecture room has a rectangular array of chairs. There are 6 boys in each row and 8 girls in each column. 15 chairs are unoccupied. What can be said about the number of rows and columns?

3. Which reals x satisfy $[x^2 - 2x] + 2[x] = [x]^2$?

4. Find all real polynomials p(x) such that $1 + p(x) \equiv (p(x-1) + p(x+1))/2$.

5. Given two positive reals a, b, how many non-similar plane quadrilaterals ABCD have $AB = a$, $BC = CD = DA = b$ and $\angle B = 90^o$?

USAMO (1972 – 2003)

1st USAMO 1972

1. Let (a, b, ... , k) denote the greatest common divisor of the integers a, b, ... k and [a, b, ... , k] denote their least common multiple. Show that for any positive integers a, b, c we have (a, b, c)2 [a, b] [b, c] [c, a] = [a, b, c]2 (a, b) (b, c) (c, a).

2. A tetrahedron has opposite sides equal. Show that all faces are acute-angled.

3. n digits, none of them 0, are randomly (and independently) generated, find the probability that their product is divisible by 10.

4. Let k be the real cube root of 2. Find integers A, B, C, a, b, c such that | $(Ax^2 + Bx + C)/(ax^2 + bx + c) - k$ | < | x - k | for all non-negative rational x.

5. A pentagon is such that each triangle formed by three adjacent vertices has area 1. Find its area, but show that there are infinitely many incongruent pentagons with this property.

2nd USAMO 1973

1. Show that if two points lie inside a regular tetrahedron the angle they subtend at a vertex is less than $\pi/3$.

2. The sequence a_n is defined by $a_1 = a_2 = 1$, $a_{n+2} = a_{n+1} + 2a_n$. The sequence b_n is defined by $b_1 = 1$, $b_2 = 7$, $b_{n+2} = 2b_{n+1} + 3b_n$. Show that the only integer in both sequences is 1.

3. Three vertices of a regular 2n+1 sided polygon are chosen at random. Find the probability that the center of the polygon lies inside the resulting triangle.

4. Find all complex numbers x, y, z which satisfy $x + y + z = x^2 + y^2 + z^2 = x^3 + y^3 + z^3 = 3$.

5. Show that the cube roots of three distinct primes cannot be terms in an arithmetic progression (whether consecutive or not).

3rd USAMO 1974

1. p(x) is a polynomial with integral coefficients. Show that there are no solutions to the equations p(a) = b, p(b) = c, p(c) = a, with a, b, c distinct integers.

2. Show that for any positive reals x, y, z we have $x^x y^y z^z \geq (xyz)^a$, where a is the arithmetic mean of x,y,z.

3. Two points in a thin spherical shell are joined by a curve shorter than the diameter of the shell. Show that the curve lies entirely in one hemisphere.

4. A, B, C play a series of games. Each game is between two players, The next game is between the winner and the person who was not playing. The series continues until one player has won two games. He wins the series. A is the weakest player, C the strongest. Each player has a fixed probability of winning against a given opponent. A chooses who plays the first game. Show that he should choose to play himself against B.

5. A point inside an equilateral triangle with side 1 is a distance a, b, c from the vertices. The triangle ABC has BC = a, CA = b, AB = c. The sides subtend equal angles at a point inside it. Show that sum of the distances of the point from the vertices is 1.

4th USAMO 1975

1. Show that for any non-negative reals x, y, [5x] + [5y] ≥ [3x+y] + [x+3y]. Hence or otherwise show that (5a)! (5b)!/(a! b! (3a+b)! (a+3b)!) is integral for any positive integers a, b.

2. Show that for any tetrahedron the sum of the squares of the lengths of two opposite edges is at most the sum of the squares of the other four.

3. A polynomial p(x) of degree n satisfies p(0) = 0, p(1) = 1/2, p(2) = 2/3, ... , p(n) = n/(n+1). Find p(n+1).

4. Two circles intersect at two points, one of them X. Find Y on one circle and Z on the other, so that X, Y and Z are collinear and XY.XZ is as large as possible.

5. A pack of n cards, including three aces, is well shuffled. Cards are turned over in turn. Show that the expected number of cards that must be turned over to reach the second ace is (n+1)/2.

5th USAMO 1976

1. The squares of a 4 x 7 chess board are colored red or blue. Show that however the coloring is done, we can find a rectangle with four distinct corner squares all the same color. Find a counter-example to show that this is not true for a 4 x 6 board.

2. AB is a fixed chord of a circle, not a diameter. CD is a variable diameter. Find the locus of the intersection of AC and BD.

3. Find all integral solutions to $a^2 + b^2 + c^2 = a^2b^2$.

4. A tetrahedron ABCD has edges of total length 1. The angles at A (BAC etc) are all 90^o. Find the maximum volume of the tetrahedron.

5. The polynomials a(x), b(x), c(x), d(x) satisfy $a(x^5) + x\, b(x^5) + x^2c(x^5) = (1 + x + x^2 + x^3 + x^4)\, d(x)$. Show that a(x) has the factor (x -1).

6th USAMO 1977

1. For which positive integers a, b does $(x^a + ... + x + 1)$ divide $(x^{ab} + x^{ab-b} + ... + x^{2b} + x^b + 1)$?

2. The triangles ABC and DEF have AD, BE and CF parallel. Show that [AEF] + [DBF] + [DEC] + [DBC] + [AEC] + [ABF] = 3 [ABC] + 3 [DEF], where [XYZ] denotes the *signed* area of the triangle XYZ. Thus [XYZ] is + area XYZ if the order X, Y, Z is anti-clockwise and - area XYZ if the order X, Y, Z is clockwise. So, in particular, [XYZ] = [YZX] = -[YXZ].

3. Prove that the product of the two real roots of $x^4 + x^3 - 1 = 0$ is a root of $x^6 + x^4 + x^3 - x^2 - 1 = 0$.

4. ABCD is a tetrahedron. The midpoint of AB is M and the midpoint of CD is N. Show that MN is perpendicular to AB and CD iff AC = BD and AD = BC.

5. The positive reals v, w, x, y, z satisfy $0 < h \le v, w, x, y, z \le k$. Show that $(v + w + x + y + z)(1/v + 1/w + 1/x + 1/y + 1/z) \le 25 + 6(\sqrt{(h/k)} - \sqrt{(k/h)})^2$. When do we have equality?

7th USAMO 1978

1. The sum of 5 real numbers is 8 and the sum of their squares is 16. What is the largest possible value for one of the numbers?

2. Two square maps cover exactly the same area of terrain on different scales. The smaller map is placed on top of the larger map and inside its borders. Show that there is a unique point on the top map which lies exactly above the corresponding point on the lower map. How can this point be constructed?

3. You are told that all integers from 33 to 73 inclusive can be expressed as a sum of positive integers whose reciprocals sum to 1. Show that the same is true for all integers greater than 73.

4. Show that if the angle between each pair of faces of a tetrahedron is equal, then the tetrahedron is regular. Does a tetrahedron have to be regular if five of the angles are equal?

5. There are 9 delegates at a conference, each speaking at most three languages. Given any three delegates, at least 2 speak a common language. Show that there are three delegates with a common language.

8th USAMO 1979

1. Find all sets of 14 or less fourth powers which sum to 1599.

2. N is the north pole. A and B are points on a great circle through N equidistant from N. C is a point on the equator. Show that the great circle through C and N bisects the angle ACB in the spherical triangle ABC (a spherical triangle has great circle arcs as sides).

3. $a_1, a_2, \ldots, a_n$ is an arbitrary sequence of positive integers. A member of the sequence is picked at random. Its value is a. Another member is picked at random, independently of the first. Its value is b. Then a third, value c. Show that the probability that a + b + c is divisible by 3 is at least 1/4.

4. P lies between the rays OA and OB. Find Q on OA and R on OB collinear with P so that 1/PQ + 1/PR is as large as possible.

5. X has n members. Given n+1 subsets of X, each with 3 members, show that we can always find two which have just one member in common.

9th USAMO 1980

1. A balance has unequal arms and pans of unequal weight. It is used to weigh three objects. The first object balances against a weight A, when placed in the left pan and against a weight a, when placed in the right pan. The corresponding weights for the second object are B and b. The third object balances against a weight C, when placed in the left pan. What is its true weight?

2. Find the maximum possible number of three term arithmetic progressions in a monotone sequence of n distinct reals.

3. A + B + C is an integral multiple of π. x, y, z are real numbers. If x sin A + y sin B + z sin C = x^2 sin 2A + y^2 sin 2B + z^2 sin 2C = 0, show that x^n sin nA + y^n sin nB + z^n sin nC = 0 for any positive integer n.

4. The insphere of a tetrahedron touches each face at its centroid. Show that the tetrahedron is regular.

5. If x, y, z are reals such that 0 ≤ x, y, z ≤ 1, show that x/(y + z + 1) + y/(z + x + 1) + z/(x + y + 1) ≤ 1 - (1 - x)(1 - y)(1 - z).

10th USAMO 1981

1. Prove that if n is not a multiple of 3, then the angle π/n can be trisected with ruler and compasses.

2. What is the largest number of towns that can meet the following criteria. Each pair is directly linked by just one of air, bus or train. At least one pair is linked by air, at least one pair by bus and at least one pair by train. No town has an air link, a bus link and a trian link. No three towns, A, B, C are such that the links between AB, AC and BC are all air, all bus or all train.

3. Show that for any triangle, $3(\sqrt{3})/2 \geq \sin 3A + \sin 3B + \sin 3C \geq -2$. When does equality hold?

4. A convex polygon has n sides. Each vertex is joined to a point P not in the same plane. If A, B, C are adjacent vertices of the polygon take the angle between the planes PBA and PBC. The sum of the n such angles equals the sum of the n angles subtended at P by the sides of the polygon (such as the angle APB). Show that n = 3.

5. Show that for any positive real x, $[nx] \geq \sum_1^n [kx]/k$.

11th USAMO 1982

1. A graph has 1982 points. Given any four points, there is at least one joined to the other three. What is the smallest number of points which are joined to 1981 points?

2. Show that if m, n are positive integers such that $(x^{m+n} + y^{m+n} + z^{m+n})/(m+n) = (x^m + y^m + z^m)/m\ (x^n + y^n + z^n)/n$ for all real x, y, z with sum 0, then {m, n} = {2, 3} or {2, 5}.

3. D is a point inside the equilateral triangle ABC. E is a point inside DBC. Show that area DBC/(perimeter DBC)^2 > area EBC/(perimeter EBC)^2.

4. Show that there is a positive integer k such that, for every positive integer n, $k\ 2^n + 1$ is composite.

5. O is the center of a sphere S. Points A, B, C are inside S, OA is perpendicular to AB and AC, and there are two spheres through A, B, and C which touch S. Show that the sum of their radii equals the radius of S.

12th USAMO 1983

1. If six points are chosen sequentially at random on the circumference of a circle, what is the probability that the triangle formed by the first three is disjoint from that formed by the second three.

2. Show that the five roots of the quintic $a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ are not all real if $2a_4^2 < 5a_5a_3$.

3. $S_1, S_2, \ldots, S_n$ are subsets of the real line. Each S_i is the union of two closed intervals. Any three S_i have a point in common. Show that there is a point which belongs to at least half the S_i.

4. Show that one can construct (with ruler and compasses) a length equal to the altitude from A of the tetrahedron ABCD, given the lengths of all the sides. [So for each pair of vertices, one is given a pair of points in the plane the appropriate distance apart.]

5. Prove that an open interval of length 1/n in the real line contains at most (n+1)/2 rational points p/q with $1 \le q \le n$.

13th USAMO 1984

1. Two roots of the real quartic $x^4 - 18x^3 + ax^2 + 200x - 1984 = 0$ have product -32. Find a.

2. Can one find a set of n distinct positive integers such that the geometric mean of any (non-empty, finite) subset is an integer? Can one find an infinite set with this property?

3. A, B, C, D, X are five points in space, such that AB, BC, CD, DA all subtend the acute angle θ at X. Find the maximum and minimum possible values of $\angle AXC + \angle BXD$ (for all such configurations) in terms of θ.

4. A maths exam has two papers, each with at least one question and 28 questions in total. Each pupil attempted 7 questions. Each pair of questions was attempted by just two pupils. Show that one pupil attempted either nil or at least 4 questions in the first paper.

5. A polynomial of degree 3n has the value 2 at 0, 3, 6, ... , 3n, the value 1 at 1, 4, 7, ... , 3n-2 and the value 0 at 2, 5, 8, ... , 3n-1. Its value at 3n+1 is 730. What is n?

14th USAMO 1985

1. Do there exist 1985 distinct positive integers such that the sum of their squares is a cube and the sum of their cubes is a square?

2. Find all real roots of the quartic $x^4 - (2N + 1)x^2 - x + N^2 + N - 1 = 0$ correct to 4 decimal places, where $N = 10^{10}$.

3. A tetrahedron has at most one edge longer than 1. What is the maximum total length of its edges?

4. A graph has $n > 2$ points. Show that we can find two points A and B such that at least $[n/2] - 1$ of the remaining points are joined to either both or neither of A and B.

5. $0 < a_1 \le a_2 \le a_3 \le \ldots$ is an unbounded sequence of integers. Let $b_n = m$ if a_m is the first member of the sequence to equal or exceed n. Given that $a_{19} = 85$, what is the maximum possible value of $a_1 + a_2 + \ldots + a_{19} + b_1 + b_2 + \ldots + b_{85}$?

15th USAMO 1986

1. Do there exist 14 consecutive positive integers each divisible by a prime less than 13? What about 21 consecutive positive integers each divisible by a prime less than 17?

2. Five professors attended a lecture. Each fell asleep just twice. For each pair there was a moment when both were asleep. Show that there was a moment when three of them were asleep.

3. What is the smallest $n > 1$ for which the average of the first n (non-zero) squares is a square?

4. A T-square allows you to construct a straight line through two points and a line perpendicular to a given line through a given point. Circles C and C' intersect at X and Y. XY is a diameter of C. P is a point on C' inside C. Using only a T-square, find points Q,R on C such that QR is perpendicular to XY and PQ is perpendicular to PR.

5. A partition of n is an increasing sequence of integers with sum n. For example, the partitions of 5 are: 1, 1, 1, 1, 1; 1, 1, 1, 2; 1, 1, 3; 1, 4; 5; 1, 2, 2; and 2, 3. If p is a partition, $f(p)$ = the number of 1s in p, and $g(p)$ = the number of distinct integers in the partition. Show that $\sum f(p) = \sum g(p)$, where the sum is taken over all partitions of n.

16th USAMO 1987

1. Find all solutions to $(m^2 + n)(m + n^2) = (m - n)^3$, where m and n are non-zero integers.

2. The feet of the angle bisectors of the triangle ABC form a right-angled triangle. If the right-angle is at X, where AX is the bisector of angle A, find all possible values for angle A.

3. X is the smallest set of polynomials p(x) such that: (1) p(x) = x belongs to X; and (2) if r(x) belongs to X, then x r(x) and (x + (1 - x) r(x)) both belong to X. Show that if r(x) and s(x) are distinct elements of X, then $r(x) \neq s(x)$ for any $0 < x < 1$.

4. M is the midpoint of XY. The points P and Q lie on a line through Y on opposite sides of Y, such that |XQ| = 2|MP| and |XY|/2 < |MP| < 3|XY|/2. For what value of |PY|/|QY| is |PQ| a minimum?

5. $a_1, a_2, \ldots, a_n$ is a sequence of 0s and 1s. T is the number of triples (a_i, a_j, a_k) with $i < j < k$ which are not equal to (0, 1, 0) or (1, 0, 1). For $1 \leq i \leq n$, f(i) is the number of $j < i$ with $a_j = a_i$ plus the number of $j > i$ with $a_j \neq a_i$. Show that T = f(1) (f(1) - 1)/2 + f(2) (f(2) - 1)/2 + ... + f(n) (f(n) - 1)/2. If n is odd, what is the smallest value of T?

17th USAMO 1988

1. The repeating decimal 0.ab ... k pq ... u = m/n, where m and n are relatively prime integers, and there is at least one decimal before the repeating part. Show that n is divisible by 2 or 5 (or both). [For example, 0.01136 = 0.01136363636 ... = 1/88 and 88 is divisible by 2.]

2. The cubic $x^3 + ax^2 + bx + c$ has real coefficients and three real roots $r \geq s \geq t$. Show that $k = a^2 - 3b \geq 0$ and that $\sqrt{k} <= r - t$.

3. Let X be the set {1, 2, ... , 20} and let P be the set of all 9-element subsets of X. Show that for any map f: P → X we can find a 10-element subset Y of X, such that f(Y - {k}) ≠ k for any k in Y.

4. ABC is a triangle with incenter I. Show that the circumcenters of IAB, IBC, ICA lie on a circle whose center is the circumcenter of ABC.

5. Let p(x) be the polynomial $(1 - x)^a (1 - x^2)^b (1 - x^3)^c \ldots (1 - x^{32})^k$, where a, b, ..., k are integers. When expanded in powers of x, the coefficient of x^1 is -2 and the coefficients of x^2, x^3, ... , x^{32} are all zero. Find k.

18th USAMO 1989

1. Let $a_n = 1 + 1/2 + 1/3 + ... + 1/n$, $b_n = a_1 + a_2 + ... + a_n$, $c_n = b_1/2 + b_2/2 + ... + b_n/(n+1)$. Find b_{1988} and c_{1988} in terms of a_{1989}.

2. In a tournament between 20 players, there are 14 games (each between two players). Each player is in at least one game. Show that we can find 6 games involving 12 different players.

3. A monic polynomial with real coefficients has modulus less than 1 at the complex number i. Show that there is a root $z = u + iv$ (with u and v real) such that $(u^2 + v^2 + 1)^2 < 4v^2 + 1$.

4. An acute-angled triangle has unequal sides. Show that the line through the circumcenter and incenter intersects the longest side and the shortest side.

5. Which is larger, the real root of $x + x^2 + ... + x^8 = 8 - 10x^9$, or the real root of $x + x^2 + ... + x^{10} = 8 - 10x^{11}$?

19th USAMO 1990

1. A license plate has six digits from 0 to 9 and may have leading zeros. If two plates must always differ in at least two places, what is the largest number of plates that is possible?

2. Define $f_1(x) = \sqrt{(x^2 + 48)}$ and $f_n(x) = \sqrt{(x^2 + 6f_{n-1}(x))}$. Find all real solutions to $f_n(x) = 2x$.

3. Show that for any odd positive integer we can always divide the set {n, n+1, n+2, ... , n+32} into two parts, one with 14 numbers and one with 19, so that the numbers in each part can be arranged in a circle, with each number relatively prime to its two neighbours. For example, for n = 1, arranging the numbers as 1, 2, 3, ... , 14 and 15, 16, 17, ... , 33, does not work, because 15 and 33 are not relatively prime.

4. How many positive integers can be written in base n so that (1) the integer has no two digits the same, and (2) each digit after the first differs by one from an earlier digit? For example, in base 3, the possible numbers are 1, 2, 10, 12, 21, 102, 120, 210.

5. ABC is acute-angled. The circle diameter AB meets the altitude from C at P and Q. The circle diameter AC meets the altitude from B at R and S. Show that P, Q, R and S lie on a circle.

20th USAMO 1991

1. An obtuse angled triangle has integral sides and one acute angle is twice the other. Find the smallest possible perimeter.

2. For each non-empty subset of {1, 2, ... , n} take the sum of the elements divided by the product. Show that the sum of the resulting quantities is $n^2 + 2n - (n + 1)s_n$, where $s_n = 1 + 1/2 + 1/3 + ... + 1/n$.

3. Define the function f on the natural numbers by f(1) = 2, $f(n) = 2^{f(n-1)}$. Show that f(n) has the same residue mod m for all sufficiently large n.

4. a and b are positive integers and $c = (a^{a+1} + b^{b+1})/(a^a + b^b)$. By considering $(x^n - n^n)/(x - n)$ or otherwise, show that $c^a + c^b \geq a^a + b^b$.

5. X is a point on the side BC of the triangle ABC. Take the other common tangent (apart from BC) to the incircles of ABX and ACX which intersects the segments AB and AC. Let it meet AX at Y. Show that the locus of Y, as X varies, is the arc of a circle.

21st USAMO 1992

1. Let a_n be the number written with 2^n nines. For example, $a_0 = 9$, $a_1 = 99$, $a_2 = 9999$. Let $b_n = \prod_0^n a_i$. Find the sum of the digits of b_n.

2. Let $k = 1^o$. Show that $\sum_0^{88} 1/(\cos nk \cos(n+1)k) = \cos k/\sin^2 k$.

3. A set of 11 distinct positive integers has the property that we can find a subset with sum n for any n between 1 and 1500 inclusive. What is the smallest possible value for the second largest element?

4. Three chords of a sphere are meet at a point X inside the sphere but are not coplanar. A sphere through an endpoint of each chord and X touches the sphere through the other endpoints and X. Show that the chords have equal length.

5. A complex polynomial has degree 1992 and distinct zeros. Show that we can find complex numbers z_n, such that if $p_1(z) = z - z_1$ and $p_n(z) = p_{n-1}(z)^2 - z_n$, then the polynomial divides $p_{1992}(z)$.

22nd USAMO 1993

1. $n > 1$, and a and b are positive real numbers such that $a^n - a - 1 = 0$ and $b^{2n} - b - 3a = 0$. Which is larger?

2. The diagonals of a convex quadrilateral meet at right angles at X. Show that the four points obtained by reflecting X in each of the sides are cyclic.

3. Let S be the set of functions f defined on reals in the closed interval [0, 1] with non-negative real values such that $f(1) = 1$ and $f(x) + f(y) \le f(x + y)$ for all x, y such that $x + y \le 1$. What is the smallest k such that $f(x) \le kx$ for all f in S and all x?

4. The sequence a_n of odd positive integers is defined as follows: $a_1 = r$, $a_2 = s$, and a_n is the greatest odd divisor of $a_{n-1} + a_{n-2}$. Show that, for sufficiently large n, a_n is constant and find this constant (in terms of r and s).

5. A sequence x_n of positive reals satisfies $x_{n-1}x_{n+1} \le x_n^2$. Let a_n be the average of the terms x_0, $x_1, \dots, x_n$ and b_n be the average of the terms $x_1, x_2, \dots, x_n$. Show that $a_nb_{n-1} \ge a_{n-1}b_n$.

23rd USAMO 1994

1. $a_1, a_2, a_3, \ldots$ are positive integers such that $a_n > a_{n-1} + 1$. Put $b_n = a_1 + a_2 + \ldots + a_n$. Show that there is always a square in the range $b_n, b_n+1, b_n+2, \ldots, b_{n+1}-1$.

2. The sequence $a_1, a_2, \ldots, a_{99}$ has $a_1 = a_3 = a_5 = \ldots = a_{97} = 1$, $a_2 = a_4 = a_6 = \ldots = a_{98} = 2$, and $a_{99} = 3$. We interpret subscripts greater than 99 by subtracting 99, so that a_{100} means a_1 etc. An allowed move is to change the value of any one of the a_n to another member of {1, 2, 3} different from its two neighbors, a_{n-1} and a_{n+1}. Is there a sequence of allowed moves which results in $a_m = a_{m+2} = \ldots = a_{m+96} = 1$, $a_{m+1} = a_{m+3} = \ldots = a_{m+95} = 2$, $a_{m+97} = 3$, $a_{n+98} = 2$ for some m? [So if m = 1, we have just interchanged the values of a_{98} and a_{99}.]

3. The hexagon ABCDEF has the following properties: (1) its vertices lie on a circle; (2) AB = CD = EF; and (3) the diagonals AD, BE, CF meet at a point. Let X be the intersection of AD and CE. Show that $CX/XE = (AC/CE)^2$.

4. x_i is a infinite sequence of positive reals such that for all n, $x_1 + x_2 + \ldots + x_n \geq \sqrt{n}$. Show that $x_1^2 + x_2^2 + \ldots + x_n^2 > (1 + 1/2 + 1/3 + \ldots + 1/n) / 4$ for all n.

5. X is a set of n positive integers with sum s and product p. Show for any integer $N \geq s$, ∑(parity(Y) (N - sum(Y))Cs) = p, where aCb is the binomial coefficient a!/(b! (a-b)!), the sum is taken over all subsets Y of X, parity(Y) = 1 if Y is empty or has an even number of elements, -1 if Y has an odd number of elements, and sum(Y) is the sum of the elements in Y.

24th USAMO 1995

1. The sequence a_0a_1, a_2, ... of non-negative integers is defined as follows. The first p-1 terms are 0, 1, 2, 3, ... , p-2. Then a_n is the least positive integer so that there is no arithmetic progression of length p in the first n+1 terms. If p is an odd prime, show that a_n is the number obtained by writing n in base p-1, then treating the result as a number in base p. For example, if p is 5, to get the 5th term one writes 5 as 11 in base 4, then treats this as a base 5 number to get 6.

2. A trigonometric map is any one of sin, cos, tan, arcsin, arccos and arctan. Show that given any positive rational number x, one can find a finite sequence of trigonometric maps which take 0 to x. [So we need to show that we can always find a sequence of trigonometric maps t_i so that: $x_1 = t_0(0)$, $x_2 = t_1(x_1)$, ... , $x_n = t_{n-1}(x_{n-1})$, $x = t_n(x_n)$.]

3. The circumcenter O of the triangle ABC does not lie on any side or median. Let the midpoints of BC, CA, AB be L, M, N respectively. Take P, Q, R on the rays OL, OM, ON respectively so that $\angle OPA = \angle OAL$, $\angle OQB = \angle OBM$ and $\angle ORC = \angle OCN$. Show that AP, BQ and CR meet at a point.

4. a_0, a_1, a_2, ... is an infinite sequence of integers such that $a_n - a_m$ is divisible by n - m for all (unequal) n and m. For some polynomial p(x) we have $p(n) > |a_n|$ for all n. Show that there is a polynomial q(x) such that $q(n) = a_n$ for all n.

5. A graph with n points and k edges has no triangles. Show that it has a point P such that there are at most $k(1 - 4k/n^2)$ edges between points not joined to P (by an edge).

25th USAMO 1996

A1. Let k = 1^o. Show that 2 sin 2k + 4 sin 4k + 6 sin 6k + ... + 180 sin 180k = 90 cot k.

A2. Let S be a set of n positive integers. Let P be the set of all integers which are the sum of one or more distinct elements of S. Show that we can find n subsets of P whose union is P such that if a, b belong to the same subset, then $a \le 2b$.

A3. Given a triangle, show that we can reflect it in some line so that the area of the intersection of the triangle and its reflection has area greater than 2/3 the area of the triangle.

B1. A type 1 sequence is a sequence with each term 0 or 1 which does not have 0, 1, 0 as consecutive terms. A type 2 sequence is a sequence with each term 0 or 1 which does not have 0, 0, 1, 1 or 1, 1, 0, 0 as consecutive terms. Show that there are twice as many type 2 sequences of length n+1 as type 1 sequences of length n.

B2. D lies inside the triangle ABC. $\angle BAC = 50^o$. $\angle DAB = 10^o$, $\angle DCA = 30^o$, $\angle DBA = 20^o$. Show that $\angle DBC = 60^o$.

B3. Does there exist a subset S of the integers such that, given any integer n, the equation n = 2s + s' has exactly one solution in S? For example, if T = {-3, 0, 1, 4), then there are unique solutions -3 = 2·0 - 3, -1 = 2·1 - 3, 0 = 2·0 + 0, 1 = 2·0 + 1, 2 = 0 + 2·1, 3 = 2·1 + 1, 4 = 2·0 + 4, 5 = 2·-3 + 1, but not for 6 = 2·1 + 4 = 2·-3 + 0, so T cannot be a subset of S.

26th USAMO 1997

A1. Let p_n be the nth prime. Let $0 < a < 1$ be a real. Define the sequence x_n by $x_0 = a$, x_n = the fractional part of p_n/x_{n-1} if $x_n \neq 0$, or 0 if $x_{n-1} = 0$. Find all a for which the sequence is eventually zero.

A2. ABC is a triangle. Take points D, E, F on the perpendicular bisectors of BC, CA, AB respectively. Show that the lines through A, B, C perpendicular to EF, FD, DE respectively are concurrent.

A3. Show that there is a unique polynomial whose coefficients are all single decimal digits which takes the value n at -2 and at -5.

B1. A sequence of polygons is derived as follows. The first polygon is a regular hexagon of area 1. Thereafter each polygon is derived from its predecessor by joining two adjacent edge midpoints and cutting off the corner. Show that all the polygons have area greater than 1/3.

B2. Show that $xyz/(x^3 + y^3 + xyz) + xyz/(y^3 + z^3 + xyz) + xyz/(z^3 + x^3 + xyz) \leq 1$ for all positive real x, y, z.

B3. The sequence of non-negative integers $c_1, c_2, \ldots, c_{1997}$ satisfies $c_1 \geq 0$ and $c_m + c_n \leq c_{m+n} <= c_m + c_n + 1$ for all $m, n > 0$ with $m + n < 1998$. Show that there is a real k such that $c_n = [nk]$ for $1 \leq n \leq 1997$.

27th USAMO 1998

A1. The sets $\{a_1, a_2, ... , a_{999}\}$ and $\{b_1, b_2, ... , b_{999}\}$ together contain all the integers from 1 to 1998. For each i, $|a_i - b_i| = 1$ or 6. For example, we might have $a_1 = 18$, $a_2 = 1$, $b_1 = 17$, $b_2 = 7$. Show that $\sum_1^{999} |a_i - b_i| = 9 \bmod 10$.

A2. Two circles are concentric. A chord AC of the outer circle touches the inner circle at Q. P is the midpoint of AQ. A line through A intersects the inner circle at R and S. The perpendicular bisectors of PR and CS meet at T on the line AC. What is the ratio AT/TC?

A3. The reals $x_1, x_2, ... , x_{n+1}$ satisfy $0 < x_i < \pi/2$ and $\sum_1^{n+1} \tan(x_i - \pi/4) \geq n-1$. Show that $\prod_1^{n+1} \tan x_i \geq n^{n+1}$.

B1. A 98 x 98 chess board has the squares colored alternately black and white in the usual way. A move consists of selecting a rectangular subset of the squares (with boundary parallel to the sides of the board) and changing their color. What is the smallest number of moves required to make all the squares black?

B2. Show that one can find a finite set of integers of any size such that for any two members the square of their difference divides their product.

B3. What is the largest number of the quadrilaterals formed by four adjacent vertices of an convex n-gon that can have an inscribed circle?

28th USAMO 1999

A1. Certain squares of an n x n board are colored black and the rest white. Every white square shares a side with a black square. Every pair of black squares can be joined by chain of black squares, so that consecutive members of the chain share a side. Show that there are at least $(n^2 - 2)/3$ black squares.

A2. For each pair of opposite sides of a cyclic quadrilateral take the larger length less the smaller length. Show that the sum of the two resulting differences is at least twice the difference in length of the diagonals.

A3. p is an odd prime. The integers a, b, c, d are not multiples of p and for any integer n not a multiple of p, we have $\{na/p\} + \{nb/p\} + \{nc/p\} + \{nd/p\} = 2$, where { } denotes the fractional part. Show that we can find at least two pairs from a, b, c, d whose sum is divisible by p.

B1. A set of $n > 3$ real numbers has sum at least n and the sum of the squares of the numbers is at least n^2. Show that the largest positive number is at least 2.

B2. Two players play a game on a line of 2000 squares. Each player in turn puts either S or O into an empty square. The game stops when three adjacent squares contain S, O, S in that order and the last player wins. If all the squares are filled without getting S, O, S, then the game is drawn. Show that the second player can always win.

B3. I is the incenter of the triangle ABC. The point D outside the triangle is such DA is parallel to BC and DB = AC, but ABCD is not a parallelogram. The angle bisector of BDC meets the line through I perpendicular to BC at X. The circumcircle of CDX meets the line BC again at Y. Show that DXY is isosceles.

29th USAMO 2000

A1. Show that there is no real-valued function f on the reals such that (f(x) + f(y))/2 ≥ f((x+y)/2) + |x - y| for all x, y.

A2. The incircle of the triangle ABC touches BC, CA, AB at D, E, F respectively. We have AF ≤ BD ≤ CE, the inradius is r and we have 2/AF + 5/BD + 5/CE = 6/r. Show that ABC is isosceles and find the lengths of its sides if r = 4.

A3. A player starts with A blue cards, B red cards and C white cards. He scores points as he plays each card. If he plays a blue card, his score is the number of white cards remaining in his hand. If he plays a red card it is three times the number of blue cards remaining in his hand. If he plays a white card, it is twice the number of red cards remaining in his hand. What is the lowest possible score as a function of A, B and C and how many different ways can it be achieved?

B1. How many squares of a 1000 x 1000 chessboard can be chosen, so that we cannot find three chosen squares with two in the same row and two in the same column?

B2. ABC is a triangle. C_1 is a circle through A and B. We can find circle C_2 through B and C touching C_1, circle C_3 through C and A touching C_2, circle C_4 through A and B touching C_3 and so on. Show that C_7 is the same as C_1.

B3. $x_1, x_2, \dots, x_n$, and $y_1, y_2, \dots, y_n$ are non-negative reals. Show that $\sum \min(x_ix_j, y_iy_j) \le \sum \min(x_iy_j, x_jy_i)$, where each sum is taken over all n^2 pairs (i, j).

30th USAMO 2001

A1. What is the smallest number of colors needed to color 8 boxes of 6 balls (one color for each ball), so that the balls in each box are all different colors and any pair of colors occurs in at most one box.

A2. The incircle of the triangle PBC touches BC at U and PC at V. The point S on BC is such that BS = CU. PS meets the incircle at two points. The nearer to P is Q. Take W on PC such that PW = CV. Let BW and PS meet at R. Show that PQ = RS.

A3. Non-negative reals x, y, z satisfy $x^2 + y^2 + z^2 + xyz = 4$. Show that $xyz \le xy + yz + zx <= xyz + 2$.

B1. ABC is a triangle and X is a point in the same plane. The three lengths XA, XB, XC can be used to form an obtuse-angled triangle. Show that if XA is the longest length, then $\angle$ BAC is acute.

B2. A set of integers is such that if a and b belong to it, then so do $a^2 - a$, and $a^2 - b$. Also, there are two members a, b whose greatest common divisor is 1 and such that a - 2 and b - 2 also have greatest common divisor 1. Show that the set contains all the integers.

B3. Every point in the plane is assigned a real number, so that for any three points which are not collinear, the number assigned to the incenter is the mean of the numbers assigned to the three points. Show that the same number is assigned to every point.

31st USAMO 2002

A1. Let S be a set with 2002 elements and P the set of all its subsets. Prove that for any n (in the range from zero to |P|) we can color n elements of P white, and the rest black, so that the union of any two elements of P with the same color has the same color.

A2. The triangle ABC satisfies the relation $\cot^2 A/2 + 4 \cot^2 B/2 + 9 \cot^2 C/2 = 9(a+b+c)^2/(49r^2)$, where r is the radius of the incircle (and a = |BC| etc, as usual). Show that ABC is similar to a triangle whose sides are integers and find the smallest set of such integers.

A3. p(x) is a polynomial of degree n with real coefficients and leading coefficient 1. Show that we can find two polynomials q(x) and r(x) which both have degree n, all roots real and leading coefficient 1, such that p(x) = q(x)/2 + r(x)/2.

B1. Find all real-valued functions f on the reals such that $f(x^2 - y^2) = x f(x) - y f(y)$ for all x, y.

B2. Show that we can link any two integers m, n greater than 2 by a chain of positive integers $m = a_1, a_2, \ldots, a_{k+1} = n$, so that the product of any two consecutive members of the chain is divisible by their sum. [For example, 7, 42, 21, 28, 70, 30, 6, 3 links 7 and 3.]

B3. A tromino is a 1 x 3 rectangle. Trominoes are placed on an n x n board. Each tromino must line up with the squares on the board, so that it covers exactly three squares. Let f(n) be the smallest number of trominoes required to stop any more being placed. Show that for all n > 0, $n^2/7 + hn \le f(n) \le n^2/5 + kn$ for some reals h and k.

33rd USAMO 2003

A1. Show that for each n we can find an n-digit number with all its digits odd which is divisible by 5^n.

A2. A convex polygon has all its sides and diagonals with rational length. It is dissected into smaller polygons by drawing all its diagonals. Show that the small polygons have all sides rational.

A3. Given a sequence S_1 of n+1 non-negative integers, $a_0, a_1, \dots , a_n$ we derive another sequence S_2 with terms $b_0, b_1, \dots , b_n$, where b_i is the number of terms preceding a_i in S_1 which are different from a_i (so $b_0 = 0$). Similarly, we derive S_2 from S_1 and so on. Show that if $a_i \le i$ for each i, then $S_n = S_{n+1}$.

B1. ABC is a triangle. A circle through A and B meets the sides AC, BC at D, E respectively. The lines AB and DE meet at F. The lines BD and CF meet at M. Show that M is the midpoint of CF iff $MB \cdot MD = MC^2$.

B2. Prove that for any positive reals x, y, z we have $(2x+y+z)^2/(2x^2 + (y+z)^2) + (2y+z+x)^2/(2y^2 + (z+x)^2) + (2z+x+y)^2/(2z^2 + (x+y)^2) \le 8$.

B3. A positive integer is written at each vertex of a hexagon. A move is to replace a number by the (non-negative) difference between the two numbers at the adjacent vertices. If the starting numbers sum to 2003^{2003}, show that it is always possible to make a sequence of moves ending with zeros at every vertex.

Vietnam (1962 – 2003)

1st Vietnam 1962 problems

1. Prove that $1/(1/a + 1/b) + 1/(1/c + 1/d) \leq 1/(1/(a+c) + 1/(b+d))$ for positive reals a, b, c, d.

2. $f(x) = (1 + x)(2 + x^2)^{1/2}(3 + x^3)^{1/3}$. Find $f'(-1)$.

3. ABCD is a tetrahedron. A' is the foot of the perpendicular from A to the opposite face, and B' is the foot of the perpendicular from B to the opposite face. Show that AA' and BB' intersect iff AB is perpendicular to CD. Do they intersect if AC = AD = BC = BD?

4. The tetrahedron ABCD has BCD equilateral and AB = AC = AD. The height is h and the angle between ABC and BCD is α. The point X is taken on AB such that the plane XCD is perpendicular to AB. Find the volume of the tetrahedron XBCD.

5. Solve the equation $\sin^6 x + \cos^6 x = 1/4$.

2nd Vietnam 1963 problems

1. A conference has 47 people attending. One woman knows 16 of the men who are attending, another knows 17, and so on up to the last woman who knows all the men who are attending. Find the number of men and women attending the conference.

2. For what values of m does the equation $x^2 + (2m + 6)x + 4m + 12 = 0$ has two real roots, both of them greater than -2.

3. Solve the equation $\sin^3 x \cos 3x + \cos^3 x \sin 3x = 3/8$.

4. The tetrahedron SABC has the faces SBC and ABC perpendicular. The three angles at S are all 60^o and SB = SC = 1. Find its volume.

5. The triangle ABC has perimeter p. Find the side length AB and the area S in terms of $\angle A$, $\angle B$ and p. In particular, find S if p = 23.6, A = 52.7 deg, B = 46 4/15 deg.

3rd Vietnam 1964 problems

1. Find cos x + cos(x + 2π/3) + cos(x + 4π/3) and sin x + sin(x + 2π/3) + sin(x + 4π/3).

2. Draw the graph of the functions $y = | x^2 - 1 |$ and $y = x + | x^2 - 1 |$. Find the number of roots of the equation $x + | x^2 - 1 | = k$, where k is a real constant.

3. Let O be a point not in the plane p and A a point in p. For each line in p through A, let H be the foot of the perpendicular from O to the line. Find the locus of H.

4. Define the sequence of positive integers f_n by $f_0 = 1$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$. Show that $f_n = (a^{n+1} - b^{n+1})/\sqrt{5}$, where a, b are real numbers such that a + b = 1, ab = -1 and a > b.

4th Vietnam 1965 problems

1. At time t = 0, a lion L is standing at point O and a horse H is at point A running with speed v perpendicular to OA. The speed and direction of the horse does not change. The lion's strategy is to run with constant speed u at an angle $0 < \varphi < \pi/2$ to the line LH. What is the condition on u and v for this strategy to result in the lion catching the horse? If the lion does not catch the horse, how close does he get? What is the choice of φ required to minimise this distance?

2. AB and CD are two fixed parallel chords of the circle S. M is a variable point on the circle. Q is the intersection of the lines MD and AB. X is the circumcenter of the triangle MCQ. Find the locus of X. What happens to X as M tends to (1) D, (2) C? Find a point E outside the plane of S such that the circumcenter of the tetrahedron MCQE has the same locus as X.

3. m an n are fixed positive integers and k is a fixed positive real. Show that the minimum value of $x_1^m + x_2^m + x_3^m + ... + x_n^m$ for real x_i satisfying $x_1 + x_2 + ... + x_n = k$ occurs at $x_1 = x_2 = ... = x_n$.

5th Vietnam 1966 problems

1. [missing]

2. a, b are two fixed lines through O. Variable lines x, y are parallel. x intersects a at A and b at C, y intersects a at B and b at D. The lines AD and BC meet at M. The line through M parallel to x meets a at L and b at N. What can you say about L, M, N? Find the locus M.

3. (1) ABCD is a rhombus. A tangent to the inscribed circle meets AB, DA, BC, CD at M, N, P, Q respectively. Find a relationship between BM and DN.

(2) ABCD is a rhombus and P a point inside. The circles through P with centers A, B, C, D meet the four sides AB, BC, CD, DA in eight points. Find a property of the resulting octagon. Use it to construct a regular octagon.

(3) [unclear].

6th Vietnam 1967 problems

1. Draw the graph of the function y = | x^3 - x^2 - 2x | /3 - | x + 1 |.

2. A river flows at speed u. A boat has speed v relative to the water. If its velocity is at an angle α relative the direction of the river, what is its speed relative to the river bank? What α minimises the time taken to cross the river?

3. (1) ABCD is a rhombus. A tangent to the inscribed circle meets AB, DA, BC, CD at M, N, P, Q respectively. Find a relationship between BM and DN.

(2) ABCD is a rhombus and P a point inside. The circles through P with centers A, B, C, D meet the four sides AB, BC, CD, DA in eight points. Find a property of the resulting octagon. Use it to construct an equiangular octagon.

(3) Rotate the figure about the line AC to form a solid. State a similar result.

7th Vietnam 1968 problems

1. The real numbers a and b satisfy $a \geq b > 0$, $a + b = 1$. Show that $a^m - a^n \geq b^m - b^n > 0$ for any positive integers $m < n$. Show that the quadratic $x^2 - b^n x - a^n$ has two real roots in the interval (-1, 1).

2. L and M are two parallel lines a distance d apart. Given r and x, construct a triangle ABC, with A on L, and B and C on M, such that the inradius is r, and angle A = x. Calculate angles B and C in terms of d, r and x. If the incircle touches the side BC at D, find a relation between BD and DC.

8th Vietnam 1969 problems

1. A graph G has n + k points. A is a subset of n points and B is the subset of the other k points. Each point of A is joined to at least k - m points of B where nm < k. Show that there is a point in B which is joined every point in A.

2. Find all real x such that $0 < x < \pi$ and $8/(3 \sin x - \sin 3x) + 3 \sin^2 x \le 5$.

3. The real numbers x_1, x_4, y_1, y_2 are positive and the real numbers x_2, x_3, y_3, y_4 are negative. We have $(x_i - a)^2 + (y_i - b)^2 \le c^2$ for i = 1, 2, 3, 4. Show that $a^2 + b^2 \le c^2$. State the result in geometric language.

4. Two circles centers O and O', radii R and R', meet at two points. A variable line L meets the circles at A, C, B, D in that order and AC/AD = CB/BD. The perpendiculars from O and O' to L have feet H and H'. Find the locus of H and H'. If $OO'^2 < R^2 + R'^2$, find a point P on L such that PO + PO' has the smallest possible value. Show that this value does not depend on the position of L. Comment on the case $OO'^2 > R^2 + R'^2$.

9th Vietnam 1970 problems

A1. ABC is a triangle. Show that sin A/2 sin B/2 sin C/2 < 1/4.

A2. Find all positive integers which divide 1890·1930·1970 and are not divisible by 45.

A3. The function f(x, y) is defined for all real numbers x, y. It satisfies f(x,0) = ax (where a is a non-zero constant) and if (c, d) and (h, k) are distinct points such that f(c, d) = f(h, k), then f(x, y) is constant on the line through (c, d) and (h, k). Show that for any real b, the set of points such that f(x, y) = b is a straight line and that all such lines are parallel. Show that f(x, y) = ax + by, for some constant b.

B1. AB and CD are perpendicular diameters of a circle. L is the tangent to the circle at A. M is a variable point on the minor arc AC. The ray BM, DM meet the line L at P and Q respectively. Show that AP·AQ = AB·PQ. Show how to construct the point M which gives BQ parallel to DP. If the lines OP and BQ meet at N find the locus of N. The lines BP and BQ meet the tangent at D at P' and Q' respectively. Find the relation between P' and Q'. The lines DP and DQ meet the line BC at P" and Q" respectively. Find the relation between P" and Q".

B2. A plane p passes through a vertex of a cube so that the three edges at the vertex make equal angles with p. Find the cosine of this angle. Find the positions of the feet of the perpendiculars from the vertices of the cube onto p. There are 28 lines through two vertices of the cube and 20 planes through three vertices of the cube. Find some relationship between these lines and planes and the plane p.

10th Vietnam 1971 problems

A1. m, n, r, s are positive integers such that: (1) $m < n$ and $r < s$; (2) m and n are relatively prime, and r and s are relatively prime; and (3) $\tan^{-1}m/n + \tan^{-1}r/s = \pi/4$. Given m and n, find r and s. Given n and s, find m and r. Given m and s, find n and r.

B1. ABCDA'B'C'D' is a cube (with ABCD and A'B'C'D' faces, and AA', BB', CC', DD' edges). L is a line which intersects or is parallel to the lines AA', BC and DB'. L meets the line BC at M (which may be the point at infinity). Let $m = |BM|$. The plane MAA' meets the line B'C' at E. Show that $|B'E| = m$. The plane MDB' meets the line A'D' at F. Show that $|D'F| = m$. Hence or otherwise show how to construct the point P at the intersection of L and the plane A'B'C'D'. Find the distance between P and the line A'B' and the distance between P and the line A'D' in terms of m. Find a relation between these two distances that does not depend on m. Find the locus of M. Let S be the envelope of the line L as M varies. Find the intersection of S with the faces of the cube.

11th Vietnam 1972 problems

A1. Let x = cos α, y = cos nα, where n is a positive integer. Show that for each x in the range [-1, 1], there is only one corresponding y. So consider y as a function of x and put $y = T_n(x)$. Find $T_1(x)$ and $T_2(x)$ and show that $T_{n+1}(x) = 2x\ T_n(x) - T_{n-1}(x)$. Show that $T_n(x)$ is a polynomial of degree n with n roots in [-1, 1].

A2. For any positive integer n, let $f(n) = \sum (-1)^{(d-1)/2}$ where the sum is taken over all odd d dividing n. Show that:

$f(2^n) = 1$

f(p) = 2 for p a prime congruent to 1 mod 4

f(p) = 0 for p a prime congruent to 3 mod 4

$f(p^n) = n+1$ for p a prime congruent to 1 mod 4

$f(p^n) = 1$ for p a prime congruent to 3 mod 4, and n even

$f(p^n) = 0$ for p a prime congruent to 3 mod 4, and n odd

Show that f(mn) = f(m) f(n) for m and n relatively prime. Find $f(5^4 11^{28} 17^{19})$ and f(1980). Show how to calculate f(n).

B1. ABC is a triangle. U is a point on the line BC. I is the midpoint of BC. The line through C parallel to AI meets the line AU at E. The line through E parallel to BC meets the line AB at F. The line through E parallel to AB meets the line BC at H. The line through H parallel to AU meets the line AB at K. The lines HK and FG meet at T. V is the point on the line AU such that A is the midpoint of UV. Show that V, T and I are collinear. [Next part unclear.]

B2. ABCD is a regular tetrahedron with side a. Take E, E' on the edge AB such that AE = a/6, AE' = 5a/6. Take F, F' on the edge AC such that AF = a/4, AF' = 3a/4. Take G, G' on the edge AD such that AG = a/3, AG' = 2a/3. Find the intersection of the planes BCD, EFG and E'F'G' and its position in the triangle BCD. Calculate the volume of EFGE'F'G' and find the angles between the lines AB, AC, AD and the plane EFG.

12th Vietnam 1974 problems

A1. Find all positive integers n and b with 0 < b < 10 such that if a_n is the positive integer with n digits, all of them 1, then a_{2n} - b a_n is a square.

A2. (1) How many positive integers n are such that n is divisible by 8 and n+1 is divisible by 25?
(2) How many positive integers n are such that n is divisible by 21 and n+1 is divisible by 165?
(3) Find all integers n such that n is divisible by 9, n+1 is divisible by 25 and n+2 is divisible by 4.

B1. ABC is a triangle. AH is the altitude. P, Q are the feet of the perpendiculars from P to AB, AC respectively. M is a variable point on PQ. The line through M perpendicular to MH meets the lines AB, AC at R, S respectively. Show that ARHS is cyclic. If M' is another position of M with corresponding points R', S', show that the ratio RR'/SS' is constant. Find the conditions on ABC such that if M moves at constant speed along PQ, then the speeds of R along AB and S along AC are the same. The point K on the line HM is on the other side of M to H and satisfies KM = HM. The line through K perpendicular to PQ meets the line RS at D. Show that if $\angle A = 90^o$, then $\angle BHR = \angle DHR$.

B2. C is a cube side 1. The 12 lines containing the sides of the cube meet at plane p in 12 points. What can you say about the 12 points?

13th Vietnam 1975 problems

A1. The roots of the equation $x^3 - x + 1 = 0$ are a, b, c. Find $a^8 + b^8 + c^8$.

A2. Find all real x which satisfy $(x^3 + a^3)/(x + a)^3 + (x^3 + b^3)/(x + b)^3 + (x^3 + c^3)/(x + c)^3 + 3(x - a)(x - b)(x - c)/(2(x + a)(x + b)(x + c)) = 3/2$.

A3. ABCD is a tetrahedron. The three edges at B are mutually perpendicular. O is the midpoint of AB and K is the foot of the perpendicular from O to CD. Show that vol KOAC/vol KOBD = AC/BD iff $2 \cdot AC \cdot BD = AB^2$.

B1. Find all terms of the arithmetic progression -1, 18, 37, 56, ... whose only digit is 5.

B2. Show that the sum of the maximum and minimum values of the function $\tan(3x)/\tan^3 x$ on the interval $(0, \pi/2)$ is rational.

B3. L is a fixed line and A a fixed point not on L. L' is a variable line (in space) through A. Let M be the point on L and N the point on L' such that MN is perpendicular to L and L'. Find the locus of M and the locus of the midpoint of MN.

14th Vietnam 1976 problems

A1. Find all integer solutions to $m^{m+n} = n^{12}$, $n^{m+n} = m^3$.

A2. Find all triangles ABC such that (a cos A + b cos B + c cos C)/(a sin A + b sin B + c sin C) = (a + b + c)/9R, where, as usual, a, b, c are the lengths of sides BC, CA, AB and R is the circumradius.

A3. P is a point inside the triangle ABC. The perpendicular distances from P to the three sides have product p. Show that $p \le 8 S^3/(27abc)$, where S = area ABC and a, b, c are the sides. Prove a similar result for a tetrahedron.

B1. Find all three digit integers abc = n, such that 2n/3 = a! b! c!

B2. L, L' are two skew lines in space and p is a plane not containing either line. M is a variable line parallel to p which meets L at X and L' at Y. Find the position of M which minimises the distance XY. L" is another fixed line. Find the line M which is also perpendicular to L".

B3. Show that $1/x_1^n + 1/x_2^n + ... + 1/x_k^n \ge k^{n+1}$ for real numbers x_i with sum 1.

15th Vietnam 1977 problems

A1. Find all real x such that $\sqrt{(x - 1/x)} + \sqrt{(1 - 1/x)} > (x - 1)/x$.

A2. Show that there are 1977 non-similar triangles such that the angles A, B, C satisfy (sin A + sin B + sin C)/(cos A + cos B + cos C) = 12/7 and sin A sin B sin C = 12/25.

A3. Into how many regions do n circles divide the plane, if each pair of circles intersects in two points and no point lies on three circles?

B1. p(x) is a real polynomial of degree 3. Find necessary and sufficient conditions on its coefficients in order that p(n) is integral for every integer n.

B2. The real numbers $a_0, a_1, \dots, a_{n+1}$ satisfy $a_0 = a_{n+1} = 0$ and $|a_{k-1} - 2a_k + a_{k+1}| \le 1$ for k = 1, 2, ... , n. Show that $|a_k| \le k(n + 1 - k)/2$ for all k.

B3. The planes p and p' are parallel. A polygon P on p has m sides and a polygon P' on p' has n sides. Find the largest and smallest distances between a vertex of P and a vertex of P'.

16th Vietnam 1978 problems

A1. Find all three digit numbers abc such that 2(abc) = bca + cab.

A2. Find all values of the parameter m such that the equations $x^2 = 2^{|x|} + |x| - y - m = 1 - y^2$ have only one root.

A3. The triangle ABC has angle A = 30° and AB = 3/4 AC. Find the point P inside the triangle which minimises 5 PA + 4 PB + 3 PC.

B1. Find three rational numbers a/d, b/d, c/d in their lowest terms such that they form an arithmetic progression and b/a = (a + 1)/(d + 1), c/b = (b + 1)/(d + 1).

B2. A river has a right-angle bend. Except at the bend, its banks are parallel lines a distance a apart. At the bend the river forms a square with the river flowing in across one side and out across an adjacent side. What is the longest boat of length c and negligible width which can pass through the bend?

B3. ABCDA'B'C'D' is a rectangular parallelepiped (so that ABCD and A'B'C'D' are faces and AA', BB', CC', DD' are edges). We have AB = a, AD = b, AA' = c. The perpendicular distances of A, A', D from the line BD' are p, q, r. Show that there is a triangle with sides p, q, r. Find a relation between a, b, c and p, q, r.

17th Vietnam 1979 problems

A1. Show that for all $x > 1$ there is a triangle with sides, $x^4 + x^3 + 2x^2 + x + 1$, $2x^3 + x^2 + 2x + 1$, $x^4 - 1$.

A2. Find all real numbers a, b, c such that $x^3 + ax^2 + bx + c$ has three real roots α,β,γ (not necessarily all distinct) and the equation $x^3 + a^3x^2 + b^3x + c^3$ has roots α^3, β^3, γ^3.

A3. ABC is a triangle. Find a point X on BC such that area ABX/area ACX = perimeter ABX/perimeter ACX.

B1. For each integer $n > 0$ show that there is a polynomial $p(x)$ such that $p(2 \cos x) = 2 \cos nx$.

B2. Find all real numbers k such that $x^2 - 2 x [x] + x - k = 0$ has at least two non-negative roots.

B3. ABCD is a rectangle with $BC/AB = \sqrt{2}$. ABEF is a congruent rectangle in a different plane. Find the angle DAF such that the lines CA and BF are perpendicular. In this configuration, find two points on the line CA and two points on the line BF so that the four points form a regular tetrahedron.

18th Vietnam 1980 problems

A1. Let $x_1, x_2, ... , x_n$ be real numbers in the interval $[0, \pi]$ such that $(1 + \cos x_1) + (1 + \cos x_2) + ... + (1 + \cos x_n)$ is an odd integer. Show that $\sin x_1 + \sin x_2 + ... + \sin x_n \geq 1$.

A2. Let $x_1, x_2, ... , x_n$ be positive reals with sum s. Show that $(x_1 + 1/x_1)^2 + (x_2 + 1/x_2)^2 + ... + (x_n + 1/x_n)^2 \geq n(n/s + s/n)^2$.

A3. P is a point inside the triangle $A_1A_2A_3$. The ray A_iP meets the opposite side at B_i. C_i is the midpoint of A_iB_i and D_i is the midpoint of PB_i. Show that area $C_1C_2C_3$ = area $D_1D_2D_3$.

B1. Show that for any tetrahedron it is possible to find two perpendicular planes such that if the projection of the tetrahedron onto the two planes has areas A and A', then $A'/A > \sqrt{2}$.

B2. Does there exist real m such that the equation $x^3 - 2x^2 - 2x + m$ has three different rational roots?

B3. Given $n > 1$ and real $s > 0$, find the maximum of $x_1x_2 + x_2x_3 + x_3x_4 + ... + x_{n-1}x_n$ for non-negative reals x_i such that $x_1 + x_2 + ... + x_n = s$.

19th Vietnam 1981 problems

A1. Show that the triangle ABC is right-angled iff sin A + sin B + sin C = cos A + cos B + cos C + 1.

A2. Find all integral values of m such that $x^3 + 2x + m$ divides $x^{12} - x^{11} + 3x^{10} + 11x^3 - x^2 + 23x + 30$.

A3. Given two points A, B not in the plane p, find the point X in the plane such that XA/XB has the smallest possible value.

B1. Find all real solutions to:

$w^2 + x^2 + y^2 + z^2 = 50$

$w^2 - x^2 + y^2 - z^2 = -24$

wx = yz

w - x + y - z = 0.

B2. $x_1, x_2, x_3, \ldots, x_n$ are reals in the interval [a, b]. $M = (x_1 + x_2 + \ldots + x_n)/n$, $V = (x_1^2 + x_2^2 + \ldots + x_n^2)/n$. Show that $M^2 \geq 4Vab/(a + b)^2$.

B3. Two circles touch externally at A. P is a point inside one of the circles, not on the line of centers. A variable line L through P meets one circle at B (and possibly another point) and the other circle at C (and possibly another point). Find L such that the circumcircle of ABC touches the line of centers at A.

20th Vietnam 1982 problems

A1. Find a quadratic with integer coefficients whose roots are cos 72^o and cos 144^o.

A2. Find all real solutions to x(x + 1)(x + 2)(x + 3) = m - 1.

A3. ABC is a triangle. A' is on the same side of BC as A, and A" is on the opposite side of BC. A'BC and A"BC are equilateral. B', B", C', C" are defined similarly. Show that area ABC + area A'B'C' = area A"B"C".

B1. Find all positive integer solutions to $2^a + 2^b + 2^c = 2336$.

B2. n is a positive integer. x and y are reals such that $0 \le x \le 1$ and $x^{n+1} \le y \le 1$. Show that the absolute value of $(y - x)(y - x^2)(y - x^3) \ldots (y - x^n)(1 + x)(1 + x^2) \ldots (1 + x^n)$ is at most $(y + x)(y + x^2) \ldots (y + x^n)(1 - x)(1 - x^2) \ldots (1 - x^n)$.

B3. ABCDA'B'C'D' is a cube (ABCD and A'B'C'D' are faces and AA', BB', CC', DD' are edges). L is the line joining the midpoints of BB' and DD'. Show that there is no line which meets L and the lines AA', BC and C'D'.

21st Vietnam 1983 problems

A1. For which positive integers m, n with n > 1 does $2^n - 1$ divides $2^m + 1$?

A2. (1) Show that $(\sin x + \cos x)\sqrt{2} \ge 2 \sin(2x)^{1/4}$ for all $0 \le x \le \pi/2$.

(2) Find all x such that $0 < x < \pi$ and $1 + 2 \cot(2x)/\cot x \ge \tan(2x)/\tan x$.

A3. P is a variable point inside the triangle ABC. D, E, F are the feet of the perpendiculars from P to the sides of the triangles. FInd the locus of P such that the area of DEF is constant.

B1. For which n can we find n different odd positive integers such that the sum of their reciprocals is 1?

B2. Let $s_n = 1/((2n-1)2n) + 2/((2n-3)(2n-1)) + 3/((2n-5)(2n-2)) + 4/((2n-7)(2n-3) + ... + n/(1(n+1))$ and $t_n = 1/1 + 1/2 + 1/3 + ... + 1/n$. Which is larger?

B3. ABCD is a tetrahedron with AB = CD. A variable plane intersects the tetrahedron in a quadrilateral. Find the positions of the plane which minimise the perimeter of the quadrilateral. Find the locus of the centroid for those quadrilaterals with minimum perimeter.

22nd Vietnam 1984 problems

A1. (1) Find a polynomial with integral coefficients which has the real number $2^{1/2} + 3^{1/3}$ as a root and the smallest possible degree.

(2) Find all real solutions to $1 + \sqrt{(1 + x^2)}\ (\sqrt{(1 + x)^3} - \sqrt{(1 - x)^3}\ ) = 2 + \sqrt{(1 - x^2)}$.

A2. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$, $a_2 = 2$, $a_{n+2} = 3a_{n+1} - a_n$. Find $\cot^{-1}a_1 + \cot^{-1}a_2 + \cot^{-1}a_3 + \ldots$.

A3. A cube side 2a has ABCD as one face. S is the other vertex (apart from B and D) adjacent to A. M, N are variable points on the lines BC, CD respectively.

(1) Find the positions of M and N such that the planes SMA and SMN are perpendicular, $BM + DN \geq 3a/2$, and $BM \cdot DN$ has the smallest value possible.

(2) Find the positions of M and N such that angle NAM = 45 deg, and the volume of SAMN is (a) a maximum, (b) a minimum, and find the maximum and minimum.

(3) Q is a variable point (in space) such that $\angle AQB = \angle AQD = 90^o$. p is the plane ABS. Q' is the intersection of DQ and p. Find the locus K of Q'. Let CQ meet K again at R. Let R' be the intersection of DR and p. Show that $\sin^2 Q'DB + \sin^2 R'DB$ is constant.

B1. (1) m, n are integers not both zero. Find the minimum value of $| 5m^2 + 11mn - 5n^2 |$. (2) Find all positive reals x such that $9x/10 = [x]/(x - [x])$.

B2. a, b are unequal reals. Find all polynomials p(x) which satisfy $x\, p(x - a) = (x - b)\, p(x)$ for all x.

B3. (1) ABCD is a tetrahedron. $\angle CAD = z$, $\angle BAC = y$, $\angle BAD = x$, the angle between the planes ACB and ACD is X, the angle between the planes ABC and ABD is Z, the angle between the planes ADB and ADC is Y. Show that $\sin x/\sin X = \sin y/\sin Y = \sin z/\sin Z$ and that $x + y = 180^o$ iff $X + Y = 180^o$.

(2) ABCD is a tetrahedron with $\angle BAC = \angle CAD = \angle DAB = 90^o$. Points A and B are fixed. C and D are variable. Show that $\angle CBD + \angle ABD + \angle ABC$ is constant. Find the locus of the center of the insphere of ABCD.

23rd Vietnam 1985 problems

A1. Find all integer solutions to $m^3 - n^3 = 2mn + 8$.

A2. Find all real-valued functions f(n) on the integers such that f(1) = 5/2, f(0) is not 0, and f(m) f(n) = f(m+n) + f(m-n) for all m, n.

A3. A parallelepiped has side lengths a, b, c. Its center is O. The plane p passes through O and is perpendicular to one of the diagonals. Find the area of its intersection with the parallelepiped.

B1. a, b, m are positive integers. Show that there is a positive integer n such that $(a^n - 1)b$ is divisible by m iff the greatest common divisor of ab and m is also the greatest common divisor of b and m.

B2. Find all real values a such that the roots of $16x^4 - ax^3 + (2a + 17)x^2 - ax + 16$ are all real and form an arithmetic progression.

B3. ABCD is a tetrahedron. The base BCD has area S. The altitude from B is at least (AC + AD)/2, the altitude from C is at least (AD + AB)/2, and the altitude from D is at least (AB + AC)/2. Find the volume of the tetrahedron.

24th Vietnam 1986 problems

A1. $a_1, a_2, \ldots, a_n$ are real numbers such that $1/2 \le a_i \le 5$ for each i. The real numbers $x_1, x_2, \ldots, x_n$ satisfy $4x_i^2 - 4a_ix_i + (a_i - 1)^2 = 0$. Let $S = (x_1 + x_2 + \ldots + x_n)/n$, $S_2 = (x_1^2 + x_2^2 + \ldots + x_n^2)/n$. Show that $\sqrt{S_2} \le S + 1$.

A2. P is a pyramid whose base is a regular 1986-gon, and whose sloping sides are all equal. Its inradius is r and its circumradius is R. Show that $R/r \ge 1 + 1/\cos(\pi/1986)$. Find the total area of the pyramid's faces when equality occurs.

A3. The polynomial p(x) has degree n and $p(1) = 2, p(2) = 4, p(3) = 8, \ldots, p(n+1) = 2^{n+1}$. Find p(n+2).

B1. ABCD is a square. ABM is an equilateral triangle in the plane perpendicular to ABCD. E is the midpoint of AB. O is the midpoint of CM. The variable point X on the side AB is a distance x from B. P is the foot of the perpendicular from M to the line CX. Find the locus of P. Find the maximum and minimum values of XO.

B2. Find all n > 1 such that $(x_1^2 + x_2^2 + \ldots + x_n^2) \ge x_n(x_1 + x_2 + \ldots + x_{n-1})$ for all real x_i.

B3. A sequence of positive integers is constructed as follows. The first term is 1. Then we take the next two even numbers: 2, 4. Then we take the next three odd numbers: 5, 7, 9. Then we take the next four even numbers: 10, 12, 14, 16. And so on. Find the nth term of the sequence.

25th Vietnam 1987 problems

A1. Let $x_n = (n+1)\pi/3974$. Find the sum of all $\cos(\pm x_1 \pm x_2 \pm ... \pm x_{1987})$.

A2. The sequences $a_0, a_1, a_2, ...$ and $b_0, b_1, b_2, ...$ are defined as follows. $a_0 = 365$, $a_{n+1} = a_n(a_n^{1986} + 1) + 1622$, $b_0 = 16$, $b_{n+1} = b_n(b_n^3 + 1) - 1952$. Show that there is no number in both sequences.

A3. There are $n > 2$ lines in the plane, no two parallel. The lines are not all concurrent. Show that there is a point on just two lines.

B1. $x_1, x_2, ... , x_n$ are positive reals with sum X and $n > 1$. $h \le k$ are two positive integers. $H = 2^h$ and $K = 2^k$. Show that $x_1^K/(X - x_1)^{H-1} + x_2^K/(X - x_2)^{H-1} + x_3^K/(X - x_3)^{H-1} + ... + x_n^K/(X - x_n)^{H-1} \ge X^{K-H+1}/((n-1)^{2H-1}n^{K-H})$. When does equality hold?

B2. The function $f(x)$ is defined and differentiable on the non-negative reals. It satisfies $| f(x) | \le 5$, $f(x) f '(x) \ge \sin x$ for all x. Show that it tends to a limit as x tends to infinity.

B3. Given 5 rays in space from the same point, show that we can always find two with an angle between them of at most 90^o.

26th Vietnam 1988 problems

A1. 994 cages each contain 2 chickens. Each day we rearrange the chickens so that the same pair of chickens are never together twice. What is the maximum number of days we can do this?

A2. The real polynomial $p(x) = x^n - nx^{n-1} + (n^2-n)/2\ x^{n-2} + a_{n-3}x^{n-3} + ... + a_1x + a_0$ (where $n > 2$) has n real roots. Find the values of $a_0, a_1, ..., a_{n-3}$.

A3. The plane is dissected into equilateral triangles of side 1 by three sets of equally spaced parallel lines. Does there exist a circle such that just 1988 vertices lie inside it?

B1. The sequence of reals $x_1, x_2, x_3, ...$ satisfies $x_{n+2} <= (x_n + x_{n+1})/2$. Show that it converges to a finite limit.

B2. ABC is an acute-angled triangle. Tan A, tan B, tan C are the roots of the equation $x^3 + px^2 + qx + p = 0$, where q is not 1. Show that $p \le \sqrt{27}$ and $q > 1$.

B3. For a line L in space let R(L) be the operation of rotation through 180 deg about L. Show that three skew lines L, M, N have a common perpendicular iff R(L) R(M) R(N) has the form R(K) for some line K.

27th Vietnam 1989 problems

A1. Show that the absolute value of sin(kx) /N + sin(kx + x) /(N+1) + sin(kx + 2x) /(N+2) + ... + sin(kx + nx) /(N+n) does not exceed the smaller of (n+1) |x| and 1/(N sin(x/2)), where N is a positive integer and k is real and satisfies $0 \le k \le N$.

A2. Let $a_1 = 1$, $a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$ be the Fibonacci sequence. Show that there are infinitely many terms of the sequence such that $1985a_n^2 + 1956a_n + 1960$ is divisible by 1989. Does there exist a term such that $1985a_n^2 + 1956a_n + 1960 + 2$ is divisible by 1989?

A3. ABCD is a square side 2. The segment AB is moved continuously until it coincides with CD (note that A is brought into coincidence with the *opposite* corner). Show that this can be done in such a way that the region passed over by AB during the motion has area $< 5\pi/6$.

B1. Do there exist integers m, n not both divisible by 5 such that $m^2 + 19n^2 = 198\cdot 10^{1989}$?

B2. Define the sequence of polynomials $p_0(x)$, $p_1(x)$, $p_2(x)$, ... by $p_0(x) = 0$, $p_{n+1}(x) = p_n(x) + (x - p_n(x)^2)/2$. Show that for any $0 \le x \le 1$, $0 < \sqrt{x} - p_n(x) \le 2/(n+1)$.

B3. ABCDA'B'C'D' is a parallelepiped (with edges AB, BC, CD, DA, AA', BB', CC', DD', A'B', B'C', C'D', D'A'). Show that if a line intersects three of the lines AB', BC', CA', AD', then it also intersects the fourth.

28th Vietnam 1990 problems

A1. $-1 < a < 1$. The sequence $x_1, x_2, x_3, \ldots$ is defined by $x_1 = a$, $x_{n+1} = (\sqrt{3 - 3x_n^2} - x_n)/2$. Find necessary and sufficient conditions on a for all members of the sequence to be positive. Show that the sequence is always periodic.

A2. n-1 or more numbers are removed from {1, 2, ... , 2n-1} so that if a is removed, so is 2a and if a and b are removed, so is a + b. What is the largest possible sum for the remaining numbers?

A3. ABCD is a tetrahedron with volume V. We wish to make three plane cuts to give a parallelepiped three of whose faces and all of whose vertices belong to the surface of the tetrahedron. Find the intersection of the three planes if the volume of the parallelepiped is 11V/50. Can it be done so that the volume of the parallelepiped is 9V/40?

B1. ABC is a triangle. P is a variable point. The feet of the perpendiculars from P to the lines BC, CA, AB are A', B', C' respectively. Find the locus of P such that PA PA' = PB PB' = PC PC'.

B2. The polynomial p(x) with degree at least 1 satisfies $p(x)\, p(2x^2) = p(3x^3 + x)$. Show that it does not have any real roots.

B3. Some children are sitting in a circle. Each has an even number of tokens (possibly zero). A child gives half his tokens to the child on his right. Then the child on his right does the same and so on. If a child about to give tokens has an odd number, then the teacher gives him an extra token. Show that after several steps, all the children will have the same number of tokens, except one who has twice the number.

29th Vietnam 1991 problems

A1. Find all real-valued functions f(x) on the reals such that f(xy)/2 + f(xz)/2 - f(x) f(yz) ≥ 1/4 for all x, y, z.

A2. For each positive integer n and odd k > 1, find the largest number N such that 2^N divides k^n - 1.

A3. The lines L, M, N in space are mutually perpendicular. A variable sphere passes through three fixed points A on L, B on M, C on N and meets the lines again at A', B', C'. Find the locus of the midpoint of the line joining the centroids of ABC and A'B'C'.

B1. 1991 students sit in a circle. Starting from student A and counting clockwise round the remaining students, every second and third student is asked to leave the circle until only one remains. (So if the students clockwise from A are A, B, C, D, E, F, ... , then B, C, E, F are the first students to leave.) Where was the surviving student originally sitting relative to A?

B2. The triangle ABC has centroid G. The lines GA, GB, GC meet the circumcircle again at D, E, F. Show that 3/R ≤ 1/GD + 1/GE + 1/GF ≤ √3 (1/AB + 1/BC + 1/CA), where R is the circumradius.

B3. Show that $x^2y/z + y^2z/x + z^2x/y \ge x^2 + y^2 + z^2$ for any non-negative reals x, y, z. **[This is false, (1,2,3), (1,1,1), (1,2,8) give >, =, < . Does anyone know the correct question?]**

30th Vietnam 1992 problems

A1. ABCD is a tetrahedron. The three face angles at A sum to 180^o, and the three face angles at B sum to 180^o. Two of the face angles at C, $\angle$ACD and $\angle$BCD, sum to 180^o. Find the sum of the areas of the four faces in terms of AC + CB = k and $\angle$ACB = x.

A2. For any positive integer n, let f(n) be the number of positive divisors of n which equal ±1 mod 10, and let g(n) be the number of positive divisors of n which equal ±3 mod 10. Show that f(n) ≥ g(n).

A3. Given a > 0, b > 0, c > 0, define the sequences a, b_n, c_n by $a_0 = a$, $b_0 = b$, $c_0 = c$, $a_{n+1} = a_n + 2/(b_n + c_n)$, $b_{n+1} = 2/(c_n + a_n)$, $c_{n+1} = c_n + 2/(a_n + b_n)$. Show that a_n tends to infinity.

B1. Label the squares of a 1991 x 1992 rectangle (m, n) with 1 ≤ m ≤ 1991 and 1 ≤ n ≤ 1992. We wish to color all the squares red. The first move is to color red the squares (m, n), (m+1, n+1), (m+2, n+1) for some m < 1990, n < 1992. Subsequent moves are to color any three (uncolored) squares in the same row, or to color any three (uncolored) squares in the same column. Can we color all the squares in this way?

B2. ABCD is a rectangle with center O and angle AOB ≤ 45^o. Rotate the rectangle about O through an angle 0 < x < 360^o. Find x such that the intersection of the old and new rectangles has the smallest possible area.

B3. Let p(x) be a polynomial with constant term 1 and every coefficient 0 or 1. Show that p(x) does not have any real roots > $(1 - \sqrt{5})/2$.

31st Vietnam 1993 problems

A1. f : [√1995, ∞) → R is defined by f(x) = x(1993 + √(1995 - x^2)). Find its maximum and minimum values.

A2. ABCD is a quadrilateral such that AB is not parallel to CD, and BC is not parallel to AD. Variable points P, Q, R, S are taken on AB, BC, CD, DA respectively so that PQRS is a parallelogram. Find the locus of its center.

A3. Find a function f(n) on the positive integers with positive integer values such that f(f(n)) = 1993 n^{1945} for all n.

B1. The tetrahedron ABCD has its vertices on the fixed sphere S. Find all configurations which minimise $AB^2 + AC^2 + AD^2 - BC^2 - BD^2 - CD^2$.

B2. 1993 points are arranged in a circle. At time 0 each point is arbitrarily labeled +1 or -1. At times n = 1, 2, 3, ... the vertices are relabeled. At time n a vertex is given the label +1 if its two neighbours had the same label at time n-1, and it is given the label -1 if its two neighbours had different labels at time n-1. Show that for some time n > 1 the labeling will be the same as at time 1.

B3. Define the sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ by $a_0 = 2$, $b_0 = 1$, $a_{n+1} = 2a_nb_n/(a_n + b_n)$, $b_{n+1} = \sqrt{(a_{n+1}b_n)}$. Show that the two sequences converge to the same limit, and find the limit.

32nd Vietnam 1994 problems

A1. Find all real solutions to:

$x^3 + 3x - 3 + \ln(x^2 - x + 1) = y$

$y^3 + 3y - 3 + \ln(y^2 - y + 1) = z$

$z^3 + 3z - 3 + \ln(z^2 - z + 1) = x$.

A2. ABC is a triangle. Reflect each vertex in the opposite side to get the triangle A'B'C'. Find a necessary and sufficient condition on ABC for A'B'C' to be equilateral.

A3. Define the sequence $x_0, x_1, x_2, \ldots$ by $x_0 = a$, where $0 < a < 1$, $x_{n+1} = 4/\pi^2 (\cos^{-1}x_n + \pi/2) \sin^{-1}x_n$. Show that the sequence converges and find its limit.

B1. There are n+1 containers arranged in a circle. One container has n stones, the others are empty. A *move* is to choose two containers A and B, take a stone from A and put it in one of the containers adjacent to B, and to take a stone from B and put it in one of the containers adjacent to A. We can take A = B. For which n is it possible by series of moves to end up with one stone in each container except that which originally held n stones.

B2. S is a sphere center O. G and G' are two perpendicular great circles on S. Take A, B, C on G and D on G' such that the altitudes of the tetrahedron ABCD intersect at a point. Find the locus of the intersection.

B3. Do there exist polynomials p(x), q(x), r(x) whose coefficients are positive integers such that $p(x) = (x^2 - 3x + 3)\, q(x)$ and $q(x) = (x^2/20 - x/15 + 1/12)\, r(x)$?

33rd Vietnam 1995 problems

A1. Find all real solutions to $x^3 - 3x^2 - 8x + 40 = 8(4x + 4)^{1/4} = 0$.

A2. The sequence $a_0, a_1, a_2, \ldots$ is defined by $a_0 = 1$, $a_1 = 3$, $a_{n+2} = a_{n+1} + 9a_n$ for n even, $9a_{n+1} + 5a_n$ for n odd. Show that $a_{1995}^2 + a_{1996}^2 + a_{1997}^2 + a_{1998}^2 + a_{1999}^2 + a_{2000}^2$ is divisible by 20, and that no a_{2n+1} is a square.

A3. ABC is a triangle with altitudes AD, BE, CF. A', B', C' are points on AD, BE, CF such that AA'/AD = BB'/BE = CC'/CF = k. Find all k such that A'B'C' is similar to ABC for all triangles ABC.

B1. ABCD is a tetrahedron. A' is the circumcenter of the face opposite A. B', C', D' are defined similarly. p_A is the plane through A perpendicular to C'D', p_B is the plane through B perpendicular to D'A', p_C is the plane through C perpendicular to A'B', and p_D is the plane through D perpendicular to B'C'. Show that the four planes have a common point. If this point is the circumcenter of ABCD, must ABCD be regular?

B2. Find all real polynomials p(x) such that p(x) = a has more than 1995 real roots, all greater than 1995, for any a > 1995. Multiple roots are counted according to their multiplicities.

B3. How many ways are there of coloring the vertices of a regular 2n-gon with n colors, such that each vertex is given one color, and every color is used for two non-adjacent vertices? Colorings are regarded as the same if one is obtained from the other by rotation.

35th Vietnam 1996 problems

A1. Find all real x, y such that $\sqrt{3x}\,(1 + 1/(x + y)) = 2$ and $\sqrt{7y}\,(1 - 1/(x + y)) = 4\sqrt{2}$.

A2. SABC is a tetrahedron. DAB, EBC, FCA are triangles in the plane of ABC congruent to SAB, SBC, SCA respectively. O is the circumcenter of DEF. Let K be the exsphere of SABC opposite O (which touches the planes SAB, SBC, SCA, ABC, lies on the opposite side of ABC to S, but on the same side of SAB as C, the same side of SBC as A, and the same side of SCA as B). Show that K touches the plane ABC at O.

A3. Let n be a positive integer and k a positive integer not greater than n. Find the number of ordered k-tuples $(a_1, a_2, \dots, a_n)$ such that: (1) all a_i are different, but all belong to $\{1, 2, \dots, n\}$; (2) $a_r > a_s$ for some $r < s$; (3) a_r has the opposite parity to r for some r.

B1. Find all functions f(n) on the positive integers with positive integer values, such that $f(n) + f(n+1) = f(n+2)\,f(n+3) - 1996$ for all n.

B2. The triangle ABC has BC = 1 and angle A = x. Let f(x) be the shortest possible distance between its incenter and its centroid. Find f(x). What is the largest value of f(x) for $60^o < x < 180^o$?

B3. Let w, x, y, z be non-negative reals such that $2(wx + wy + wz + xy + xz + yz) + wxy + xyz + yzw + zwx = 16$. Show that $3(w + x + y + z) \ge 2(wx + wy + wz + xy + xz + yz)$.

35th Vietnam 1997 problems

A1. S is a fixed circle with radius R. P is a fixed point inside the circle with OP = d < R. ABCD is a variable quadrilateral, such that A, B, C, D lie on S, AC intersects BD at P, and AC is perpendicular to BD. Find the maximum and minimum values of the perimeter of ABCD in terms of R and d.

A2. n > 1 is any integer not divisible by 1997. Put $a_m = m + mn/1997$ for m = 1, 2, ... , 1996 and $b_m = m + 1997m/n$ for m = 1, 2, ... , n-1. Arrange all the terms a_i, b_j in a single sequence in ascending order. Show that the difference between any two consecutive terms is less than 2.

A3. How many functions f(n) defined on the positive integers with positive integer values satisfy f(1) = 1 and $f(n)\,f(n+2) = f(n+1)^2 + 1997$ for all n?

B1. Let $k = 3^{1/3}$. Find a polynomial p(x) with rational coefficients and degree as small as possible such that $p(k + k^2) = 3 + k$. Does there exist a polynomial q(x) with integer coefficients such that $q(k + k^2) = 3 + k$?

B2. Show that for any positive integer n, we can find a positive integer f(n) such that $19^{f(n)} - 97$ is divisible by 2^n.

B3. Given 75 points in a unit cube, no three collinear, show that we can choose three points which form a triangle with area at most 7/72.

36th Vietnam 1998 problems

A1. Define the sequence $x_1, x_2, x_3, ...$ by $x_1 = a \geq 1$, $x_{n+1} = 1 + \ln(x_n(x_n^2+3)/(1 + 3x_n^2))$. Show that the sequence converges and find the limit.

A2. Let O be the circumcenter of the tetrahedron ABCD. Let A', B', C', D' be points on the circumsphere such that AA', BB', CC' and DD' are diameters. Let A" be the centroid of the triangle BCD. Define B", C", D" similarly. Show that the lines A'A", B'B", C'C", D'D" are concurrent. Suppose they meet at X. Show that the line through X and the midpoint of AB is perpendicular to CD.

A3. The sequence $a_0, a_1, a_2, ...$ is defined by $a_0 = 20$, $a_1 = 100$, $a_{n+2} = 4a_{n+1} + 5a_n + 20$. Find the smallest m such that $a_m - a_0$, $a_{m+1} - a_1$, $a_{m+2} - a_2$, ... are all divisible by 1998.

B1. Does there exist an infinite real sequence $x_1, x_2, x_3, ...$ such that $| x_n | \leq 0.666$, and $| x_m - x_n | \geq 1/(n^2 + n + m^2 + m)$ for all distinct m, n?

B2. What is the minimum value of $\sqrt{(x+1)^2 + (y-1)^2} + \sqrt{(x-1)^2 + (y+1)^2} + \sqrt{(x+2)^2 + (y+2)^2}$?

B3. Find all positive integers n for which there is a polynomial p(x) with real coefficients such that $p(x^{1998} - x^{-1998}) = (x^n - x^{-n})$ for all x.

37th Vietnam 1999 problems

A1. Find all real solutions to $(1 + 4^{2x-y})(5^{y-2x+1}) = 2^{2x-y+1} + 1$, $y^3 + 4x + \ln(y^2 + 2x) + 1 = 0$.

A2. ABC is a triangle. A' is the midpoint of the arc BC of the circumcircle not containing A. B' and C' are defined similarly. The segments A'B', B'C', C'A' intersect the sides of the triangle in six points, two on each side. These points divide each side of the triangle into three parts. Show that the three middle parts are equal iff ABC is equilateral.

A3. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$, $a_2 = 2$, $a_{n+2} = 3a_{n+1} - a_n$. The sequence $b_1, b_2, b_3, \ldots$ is defined by $b_1 = 1$, $b_2 = 4$, $b_{n+2} = 3b_{n+1} - b_n$. Show that the positive integers a, b satisfy $5a^2 - b^2 = 4$ iff $a = a_n$, $b = b_n$ for some n.

B1. Find the maximum value of $2/(x^2 + 1) - 2/(y^2 + 1) + 3/(z^2 + 1)$ for positive reals x, y, z which satisfy $xyz + x + z = y$.

B2. OA, OB, OC, OD are 4 rays in space such that the angle between any two is the same. Show that for a variable ray OX, the sum of the cosines of the angles XOA, XOB, XOC, XOD is constant and the sum of the squares of the cosines is also constant.

B3. Find all functions f(n) defined on the non-negative integers with values in the set {0, 1, 2, ... , 2000} such that: (1) $f(n) = n$ for $0 \le n \le 2000$; and (2) $f(f(m) + f(n)) = f(m + n)$ for all m, n.

38th Vietnam 2000 problems

A1. Define a sequence of positive reals $x_0, x_1, x_2, \ldots$ by $x_0 = b$, $x_{n+1} = \sqrt{(c - \sqrt{(c + x_n)})}$. Find all values of c such that for all b in the interval (0, c), such a sequence exists and converges to a finite limit as n tends to infinity.

A2. C and C' are circles centers O and O' respectively. X and X' are points on C and C' respectively such that the lines OX and O'X' intersect. M and M' are variable points on C and C' respectively, such that $\angle XOM = \angle X'O'M'$ (both measured clockwise). Find the locus of the midpoint of MM'. Let OM and O'M' meet at Q. Show that the circumcircle of QMM' passes through a fixed point.

A3. Let $p(x) = x^3 + 153x^2 - 111x + 38$. Show that p(n) is divisible by 3^{2000} for at least nine positive integers n less than 3^{2000}. For how many such n is it divisible?

B1. Given an angle α such that $0 < \alpha < \pi$, show that there is a unique real monic quadratic $x^2 + ax + b$ which is a factor of $p_n(x) = \sin \alpha\, x^n - \sin(n\alpha)\, x + \sin(n\alpha-\alpha)$ for all $n > 2$. Show that there is no linear polynomial $x + c$ which divides $p_n(x)$ for all $n > 2$.

B2. Find all n > 3 such that we can find n points in space, no three collinear and no four on the same circle, such that the circles through any three points all have the same radius.

B3. p(x) is a polynomial with real coefficients such that $p(x^2 - 1) = p(x)\, p(-x)$. What is the largest number of real roots that p(x) can have?

39th Vietnam 2001 problems

A1. A circle center O meets a circle center O' at A and B. The line TT' touches the first circle at T and the second at T'. The perpendiculars from T and T' meet the line OO' at S and S'. The ray AS meets the first circle again at R, and the ray AS' meets the second circle again at R'. Show that R, B and R' are collinear.

A2. Let $N = 6^n$, where n is a positive integer, and let $M = a^N + b^N$, where a and b are relatively prime integers greater than 1. M has at least two odd divisors greater than 1. Find the residue of M mod $6\ 12^n$.

A3. For real a, b define the sequence $x_0, x_1, x_2, \ldots$ by $x_0 = a$, $x_{n+1} = x_n + b \sin x_n$. If b = 1, show that the sequence converges to a finite limit for all a. If b > 2, show that the sequence diverges for some a.

B1. Find the maximum value of $1/\sqrt{x} + 2/\sqrt{y} + 3/\sqrt{z}$, where x, y, z are positive reals satisfying $1/\sqrt{2} \le z \le \min(x\sqrt{2}, y\sqrt{3})$, $x + z\sqrt{3} \ge \sqrt{6}$, $y\sqrt{3} + z\sqrt{10} \ge= 2\sqrt{5}$.

B2. Find all real-valued continuous functions defined on the interval (-1, 1) such that $(1 - x^2) f(2x/(1 + x^2)) = (1 + x^2)^2 f(x)$ for all x.

B3. $a_1, a_2, \ldots, a_{2n}$ is a permutation of 1, 2, ... , 2n such that $|a_i - a_{i+1}| \ne |a_j - a_{j+1}|$ for $i \ne j$. Show that $a_1 = a_{2n} + n$ iff $1 \le a_{2i} \le n$ for i = 1, 2, ... n.

40th Vietnam 2002 problems

A1. Solve the following equation: $\sqrt{(4 - 3\sqrt{(10 - 3x)})} = x - 2$.

A2. ABC is an isosceles triangle with AB = AC. O is a variable point on the line BC such that the circle center O radius OA does not have the lines AB or AC as tangents. The lines AB, AC meet the circle again at M, N respectively. Find the locus of the orthocenter of the triangle AMN.

A3. m < 2001 and n < 2002 are fixed positive integers. A set of distinct real numbers are arranged in an array with 2001 rows and 2002 columns. A number in the array is *bad* if it is smaller than at least m numbers in the same column and at least n numbers in the same row. What is the smallest possible number of bad numbers in the array?

B1. If all the roots of the polynomial $x^3 + a x^2 + bx + c$ are real, show that $12ab + 27c \le 6a^3 + 10(a^2 - 2b)^{3/2}$. When does equality hold?

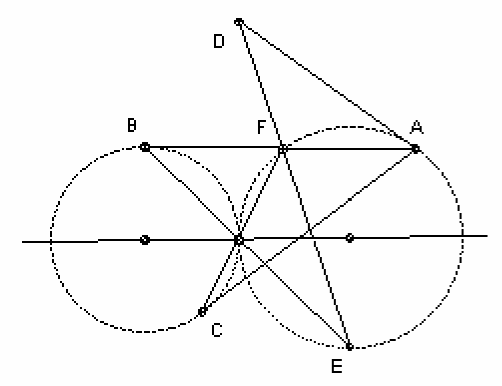
B2. Find all positive integers n for which the equation $a + b + c + d = n\sqrt{(abcd)}$ has a solution in positive integers.

B3. n is a positive integer. Show that the equation $1/(x - 1) + 1/(2^2x - 1) + ... + 1/(n^2x - 1) = 1/2$ has a unique solution $x_n > 1$. Show that as n tends to infinity, x_n tends to 4.

41st Vietnam 2003 problems

A1. Let R be the reals and f: R → R a function such that f(cot x) = cos 2x + sin 2x for all 0 < x < π. Define g(x) = f(x) f(1-x) for -1 ≤ x ≤ 1. Find the maximum and minimum values of g on the closed interval [-1, 1].

A2. The circles C_1 and C_2 touch externally at M and the radius of C_2 is larger than that of C_1. A is any point on C_2 which does not lie on the line joining the centers of the circles. B and C are points on C_1 such that AB and AC are tangent to C_1. The lines BM, CM intersect C_2 again at E, F respectively. D is the intersection of the tangent at A and the line EF. Show that the locus of D as A varies is a straight line.



A3. Let S_n be the number of permutations $(a_1, a_2, ... , a_n)$ of (1, 2, ... , n) such that $1 \le |a_k - k| \le 2$ for all k. Show that $(7/4) S_{n-1} < S_n < 2 S_{n-1}$ for n > 6.

B1. Find the largest positive integer n such that the following equations have integer solutions in x, $y_1, y_2, ... , y_n$: $(x + 1)^2 + y_1^2 = (x + 2)^2 + y_2^2 = ... = (x + n)^2 + y_n^2$.

B2. Define $p(x) = 4x^3 - 2x^2 - 15x + 9$, $q(x) = 12x^3 + 6x^2 - 7x + 1$. Show that each polynomial has just three distinct real roots. Let A be the largest root of p(x) and B the largest root of q(x). Show that $A^2 + 3 B^2 = 4$.

B3. Let R^+ be the set of positive reals and let F be the set of all functions $f : R^+ \to R^+$ such that $f(3x) \ge f(f(2x)) + x$ for all x. Find the largest A such that $f(x) \ge A x$ for all f in F and all x in R^+.

PART II. International/Regional Olympiad problems

Iberoamerican

(1985 – 2003)

1st Iberoamerican 1985 problems

A1. Find all integer solutions to: a + b + c = 24, $a^2 + b^2 + c^2 = 210$, abc = 440.

A2. P is a point inside the equilateral triangle ABC such that PA = 5, PB = 7, PC = 8. Find AB.

A3. Find the roots r_1, r_2, r_3, r_4 of the equation $4x^4 - ax^3 + bx^2 - cx + 5 = 0$, given that they are positive reals satisfying $r_1/2 + r_2/4 + r_3/5 + r_4/8 = 1$.

B1. The reals x, y, z satisfy x ≠ 1, y ≠ 1, x ≠ y, and $(yz - x^2)/(1 - x) = (xz - y^2)/(1 - y)$. Show that $(yx - x^2)/(1 - x) = x + y + z$.

B2. The function f(n) is defined on the positive integers and takes non-negative integer values. It satisfies (1) f(mn) = f(m) + f(n), (2) f(n) = 0 if the last digit of n is 3, (3) f(10) = 0. Find f(1985).

B3. O is the circumcenter of the triangle ABC. The lines AO, BO, CO meet the opposite sides at D, E, F respectively. Show that 1/AD + 1/BE + 1/CF = 2/AO.

2nd Iberoamerican 1987 problems

A1. Find f(x) such that $f(x)^2f((1-x)/(1+x)) = 64x$ for x not 0, ±1.

A2. In the triangle ABC, the midpoints of AC and AB are M and N respectively. BM and CN meet at P. Show that if it is possible to inscribe a circle in the quadrilateral AMPN (touching every side), then ABC is isosceles.

A3. Show that if $(2 + \sqrt{3})^k = 1 + m + n\sqrt{3}$, for positive integers m, n, k with k odd, then m is a perfect square.

B1. Define the sequence $p_1, p_2, p_3, \ldots$ as follows. $p_1 = 2$, and p_n is the largest prime divisor of $p_1p_2 \ldots p_{n-1} + 1$. Prove that 5 does not occur in the sequence.

B2. Show that the roots r, s, t of the equation $x(x - 2)(3x - 7) = 2$ are real and positive. Find $\tan^{-1}r + \tan^{-1}s + \tan^{-1}t$.

B3. ABCD is a convex quadrilateral. P, Q are points on the sides AD, BC respectively such that AP/PD = BQ/QC = AB/CD. Show that the angle between the lines PQ and AB equals the angle between the lines PQ and CD.

3rd Iberoamerican 1988 problems

A1. The sides of a triangle form an arithmetic progression. The altitudes also form an arithmetic progression. Show that the triangle must be equilateral.

A2. The positive integers a, b, c, d, p, q satisfy ad - bc = 1 and a/b > p/q > c/d. Show that q ≥ b + d and that if q = b + d, then p = a + c.

A3. P is a fixed point in the plane. Show that amongst triangles ABC such that PA = 3, PB = 5, PC = 7, those with the largest perimeter have P as incenter.

B1. Points A_1, A_2, ... , A_n are equally spaced on the side BC of the triangle ABC (so that $BA_1 = A_1A_2 = ... = A_{n-1}A_n = A_nC$). Similarly, points B_1, B_2, ... , B_n are equally spaced on the side CA, and points C_1, C_2, ... , C_n are equally spaced on the side AB. Show that $(AA_1^2 + AA_2^2 + ... + AA_n^2 + BB_1^2 + BB_2^2 + ... + BB_n^2 + C_1^2 + ... + CC_n^2)$ is a rational multiple of $(AB^2 + BC^2 + CA^2)$.

B2. Let $k^3 = 2$ and let x, y, z be any rational numbers such that $x + y k + z k^2$ is non-zero. Show that there are rational numbers u, v, w such that $(x + y k + z k^2)(u + v k + w k^2) = 1$.

B3. Let S be the collection of all sets of n distinct positive integers, with no three in arithmetic progression. Show that there is a member of S which has the largest sum of the inverses of its elements (you do not have to find it or to show that it is unique).

4th Iberoamerican 1989 problems

A1. Find all real solutions to: $x + y - z = -1$; $x^2 - y^2 + z^2 = 1$, $-x^3 + y^3 + z^3 = -1$.

A2. Given positive real numbers x, y, z each less than $\pi/2$, show that $\pi/2 + 2 \sin x \cos y + 2 \sin y \cos z > \sin 2x + \sin 2y + \sin 2z$.

A3. If a, b, c, are the sides of a triangle, show that $(a - b)/(a + b) + (b - c)/(b + c) + (c - a)/(a + c) < 1/16$.

B1. The incircle of the triangle ABC touches AC at M and BC at N and has center O. AO meets MN at P and BO meets MN at Q. Show that $MP \cdot OA = BC \cdot OQ$.

B2. The function f on the positive integers satisfies $f(1) = 1$, $f(2n + 1) = f(2n) + 1$ and $f(2n) = 3 f(n)$. Find the set of all m such that $m = f(n)$ for some n.

B3. Show that there are infinitely many solutions in positive integers to $2a^2 - 3a + 1= 3b^2 + b$.

5th Iberoamerican 1990 problems

A1. The function f is defined on the non-negative integers. $f(2^n - 1) = 0$ for n = 0, 1, 2, ... If m is not of the form $2^n - 1$, then f(m) = f(m+1) + 1. Show that $f(n) + n = 2^k - 1$ for some k, and find $f(2^{1990})$.

A2. I is the incenter of the triangle ABC and the incircle touches BC, CA, AB at D, E, F respectively. AD meets the incircle again at P. M is the midpoint of EF. Show that PMID is cyclic (or the points are collinear).

A3. $f(x) = (x + b)^2 + c$, where b and c are integers. If the prime p divides c, but p^2 does not divide c, show that f(n) is not divisible by p^2 for any integer n. If an odd prime q does not divide c, but divides f(n) for some n, show that for any r, we can find N such that q^r divides f(N).

B1. The circle C has diameter AB. The tangent at B is T. For each point M (not equal to A) on C there is a circle C' which touches T and touches C at M. Find the point at which C' touches T and find the locus of the center of C' as M varies. Show that there is a circle orthogonal to all the circles C'.

B2. A and B are opposite corners of an n x n board, divided into n^2 squares by lines parallel to the sides. In each square the diagonal parallel to AB is drawn, so that the board is divided into $2n^2$ small triangles. The board has $(n + 1)^2$ nodes and large number of line segments, each of length 1 or √2. A piece moves from A to B along the line segments. It never moves along the same segment twice and its path includes exactly two sides of every small triangle on the board. For which n is this possible?

B3. f(x) is a polynomial of degree 3 with rational coefficients. If its graph touches the x-axis, show that it has three rational roots.

6th Iberoamerican 1991 problems

A1. The number 1 or the number -1 is assigned to each vertex of a cube. Then each face is given the product of its four vertices. What are the possible totals for the resulting 14 numbers?

A2. Two perpendicular lines divide a square into four parts, three of which have area 1. Show that the fourth part also has area 1.

A3. f is a function defined on all reals in the interval [0, 1] and satisfies f(0) = 0, f(x/3) = f(x)/2, f(1 - x) = 1 - f(x). Find f(18/1991).

B1. Find a number N with five digits, all different and none zero, which equals the sum of all distinct three digit numbers whose digits are all different and are all digits of N.

B2. Let p(m, n) be the polynomial 2m2 - 6mn + 5n2. The range of p is the set of all integers k such that k = p(m, n) for some integers m, n. Find which members of {1, 2, ... , 100} are in the range of p. Show that if h and k are in the range of p, then so is hk.

B3. Given three non-collinear points M, N, H show how to construct a triangle which has H as orthocenter and M and N as the midpoints of two sides.

7th Iberoamerican 1992 problems

A1. a_n is the last digit of 1 + 2 + ... + n. Find $a_1 + a_2 + ... + a_{1992}$.

A2. Let $f(x) = a_1/(x + a_1) + a_2/(x + a_2) + ... + a_n/(x + a_n)$, where a_i are unequal positive reals. Find the sum of the lengths of the intervals in which $f(x) \geq 1$.

A3. ABC is an equilateral triangle with side 2. Show that any point P on the incircle satisfies $PA^2 + PB^2 + PC^2 = 5$. Show also that the triangle with side lengths PA, PB, PC has area $(\sqrt{3})/4$.

B1. Let a_n, b_n be two sequences of integers such that: (1) $a_0 = 0$, $b_0 = 8$; (2) $a_{n+2} = 2 a_{n+1} - a_n + 2$, $b_{n+2} = 2 b_{n+1} - b_n$, (3) $a_n^2 + b_n^2$ is a square for n > 0. Find at least two possible values for (a_{1992}, b_{1992}).

B2. Construct a cyclic trapezium ABCD with AB parallel to CD, perpendicular distance h between AB and CD, and AB + CD = m.

B3. Given a triangle ABC, take A' on the ray BA (on the opposite side of A to B) so that AA' = BC, and take A" on the ray CA (on the opposite side of A to C) so that AA" = BC. Similarly take B', B" on the rays CB, AB respectively with BB' = BB" = CA, and C', C" on the rays AB, CB. Show that the area of the hexagon A"A'B"B'C"C' is at least 13 times the area of the triangle ABC.

8th Iberoamerican 1993 problems

A1. A palindrome is a positive integers which is unchanged if you reverse the order of its digits. For example, 23432. If all palindromes are written in increasing order, what possible prime values can the difference between successive palindromes take?

A2. Show that any convex polygon of area 1 is contained in some parallelogram of area 2.

A3. Find all functions f on the positive integers with positive integer values such that (1) if $x < y$, then $f(x) < f(y)$, and (2) $f(y f(x)) = x^2 f(xy)$.

B1. ABC is an equilateral triangle. D is on the side AB and E is on the side AC such that DE touches the incircle. Show that AD/DB + AE/EC = 1.

B2. If P and Q are two points in the plane, let m(PQ) be the perpendicular bisector of PQ. S is a finite set of $n > 1$ points such that: (1) if P and Q belong to S, then some point of m(PQ) belongs to S, (2) if PQ, P'Q', P"Q" are three distinct segments, whose endpoints are all in S, then if there is a point in all of m(PQ), m(P'Q'), m(P"Q") it does not belong to S. What are the possible values of n?

B3. We say that two non-negative integers are *related* if their sum uses only the digits 0 and 1. For example 22 and 79 are related. Let A and B be two infinite sets of non-negative integers such that: (1) if $a \in A$ and $b \in B$, then a and b are related, (2) if c is related to every member of A, then it belongs to B, (3) if c is related to every member of B, then it belongs to A. Show that in one of the sets A, B we can find an infinite number of pairs of consecutive numbers.

9th Iberoamerican 1994 problems

A1. Show that there is a number $1 < b < 1993$ such that if 1994 is written in base b then all its digits are the same. Show that there is no number $1 < b < 1992$ such that if 1993 is written in base b then all its digits are the same.

A2. ABCD is a cyclic quadrilateral. A circle whose center is on the side AB touches the other three sides. Show that AB = AD + BC. What is the maximum possible area of ABCD in terms of |AB| and |CD|?

A3. There is a bulb in each cell of an n x n board. Initially all the bulbs are off. If a bulb is touched, that bulb and all the bulbs in the same row and column change state (those that are on, turn off, and those that are off, turn on). Show that it is possible by touching m bulbs to turn all the bulbs on. What is the minimum possible value of m?

B1. ABC is an acute-angled triangle. P is a point inside its circumcircle. The rays AP, BP, CP intersect the circle again at D, E, F. Find P so that DEF is equilateral.

B2. n and r are positive integers. Find the smallest k for which we can construct r subsets A_1, A_2, ... , A_r of {0, 1, 2, ... , n-1} each with k elements such that each integer $0 \le m < n$ can be written as a sum of one element from each of the r subsets.

B3. Show that given any integer $0 < n \le 2^{1000000}$ we can find at set S of at most 1100000 positive integers such that S includes 1 and n and every element of S except 1 is a sum of two (possibly equal) smaller elements of S.

10th Iberoamerican 1995 problems

A1. Find all possible values for the sum of the digits of a square.

A2. $n > 1$. Find all solutions in real numbers $x_1, x_2, \ldots, x_{n+1}$ all at least 1 such that: (1) $x_1^{1/2} + x_2^{1/3} + x_3^{1/4} + \ldots + x_n^{1/(n+1)} = n\, x_{n+1}^{1/2}$; and (2) $(x_1 + x_2 + \ldots + x_n)/n = x_{n+1}$.

A3. L and L' are two perpendicular lines not in the same plane. AA' is perpendicular to both lines, where A belongs to L and A' belongs to L'. S is the sphere with diameter AA'. For which points P on S can we find points X on L and X' on L' such that XX' touches S at P?

B1. ABCD is an n x n board. We call a diagonal row of cells a positive diagonal if it is parallel to AC. How many coins must be placed on an n x n board such that every cell either has a coin or is in the same row, column or positive diagonal as a coin?

B2. The incircle of the triangle ABC touches the sides BC, CA, AB at D, E, F respectively. AD meets the circle again at X and AX = XD. BX meets the circle again at Y and CX meets the circle again at Z. Show that EY = FZ.

B3. f is a function defined on the positive integers with positive integer values. Use $f^m(n)$ to mean $f(f(\ldots f(n) \ldots)) = n$ where f is taken m times, so that $f^2(n) = f(f(n))$, for example. Find the largest possible $0 < k < 1$ such that for some function f, we have $f^m(n) \neq n$ for $m = 1, 2, \ldots, [kn]$, but $f^m(n) = n$ for some m (which may depend on n).

11th Iberoamerican 1996 problems

A1. Find the smallest positive integer n so that a cube with side n can be divided into 1996 cubes each with side a positive integer.

A2. M is the midpoint of the median AD of the triangle ABC. The ray BM meets AC at N. Show that AB is tangent to the circumcircle of NBC iff $BM/MN = (BC/BN)^2$.

A3. $n = k^2 - k + 1$, where k is a prime plus one. Show that we can color some squares of an n x n board black so that each row and column has exactly k black squares, but there is no rectangle with sides parallel to the sides of the board which has its four corner squares black.

B1. $n > 2$ is an integer. Consider the pairs (a, b) of relatively prime positive integers, such that $a < b \le n$ and $a + b > n$. Show that the sum of 1/ab taken over all such pairs is 1/2.

B2. An equilateral triangle of side n is divided into n^2 equilateral triangles of side 1 by lines parallel to the sides. Initially, all the sides of all the small triangles are painted blue. Three coins A, B, C are placed at vertices of the small triangles. Each coin in turn is moved a distance 1 along a blue side to an adjacent vertex. The side it moves along is painted red, so once a coin has moved along a side, the side cannot be used again. More than one coin is allowed to occupy the same vertex. The coins are moved repeatedly in the order A, B, C, A, B, C, ... Show that it is possible to paint all the sides red in this way.

B3. $A_1, A_2, \dots, A_n$ are points in the plane. A non-zero real number k_i is assigned to each point, so that the square of the distance between A_i and A_j (for $i \ne j$) is $k_i + k_j$. Show that n is at most 4 and that if n = 4, then $1/k_1 + 1/k_2 + 1/k_3 + 1/k_4 = 0$.

12th Iberoamerican 1997 problems

A1. $k \geq 1$ is a real number such that if m is a multiple of n, then [mk] is a multiple of [nk]. Show that k is an integer.

A2. I is the incenter of the triangle ABC. A circle with center I meets the side BC at D and P, with D nearer to B. Similarly, it meets the side CA at E and Q, with E nearer to C, and it meets AB at F and R, with F nearer to A. The lines EF and QR meet at S, the lines FD and RP meet at T, and the lines DE and PQ meet at U. Show that the circumcircles of DUP, ESQ and FTR have a single point in common.

A3. $n > 1$ is an integer. D_n is the set of lattice points (x, y) with $|x|, |y| \leq n$. If the points of D_n are colored with three colors (one for each point), show that there are always two points with the same color such that the line containing them does not contain any other points of D_n. Show that it is possible to color the points of D_n with four colors (one for each point) so that if any line contains just two points of D_n then those two points have different colors.

B1. Let o(n) be the number of 2n-tuples $(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$ such that each $a_i, b_j = 0$ or 1 and $a_1b_1 + a_2b_2 + \ldots + a_nb_n$ is odd. Similarly, let e(n) be the number for which the sum is even. Show that $o(n)/e(n) = (2^n - 1)/(2^n + 1)$.

B2. ABC is an acute-angled triangle with orthocenter H. AE and BF are altitudes. AE is reflected in the angle bisector of angle A and BF is reflected in the angle bisector of angle B. The two reflections intersect at O. The rays AE and AO meet the circumcircle of ABC at M and N respectively. P is the intersection of BC and HN, R is the intersection of BC and OM, and S is the intersection of HR and OP. Show that AHSO is a parallelogram.

B3. Given 1997 points inside a circle of radius 1, one of them the center of the circle. For each point take the distance to the closest (distinct) point. Show that the sum of the squares of the resulting distances is at most 9.

13th Iberoamerican 1998 problems

A1. There are 98 points on a circle. Two players play alternately as follows. Each player joins two points which are not already joined. The game ends when every point has been joined to at least one other. The winner is the last player to play. Does the first or second player have a winning strategy?

A2. The incircle of the triangle ABC touches BC, CA, AB at D, E, F respectively. AD meets the circle again at Q. Show that the line EQ passes through the midpoint of AF iff AC = BC.

A3. Find the smallest number n such that given any n distinct numbers from {1, 2, 3, ... , 999}, one can choose four different numbers a, b, c, d such that a + 2b + 3c = d.

B1. Representatives from n > 1 different countries sit around a table. If two people are from the same country then their respective right hand neighbors are from different countries. Find the maximum number of people who can sit at the table for each n.

B2. P_1, P_2, ... , P_n are points in the plane and r_1, r_2, ... , r_n are real numbers such that the distance between P_i and P_j is $r_i + r_j$ (for i not equal to j). Find the largest n for which this is possible.

B3. k is the positive root of the equation $x^2 - 1998x - 1 = 0$. Define the sequence x_0, x_1, x_2, ... by $x_0 = 1$, $x_{n+1} = [k\, x_n]$. Find the remainder when x_{1998} is divided by 1998.

14th Iberoamerican 1999 problems

A1. Find all positive integers $n < 1000$ such that the cube of the sum of the digits of n equals n^2.

A2. Given two circles C and C' we say that C *bisects* C' if their common chord is a diameter of C'. Show that for any two circles which are not concentric, there are infinitely many circles which bisect them both. Find the locus of the centers of the bisecting circles.

A3. Given points $P_1, P_2, \ldots, P_n$ on a line we construct a circle on diameter P_iP_j for each pair i, j and we color the circle with one of k colors. For each k, find all n for which we can always find two circles of the same color with a common external tangent.

B1. Show that any integer greater than 10 whose digits are all members of {1, 3, 7, 9} has a prime factor ≥ 11.

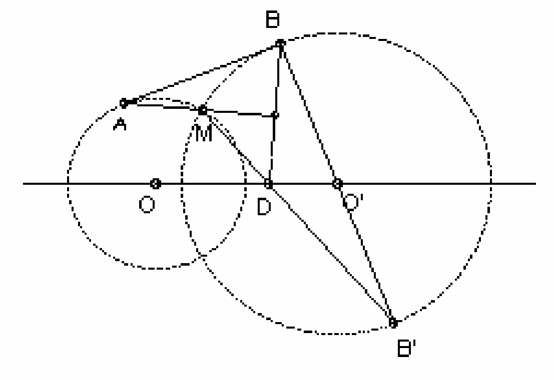
B2. O is the circumcenter of the acute-angled triangle ABC. The altitudes are AD, BE and CF. The line EF cuts the circumcircle at P and Q. Show that OA is perpendicular to PQ. If M is the midpoint of BC, show that $AP^2 = 2\, AD \cdot OM$.

B3. Given two points A and B, take C on the perpendicular bisector of AB. Define the sequence $C_1, C_2, C_3, \ldots$ as follows. $C_1 = C$. If C_n is not on AB, then C_{n+1} is the circumcenter of the triangle ABC_n. If C_n lies on AB, then C_{n+1} is not defined and the sequence terminates. Find all points C such that the sequence is periodic from some point on.

15th Iberoamerican 2000 problems

A1. Label the vertices of a regular n-gon from 1 to n > 3. Draw all the diagonals. Show that if n is odd then we can label each side and diagonal with a number from 1 to n different from the labels of its endpoints so that at each vertex the sides and diagonals all have different labels.

A2. Two circles C and C' have centers O and O' and meet at M and N. The common tangent closer to M touches C at A and C' at B. The line through B perpendicular to AM meets the line OO' at D. BO'B' is a diameter of C'. Show that M, D and B' are collinear.



A3. Find all solutions to $(m + 1)^a = m^b + 1$ in integers greater than 1.

B1. Some terms are deleted from an infinite arithmetic progression 1, x, y, ... of real numbers to leave an infinite geometric progression 1, a, b, ... Find all possible values of a.

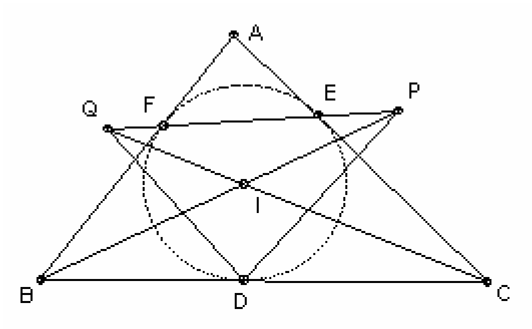
B2. Given a pile of 2000 stones, two players take turns in taking stones from the pile. Each player must remove 1, 2, 3, 4, or 5 stones from the pile at each turn, but may not take the same number as his opponent took on his last move. The player who takes the last stone wins. Does the first or second player have a winning strategy?

B3. A convex hexagon is called a *unit* if it has four diagonals of length 1, whose endpoints include all the vertices of the hexagon. Show that there is a unit of area k for any $0 < k \leq 1$. What is the largest possible area for a unit?

16th Iberoamerican 2001 problems

A1. Show that there are arbitrarily large numbers n such that: (1) all its digits are 2 or more; and (2) the product of any four of its digits divides n.

A2. ABC is a triangle. The incircle has center I and touches the sides BC, CA, AB at D, E, F respectively. The rays BI and CI meet the line EF at P and Q respectively. Show that if DPQ is isosceles, then ABC is isosceles.



A3. Let X be a set with n elements. Given k > 2 subsets of X, each with at least r elements, show that we can always find two of them whose intersection has at least r - nk/(4k - 4) elements.

B1. Call a set of 3 distinct elements which are in arithmetic progression a *trio*. What is the largest number of trios that can be subsets of a set of n distinct real numbers?

B2. Two players play a game on a 2000 x 2001 board. Each has one piece and the players move their pieces alternately. A *short move* is one square in any direction (including diagonally) or no move at all. On his first turn each player makes a short move. On subsequent turns a player must make the same move as on his previous turn followed by a short move. This is treated as a single move. The board is assumed to wrap in both directions so a player on the edge of the board can move to the opposite edge. The first player wins if he can move his piece onto the same square as his opponent's piece. For example, suppose we label the squares from (0, 0) to (1999, 2000), and the first player's piece is initially at (0, 0) and the second player's at (1996, 3). The first player could move to (1999, 2000), then the second player to (1996, 2). Then the first player could move to (1998, 1998), then the second player to (1995, 1). Can the first player always win irrespective of the initial positions of the two pieces?

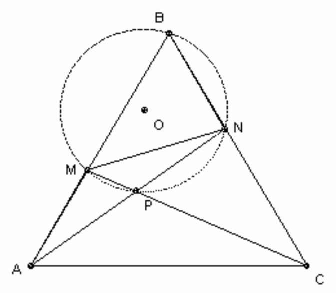
B3. Show that a square with side 1 cannot be covered by five squares with side less than 1/2.

17th Iberoamerican 2002 problems

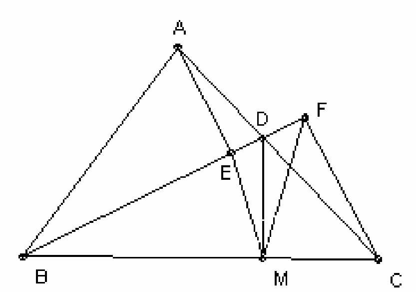
A1. The numbers 1, 2, ... , 2002 are written in order on a blackboard. Then the 1st, 4th, 7th, ... , 3k+1th, ... numbers in the list are erased. Then the 1st, 4th, 7th, ... 3k+1th numbers in the remaining list are erased (leaving 3, 5, 8, 9, 12, ...). This process is carried out repeatedly until there are no numbers left. What is the last number to be erased?

A2. Given a set of 9 points in the plane, no three collinear, show that for each point P in the set, the number of triangles containing P formed from the other 8 points in the set must be even.

A3. ABC is an equilateral triangle. P is a variable interior point such that angle APC = 120^o. The ray CP meets AB at M, and the ray AP meets BC at N. What is the locus of the circumcenter of the triangle MBN as P varies?



B1. ABC is a triangle. BD is the an angle bisector. E, F are the feet of the perpendiculars from A, C respectively to the line BD. M is the foot of the perpendicular from D to the line BC. Show that $\angle$DME = $\angle$DMF.



B2. The sequence a_n is defined as follows: $a_1 = 56$, $a_{n+1} = a_n - 1/a_n$. Show that $a_n < 0$ for some n such that $0 < n < 2002$.

B3. A game is played on a 2001 x 2001 board as follows. The first player's piece is the policeman, the second player's piece is the robber. Each piece can move one square south, one square east or one square northwest. In addition, the policeman (but not the robber) can move from the bottom right to the top left square in a single move. The policeman starts in the central square, and the robber starts one square diagonally northeast of the policeman. If the policeman moves onto the same square as the robber, then the robber is captured and the first player wins. However, the robber may move onto the same square as the policeman without being captured (and play continues). Show that the robber can avoid capture for at least 10000 moves, but that the policeman can ultimately capture the robber.

18th Iberoamerican 2003 problems

A1. Let A, B be two sets of N consecutive integers. If N = 2003, can we form N pairs (a, b) with $a \in A$, $b \in B$ such that the sums of the pairs are N consecutive integers? What about N = 2004?

A2. C is a point on the semicircle with diameter AB. D is a point on the arc BC. M, P, N are the midpoints of AC, CD and BD. The circumcenters of ACP and BDP are O, O'. Show that MN and OO' are parallel.

A3. Pablo was trying to solve the following problem: find the sequence $x_0, x_1, x_2, \dots, x_{2003}$ which satisfies $x_0 = 1$, $0 \le x_i \le 2 x_{i-1}$ for $1 \le i \le 2003$ and which maximises S. Unfortunately he could not remember the expression for S, but he knew that it had the form $S = \pm x_1 \pm x_2 \pm \dots \pm x_{2002} + x_{2003}$. Show that he can still solve the problem.

B1. A □ {1, 2, 3, ... , 49} does not contain six consecutive integers. Find the largest possible value of |A|. How many such subsets are there (of the maximum size)?

B2. ABCD is a square. P, Q are points on the sides BC, CD respectively, distinct from the endpoints such that BP = CQ. X, Y are points on AP, AQ respectively. Show that there is a triangle with side lengths BX, XY, YD.

B3. The sequences $a_0, a_1, a_2, \dots$ and $b_0, b_1, b_2, \dots$ are defined by $a_0 = 1$, $b_0 = 4$, $a_{n+1} = a_n^{2001} + b_n$, $b_{n+1} = b_n^{2001} + a_n$. Show that no member of either sequence is divisible by 2003.

Balkan (1984 – 2003)

1st Balkan 1984

A1. Let $x_1, x_2, \ldots, x_n$ be positive reals with sum 1. Prove that: $x_1/(2 - x_1) + x_2/(2 - x_2) + \ldots + x_n/(2 - x_n) \geq n/(2n - 1)$.

A2. ABCD is a cyclic quadrilateral. A' is the orthocenter (point where the altitudes meet) of BCD, B' is the orthocenter of ACD, C' is the orthocenter of ABD, and D' is the orthocenter of ABC. Prove that ABCD and A'B'C'D' are congruent.

A3. Prove that given any positive integer n we can find a larger integer m such that the decimal expansion of 5^m can be obtained from that for 5^n by adding additional digits on the left.

A4. Given positive reals a, b, c find all real solutions (x, y, z) to the equations $ax + by = (x - y)^2$, $by + cz = (y - z)^2$, $cz + ax = (z - x)^2$.

2nd Balkan 1985

A1. ABC is a triangle. O is the circumcenter, D is the midpoint of AB, and E is the centroid of ACD. Prove that OE is perpendicular to CD iff AB = AC.

A2. The reals w, x, y, z all lie between $-\pi/2$ and $\pi/2$ and satisfy $\sin w + \sin x + \sin y + \sin z = 1$, $\cos 2w + \cos 2x + \cos 2y + \cos 2z \geq 10/3$. Prove that they are all non-negative and at most $\pi/6$.

A3. Can we find an integer N such that if a and b are integers which are equally spaced either side of N/2 (so that $N/2 - a = b - N/2$), then exactly one of a, b can be written as $19m + 85n$ for some positive integers m, n?

A4. There are 1985 people in a room. Each speaks at most 5 languages. Given any three people, at least two of them have a language in common. Prove that there is a language spoken by at least 200 people in the room.

3rd Balkan 1986

A1. A line through the incenter of a triangle meets the circumcircle and incircle in the points A, B, C, D (in that order). Show that $AB \cdot CD \geq BC^2/4$. When do you have equality?

A2. A point is chosen on each edge of a tetrahedron so that the product of the distances from the point to each end of the edge is the same for each of the 6 points. Show that the 6 points lie on a sphere.

A3. The integers r, s are non-zero and k is a positive real. The sequence a_n is defined by $a_1 = r$, $a_2 = s$, $a_{n+2} = (a_{n+1}^2 + k)/a_n$. Show that all terms of the sequence are integers iff $(r^2 + s^2 + k)/(rs)$ is an integer.

A4. A point P lies inside the triangle ABC and the triangles PAB, PBC, PCA all have the same area and the same perimeter. Show that the triangle is equilateral. If P lies outside the triangle, show that the triangle is right-angled.

4th Balkan 1987

A1. f is a real valued function on the reals satisfying (1) f(0) = 1/2, (2) for some real a we have f(x+y) = f(x) f(a-y) + f(y) f(a-x) for all x, y. Prove that f is constant.

A2. Find all real numbers $x \geq y \geq 1$ such that $\sqrt{x - 1} + \sqrt{y - 1}$ and $\sqrt{x + 1} + \sqrt{y + 1}$ are consecutive integers.

A3. ABC is a triangle with BC = 1. We have $(\sin A/2)^{23}/(\cos A/2)^{48} = (\sin B/2)^{23}/(\cos B/2)^{48}$. Find AC.

A4. Two circles have centers a distance 2 apart and radii 1 and $\sqrt{2}$. X is one of the points on both circles. M lies on the smaller circle, Y lies on the larger circle and M is the midpoint of XY. Find the distance XY.

5th Balkan 1988

A1. ABC is a triangle area 1. AH is an altitude, M is the midpoint of BC and K is the point where the angle bisector at A meets the segment BC. The area of the triangle AHM is 1/4 and the area of AKM is 1 - (√3)/2. Find the angles of the triangle.

A2. Find all real polynomials p(x, y) such that p(x, y) p(u, v) = p(xu + yv, xv + yu) for all x, y, u, v.

A3. The sum of the squares of the edges of a tetrahedron is S. Prove that the tetrahedron can be fitted between two parallel planes a distance √(S/12) apart.

A4. xn is the sequence 51, 53, 57, 65, ... , 2n + 49, ... Find all n such that xn and xn+1 are each the product of just two distinct primes with the same difference.

6th Balkan 1989

A1. Find all integers which are the sum of the squares of their four smallest positive divisors.

A2. A prime p has decimal digits $p_np_{n-1}...p_0$ with $p_n > 1$. Show that the polynomial $p_nx^n + p_{n-1}x^{n-1} + ... + p_1x + p_0$ has no factors which are polynomials with integer coefficients and degree strictly between 0 and n.

A3. The triangle ABC has area 1. Take X on AB and Y on AC so that the centroid G is on the opposite of XY to B and C. Show that area BXGY + area CYGX ≥ 4/9. When do we have equality?

A4. S is a collection of subsets of {1, 2, ... , n} of size 3. Any two distinct elements of S have at most one common element. Show that S cannot have more than n(n-1)/6 elements. Find a set S with n(n-4)/6 elements.

7th Balkan 1990

A1. The sequence un is defined by u1 = 1, u2 = 3, un = (n+1) un-1 - n un-2. Which members of the sequence which are divisible by 11?

A2. Expand (x + 2x2 + 3x3 + ... + nxn)2 and add the coefficients of xn+1 through x2n. Show that the result is n(n+1)(5n2 + 5n + 2)/24.

A3. The feet of the altitudes of the triangle ABC are D, E, F. The incircle of DEF meets its sides at G, H, I. Prove that ABC and GHI have the same Euler line (the line through the circumcenter and the centroid).

A4. The function f is defined on the positive integers and f(m) ≠ f(n) if m - n is prime. What is the smallest possible size of the image of f.

8th Balkan 1991

A1. The circumcircle of the acute-angled triangle ABC has center O. M lies on the minor arc AB. The line through M perpendicular to OA cuts AB at K and AC at L. The line through M perpendicular to OB cuts AB at N and BC at P. MN = KL. Find angle MLP in terms of angles A, B and C.

A2. Find an infinite set of incongruent triangles each of which has integral area and sides which are relatively prime integers, but none of whose altitudes are integral.

A3. A regular hexagon area H has its vertices on the perimeter of a convex polygon of area A. Prove that $2A \leq 3H$. When do we have equality?

A4. A is the set of positive integers and B is A □ {0}. Prove that no bijection f: A → B can satisfy $f(mn) = f(m) + f(n) + 3 f(m) f(n)$ for all m, n.

9th Balkan 1992

A1. Let $a(n) = 3^{4n}$. For which n is $(m^{a(n)+6} - m^{a(n)+4} - m^5 + m^3)$ always divisible by 1992?

A2. Prove that $(2n^2 + 3n + 1)^n \geq 6^n n!\, n!$ for all positive integers.

A3. ABC is a triangle area 1. Take D on BC, E on CA, F on AB, so that AFDE is cyclic. Prove that: area DEF $\leq EF^2/(4\, AD^2)$.

A4. For each n > 2 find the smallest f(n) such that any subset of {1, 2, 3, ... , n} with f(n) elements must have three which are relatively prime (in pairs).

10th Balkan 1993

A1. Given reals $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le a_6$ satisfying $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 10$ and $(a_1 - 1)^2 + (a_2 - 1)^2 + (a_3 - 1)^2 + (a_4 - 1)^2 + (a_5 - 1)^2 + (a_6 - 1)^2 = 6$, what is the largest possible a_6?

A2. How many non-negative integers with not more than 1993 decimal digits have non-decreasing digits? [For example, 55677 is acceptable, 54 is not.]

A3. Two circles centers A and B lie outside each other and touch at X. A third circle center C encloses both and touches them at Y and Z respectively. The tangent to the first two circles at X forms a chord of the third circle with midpoint M. Prove that $\angle YMZ = \angle ACB$.

A4. p is prime and k is a positive integer > 1. Show that we can find positive integers (m, n) ≠ (1, 1) such that $(m^p + n^p)/2 = ((m + n)/2)^k$ iff k = p.

11th Balkan 1994

A1. Given a point P inside an acute angle XAY, show how to construct a line through P meeting the line AX at B and the line AY at C such that the area of the triangle ABC is AP^2.

A2. Show that $x^4 - 1993x^3 + (1993 + n)x^2 - 11x + n = 0$ has at most one integer root if n is an integer.

A3. What is the maximum value f(n) of $|s_1 - s_2| + |s_2 - s_3| + ... + |s_{n-1} - s_n|$ over all permutations $s_1, s_2, ..., s_n$ of 1, 2, ... , n?

A4. Find the smallest n > 4 for which we can find a graph on n points with no triangles and such that for every two unjoined points we can find just two points joined to both of them.

12th Balkan 1995

A1. Define a_n by $a_3 = (2 + 3)/(1 + 6)$, $a_n = (a_{n-1} + n)/(1 + n\ a_{n-1})$. Find a_{1995}.

A2. Two circles centers O and O' meet at A and B, so that OA is perpendicular to O'A. OO' meets the circles at C, E, D, F, so that the points C, O, E, D, O', F lie on the line in that order. BE meets the circle again at K and meets CA at M. BD meets the circle again at L and AF at N. Show that (KE/KM) (LN/LD) = (O'E/OD).

A3. m and n are positive integers with m > n and m + n even. Prove that the roots of $x^2 - (m^2 - m + 1)(x - n^2 - 1) - (n^2 + 1)^2 = 0$ are positive integers, but not squares.

A4. Let S be an n x n array of lattice points. Let T be the set of all subsets of S of size 4 which form squares. Let A, B and C be the number of pairs {P, Q} of points of S which belong to respectively no, just two and just three elements of T. Show that A = B + 2C. [Note that there are plenty of squares tilted at an angle to the lattice and that the pair can be adjacent corners or opposite corners of the square.]

13th Balkan 1996

A1. Let d be the distance between the circumcenter and the centroid of a triangle. Let R be its circumradius and r the radius of its inscribed circle. Show that $d^2 \le R(R - 2r)$.

A2. $p > 5$ is prime. $A = \{p - n^2$ where $n^2 < p\}$. Show that we can find integers a and b in A such that $a > 1$ and a divides b.

A3. In a convex pentagon consider the five lines joining a vertex to the midpoint of the opposite side. Show that if four of these lines pass through a point, then so does the fifth.

A4. Can we find a subset X of $\{1, 2, 3, \dots, 2^{1996}-1\}$ with at most 2012 elements such that 1 and $2^{1996}-1$ belong to X and every element of X except 1 is the sum of two distinct elements of X or twice an element of X?

14th Balkan 1997

A1. ABCD is a convex quadrilateral. X is a point inside it. $XA^2 + XB^2 + XC^2 + XD^2$ is twice the area of the quadrilateral. Show that it is a square and that X is its center.

A2. A collection of m subsets of X = {1, 2, ... , n} has the property that given any two elements of X we can find a subset in the collection which contains just one of the two. Prove that $n \le 2^m$.

A3. Two circles C and C' lying outside each other touch at T. They lie inside a third circle and touch it at X and X' respectively. Their common tangent at T intersects the third circle at S. SX meets C again at P and XX' meets C again at Q. SX' meets C' again at U and XX' meets C' again at V. Prove that the lines ST, PQ and UV are concurrent.

A4. Find all real-valued functions on the reals which satisfy $f(xf(x) + f(y)) = f(x)^2 + y$ for all x, y.

15th Balkan 1998

A1. How many different integers can be written as $[n^2/1998]$ for $n = 1, 2, ... , 1997$?

A2. x_i are distinct positive reals satisfying $x_1 < x_2 < ... < x_{2n+1}$. Show that $x_1 - x_2 + x_3 - x_4 + ... - x_{2n} + x_{2n+1} < (x_1^n - x_2^n + ... - x_{2n}^n + x_{2n+1}^n)^{1/n}$.

A3. Let S be the set of all points inside or on a triangle. Let T be the set S with one interior point excluded. Show that one can find points P_i, Q_i such that P_i and Q_i are distinct and the closed segments P_iQ_i are all disjoint and have union T.

A4. Prove that there are no integers m, n satisfying $m^2 = n^5 - 4$.

16th Balkan 1999

A1. O is the circumcenter of the triangle ABC. XY is the diameter of the circumcircle perpendicular to BC. It meets BC at M. X is closer to M than Y. Z is the point on MY such that MZ = MX. W is the midpoint of AZ. Show that W lies on the circle through the midpoints of the sides of ABC. Show that MW is perpendicular to AY.

A2. p is an odd prime congruent to 2 mod 3. Prove that at most p-1 members of the set $\{m^2 - n^3 - 1 : 0 < m, n < p\}$ are divisible by p.

A3. ABC is an acute-angled triangle area 1. Show that the triangle whose vertices are the feet of the perpendiculars from the centroid to AB, BC, CA has area between 4/27 and 1/4.

A4. $0 = a_1, a_2, a_3, \ldots$ is a non-decreasing, unbounded sequence of non-negative integers. Let the number of members of the sequence not exceeding n be b_n. Prove that $(x_0 + x_1 + \ldots + x_m)(y_0 + y_1 + \ldots + y_n) \geq (m + 1)(n + 1)$.

17th Balkan 2000

A1. Find all real-valued functions on the reals which satisfy $f(xf(x) + f(y)) = f(x)^2 + y$ for all x, y.

A2. ABC is an acute-angled triangle which is not isosceles. M is the midpoint of BC. X is any point on the segment AM. Y is the foot of the perpendicular from X to BC. Z is any point on the segment XY. U and V are the feet of the perpendiculars from Z to AB and AC. Show that the bisectors of angles UZV and UXV are parallel.

A3. How many 1 by $10\sqrt{2}$ rectangles can be cut from a 50 x 90 rectangle using cuts parallel to its edges.

A4. Show that for any n we can find a set X of n distinct integers greater than 1, such that the average of the elements of any subset of X is a square, cube or higher power.

18th Balkan 2001

A1. If $2^n - 1 = ab$ and 2^k is the highest power of 2 dividing $2^n - 2 + a - b$ then k is even.

A2. A convex pentagon has rational sides and equal angles. Show that it is regular.

A3. a, b, c are positive reals whose product does not exceed their sum. Show that $a^2 + b^2 + c^2 \geq (\sqrt{3})\ abc$.

A4. A cube side 3 is divided into 27 unit cubes. The unit cubes are arbitrarily labeled 1 to 27 (each cube is given a different number). A move consists of swapping the cube labeled 27 with one of its neighbours. Is it possible to find a finite sequence of moves at the end of which cube 27 is in its original position, but cube n has moved to the position originally occupied by 27-n (for n = 1, 2, ... , 26)?

19th Balkan 2002

A1. Show that a finite graph in which every point has at least three edges contains an even cycle.

A2. The sequence a_n is defined by $a_1 = 20$, $a_2 = 30$, $a_{n+1} = 3a_n - a_{n-1}$. Find all n for which $5a_{n+1}a_n + 1$ is a square.

A3. Two unequal circles intersect at A and B. The two common tangents touch one circle at P, Q and the other at R, S. Show that the orthocenters of APQ, BPQ, ARS, BRS form a rectangle.

A4. N is the set of positive integers. Find all functions f: N → N such that f(f(n)) + f(n) = 2n + 2001 or 2n + 2002.

20th Balkan 2003

A1. Is there a set of 4004 positive integers such that the sum of each subset of 2003 elements is not divisible by 2003?

A2. ABC is a triangle. The tangent to the circumcircle at A meets the line BC at D. The perpendicular to BC at B meets the perpendicular bisector of AB at E, and the perpendicular to BC at C meets the perpendicular bisector of AC at F. Show that D, E, F are collinear.

A3. Find all real-valued functions f(x) on the rationals such that:
(1) $f(x + y) - y f(x) - x f(y) = f(x) f(y) - x - y + xy$, for all x, y
(2) $f(x) = 2 f(x+1) + 2 + x$, for all x and
(3) $f(1) + 1 > 0$.

A4. A rectangle ABCD has side lengths AB = m, AD = n, with m and n relatively prime and both odd. It is divided into unit squares and the diagonal AC intersects the sides of the unit squares at the points $A_1 = A, A_2, A_3, \dots, A_N = C$. Show that $A_1A_2 - A_2A_3 + A_3A_4 - \dots \pm A_{N-1}A_N = AC/mn$.

Austrian-Polish
(1978 – 2003)

1st Austrian-Polish 1978

1. Find all real-valued functions f on the positive reals which satisfy $f(x + y) = f(x^2 + y^2)$ for all positive x, y.

2. A parallelogram has its vertices on the boundary of a regular hexagon and its center at the center of the hexagon. Show that its area is at most 2/3 the area of the hexagon.

3. Let $x = 1^o$. Show that $(\tan x \tan 2x \ldots \tan 44x)^{1/44} < \sqrt{2} - 1 < (\tan x + \tan 2x + \ldots + \tan 44x)/44$.

4. Given a positive rational k not equal to 1, show that we can partition the positive integers into sets A_k and B_k, so that if m and n are both in A_k or both in B_k then m/n does not equal k.

5. The sets $A_1, A_2, \ldots, A_{1978}$ each have 40 elements and the intersection of any two distinct sets has just one element. Show that the intersection of all the sets has one element.

6. S is a set of disks in the plane. No point belongs to the interior of more than one disk. Each disk has a point in common with at least 6 other disks. Show that S is infinite.

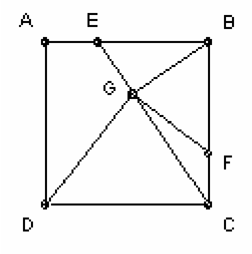
7. S is a finite set of lattice points in the plane such that we can find a bijection $f: S \to S$ satisfying $|P - f(P)| = 1$ for all P in S. Show that we can find a bijection $g: S \to S$ such that $|P - g(P)| = 1$ for all P in S and $g(g(P)) = P$ for all P in S.

8. k is a positive integer. Define $a_1 = \sqrt{k}$, $a_{n+1} = \sqrt{(k + a_n)}$. Show that the sequence a_n converges. Find all k such that the limit is an integer. Show that if k is odd, then the limit is irrational.

9. P is a convex polygon. Some of the diagonals are drawn, so that no interior point of P lies on more than one diagonal. Show that at least two vertices of P do not lie on any diagonals.

2nd Austrian-Polish 1979

1. ABCD is a square. E is any point on AB. F is the point on BC such that BF = BE. The perpendicular from B meets EF at G. Show that $\angle DGF = 90^o$.



2. Find all polynomials of degree n with real roots $x_1 \le x_2 \le ... \le x_n$ such that x_k belongs to the closed interval [k, k+1] and the product of the roots is (n+1)/(n-1)! .

3. Find all positive integers n such that for all real numbers $x_1, x_2, ... , x_n$ we have $S_2S_1 - S_3 \ge 6P$, where $S_k = \sum x_i^k$, and $P = x_1x_2 ... x_n$.

4. Let N_0 = {0, 1, 2, 3, ... } and R be the reals. Find all functions f: $N_0 \to R$ such that f(m + n) + f(m - n) = f(3m) for all m, n.

5. A tetrahedron has circumcenter O and incenter I. If O = I, show that the faces are all congruent.

6. k is real, n is a positive integer. Find all solutions $(x_1, x_2, ... , x_n)$ to the n equations:

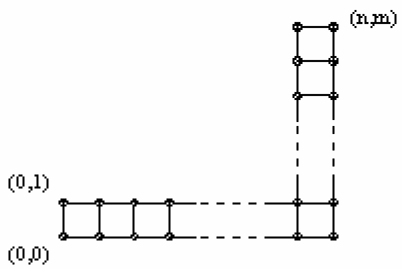
$x_1 + x_2 + ... + x_n = k$

$x_1^2 + x_2^2 + ... + x_n^2 = k^2$

...

$x_1^n + x_2^n + ... + x_n^n = k^n$

7. Find the number of paths from (0, 0) to (n, m) which pass through each node at most once.



8. ABCD is a tetrahedron. M is the midpoint of AC and N is the midpoint of BD. Show that $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4\ MN^2$.

9. Find the largest power of 2 that divides $[(3 + \sqrt{11})^{2n+1}]$.

3rd Austrian-Polish 1980

1. A, B, C are infinite arithmetic progressions of integers. {1, 2, 3, 4, 5, 6, 7, 8} is a subset of their union. Show that 1980 also belongs to their union.

2. $1 = a_1 < a_2 < a_3 < \dots$ is an infinite sequence of integers such that $a_n < 2n-1$. Show that every positive integer is the difference of two members of the sequence.

3. P is an interior point of a tetrahedron. Show that the sum of the six angles subtended by the sides at P is greater than 540^o.

4. If S is a non-empty set of positive integers, let p(S) be the reciprocal of the product of the members of S. Show that $\sum p(S) = n$, where there sum is taken over all non-empty subsets of {1, 2, ... , n}.

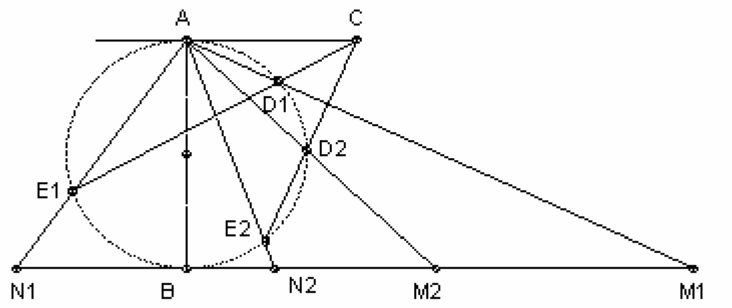
5. ABC is a triangle. A' is any point on the segment BC other than B and C. P_A is the perpendicular bisector of AA'. The lines P_B and P_C are defined similarly. Show that no point can lie on all of P_A, P_B and P_C.

6. The real numbers $x_1, x_2, x_3, \dots$ satisfy $|x_{m+n} - x_m - x_n| \le 1$ for all m, n. Show that $|x_m/m - x_n/n| < 1/m + 1/n$ for all m, n.

7. Find the largest n for which we can find n-1 distinct positive integers a_i such that $a_i = b_i + 1980/b_i$ for some integer b_i, where $b_1b_2 \dots b_{n-1}$ divides 1980.

8. Given 1980 points in the plane, no two a distance < 1 apart, show that we can find a subset of 220 points, no two a distance < √3 apart.

9. C is an arbitrary point on the tangent to the circle K at A. D_1, D_2, E_1, E_2 are any points on the circle such that C, D_1, E_1 are collinear in that order, and C, D_2, E_2 are collinear in that order. AB is a diameter of the circle and the tangent at B meets the lines AD_1, AD_2, AE_1, AE_2 at M_1, M_2, N_1, N_2 respectively. Show that $M_1M_2 = N_1N_2$.

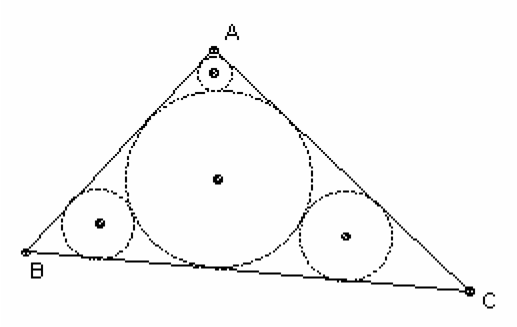


4th Austrian-Polish 1981

1. Find the smallest n for which we can find 15 distinct elements $a_1, a_2, \dots, a_{15}$ of {16, 17, ... , n} such that a_k is a multiple of k.

2. The rational sequence $a_0, a_1, a_2, \dots$ satisfies $a_{n+1} = 2a_n^2 - 2a_n + 1$. Find all a_0 for which there are four distinct integers r, s, t, u such that $a_r - a_s = a_t - a_u$.

3. The diagram shows the incircle of ABC and three other circles inside the triangle, each touching the incircle and two sides of the triangle. The radius of the incircle is r and the radius of the circle nearest to A, B, C is r_A, r_B, r_C respectively. Show that $r_A + r_B + r_C \geq r$, with equality iff ABC is equilateral.



4. n symbols are arranged in a circle. Each symbol is 0 or 1. A valid move is to change any 1 to 0 and to change the two adjacent symbols. For example one could change ... 01101 ... to ... 10001 ... . The initial configuration has just one 1. For which n can one obtain all 0s by a sequence of valid moves?

5. A quartic with rational coefficients has just one real root. Show that the root must be rational.

6. The real sequences $x_1, x_2, x_3, \dots, y_1, y_2, y_3, \dots, z_1, z_2, z_3, \dots$ satisfy $x_{n+1} = y_n + 1/z_n$, $y_{n+1} = z_n + 1/x_n$, $z_{n+1} = x_n + 1/y_n$. Show that the sequences are all unbounded.

7. If $N = 2^n > 1$ and $k > 3$ is odd, show that $k^N - 1$ has at least n+1 distinct prime factors.

8. A is a set of r parallel lines, B is a set of s parallel lines, and C is a set of t parallel lines. What is the smallest value of r + s + t such that the r + s + t lines divide the plane into at least 1982 regions?

9. Let X be the closed interval [0, 1]. Let f: X → X be a function. Define $f^1 = f$, $f^{n+1}(x) = f(f^n(x))$. For some n we have $|f^n(x) - f^n(y)| < |x - y|$ for all distinct x, y. Show that f has a unique fixed point.

5th Austrian-Polish 1982

1. Find all positive integers m, n for which $(n+1)^m - n$ and $(n+1)^{m+3} - n$ have a common factor (greater than 1).

2. C is a circle center O radius 1, and D is the interior of C (so D is the open disk center O radius 1). F is a closed convex subset of D. From any point of C there are two tangents to F, which are at an angle 60^o. Show that F must be the closed disk center O radius 1/2.

3. Let $n > 1$ be an integer. Let $f(k) = 1 + 3^k/(3^n - 1)$, $g(k) = 1 - 3^k/(3^n - 1)$. Show that $\tan(f(1)\pi/3) \tan(f(2)\pi/3) \dots \tan(f(n)\pi/3) \tan(g(1)\pi/3) \tan(g(2)\pi/3) \dots \tan(g(n)\pi/3) = 1$.

4. The sequence $a_1, a_2, a_3, \dots$ satisfies $a_{n+1} = a_n + f(a_n)$, where f(m) is the product of the (decimal) digits of n. Is the sequence bounded for all a_1?

5. The closed interval [0, 1] is the union of two disjoint sets A, B. Show that we cannot find a real number k such that B = {x + k | x belongs to A}.

6. k is a fixed integer. Let $N_k = \{k, k+1, k+2, \dots\}$. Find all real-valued functions f on N_k such that $f(m + n) = f(m) f(n)$ for all m, n such that m, n and m+n are all in N_k.

7. Find positive integers r, s, t, u, v, w such that (1) $r > t > v$, (2) $r^s = t^u = v^w$, (3) $tu = vw$, (4) $r + s = t + u$, and (5) tu is as small as possible.

8. P is a point inside a regular tetrahedron with side 1. Show that the sum of the perpendicular distances from P to each edge is at least $3/\sqrt{2}$, with equality iff P is the center.

9. $n > 2$ is an integer. Let S_n be the sum of the n^2 values $1/\sqrt{(i^2 + j^2)}$ for i = 1, 2, ... , n, and j = 1, 2, ... , n. Show that $S_n \geq n$. Find a constant k as small as possible such that $S_n \leq kn$.

6th Austrian-Polish 1983

1. The non-negative reals a, b, c, d satisfy $a^5 + b^5 \le 1$, $c^5 + d^5 \le 1$. Show that $a^2c^3 + b^2d^3 \le 1$.

2. Find all primes p, q such that p(p+1) + q(q+1) = n(n+1) for some positive integer n.

3. A finite set of closed disks in the plane cover an area A (some of the disks may overlap). Show that we can find a subset of non-overlapping disks which cover an area of at least A/9.

4. The disjoint sets A and B together contain all the positive integers. Show that given any integer n, we can find integers a > b > n such that a, b and a+b are all in A or all in B.

5. Given reals $0 < a_1 < a_2 < a_3 < a_4$, for which real k do the equations

$x_1 + x_2 + x_3 + x_4 = 1$

$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = k$

$a_1^2x_1 + a_2^2x_2 + a_3^2x_3 + a_4^2x_4 = k^2$

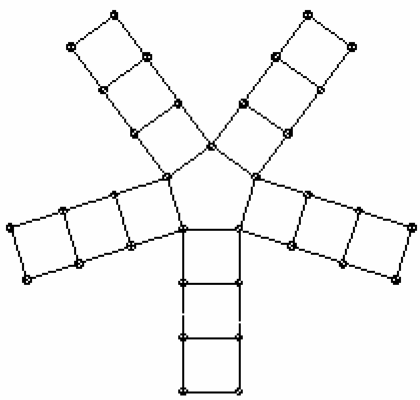
have a solution in non-negative reals x_i?

6. There are six lines in space such that given any three we can find two which are perpendicular. Show that we can divide the lines into two groups of three, so that the lines in each group are all perpendicular to each other.

7. Let $C_1, C_2, C_3, C_4, C_5, C_6$ be closed line segments in the plane. Each pair has at most one common point. S is the union of the six segments. There are four distinct points P_1, P_2, P_3, P_4, such that any straight line through at least one of the points P_i intersects S in exactly two points. Is S necessarily a hexagon?

8. (1) Show that $(2^{n+1} - 1)(2^n - 1)^2(2^{n-1} - 1)^4(2^{n-2} - 1)^8 \dots (2^2 - 1)^N$ divides $(2^{n+1} - 1)!$ (where $N = 2^{n-1}$).
(2) The sequence $a_1 = 1, a_2, a_3, \dots$ satisfies $a_n = (4n - 6)a_{n-1}/n$. Show that all terms are integers.

9. A regular p-gon (p prime) has 1 x k rectangles on the outside of each edge. Each rectangle is divided into k unit squares, so that the figure is divided into pk + 1 regions. How many ways can the figure be colored with three colors, so that adjacent regions have different colors and there is no symmetry axis?



7th Austrian-Polish 1984

1. A tetrahedron is such that the foot of the altitude from each vertex is the incenter of the opposite face. Show that the tetrahedron is regular.

2. Find the 4 digit number N which uses only two digits, neither of them 0, such that the greatest common divisor of N and the number obtained from N by interchanging the two digits is as large as possible. For example, if N was 2444, then the greatest common divisor of 2444 and 4222 is 2, which is not maximal.

3. Show that for $n > 1$ and any positive real numbers $k, x_1, x_2, \ldots, x_n$: $f(x_1 - x_2)/(x_1 + x_2) + f(x_2 - x_3)/(x_2 + x_3) + \ldots + f(x_n - x_1)/(x_n + x_1) \geq n^2/(2(x_1 + \ldots + x_n))$, where $f(x) = k^x$. When does equality hold?

4. $A_1A_2A_3A_4A_5A_6A_7$ is a regular heptagon and the point P lies on the circumcircle between A_7 and A_1. Show that $PA_1 + PA_3 + PA_5 + PA_7 = PA_2 + PA_4 + PA_6$.

5. Given $n > 2$ distinct integers $a_1, a_2, \ldots, a_n$, find all solutions $(x_1, x_2, \ldots, x_n, y)$ in non-negative integers to:

$a_1x_1 + a_2x_2 + \ldots + a_nx_n = yx_1$

$a_2x_1 + a_3x_2 + \ldots + a_1x_n = yx_2$

...

$a_nx_1 + a_1x_2 + \ldots + a_{n-1}x_n = yx_n$

such that the greatest common divisor of all the x_i is 1.

6. The points of a graph are labeled $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$. From A_i we must draw a red arrow to one of B_i, A_{i+1} or B_{i+1}. and a blue arrow to one of B_i, A_{i-1} or B_{i-1}, but the two arrows must go to different points. From B_i we must draw a red arrow to one of A_i, A_{i-1} or B_{i-1} and a blue arrow to one of A_i, A_{i+1} or B_{i+1}. Again the two arrows must go to different points. Note that A_0, B_0, A_{n+1}, B_{n+1} do not exist, so (for example) the blue arrow from A_1 must go to B_1. How many different possible configurations are there?

7. An m x n array of real numbers, each with absolute value at most 1, has all its column sums zero. Show that we can rearrange the numbers in each column so that the absolute value of each resulting row sum is less than 2.

8. Let X be the set of real numbers > 1. Define $f: X \to X$ and $g: X \to X$ by $f(x) = 2x$ and $g(x) = x/(x-1)$. Show that given any real numbers $1 < A < B$ we can find a finite sequence $x_1 = 2, x_2, \ldots, x_n$ such that $A < x_n < B$ and $x_i = f(x_{i-1})$ or $g(x_{i-1})$.

9. Find all functions f which are defined on the rationals, take real values and satisfy $f(x + y) = f(x) f(y) - f(xy) + 1$ for all x, y.

8th Austrian-Polish 1985

1. a, b, c are distinct non-zero reals with sum zero. Let x = (b-c)/a, y = (c-a)/b, z = (a-b)/c. Show that (x + y + z)(1/x + 1/y + 1/z) = 9.

2. For which n > 7 is there a graph with n points, such that there are 3 points of degree n-1, 3 points of degree n-2 and one person of degree k for k = 4, 5, 6 ... , n-3?

3. Four points form a convex quadrilateral with area 1, show that the sum of the six distances between each pair of points is at least $4 + \sqrt{8}$.

4. Find all real solutions to:

$x^4 + y^2 - xy^3 = 9x/8$

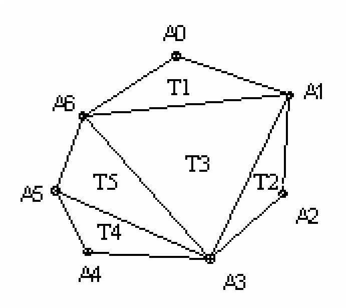
$y^4 + x^2 - x^3y = 9y/8$.

5. We have N identical sets of weights. Each set has four weights, each a different natural number. There is a subset of the 4N weights which weighs k for k = 1, 2, 3, ... , 1985. The weights and N are chosen so that the total weight of the 4N weights is as small as possible. How many such minimal sets are there?

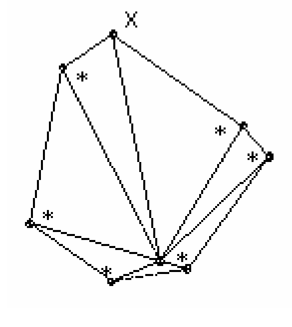
6. ABCD is a tetrahedron. P is a point inside. The centroids of PBCD, APCD, ABPD, ABCP are A', B', C', D' respectively. Show that the volume of A'B'C'D' is 1/64 the volume of ABCD.

7. Find the least upper bound for the set of values $(x_1x_2 + 2x_2x_3 + x_3x_4)/(x_1^2 + x_2^2 + x_3^2 + x_4^2)$, where x_i are reals, not all zero.

8. A convex n-gon $A_0A_1 \ldots A_n$ is divided into n-2 triangles by diagonals which do not intersect (except possibly at vertices of the n-gon). Find the number of ways of labeling the triangles $T_1, T_2, \ldots, T_{n-2}$, so that A_i is a vertex of T_i for each i. The diagram shows a possible labeling.

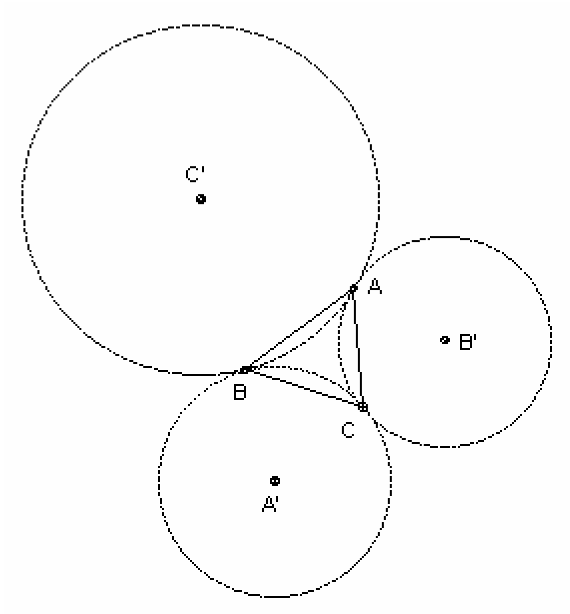


9. Given any convex polygon, show that we can find a point P inside the polygon and three vertices X, Y, Z, such that each of the two angles between PX and a side at X is acute, and similarly for Y and Z. The diagram below shows a poor choice of P. There is only vertex X satisfying the condition. Asterisks mark obtuse angles.



9th Austrian-Polish 1986

1. ABC is a triangle which does not contain a right-angle. A' is the center of a circle through B and C. Similarly B' is the center of a circle through C and A, and C' is the center of a circle through A and B. Each pair of circles touches. The triangles ABC and A'B'C' are similar. Find their angles.



2. The monic polynomial p(x) has degree $n > 1$ and all its roots distinct negative reals. The coefficient of x is A and the constant term is B. Show that $Ap(1) > 2n^2B$. (A monic polynomial has leading coefficient 1).

3. Every point in space is colored red or blue. Show that we can either find a unit square with red vertices, or a unit square with blue vertices, or a unit square with just one blue vertex.

4. Find all positive integer solutions (m, n, N) to $m^N - n^N = 2^{100}$ with $N > 1$.

5. Find all real solutions to:

$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4;$

$x_1x_3 + x_2x_4 + x_3x_2 + x_4x_1 = 0;$

$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = -2;$

$x_1x_2x_3x_4 = -1.$

6. The inscribed and circumscribed spheres of a tetrahedron have radii r and R and are concentric. Find the range of possible values for R/r.

7. $k < n^2/4$ is a positive integer with no prime divisor greater than n. Show that k divides n! .

8. mn distinct reals are arranged in an m x n array so that the entries in each row increase from left to right. Each column is then rearranged so that the entries increase from bottom to top. Show that the elements in each row still increase from left to right.

9. Find all continuous real-valued functions f on the reals such that (1) $f(1) = 1$, (2) $f(f(x)) = f(x)^2$ for all x, (3) either $f(x) \geq f(y)$ for all $x \geq y$, or $f(x) \leq f(y)$ for all $x \leq y$.

10th Austrian-Polish 1987

1. P is a point inside a sphere. Three chords through P are mutually perpendicular. Show that the sum of the squares of their lengths is independent of their directions.

2. n is a square such that if a prime p divides n, then p has an even number of digits. Show that if the rationals x, y satisfy $x^n - 1987x = y^n - 1987y$, then x = y.

3. f is a real-valued function on the reals such that f(x+1) = f(x) + 1. The sequence $x_0, x_1, x_2, ...$ satisfies $x_n = f(x_{n-1})$ for all positive n. For some n > 0, $x_n - x_0 = k$, an integer. Show that lim x_n/n exists and find it.

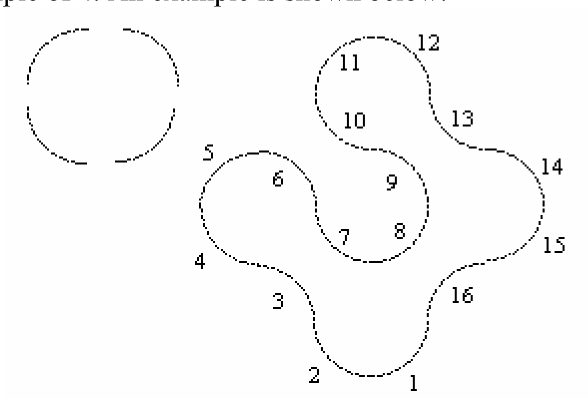
4. Is there a subset of {1, 2, ... , 3000} with 2000 elements such that n and 2n do not both belong to the subset for any n?

5. Space is partitioned into three disjoint sets. Show that for each d > 0 we can find two points a distance d apart in one of the sets.

6. C is a circle radius 1 and n is a fixed positive integer. Let F be the set of all sets S of n points on C and let D be the set of all diameters of C. Given any element S of F and any element d of D, let f(S, d) be the shortest (perpendicular) distance from a member of S to d. Find $g(n) = \min_F \max_D f(S, d)$, and find all sets S for which $\max_D f(S, d) = g(n)$.

7. A palindrome is a number which is the same read backwards (for example, 43534). Show that there are infinitely many palindromes whose digit sum is 1 more than one-third of the product of their digits. Show that only finitely many of these have all digits > 1 and find them.

8. A closed contour is made up of translates of the four quarters of a circle. The translates are fitted together smoothly (so that the tangents are the same at the join). Show that the number of translates is a multiple of 4. An example is shown below:



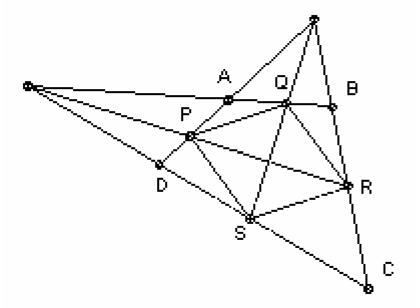
9. Let X be the set of points { (x, y): x = 1, 2, ... , 12; y = 1, 2, ... , 13}. Show that every subset of X with 49 elements has 4 points which are the vertices of a rectangle with sides parallel to the axes. Show that there is a subset of X with 48 elements which does not contain such points.

11th Austrian-Polish 1988

1. p(x) is a polynomial with integer coefficients and at least 6 distinct integer roots. Show that p(x) - 12 has no integer roots.

2. $1 \le a_1 \le a_2 \le ... \le a_n$ are positive integers. If $a_2 \ge 2$, show that $(a_1x_1^2 + ... + a_nx_n^2) + 2(x_1x_2 + x_2x_3 + ... + x_{n-1}x_n) > 0$ for all real x_i which are not all zero. If $a_2 < 2$, show that we can find x_i not all zero for which the inequality is false.

3. ABCD is a convex quadrilateral with no two sides parallel. The bisectors of the angles formed by the two pairs of opposite sides meet the sides of ABCD at the points P, Q, R, S, so that PQRS is convex. Show that ABCD has a circumcircle iff PQRS is a rhombus.



4. Find all strictly increasing real-valued functions on the reals such that f(f(x) + y) = f(x + y) + f(0) for all x, y.

5. The integer sequences $a_1, a_2, a_3, ...$ and $b_1, b_2, b_3, ...$ satisfy $b_n = a_n + 9$, $a_{n+1} = 8b_n + 8$. The number 1988 appears in at least one of the sequences. Show that the sequence a_n does not contain a square.

6. R_1, R_2, R_3 are three rays (in space) with endpoint O such that if A_i is any point on R_i except O, then the triangle $A_1A_2A_3$ is acute-angled. Show that the three rays are mutually perpendicular.

7. 8 points are arranged in a circle. Each point is colored yellow or blue. Once a minute the colors are all changed simultaneously. If the two neighbors of a point were the same color, then the point becomes yellow. If the two neighbors are opposite colors, then the point becomes blue. Show that whatever the initial colors, all points eventually become yellow. What is the maximum time required to achieve this state?

8. Using at most 1988 unit cubes form three boards: A measuring 1 x a x a, B 1 x b x b and C 1 x c x c, where a ≤ b ≤ c. Let d(a, b, c) be the number of ways of placing B flat on C and A flat on B, so that the small cubes line up, the footprint of B lies within C (so it does not overhang), and similarly the footprint of A lies within B. We regard arrangements as distinct even if one can be rotated or reflected into the other. Find a, b, c which maximize d(a, b, c).

9. The rectangle R has integral sides a and b. Let D(a, b) be the number of ways of tiling it with congruent rectangular tiles which are all similarly oriented. Find the value of a+b which maximizes D(a, b)/(a+b).

12th Austrian-Polish 1989

1. Show that $(\sum x_iy_iz_i)^2 \le (\sum x_i^3)(\sum y_i^3)(\sum z_i^3)$ for any positive reals x_i, y_i, z_i, where i runs from 1 to n.

2. Each point of the plane is colored red or blue. Show that there are either three blue points forming an equilateral triangle, or three red points forming an equilateral triangle.

3. Find all positive integers n with 4-digits $n_1n_2n_3n_4$ such that: (1) $n_1 = n_2$ and $n_3 = n_4$; (2) the three digit numbers $n_1n_1n_3$ and $n_1n_3n_3$ are both prime; (3) n is the product of a one-digit prime, a two-digit prime and a three-digit prime.

4. Show that for any convex polygon we can find a circle through three adjacent vertices such that all points of the polygon lie inside or on the circle.

5. C is a cube side 1 and S its inscribed sphere. X is a vertex of the cube. Let L be a line through X which intersects S. Let Y be the point belonging to S and L for which XY is a minimum, and let Z be the point belonging to C and L for which XZ is a maximum. Let d(L) be the product XY.XZ. Find the maximum value of d(L) for all such lines L. Which lines give the maximum value?

6. Find the longest strictly increasing sequence of squares such that the difference between any two adjacent terms is a prime or the square of a prime.

7. Define $f(1) = 2$, $f(2) = 3^{f(1)}$, $f(3) = 2^{f(2)}$, $f(4) = 3^{f(3)}$, $f(5) = 2^{f(4)}$ and so on. Similarly, define $g(1) = 3$, $g(2) = 2^{g(1)}$, $g(3) = 3^{g(2)}$, $g(4) = 2^{g(3)}$, $g(5) = 3^{g(4)}$ and so on. Which is larger, f(10) or g(10)?

8. ABC is an acute-angled triangle and P a point inside or on the boundary. The feet of the perpendiculars from P to BC, CA, AB are A', B', C' respectively. Show that if ABC is equilateral, then (AC' + BA' + CB')/(PA' + PB' + PC') is the same for all positions of P, but that for any other triangle it is not.

9. Call a positive integer *blue* if for some odd k > 1, it is a sum of the squares of k consecutive positive integers. For example, $14 = 1^2 + 2^2 + 3^2$ and $415 = 7^2 + 8^2 + 9^2 + 10^2 + 11^2$ are blue. Find the smallest blue integer which is also an odd square.

13th Austrian-Polish 1990

1. The distinct points X_1, X_2, X_3, X_4, X_5, X_6 all lie on the same side of the line AB. The six triangles ABX_i are all similar. Show that the X_i lie on a circle.

2. Find all solutions in positive integers to $a^A = b^B = c^C = 1990^{1990}abc$, where $A = b^c$, $B = c^a$, $C = a^b$.

3. Show that there are two real solutions to:

$x + y^2 + z^4 = 0$

$y + z^2 + x^4 = 0$

$z + x^2 + y^5 = 0$.

4. Find all solutions in positive integers to:

$x_1^4 + 14x_1x_2 + 1 = y_1^4$

$x_2^4 + 14x_2x_3 + 1 = y_2^4$

...

$x_n^4 + 14x_nx_1 + 1 = y_n^4$.

5. If $a_1, \dots , a_n$ is a permutation of 1, 2, ... , n, call $\sum |a_i - i|$ its *modsum*. Find the average modsum of all n! permutations.

6. p(x) is a polynomial with integer coefficients. The sequence of integers $a_1, a_2, \dots , a_n$ (where $n > 2$) satisfies $a_2 = p(a_1)$, $a_3 = p(a_2)$, ... , $a_n = p(a_{n-1})$, $a_1 = p(a_n)$. Show that $a_1 = a_3$.

7. D_n is a set of domino pieces. For each pair of non-negative integers (a, b) with $a \le b \le n$, there is one domino, denoted [a, b] or [b, a] in D_n. A ring is a sequence of dominoes $[a_1, b_1]$, $[a_2, b_2]$, ... , $[a_k, b_k]$ such that $b_1 = a_2$, $b_2 = a_3$, ... , $b_{k-1} = a_k$ and $b_k = a_1$. Show that if n is even there is a ring which uses all the pieces. Show that for n odd, at least (n+1)/2 pieces are not used in any ring. For n odd, how many different sets of (n+1)/2 are there, such that the pieces not in the set can form a ring?

8. We are given a supply of a x b tiles with a and b distinct positive integers. The tiles are to be used to tile a 28 x 48 rectangle. Find a, b such that the tile has the smallest possible area and there is only one possible tiling. (If there are two distinct tilings, one of which is a reflection of the other, then we treat that as more than one possible tiling. Similarly for other symmetries.) Find a, b such that the tile has the largest possible area and there is more than one possible tiling.

9. $a_1, a_2, \dots , a_n$ is a sequence of integers such that every non-empty subsequence has non-zero sum. Show that we can partition the positive integers into a finite number of sets such that if x_i all belong to the same set, then $a_1x_1 + a_2x_2 + \dots + a_nx_n$ is non-zero.

14th Austrian-Polish 1991

1. Show that there are infinitely many integers m > 1 such that mC2 = 3(nC4) for some integer n > 3, where aCb denotes the binomial coefficient a!/(b! (a-b)!). Find all such m.

2. Find all real solutions to:

$(x^2 - 6x + 13)y = 20$

$(y^2 - 6y + 13)z = 20$

$(z^2 - 6z + 13)x = 20$.

3. A_1, A_2 are distinct points in the plane. Find all points A_3 for which we can find n > 2 and points $P_1, P_2, \dots, P_n$ (not necessarily distinct) such that the midpoints of P_1P_2, P_2P_3, P_3P_4, ... , $P_{n-1}P_n$, P_nP_1 are A_1, A_2, A_3, A_1, A_2, A_3, A_1, ... respectively.

4. The polynomial p(x) is non-negative for all $0 \le x \le 1$. Show that there are polynomials q(x), r(x), s(x) which are non-negative for all x such that p(x) = q(x) + x r(x) + (1 - x) s(x).

5. Show that if the real numbers x, y, z satisfy xyz = 1, then $x^2 + y^2 + z^2 + xy + yz + zx \ge 2(\sqrt{x} + \sqrt{y} + \sqrt{z})$.

6. ABCD is a convex quadrilateral. P is a point inside ABCD such that PAB, PBC, PCD, PDA have equal area. Show that area ABC = area ADC, or area BCD = area BAD.

7. Find the maximum value of $(x + x^2 + x^3 + \dots + x^{2n-1})/(1 + x^n)^2$ for positive real x and the values of x at which the maximum is achieved.

8. Find all solutions to ab = -1 mod c, bc = 1 mod a, ca = 1 mod b, such that a, b, c are all distinct positive integers and one of them is 19.

9. Let X be the set {1, 2, 3, ... , 2n}. g is a function X → X such that g(k) ≠ k and g(g(k)) = k for all k. How many functions f: X → X are there such that f(k) ≠ k and f(f(f(k))) = g(k) for all k?

15th Austrian-Polish 1992

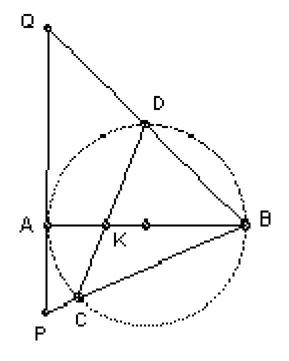
1. Given a positive integer n, let s(n) be the sum of the positive divisors of n. For example s(5) = 6. Given any three consecutive integers, show that at least one has s(n) even.

2. Each point on the boundary of a square is to be colored with one of n colors. Find the smallest n such that we can color the points in such a way that there is no right-angled triangle with its vertices on the boundary and all the same color.

3. Show that $2(xy + yz + zx)^{1/2} \le 3^{1/2}(x + y)^{1/3}(y + z)^{1/3}(z + x)^{1/3}$ for all positive reals x, y, z.

4. The cubic $x^3 + ax^2 + bx + c$ has roots uv, u^k, v^k, where u and v are real and k is a positive integer. Show that if a, b, c are rational and k = 2, then uv is rational. Is the same true if k = 3?

5. K is a point on the diameter AB closer to A than B. CD is a variable chord through K. The lines BC and BD meet the tangent at A at P and Q respectively. Show that AP·AQ is constant.



6. Does there exist a function f: Z → Z (where Z is the set of integers) such that: (1) f(Z) includes the values 1, 2, 4, 23, 92; (2) f(92 + n) = f(92 - n) for all n; (3) f(1748 + n) = f(1748 - n) for all n; (4) f(1992 + n) = f(1992 - n) for all n?

7. What conditions must the angles of the triangle ABC satisfy if there is a point X in space such that the angles AXB, BXC, CXA are all 90^o? If the point X exists, d = max(XA, XB, XC), and h = length of longest altitude of ABC, show that $(\sqrt{(2/3)})\, h \le d \le h$.

8. $x_1, x_2, \dots, x_n$ are non-zero real numbers with sum s such that $(s - 2x_1 - 2x_2)/x_1 = (s - 2x_2 - 2x_3)/x_2 = \dots = (s - 2x_{n-1} - 2x_n)/x_{n-1} = (s - 2x_n - 2x_1)/x_n$. What possible values can be taken by $(s - x_1)(s - x_2) \dots (s - x_n)/(x_1 \dots x_n)$?

9. n is an integer > 1. A word is a sequence $X_1, X_2, \dots, X_{2n}$ of 2n symbols, n of which are A and n of which are B. Let r(n) be the number of words such that only one of the sequences $X_1, X_2, \dots, X_k$ have equal numbers of As and Bs (namely the sequence with k = 2n). Let s(n) be the number of words such that just two of the sequences have equal numbers of As and Bs. Find s(n)/r(n).

16th Austrian-Polish 1993

1. Find all positive integers m, n such that $2^m - 3^n = 7$.

2. Find all tetrahedra ABCD such that area ABD + area ACD + area BCD ≤ 1 and the volume of the tetrahedron is as large as possible.

3. Let N be the set of positive integers. Define f: N → N by f(n) = n+1 if n is a prime power and = $w_1 + ... + w_k$ when n is a product of the coprime prime powers $w_1, w_2, ... , w_k$. For example f(12) = 7. Find the smallest term of the infinite sequence m, f(m), f(f(m)), f(f(f(m))), ...

4. The Fibonacci numbers are defined by $F_0 = 1$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$. The positive integers A, B are such that A^{19} divides B^{93} and B^{19} divides A^{93}. Show that if h < k are consecutive Fibonacci numbers then $(AB)^h$ divides $(A^4 + B^8)^k$.

5. Find all real solutions to:

$x + y = 3x + 4$

$2y^3 + z = 6y + 6$

$3z^3 + x = 9z + 8$.

6. Show that for non-negative real numbers x, y we have: $((\sqrt{x} + (\sqrt{y})/2)^2 \leq (x + (x^2y)^{1/3} + (xy^2)^{1/3} + y)/4 \leq (x + (\sqrt{(xy)} + y)/3 \leq ((x^{2/3} + y^{2/3})/2)^{3/2}$.

7. The integer sequence $a_0 = 1, a_1, a_2, ...$ is defined by $a_{n+1} = [(a_n + n)^{1/3}]^3$. Find an explicit formula for a_n and find all n such that $a_n = n$.

8. Find all real polynomials p(x) such that there is a unique real polynomial q(x) with (1) q(0) = 0, (2) x + q(y + p(x)) = y + q(x + p(y)) for all x, y.

9. ABC is an equilateral triangle. On the line AB take a point P such that A is between P and B. Let r be the inradius of the triangle PAC and r' the exradius of triangle PBC opposite P. Find r + r' in terms of the side length a of ABC.

17th Austrian-Polish 1994

1. f is a real-valued function on the reals such that $f(x + 19) \le f(x) + 19$ and $f(x + 94) \ge f(x) + 94$ for all x. Show that $f(x + 1) = f(x) + 1$ for all x.

2. The sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ are defined by $a_0 = 1/2$, $a_{n+1} = 2a_n/(1 + a_n^2)$, $b_0 = 4$, $b_{n+1} = b_n^2 - 2b_n + 2$. Show that $a_{n+1}b_{n+1} = 2b_0b_1 \ldots b_n$.

3. Each cell of a 2 x 15 board is made into a room. The 43 interior walls are labeled from 1 to 43. Wall n has d_n doors, where $d_n = 0, 1, 2$ or 3. The doors are arranged so that each room has a total of three doors and it is possible to get from any room to any other room. How many possible arrangements are there for $(d_1, d_2, \ldots, d_{43})$.

4. P is a regular (n+1)-gon. One vertex is labeled 0. There are n! ways of labeling the other vertices 1, 2, ... , n. For each such labeling take the sum of the (positive) difference between the labels at the end of each side. For example, if n = 5 and the vertices are labeled 0, 3, 1, 4, 2, 5, then the sum is 3 + 2 + 3 + 2 + 3 + 5 = 18. Find the smallest possible such sum f(n) and the number of possible labelings which give it.

5. Find all integer solutions to $(a + b)(b + c)(c + a)/2 + (a + b + c)^3 = 1 - abc$.

6. n is an odd integer and the non-negative integers $a_1, a_2, \ldots, a_n$ satisfy:

$(a_2 - a_1)^2 + 2(a_2 + a_1) + 1 = n^2$

$(a_3 - a_2)^2 + 2(a_3 + a_2) + 1 = n^2$

...

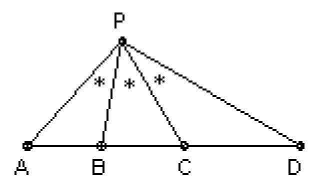
$(a_1 - a_n)^2 + 2(a_1 + a_n) + 1 = n^2$.

Show that there are two consecutive a_i which are equal (we treat a_n and a_1 as consecutive).

7. Find all two digit numbers n = ab such that $(x^a - x^b)$ is divisible by n for all integers x.

8. Let R be the reals. For each real a, b, find all functions $f: R^2 \to R$ which satisfy $f(x, y) = a f(x, z) + b f(y, z)$ for all x, y, z.

9. The points A, B, C, D lie on a line in that order, with AB = a, BC = b, CD = c. Show that a point P exists such that $\angle APB = \angle BPC = \angle CPD$ iff $(a + b)(b + c) < 4ac$. If the point exists, show how to construct it.



18th Austrian-Polish 1995

1. Find all real solutions to:

$x_1 = x_n + x_{n-1}$

$x_2 = x_1 + x_n$

$x_3 = x_2 + x_1$

...

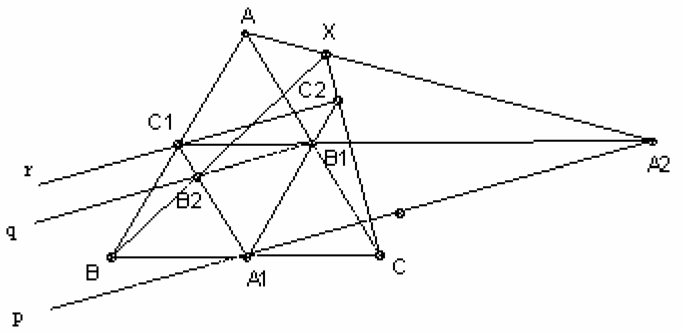
$x_n = x_{n-1} + x_{n-2}$.

2. Show that for any 4 points in the plane, we can choose a non-empty subset X such that no disk contains the points in X and not the points not in X.

3. Factor $1 + 5y^2 + 25y^4 + 125y^6 + 625y^8$.

4. Find all real polynomials p(x) such that $p(x)^2 + 2p(x)p(1/x) + p(1/x)^2 = p(x^2)p(1/x^2)$ for all non-zero x.

5. ABC is an equilateral triangle. A_1, B_1, C_1 are the midpoints of BC, CA, AB respectively. p is an arbitrary line through A_1. q and r are lines parallel to p through B_1 and C_1 respectively. p meets the line B_1C_1 at A_2. Similarly, q meets C_1A_1 at B_2, and r meets A_1B_1 at C_2. Show that the lines AA_2, BB_2, CC_2 meet at some point X, and that X lies on the circumcircle of ABC.



6. Find the number of 4-tuples (A, B, C, D), where A, B, C, D are subsets of {1, 2, 3, ... , n} such that A and B have at least one common element, B and C have at least one common element, and C and D have at least one common element.

7. Let s(n) be the number of integer solutions (x, y) to the equation $3y^4 + 4ny^3 + 2xy + 48 = 0$ such that |x| is a square and y is square-free (so there is no prime p such that p^2 divides y). Let s be the maximal value of s(n) (as n varies). Find all n such that s(n) = s.

8. C is the cube { (x, y, z) such that $|x| \le 1$, $|y| \le 1$, $|z| \le 1$}. P_n, where n = 1, 2, ... , 95, are any points of the cube. v_n is the vector from (0, 0, 0) to P_n. S is the set of 2^{95} v vectors of the form $\pm v_1 \pm v_2 \pm ... \pm v_{95}$. Each vector v in S can be regarded as a vector from (0, 0, 0) to some point (a, b, c). Show that some vector in S has $a^2 + b^2 + c^2 \le d$, where d = 48. Find the smallest d for which we can always find a vector in S with $a^2 + b^2 + c^2 \le d$.

9. Show that $(n-1)(m-1)(x^{n+m} + y^{n+m}) + (n+m-1)(x^ny^m + x^my^n) \ge mn(x^{n+m-1}y + xy^{n+m-1})$ for all positive integers m, n and all positive real numbers x, y.

19th Austrian-Polish 1996

1. Show that there are 3^{k-1} positive integers n such that n has k digits, all odd, n is divisible by 5 and n/5 has k odd digits. [For example, for k = 2, the possible numbers are 55, 75 and 95.]

2. ABCDEF is a convex hexagon is such that opposite sides are parallel and the perpendicular distance between each pair of opposite sides is equal. The angles at A and D are 90^o. Show that the diagonals BE and CF are at 45^o.

3. The polynomials $p_n(x)$ are defined by $p_0(x) = 0$, $p_1(x) = x$, $p_{n+2}(x) = x\, p_{n+1}(x) + (1 - x)\, p_n(x)$. Find the real roots of each $p_n(x)$.

4. The real numbers w, x, y, z have zero sum and sum of squares 1. Show that the sum wx + xy + yz + zw lies between -1 and 0.

5. P is a convex polyhedron. S is a sphere which meets each edge of P at the two points which divide the edge into three equal parts. Show that there is a sphere which touches every edge of P.

6. k, n are positive integers such that n > k > 1. Find all real solutions $x_1, x_2, \dots, x_n$ to $x_i^3(x_i^2 + x_{i+1}^2 + \dots + x_{i+k-1}^2) = x_{i-1}^2$ for i = 1, 2, ... , n. [Note that we take x_0 to mean x_n, and x_{n+j} to mean x_j.]

7. Show that there are no non-negative integers m, n satisfying $m! + 48 = 48 (m + 1)^n$.

8. Show that there is no real polynomial of degree 998 such that $p(x)^2 - 1 = p(x^2 + 1)$ for all x.

9. A *block* is a rectangular parallelepiped with integer sides a, b, c which is not a cube. N blocks are used to form a 10 x 10 x 10 cube. The blocks may be different sizes. Show that if N ≥ 100, then at least two of the blocks must have the same dimensions and be placed with corresponding edges parallel. Prove the same for some number smaller than 100.

20th Austrian-Polish 1997

1. Four circles, none of which lies inside another, pass through the point P. Two circles touch the line L at P and the other two touch the line M at P. The other points of intersection of the circles are A, B, C, D. Show that A, B, C, D lie on a circle iff L and M are perpendicular.

2. A piece is on each square of an m x n board. The allowed move for each piece is h squares parallel to the bottom edge of the board and k squares parallel to the sides. How many ways can we move every piece simultaneously so that after the move there is still one piece on each square?

3. The 97 numbers 49/1, 49/2, 49/3, ... , 49/97 are written on a blackboard. We repeatedly pick two numbers a, b on the board and replace them by 2ab - a - b + 1 until only one number remains. What are the possible values of the final number?

4. ABCD is a convex quadrilateral with AB parallel to CD. The diagonals meet at E. X is the midpoint of the line joining the orthocenters of BEC and AED. Show that X lies on the perpendicular to AB through E.

5. Show that no cubic with integer coefficients can take the value ±3 at each of four distinct primes.

6. Show that there is no integer-valued function on the integers such that f(m + f(n)) = f(m) - n for all m, n.

7. Show that $x^2 + y^2 + 1 > x(y + 1)$ for all reals x, y. Find the largest k such that $x^2 + y^2 + 1 \geq kx(y + 1)$ for all reals x, y. Find the largest k such that $m^2 + n^2 + 1 \geq km(n + 1)$ for all integers m, n.

8. Let X be a set with n members. Find the largest number of subsets of X each with 3 members so that no two are disjoint.

9. k > 0 and P is a solid parallelepiped. S is the set of all points X for which there is a point Y in P such that XY ≤ k. Show that the volume of $S = V + Fk + \pi Ek^2/4 + 4\pi k^3/3$, where V, F, E are respectively the volume, surface area and total edge length of P.

21st Austrian-Polish 1998

1. Show that $(wx + yz - 1)^2 \geq (w^2 + y^2 - 1)(x^2 + z^2 - 1)$ for reals w, x, y, z such that $w^2 + y^2 \leq 1$.

2. n points lie in a line. How many ways are there of coloring the points with 5 colors, one of them red, so that every two adjacent points are either the same color or one or more of them is red?

3. Find all real solutions to $x^3 = 2 - y$, $y^3 = 2 - x$.

4. Show that $[1^{m/1}] + [2^{m/4}] + [3^{m/9}] + ... + [n^{m/N}] \leq n + m(2^{m/4} - 1)$, where $N = n^2$, and m, n are any positive integers.

5. Find all positive integers m, n such that the roots of $x^3 - 17x^2 + mx - n^2$ are all integral.

6. A, B, C, D, E, F lie on a circle in that order. The tangents at A and D meet at P and the lines BF and CE pass through P. Show that the lines AD, BC, EF are parallel or concurrent.

7. Find positive integers m, n with the smallest possible product mn such that the number $m^m n^n$ ends in exactly 98 zeros.

8. Given an infinite sheet of squared paper. A positive integer is written in each small square. Each small square has area 1. For some n > 2, every two congruent polygons (even if mirror images) with area n and sides along the rulings on the paper have the same sum for the numbers inside. Show that all the numbers in the squares must be equal.

9. ABC is a triangle. K, L, M are the midpoints of the sides BC, CA, AB respectively, and D, E, F are the midpoints of the arcs BC (not containing A), CA (not containing B), AB (not containing C) respectively. Show that r + KD + LE + MF = R, where r is the inradius and R the circumradius.

22nd Austrian-Polish 1999

1. X is the set {1, 2, 3, ... , n}. How many ordered 6-tuples (A_1, A_2, ... , A_6) of subsets of X are there such that every element of X belongs to 0, 3 or 6 subsets in the 6-tuple?

2. Find the best possible k, k' such that k < v/(v + w) + w/(w + x) + x/(x + y) + y/(y + z) + z/(z + v) < k' for all positive reals v, w, x, y, z.

3. Given n > 1, find all real-valued functions $f_i(x)$ on the reals such that for all x, y we have:
$f_1(x) + f_1(y) = f_2(x) f_2(y)$

$f_2(x^2) + f_2(y^2) = f_3(x)f_3(y)$

$f_3(x^3) + f_3(y^3) = f_4(x)f_4(y)$

...
$f_n(x^n) + f_n(y^n) = f_1(x)f_1(y)$.

4. P is a point inside the triangle ABC. Show that there are unique points A_1 on the line AB and A_2 on the line CA such that P, A_1, A_2 are collinear and $PA_1 = PA_2$. Similarly, take B_1, B_2, C_1, C_2, so that P, B_1, B_2 are collinear, with B_1 on the line BC, B_2 on the line AB and $PB_1 = PB_2$, and P, C_1, C_2 are collinear, with C_1 on the line CA, C_2 on the line BC and $PC_1 = PC_2$. Find the point P such that the triangles AA_1A_2, BB_1B_2, CC_1C_2 have equal area, and show it is unique.

5. The integer sequence a_n satisfies $a_{n+1} = a_n^3 + 1999$. Show that it contains at most one square.

6. Find all non-negative real solutions to

$x_2^2 + x_1x_2 + x_1^4 = 1$

$x_3^2 + x_2x_3 + x_2^4 = 1$

$x_4^2 + x_3x_4 + x_3^4 = 1$

...
$x_{1999}^2 + x_{1998}x_{1999} + x_{1998}^4 = 1$

$x_1^2 + x_{1999}x_1 + x_1^4 = 1$.

7. Find all positive integers m, n such that $m^{n+m} = n^{n-m}$.

8. P and Q are on the same side of the line L. The feet of the perpendiculars from P, Q to L are M, N respectively. The point S is such that PS = PM and QS = QN. The perpendicular bisectors of SM and SN meet at R. The ray RS meets the circumcircle of PQR again at T. Show that S is the midpoint of RT.

9. A *valid* set is a finite set of plane lattice points and segments such that: (1) the endpoints of each segment are lattice points and it is parallel to x = 0, y = 0, y = x or y = -x; (2) two segments have at most one common point; (3) each segment has just five points in the set. Does there exist an infinite sequence of valid sets, S_1, S_2, S_3, ... such that S_{n+1} is formed by adding one segment and one lattice point to S_n?

23rd Austrian-Polish 2000

1. Find all polynomials p(x) with real coefficients such that for some $n > 0$, $p(x+2) - p(x+3) + 2p(x+4) - 2p(x+5) + ... - n\,p(x+2n) + n\,p(x+2n+1) = 0$ holds for infinitely many real x.

2. O is a vertex of a cube side 1. OABC and OADE are faces of the cube. Find the shortest distance between a point of the circle inscribed in OABC and a point of the circumcircle of OAD.

3. For $n > 2$ find all real solutions to: $x_1^3 = x_2 + x_3$, $x_2^3 = x_3 + x_4$, ... , $x_{n-2}^3 = x_{n-1} + x_n$, $x_{n-1}^3 = x_n + x_1$, $x_n^3 = x_1 + x_2$.

4. Find all positive integers n, not divisible by any primes except (possibly) 2 and 5, such that $n + 25$ is a square.

5. For which $n > 4$ can we color the vertices of a regular n-gon with 6 colors so that every 5 adjacent vertices have different colors?

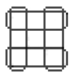
6. A unit cube is glued onto each face of a central unit cube (so that the glued faces coincide). Can copies of the resulting solid fill space?

7. ABC is a triangle. The points A', B', C' lie on the lines BC, CA, AB respectively and A'B'C' is similar to ABC. Find all possible positions for the circumcenter of A'B'C'.

8. Given 27 points in the plane. Four of the points form a unit square. The other points are all inside the square. No three points are collinear. Show that we can find three points forming a triangle with area at most 1/48.

9. Show that $2 \le (1 - x^2)^2 + (1 - y^2)^2 + (1 - z^2)^2 \le (1 + x)(1 + y)(1 + z)$ for non-negative reals x, y, z with sum 1.

10. The diagram shows the plan of the castle. There are 16 nodes. Eight pairs are connected by two links each (at the four corners). How many closed paths pass through each node just once (only count a path once irrespective of whether it is traversed clockwise or counter-clockwise)? How many closed paths pass through each link once? In this case, treat paths as different if a link is traversed in opposite directions.



24th Austrian-Polish 2001

1. How many positive integers n have a non-negative power which is a sum of 2001 non-negative powers of n?

2. Take n > 2. Solve $x_1 + x_2 = x_3^2$, $x_2 + x_3 = x_4^2$, ... , $x_{n-2} + x_{n-1} = x_n^2$, $x_{n-1} + x_n = x_1^2$, $x_n + x_1 = x_2^2$ for the real numbers $x_1, x_2, ... , x_n$.

3. Show that $2 < (a + b)/c + (b + c)/a + (c + a)/b - (a^3 + b^3 + c^3)/(abc) \le 3$, where a, b, c are the sides of a triangle.

4. Show that the area of a quadrilateral is at most (ac + bd)/2, where the side lengths are a, b, c, d (with a opposite c). When does equality hold?

5. Label the squares of a chessboard according to a knights' tour. So for i = 1, 2, ... , 63, the square labeled i+1 is one away from i in the direction parallel to one side of the board and two away in the perpendicular direction. Take any positive numbers $x_1, x_2, ... , x_{64}$ and let $y_i = 1 + x_i^2 - (x_{i-1}^2 x_{i+1})^{1/3}$ for i a white square and $1 + x_i^2 - (x_{i-1} x_{i+1}^2)^{1/3}$ for i a black square (x_0 means x_{64} and x_{65} means x_1). Show that $y_1 + y_2 + ... + y_{64} \ge 48$.

6. Define $a_0 = 1$, $a_{n+1} = a_n + [a_n^{1/k}]$, where k is a positive integer. Find $S_k = \{n \mid n = a_m$ for some m$\}$.

7. Show that there are infinitely many positive integers n which do not contain the digit 0, whose digit sum divides n and in which each digit that does occur occurs the same number of times. Show that there is a positive integer n which does not contain the digit 0, whose digit sum divides n, and which has k digits.

8. The top and bottom faces of a prism are regular octagons and the sides are squares. Every edge has length 1. The midpoints of the faces are M_i. A point P inside the prism is such that each ray M_iP meets the prism again in a different face at N_i. Show that $PM_1/M_1N_1 + PM_2/M_2N_2 + ... + PM_{10}/M_{10}M_{10} = 5$.

9. Find the largest possible number of subsets of {1, 2, ... , 2n} each with n elements such that the intersection of any three distinct subsets has at most one element.

10. A sequence of real numbers is such that the product of each pair of consecutive terms lies between -1 and 1 and the sum of every twenty consecutive terms is non-negative. What is the largest possible value for the sum of the first 2010 terms?

25th Austrian-Polish 2002

1. Find all triples (a, b, c) of non-negative integers such that $2^a + 2^b + 1$ is a multiple of $2^c - 1$.

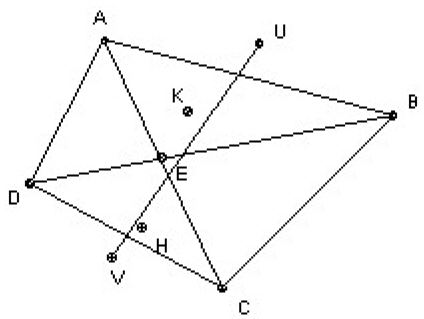
2. Show that any convex polygon with an even number of vertices has a diagonal which is not parallel to any of its edges.

3. A line through the centroid of a tetrahedron meets its surface at X and Y, so the centroid divides the segment XY into two parts. Show that the shorter part is at least one-third of the length of the longer part.

4. Given a positive integer n, find the largest set of real numbers such that $n + (x_1^{n+1} + x_2^{n+1} + ... + x_n^{n+1}) \geq n\, x_1 ... x_n + (x_1 + ... + x_n)$ for all x_i in the set. When do we have equality?

5. p(x) is a polynomial with integer coefficients. Every value p(n) for n an integer is divisible by at least one of 2, 7, 11, 13. Show that every coefficient of p(x) is divisible by 2, or every coefficient is 7, or every coefficient is divisible by 11, or every coefficient is divisible by 13.

6. ABCD is a convex quadrilateral whose diagonals meet at E. The circumcenters of ABE, CDE are U, V respectively, and the orthocenters of ABE, CDE are H, K respectively. Show that E lies on the line UK iff it lies on the line VH.



7. Let N be the set of positive integers and R the set of reals. Find all functions $f : N \to R$ such that $f(x + 22) = f(x)$ and $f(x^2y) = f(x)^2f(y)$ for all x, y.

8. How many real n-tuples $(x_1, x_2, ... , x_n)$ satisfy the equations $\cos x_1 = x_2$, $\cos x_2 = x_3$, ... , $\cos x_{n-1} = x_n$, $\cos x_n = x_1$?

9. A graph G has 2002 points and at least one edge. Every subgraph of 1001 points has the same number of edges. Find the smallest possible number of edges in the graph, or failing that the best lower bound you can.

10. Is it true that given any positive integer N, we can find X such that (1) any real sequence $x_0, x_1, x_2, ...$ satisfying $x_{n+1} = X - 1/x_n$ satisfies $x_k = x_{k+N}$ for all k, and (2) given a positive integer M < N, we can always find some real sequence $x_0, x_1, x_2, ...$ satisfying $x_{n+1} = X - 1/x_n$ such that $x_k = x_{k+M}$ does not hold for all k?

26th Austrian-Polish 2003

1. Find all real polynomials p(x) such that $p(x-1)p(x+1) \equiv p(x^2-1)$.

2. The sequence $a_0, a_1, a_2, \ldots$ is defined by $a_0 = a$, $a_{n+1} = a_n + L(a_n)$, where L(m) is the last digit of m (eg L(14) = 4). Suppose that the sequence is strictly increasing. Show that infinitely many terms must be divisible by d = 3. For what other d is this true?

3. ABC is a triangle. Take a = BC etc as usual. Take points T_1, T_2 on the side AB so that $AT_1 = T_1T_2 = T_2B$. Similarly, take points T_3, T_4 on the side BC so that $BT_3 = T_3T_4 = T_4C$, and points T_5, T_6 on the side CA so that $CT_5 = T_5T_6 = T_6A$. Show that if $a' = BT_5$, $b' = CT_1$, $c' = AT_3$, then there is a triangle A'B'C' with sides a', b', c' (a' = B'C' etc). In the same way we take points T_i' on the sides of A'B'C' and put $a'' = B'T_6'$, $b'' = C'T_2'$, $c'' = A'T_4'$. Show that there is a triangle A"B"C" with sides a", b", c" and that it is similar to ABC. Find a"/a.

4. A positive integer m is alpine if m divides $2^{2n+1} + 1$ for some positive integer n. Show that the product of two alpine numbers is alpine.

5. A triangle with sides a, b, c has area F. The distances of its centroid from the vertices are x, y, z. Show that if $(x + y + z)^2 \le (a^2 + b^2 + c^2)/2 + 2F\sqrt{3}$, then the triangle is equilateral.

6. ABCD is a tetrahedron such that we can find a sphere k(A,B,C) through A, B, C which meets the plane BCD in the circle diameter BC, meets the plane ACD in the circle diameter AC, and meets the plane ABD in the circle diameter AB. Show that there exist spheres k(A,B,D), k(B,C,D) and k(C,A,D) with analogous properties.

7. Put $f(n) = (n^n - 1)/(n - 1)$. Show that $n!^{f(n)}$ divides $(n^n)!$. Find as many positive integers as possible for which $n!^{f(n)+1}$ does not divide $(n^n)!$.

8. Given reals $x_1 \ge x_2 \ge \ldots \ge x_{2003} \ge 0$, show that $x_1^n - x_2^n + x_3^n - \ldots - x_{2002}^n + x_{2003}^n \ge (x_1 - x_2 + x_3 - x_4 + \ldots - x_{2002} + x_{2003})^n$ for any positive integer n.

9. Take any 26 distinct numbers from {1, 2, ... , 100}. Show that there must be a non-empty subset of the 26 whose product is a square.

10. What is the smallest number of 5x1 tiles which must be placed on a 31x5 rectangle (each covering exactly 5 unit squares) so that no further tiles can be placed? How many different ways are there of placing the minimal number (so that further tiles are blocked)? What are the answers for a 52x5 rectangle?

APMO (1989 – 2003)

1st APMO 1989

A1. a_i are positive reals. $s = a_1 + ... + a_n$. Prove that for any integer $n > 1$ we have $(1 + a_1) ... (1 + a_n) < 1 + s + s^2/2! + ... + s^n/n!$.

A2. Prove that $5n^2 = 36a^2 + 18b^2 + 6c^2$ has no integer solutions except $a = b = c = n = 0$.

A3. ABC is a triangle. X lies on the segment AB so that AX/AB = 1/4. CX intersects the median from A at A' and the median from B at B''. Points B', C', A'', C'' are defined similarly. Find the area of the triangle A''B''C'' divided by the area of the triangle A'B'C'.

A4. Show that a graph with n vertices and k edges has at least $k(4k - n^2)/3n$ triangles.

A5. f is a strictly increasing real-valued function on the reals. It has inverse f^{-1}. Find all possible f such that $f(x) + f^{-1}(x) = 2x$ for all x.

2nd APMO 1990

A1. Given θ in the range (0, π) how many (incongruent) triangles ABC have angle A = θ, BC = 1, and the following four points concyclic: A, the centroid, the midpoint of AB and the midpoint of AC?

A2. $x_1, \dots, x_n$ are positive reals. s_k is the sum of all products of k of the x_i (for example, if n = 3, $s_1 = x_1 + x_2 + x_3$, $s_2 = x_1x_2 + x_2x_3 + x_3x_1$, $s_3 = x_1x_2x_3$). Show that $s_k s_{n-k} \geq (nCk)^2 s_n$ for 0 < k < n.

A3. A triangle ABC has base AB = 1 and the altitude from C length h. What is the maximum possible product of the three altitudes? For which triangles is it achieved?

A4. A graph with n > 1 points satisfies the following conditions: (1) no point has edges to all the other points, (2) there are no triangles, (3) given any two points A, B such that there is no edge AB, there is exactly one point C such that there are edges AC and BC. Prove that each point has the same number of edges. Find the smallest possible n.

A5. Show that for any n ≥ 6 we can find a convex hexagon which can be divided into n congruent triangles.

3rd APMO 1991

A1. ABC is a triangle. G is the centroid. The line parallel to BC through G meets AB at B' and AC at C'. Let A" be the midpoint of BC, C" the intersection of B'C and BG, and B" the intersection of C'B and CG. Prove that A"B"C" is similar to ABC.

A2. There are 997 points in the plane. Show that they have at least 1991 distinct midpoints. Is it possible to have exactly 1991 midpoints?

A3. x_i and y_i are positive reals with $\sum_1^n x_i = \sum_1^n y_i$. Show that $\sum_1^n x_i^2/(x_i + y_i) \geq (\sum_1^n x_i)/2$.

A4. A sequence of values in the range 0, 1, 2, ... , k-1 is defined as follows: $a_1 = 1$, $a_n = a_{n-1} + n \pmod{k}$. For which k does the sequence assume all k possible values?

A5. Circles C and C' both touch the line AB at B. Show how to construct all possible circles which touch C and C' and pass through A.

4th APMO 1992

A1. A triangle has sides a, b, c. Construct another triangle sides (-a + b + c)/2, (a - b + c)/2, (a + b - c)/2. For which triangles can this process be repeated arbitrarily many times?

A2. Given a circle C centre O. A circle C' has centre X inside C and touches C at A. Another circle has centre Y inside C and touches C at B and touches C' at Z. Prove that the lines XB, YA and OZ are concurrent.

A3. Given three positive integers a, b, c, we can derive 8 numbers using one addition and one multiplication and using each number just once: a+b+c, a+bc, b+ac, c+ab, (a+b)c, (b+c)a, (c+a)b, abc. Show that if a, b, c are distinct positive integers such that $n/2 < a, b, c, \le n$, then the 8 derived numbers are all different. Show that if p is prime and $n \ge p^2$, then there are just d(p-1) ways of choosing two distinct numbers b, c from {p+1, p+2, ... , n} so that the 8 numbers derived from p, b, c are *not* all distinct, where d(p-1) is the number of positive divisors of p-1.

A4. Find all possible pairs of positive integers (m, n) so that if you draw n lines which intersect in n(n-1)/2 distinct points and m parallel lines which meet the n lines in a further mn points (distinct from each other and from the first n(n-1)/2) points, then you form exactly 1992 regions.

A5. $a_1, a_2, a_3, \ldots a_n$ is a sequence of non-zero integers such that the sum of any 7 consecutive terms is positive and the sum of any 11 consecutive terms is negative. What is the largest possible value for n?

5th APMO 1993

A1. A, B, C is a triangle. X, Y, Z lie on the sides BC, CA, AB respectively, so that AYZ and XYZ are equilateral. BY and CZ meet at K. Prove that $YZ^2 = YK.YB$.

A2. How many different values are taken by the expression [x] + [2x] + [5x/3] + [3x]+ [4x] for real x in the range $0 \le x \le 100$?

A3. p(x) = (x + a) q(x) is a real polynomial of degree n. The largest absolute value of the coefficients of p(x) is h and the largest absolute value of the coefficients of q(x) is k. Prove that $k \le hn$.

A4. Find all positive integers n for which $x^n + (x+2)^n + (2-x)^n = 0$ has an integral solution.

A5. C is a 1993-gon of lattice points in the plane (not necessarily convex). Each side of C has no lattice points except the two vertices. Prove that at least one side contains a point (x, y) with 2x and 2y both odd integers.

6th APMO 1994

A1. Find all real-valued functions f on the reals such that (1) $f(1) = 1$, (2) $f(-1) = -1$, (3) $f(x) \leq f(0)$ for $0 < x < 1$, (4) $f(x + y) \geq f(x) + f(y)$ for all x, y, (5) $f(x + y) \leq f(x) + f(y) + 1$ for all x, y.

A2. ABC is a triangle and A, B, C are not collinear. Prove that the distance between the orthocenter and the circumcenter is less than three times the circumradius.

A3. Find all positive integers n such that $n = a^2 + b^2$, where a and b are relatively prime positive integers, and every prime not exceeding $\sqrt{n}$ divides ab.

A4. Can you find infinitely many points in the plane such that the distance between any two is rational and no three are collinear?

A5. Prove that for any $n > 1$ there is either a power of 10 with n digits in base 2 or a power of 10 with n digits in base 5, but not both.

7th APMO 1995

A1. Find all real sequences $x_1, x_2, \ldots, x_{1995}$ which satisfy $2\sqrt{(x_n - n + 1)} \geq x_{n+1} - n + 1$ for n = 1, 2, ... , 1994, and $2\sqrt{(x_{1995} - 1994)} \geq x_1 + 1$.

A2. Find the smallest n such that any sequence $a_1, a_2, \ldots, a_n$ whose values are relatively prime square-free integers between 2 and 1995 must contain a prime. [An integer is square-free if it is not divisible by any square except 1.]

A3. ABCD is a fixed cyclic quadrilateral with AB not parallel to CD. Find the locus of points P for which we can find circles through AB and CD touching at P.

A4. Take a fixed point P inside a fixed circle. Take a pair of perpendicular chords AC, BD through P. Take Q to be one of the four points such that AQBP, BQCP, CQDP or DQAP is a rectangle. Find the locus of all possible Q for all possible such chords.

A5. f is a function from the integers to {1, 2, 3, ... , n} such that f(A) and f(B) are unequal whenever A and B differ by 5, 7 or 12. What is the smallest possible n?

8th APMO 1996

A1. ABCD is a fixed rhombus. P lies on AB and Q on BC, so that PQ is perpendicular to BD. Similarly P' lies on AD and Q' on CD, so that P'Q' is perpendicular to BD. The distance between PQ and P'Q' is more than BD/2. Show that the perimeter of the hexagon APQCQ'P' depends only on the distance between PQ and P'Q'.

A2. Prove that $(n+1)^m n^m \geq (n+m)!/(n-m)! \geq 2^m m!$ for all positive integers n, m with $n \geq m$.

A3. Given four concyclic points. For each subset of three points take the incenter. Show that the four incenters from a rectangle.

A4. For which n in the range 1 to 1996 is it possible to divide n married couples into exactly 17 single sex groups, so that the size of any two groups differs by at most one.

A5. A triangle has side lengths a, b, c. Prove that $\sqrt{a + b - c} + \sqrt{b + c - a} + \sqrt{c + a - b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$. When do you have equality?

9th APMO 1997

A1. Let $T_n = 1 + 2 + ... + n = n(n+1)/2$. Let $S_n = 1/T_1 + 1/T_2 + ... + 1/T_n$. Prove that $1/S_1 + 1/S_2 + ... + 1/S_{1996} > 1001$.

A2. Find an n in the range 100, 101, ... , 1997 such that n divides $2^n + 2$.

A3. ABC is a triangle. The bisector of A meets the segment BC at X and the circumcircle at Y. Let $r_A = AX/AY$. Define r_B and r_C similarly. Prove that $r_A/\sin^2 A + r_B/\sin^2 B + r_C/\sin^2 C \geq 3$ with equality iff the triangle is equilateral.

A4. P_1 and P_3 are fixed points. P_2 lies on the line perpendicular to P_1P_3 through P_3. The sequence $P_4, P_5, P_6, ...$ is defined inductively as follows: P_{n+1} is the foot of the perpendicular from P_n to $P_{n-1}P_{n-2}$. Show that the sequence converges to a point P (whose position depends on P_2). What is the locus of P as P_2 varies?

A5. n people are seated in a circle. A total of nk coins are distributed amongst the people, but not necessarily equally. A *move* is the transfer of a single coin between two adjacent people. Find an algorithm for making the minimum number of moves which result in everyone ending up with the same number of coins?

10th APMO 1998

A1. S is the set of all possible n-tuples (X_1, X_2, ... , X_n) where each X_i is a subset of {1, 2, ... , 1998}. For each member k of S let f(k) be the number of elements in the union of its n elements. Find the sum of f(k) over all k in S.

A2. Show that (36m + n)(m + 36n) is not a power of 2 for any positive integers m, n.

A3. Prove that (1 + x/y)(1 + y/z)(1 + z/x) ≥ 2 + 2(x + y + z)/w for all positive reals x, y, z, where w is the cube root of xyz.

A4. ABC is a triangle. AD is an altitude. X lies on the circle ABD and Y lies on the circle ACD. X, D and Y are collinear. M is the midpoint of XY and M' is the midpoint of BC. Prove that MM' is perpendicular to AM.

A5. What is the largest integer divisible by all positive integers less than its cube root.

11th APMO 1999

A1. Find the smallest positive integer n such that no arithmetic progression of 1999 reals contains just n integers.

A2. The real numbers $x_1, x_2, x_3, \dots$ satisfy $x_{i+j} \leq x_i + x_j$ for all i, j. Prove that $x_1 + x_2/2 + \dots + x_n/n \geq x_n$.

A3. Two circles touch the line AB at A and B and intersect each other at X and Y with X nearer to the line AB. The tangent to the circle AXY at X meets the circle BXY at W. The ray AX meets BW at Z. Show that BW and BX are tangents to the circle XYZ.

A4. Find all pairs of integers m, n such that $m^2 + 4n$ and $n^2 + 4m$ are both squares.

A5. A set of 2n+1 points in the plane has no three collinear and no four concyclic. A circle is said to divide the set if it passes through 3 of the points and has exactly n - 1 points inside it. Show that the number of circles which divide the set is even iff n is even.

12th APMO 2000

A1. Find $a_1^3/(1 - 3a_1 + 3a_1^2) + a_2^3/(1 - 3a_2 + 3a_2^2) + ... + a_{101}^3/(1 - 3a_{101} + 3a_{101}^2)$, where $a_n = n/101$.

A2. Find all permutations $a_1, a_2, ... , a_9$ of 1, 2, ... , 9 such that $a_1 + a_2 + a_3 + a_4 = a_4 + a_5 + a_6 + a_7 = a_7 + a_8 + a_9 + a_1$ and $a_1^2 + a_2^2 + a_3^2 + a_4^2 = a_4^2 + a_5^2 + a_6^2 + a_7^2 = a_7^2 + a_8^2 + a_9^2 + a_1^2$.

A3. ABC is a triangle. The angle bisector at A meets the side BC at X. The perpendicular to AX at X meets AB at Y. The perpendicular to AB at Y meets the ray AX at R. XY meets the median from A at S. Prove that RS is perpendicular to BC.

A4. If $m < n$ are positive integers prove that $n^n/(m^m (n-m)^{n-m}) > n!/(m! (n-m)!) > n^n/(m^m(n+1) (n-m)^{n-m})$.

A5. Given a permutation $s_0, s_2, ... , s_n$ of 0, 1, 2, ... , n, we may *transform* it if we can find i, j such that $s_i = 0$ and $s_j = s_{i-1} + 1$. The new permutation is obtained by transposing s_i and s_j. For which n can we obtain (1, 2, ... , n, 0) by repeated transformations starting with (1, n, n-1, .. , 3, 2, 0)?

13th APMO 2001

A1. If n is a positive integer, let d be the number of digits in n (in base 10) and s be the sum of the digits. Let n(k) be the number formed by deleting the last k digits of n. Prove that n = s + 9 n(1) + 9 n(2) + ... + 9 n(d).

A2. Find the largest n so that the number of integers less than or equal to n and divisible by 3 equals the number divisible by 5 or 7 (or both).

A3. Two equal-sized regular n-gons intersect to form a 2n-gon C. Prove that the sum of the sides of C which form part of one n-gon equals half the perimeter of C.

A4. Find all real polynomials p(x) such that x is rational iff p(x) is rational.

A5. What is the largest n for which we can find n + 4 points in the plane, A, B, C, D, X_1, ... , X_n, so that AB is not equal to CD, but for each i the two triangles ABX_i and CDX_i are congruent?

14th APMO 2002

A1. x_i are non-negative integers. Prove that $x_1!\ x_2!\ ...\ x_n! \geq ([(x_1 + ... + x_n)/n] !)^n$ (where [y] denotes the largest integer not exceeding y). When do you have equality?

A2. Find all pairs m, n of positive integers such that $m^2 - n$ divides $m + n^2$ and $n^2 - m$ divides $m^2 + n$.

A3. ABC is an equilateral triangle. M is the midpoint of AC and N is the midpoint of AB. P lies on the segment MC, and Q lies on the segment NB. R is the orthocenter of ABP and S is the orthocenter of ACQ. The lines BP and CQ meet at T. Find all possible values for angle BCQ such that RST is equilateral.

A4. The positive reals a, b, c satisfy $1/a + 1/b + 1/c = 1$. Prove that $\sqrt{(a + bc)} + \sqrt{(b + ca)} + \sqrt{(c + ab)} \geq \sqrt{(abc)} + \sqrt{a} + \sqrt{b} + \sqrt{c}$.

A5. Find all real-valued functions f on the reals which have at most finitely many zeros and satisfy $f(x^4 + y) = x^3f(x) + f(f(y))$ for all x, y.

15th APMO 2003

1. The polynomial $a_8x^8 + a_7x^7 + ... + a_0$ has $a_8 = 1$, $a_7 = -4$, $a_6 = 7$ and all its roots positive and real. Find the possible values for a_0.

2. A unit square lies across two parallel lines a unit distance apart, so that two triangular areas of the square lie outside the lines. Show that the sum of the perimeters of these two triangles is independent of how the square is placed.

3. $k > 14$ is an integer and p is the largest prime smaller than k. k is chosen so that $p \geq 3k/4$. Prove that $2p$ does not divide $(2p - k)!$, but that n does divide $(n - k)!$ for composite $n > 2p$.

4. Show that $(a^n + b^n)^{1/n} + (b^n + c^n)^{1/n} + (c^n + a^n)^{1/n} < 1 + (2^{1/n})/2$, where $n > 1$ is an integer and a, b, c are the sides of a triangle with unit perimeter.

5. Find the smallest positive integer k such that among any k people, either there are $2m$ who can be divided into m pairs of people who know each other, or there are $2n$ who can be divided into n pairs of people who do not know each other.

16th APMO 2004

1. Find all non-empty finite sets S of positive integers such that if m,n ∈ S, then (m+n)/gcd(m,n) ∈ S.

2. ABC is an acute-angled triangle with circumcenter O and orthocenter H (and O ≠ H). Show that one of area AOH, area BOH, area COH is the sum of the other two.

3. 2004 points are in the plane, no three collinear. S is the set of lines through any two of the points. Show that the points can be colored with two colors so that any two of the points have the same color iff there are an odd number of lines in S which separate them (a line separates them if they are on opposite sides of it).

4. Show that $[(n-1)!/(n^2+n)]$ is even for any positive integer n.

5. Show that $(x^2 + 2)(y^2 + 2)(z^2 + 2) \geq 9(xy + yz + zx)$ for any positive reals x, y, z.

IMO (1959 – 2003)

1st IMO 1959

A1. Prove that (21n+4)/(14n+3) is irreducible for every natural number n.

A2. For what real values of x is $\sqrt{x + \sqrt{2x-1}} + \sqrt{x - \sqrt{2x-1}} = A$ given (a) $A = \sqrt{2}$, (b) A = 1, (c) A = 2, where only non-negative real numbers are allowed in square roots and the root always denotes the non-negative root?

A3. Let a, b, c be real numbers. Given the equation for cos x: $a \cos^2 x + b \cos x + c = 0$, form a quadratic equation in cos 2x whose roots are the same values of x. Compare the equations in cos x and cos 2x for a=4, b=2, c=-1.

B1. Given the length |AC|, construct a triangle ABC with $\angle ABC = 90^o$, and the median BM satisfying $BM^2 = AB \cdot BC$.

B2. An arbitrary point M is taken in the interior of the segment AB. Squares AMCD and MBEF are constructed on the same side of AB. The circles circumscribed about these squares, with centers P and Q, intersect at M and N.

(a) prove that AF and BC intersect at N;

(b) prove that the lines MN pass through a fixed point S (independent of M);

(c) find the locus of the midpoints of the segments PQ as M varies.

B3. The planes P and Q are not parallel. The point A lies in P but not Q, and the point C lies in Q but not P. Construct points B in P and D in Q such that the quadrilateral ABCD satisfies the following conditions: (1) it lies in a plane, (2) the vertices are in the order A, B, C, D, (3) it is an isosceles trapezoid with AB parallel to CD (meaning that AD = BC, but AD is not parallel to BC unless it is a square), and (4) a circle can be inscribed in ABCD touching the sides.

2nd IMO 1960

A1. Determine all 3 digit numbers N which are divisible by 11 and where N/11 is equal to the sum of the squares of the digits of N.

A2. For what real values of x does the following inequality hold: $4x^2/(1 - \sqrt{(1 + 2x)})^2 < 2x + 9$?

A3. In a given right triangle ABC, the hypoteneuse BC, length a, is divided into n equal parts with n an odd integer. The central part subtends an angle α at A. h is the perpendicular distance from A to BC. Prove that: $\tan \alpha = 4nh/(an^2 - a)$.

B1. Construct a triangle ABC given the lengths of the altitudes from A and B and the length of the median from A.

B2. The cube ABCDA'B'C'D' has A above A', B above B' and so on. X is any point of the face diagonal AC and Y is any point of B'D'.

(a) find the locus of the midpoint of XY;

(b) find the locus of the point Z which lies one-third of the way along XY, so that $ZY=2\cdot XZ$.

B3. A cone of revolution has an inscribed sphere tangent to the base of the cone (and to the sloping surface of the cone). A cylinder is circumscribed about the sphere so that its base lies in the base of the cone. The volume of the cone is V_1 and the volume of the cylinder is V_2.

(a) Prove that $V_1 \neq V_2$;

(b) Find the smallest possible value of V_1/V_2. For this case construct the half angle of the cone.

B4. In the isosceles trapezoid ABCD (AB parallel to DC, and BC = AD), let AB = a, CD = c and let the perpendicular distance from A to CD be h. Show how to construct all points X on the axis of symmetry such that $\angle BXC = \angle AXD = 90^o$. Find the distance of each such X from AB and from CD. What is the condition for such points to exist?

3rd IMO 1961

A1. Solve the following equations for x, y and z:

$x + y + z = a$;

$x^2 + y^2 + z^2 = b^2$;

$xy = z^2$

What conditions must a and b satisfy for x, y and z to be distinct positive numbers?

A2. Let a, b, c be the sides of a triangle and A its area. Prove that: $a^2 + b^2 + c^2 \geq 4\sqrt{3}$ A. When do we have equality?

A3. Solve the equation $\cos^n x - \sin^n x = 1$, where n is a natural number.

B1. P is inside the triangle ABC. PA intersects BC in D, PB intersects AC in E, and PC intersects AB in F. Prove that at least one of AP/PD, BP/PE, CP/PF does not exceed 2, and at least one is not less than 2.

B2. Construct the triangle ABC, given the lengths AC = b, AB =c and the acute $\angle AMB = \alpha$, where M is the midpoint of BC. Prove that the construction is possible if and only if $b \tan(\alpha/2) \leq c < b$. When does equality hold?

B3. Given 3 non-collinear points A, B, C and a plane p not parallel to ABC and such that A, B, C are all on the same side of p. Take three arbitrary points A', B', C' in p. Let A'', B'', C'' be the midpoints of AA', BB', CC' respectively, and let O be the centroid of A'', B'', C''. What is the locus of O as A', B', C' vary?

4th IMO 1962

A1. Find the smallest natural number with 6 as the last digit, such that if the final 6 is moved to the front of the number it is multiplied by 4.

A2. Find all real x satisfying: $\sqrt{3 - x} - \sqrt{x + 1} > 1/2$.

A3. The cube ABCDA'B'C'D' has upper face ABCD and lower face A'B'C'D' with A directly above A' and so on. The point x moves at constant speed along the perimeter of ABCD, and the point Y moves at the same speed along the perimeter of B'C'CB. X leaves A towards B at the same moment as Y leaves B' towards C'. What is the locus of the midpoint of XY?

B1. Find all real solutions to $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$.

B2. Given three distinct points A, B, C on a circle K, construct a point D on K, such that a circle can be inscribed in ABCD.

B3. The radius of the circumcircle of an isosceles triangle is R and the radius of its inscribed circle is r. Prove that the distance between the two centers is $\sqrt{R(R - 2r)}$.

B4. Prove that a regular tetrahedron has five distinct spheres each tangent to its six extended edges. Conversely, prove that if a tetrahedron has five such spheres then it is regular.

5th IMO 1963

A1. For which real values of p does the equation $\sqrt{(x^2 - p)} + 2\sqrt{(x^2 - 1)} = x$ have real roots? What are the roots?

A2. Given a point A and a segment BC, determine the locus of all points P in space for which $\angle APX = 90^o$ for some X on the segment BC.

A3. An n-gon has all angles equal and the lengths of consecutive sides satisfy $a_1 \geq a_2 \geq ... \geq a_n$. Prove that all the sides are equal.

B1. Find all solutions $x_1, ... , x_5$ to the five equations $x_i + x_{i+2} = y\, x_{i+1}$ for i = 1, ... , 5, where subscripts are reduced by 5 if necessary.

B2. Prove that $\cos \pi/7 - \cos 2\pi/7 + \cos 3\pi/7 = 1/2$.

B3. Five students A, B, C, D, E were placed 1 to 5 in a contest with no ties. One prediction was that the result would be the order A, B, C, D, E. But no student finished in the position predicted and no two students predicted to finish consecutively did so. For example, the outcome for C and D was not 1, 2 (respectively), or 2, 3, or 3, 4 or 4, 5. Another prediction was the order D, A, E, C, B. Exactly two students finished in the places predicted and two disjoint pairs predicted to finish consecutively did so. Determine the outcome.

6th IMO 1964

A1. (a) Find all natural numbers n for which 7 divides $2^n - 1$.

(b) Prove that there is no natural number n for which 7 divides $2^n + 1$.

A2. Suppose that a, b, c are the sides of a triangle. Prove that: $a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc$.

A3. Triangle ABC has sides a, b, c. Tangents to the inscribed circle are constructed parallel to the sides. Each tangent forms a triangle with the other two sides of the triangle and a circle is inscribed in each of these three triangles. Find the total area of all four inscribed circles.

B1. Each pair from 17 people exchange letters on one of three topics. Prove that there are at least 3 people who write to each other on the same topic. [In other words, if we color the edges of the complete graph K_{17} with three colors, then we can find a triangle all the same color.]

B2. 5 points in a plane are situated so that no two of the lines joining a pair of points are coincident, parallel or perpendicular. Through each point lines are drawn perpendicular to each of the lines through two of the other 4 points. Determine the maximum number of intersections these perpendiculars can have.

B3. ABCD is a tetrahedron and D_0 is the centroid of ABC. Lines parallel to DD_0 are drawn through A, B and C and meet the planes BCD, CAD and ABD in A_0, B_0, and C_0 respectively. Prove that the volume of ABCD is one-third of the volume of $A_0B_0C_0D_0$. Is the result true if D_0 is an arbitrary point inside ABC?

7th IMO 1965

A1. Find all x in the interval $[0, 2\pi]$ which satisfy: $2 \cos x \le |\sqrt{(1 + \sin 2x)} - \sqrt{(1 - \sin 2x)}| \le \sqrt{2}$.

A2. The coefficients a_{ij} of the following equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0$$

satisfy the following: (a) a_{11}, a_{22}, a_{33} are positive, (b) other a_{ij} are negative, (c) the sum of the coefficients in each equation is positive. Prove that the only solution is $x_1 = x_2 = x_3 = 0$.

A3. The tetrahedron ABCD is divided into two parts by a plane parallel to AB and CD. The distance of the plane from AB is k times its distance from CD. Find the ratio of the volumes of the two parts.

B1. Find all sets of four real numbers such that the sum of any one and the product of the other three is 2.

B2. The triangle OAB has $\angle O$ acute. M is an arbitrary point on AB. P and Q are the feet of the perpendiculars from M to OA and OB respectively. What is the locus of H, the orthocenter of the triangle OPQ (the point where its altitudes meet)? What is the locus if M is allowed to vary over the interior of OAB?

B3. Given $n > 2$ points in the plane, prove that at most n pairs of points are the maximum distance apart (of any two points in the set).

8th IMO 1966

A1. Problems A, B and C were posed in a mathematical contest. 25 competitors solved at least one of the three. Amongst those who did not solve A, twice as many solved B as C. The number solving only A was one more than the number solving A and at least one other. The number solving just A equalled the number solving just B plus the number solving just C. How many solved just B?

A2. Prove that if BC + AC = tan C/2 (BC tan A + AC tan B), then the triangle ABC is isosceles.

A3. Prove that a point in space has the smallest sum of the distances to the vertices of a regular tetrahedron iff it is the center of the tetrahedron.

B1. Prove that $1/\sin 2x + 1/\sin 4x + ... + 1/\sin 2^n x = \cot x - \cot 2^n x$ for any natural number n and any real x (with $\sin 2^n x$ non-zero).

B2. Solve the equations:
$|a_i - a_1|\, x_1 + |a_i - a_2|\, x_2 + |a_i - a_3|\, x_3 + |a_i - a_4|\, x_4 = 1$, $i = 1, 2, 3, 4$, where a_i are distinct reals.

B3. Take any points K, L, M on the sides BC, CA, AB of the triangle ABC. Prove that at least one of the triangles AML, BKM, CLK has area ≤ 1/4 area ABC.

9th IMO 1967

A1. The parallelogram ABCD has AB = a, AD = 1, angle BAD = A, and the triangle ABD has all angles acute. Prove that circles radius 1 and center A, B, C, D cover the parallelogram iff $a \le \cos A + \sqrt{3} \sin A$.

A2. Prove that a tetrahedron with just one edge length greater than 1 has volume at most 1/8.

A3. Let k, m, n be natural numbers such that m + k + 1 is a prime greater than n + 1. Let $c_s = s(s+1)$. Prove that: $(c_{m+1} - c_k)(c_{m+2} - c_k) \dots (c_{m+n} - c_k)$ is divisible by the product $c_1 c_2 \dots c_n$.

B1. $A_0B_0C_0$ and $A_1B_1C_1$ are acute-angled triangles. Construct the triangle ABC with the largest possible area which is circumscribed about $A_0B_0C_0$ (BC contains A_0, CA contains B_0, and AB contains C_0) and similar to $A_1B_1C_1$.

B2. $a_1, \dots , a_8$ are reals, not all zero. Let $c_n = a_1^n + a_2^n + \dots + a_8^n$ for n = 1, 2, 3, ... Given that an infinite number of c_n are zero, find all n for which c_n is zero.

B3. In a sports contest a total of m medals were awarded over n days. On the first day one medal and 1/7 of the remaining medals were awarded. On the second day two medals and 1/7 of the remaining medals were awarded, and so on. On the last day, the remaining n medals were awarded. How many medals were awarded, and over how many days?

10th IMO 1968

A1. Find all triangles whose side lengths are consecutive integers, and one of whose angles is twice another.

A2. Find all natural numbers n the product of whose decimal digits is $n^2 - 10n - 22$.

A3. a, b, c are real with a non-zero. $x_1, x_2, \ldots, x_n$ satisfy the n equations:

$$ax_i^2 + bx_i + c = x_{i+1}, \text{ for } 1 \le i < n$$

$$ax_n^2 + bx_n + c = x_1$$

Prove that the system has zero, 1 or >1 real solutions according as $(b - 1)^2 - 4ac$ is <0, =0 or >0.

B1. Prove that every tetrahedron has a vertex whose three edges have the right lengths to form a triangle.

B2. Let f be a real-valued function defined for all real numbers, such that for some a > 0 we have $f(x + a) = 1/2 + \sqrt{f(x) - f(x)^2}$ for all x. Prove that f is periodic, and give an example of such a non-constant f for a = 1.

B3. For every natural number n evaluate the sum $[(n+1)/2] + [(n+2)/4] + [(n+4)/8] + \ldots + [(n+2^k)/2^{k+1}] + \ldots$, where [x] denotes the greatest integer ≤ x.

11th IMO 1969

A1. Prove that there are infinitely many positive integers m, such that $n^4 + m$ is not prime for any positive integer n.

A2. Let $f(x) = \cos(a_1 + x) + 1/2 \cos(a_2 + x) + 1/4 \cos(a_3 + x) + ... + 1/2^{n-1} \cos(a_n + x)$, where a_i are real constants and x is a real variable. If $f(x_1) = f(x_2) = 0$, prove that $x_1 - x_2$ is a multiple of π.

A3. For each of k = 1, 2, 3, 4, 5 find necessary and sufficient conditions on a > 0 such that there exists a tetrahedron with k edges length a and the remainder length 1.

B1. C is a point on the semicircle diameter AB, between A and B. D is the foot of the perpendicular from C to AB. The circle K_1 is the in-circle of ABC, the circle K_2 touches CD, DA and the semicircle, the circle K_3 touches CD, DB and the semicircle. Prove that K_1, K_2 and K_3 have another common tangent apart from AB.

B2. Given n > 4 points in the plane, no three collinear. Prove that there are at least (n-3)(n-4)/2 convex quadrilaterals with vertices amongst the n points.

B3. Given real numbers $x_1, x_2, y_1, y_2, z_1, z_2$, satisfying $x_1 > 0$, $x_2 > 0$, $x_1y_1 > z_1^2$, and $x_2y_2 > z_2^2$, prove that: $8/((x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2) \le 1/(x_1y_1 - z_1^2) + 1/(x_2y_2 - z_2^2)$. Give necessary and sufficient conditions for equality.

12th IMO 1970

A1. M is any point on the side AB of the triangle ABC. r, r_1, r_2 are the radii of the circles inscribed in ABC, AMC, BMC. q is the radius of the circle on the opposite side of AB to C, touching the three sides of AB and the extensions of CA and CB. Similarly, q_1 and q_2. Prove that $r_1r_2q = rq_1q_2$.

A2. We have $0 \le x_i < b$ for i = 0, 1, ... , n and $x_n > 0$, $x_{n-1} > 0$. If a > b, and $x_nx_{n-1}...x_0$ represents the number A base a and B base b, whilst $x_{n-1}x_{n-2}...x_0$ represents the number A' base a and B' base b, prove that A'B < AB'.

A3. The real numbers a_0, a_1, a_2, ... satisfy $1 = a_0 \le a_1 \le a_2 \le ...$. b_1, b_2, b_3, ... are defined by $b_n = \sum_{1\le k\le n} (1 - a_{k-1}/a_k)/\sqrt{a_k}$.
(a) Prove that $0 \le b_n < 2$.
(b) Given c satisfying $0 \le c < 2$, prove that we can find a_n so that $b_n > c$ for all sufficiently large n.

B1. Find all positive integers n such that the set {n, n+1, n+2, n+3, n+4, n+5} can be partitioned into two subsets so that the product of the numbers in each subset is equal.

B2. In the tetrahedron ABCD, angle BDC = 90^o and the foot of the perpendicular from D to ABC is the intersection of the altitudes of ABC. Prove that:
$(AB + BC + CA)^2 \le 6(AD^2 + BD^2 + CD^2)$.
When do we have equality?

B3. Given 100 coplanar points, no 3 collinear, prove that at most 70% of the triangles formed by the points have all angles acute.

13th IMO 1971

A1. Let $E_n = (a_1 - a_2)(a_1 - a_3) \dots (a_1 - a_n) + (a_2 - a_1)(a_2 - a_3) \dots (a_2 - a_n) + \dots + (a_n - a_1)(a_n - a_2) \dots (a_n - a_{n-1})$. Let S_n be the proposition that $E_n \geq 0$ for all real a_i. Prove that S_n is true for n = 3 and 5, but for no other n > 2.

A2. Let P_1 be a convex polyhedron with vertices $A_1, A_2, \dots , A_9$. Let P_i be the polyhedron obtained from P_1 by a translation that moves A_1 to A_i. Prove that at least two of the polyhedra $P_1, P_2, \dots , P_9$ have an interior point in common.

A3. Prove that we can find an infinite set of positive integers of the form $2^n - 3$ (where n is a positive integer) every pair of which are relatively prime.

B1. All faces of the tetrahedron ABCD are acute-angled. Take a point X in the interior of the segment AB, and similarly Y in BC, Z in CD and T in AD.
(a) If angle DAB + angle BCD ≠ angle CDA + angle ABC, then prove that none of the closed paths XYZTX has minimal length;
(b) If angle DAB + angle BCD =angle CDA + angle ABC, then there are infinitely many shortest paths XYZTX, each with length 2 AC sin k, where 2k = angle BAC + angle CAD + angle DAB

B2. Prove that for every positive integer m we can find a finite set S of points in the plane, such that given any point A of S, there are exactly m points in S at unit distance from A.

B3. Let $A = (a_{ij})$, where i, j = 1, 2, ... , n, be a square matrix with all a_{ij} non-negative integers. For each i, j such that $a_{ij} = 0$, the sum of the elements in the ith row and the jth column is at least n. Prove that the sum of all the elements in the matrix is at least $n^2/2$.

14th IMO 1972

A1. Given any set of ten distinct numbers in the range 10, 11, ... , 99, prove that we can always find two disjoint subsets with the same sum.

A2. Given $n > 4$, prove that every cyclic quadrilateral can be dissected into n cyclic quadrilaterals.

A3. Prove that $(2m)!(2n)!$ is a multiple of $m!n!(m+n)!$ for any non-negative integers m and n.

B1. Find all positive real solutions to:

$(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) \le 0$

$(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) \le 0$

$(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) \le 0$

$(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) \le 0$

$(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) \le 0$

B2. f and g are real-valued functions defined on the real line. For all x and y, $f(x + y) + f(x - y) = 2f(x)g(y)$. f is not identically zero and $|f(x)| \le 1$ for all x. Prove that $|g(x)| \le 1$ for all x.

B3. Given four distinct parallel planes, prove that there exists a regular tetrahedron with a vertex on each plane.

15th IMO 1973

A1. $OP_1, OP_2, \ldots, OP_{2n+1}$ are unit vectors in a plane. $P_1, P_2, \ldots, P_{2n+1}$ all lie on the same side of a line through O. Prove that $|OP_1 + \ldots + OP_{2n+1}| \geq 1$.

A2. Can we find a finite set of non-coplanar points, such that given any two points, A and B, there are two others, C and D, with the lines AB and CD parallel and distinct?

A3. a and b are real numbers for which the equation $x^4 + ax^3 + bx^2 + ax + 1 = 0$ has at least one real solution. Find the least possible value of $a^2 + b^2$.

B1. A soldier needs to sweep a region with the shape of an equilateral triangle for mines. The detector has an effective radius equal to half the altitude of the triangle. He starts at a vertex of the triangle. What path should he follow in order to travel the least distance and still sweep the whole region?

B2. G is a set of non-constant functions f. Each f is defined on the real line and has the form $f(x) = ax + b$ for some real a, b. If f and g are in G, then so is fg, where fg is defined by $fg(x) = f(g(x))$. If f is in G, then so is the inverse f^{-1}. If $f(x) = ax + b$, then $f^{-1}(x) = x/a - b/a$. Every f in G has a fixed point (in other words we can find x_f such that $f(x_f) = x_f$. Prove that all the functions in G have a common fixed point.

B3. $a_1, a_2, \ldots, a_n$ are positive reals, and q satisfies $0 < q < 1$. Find $b_1, b_2, \ldots, b_n$ such that:
(a) $a_i < b_i$ for $i = 1, 2, \ldots, n$,
(b) $q < b_{i+1}/b_i < 1/q$ for $i = 1, 2, \ldots, n-1$,
(c) $b_1 + b_2 + \ldots + b_n < (a_1 + a_2 + \ldots + a_n)(1 + q)/(1 - q)$.

16th IMO 1974

A1. Three players play the following game. There are three cards each with a different positive integer. In each round the cards are randomly dealt to the players and each receives the number of counters on his card. After two or more rounds, one player has received 20, another 10 and the third 9 counters. In the last round the player with 10 received the largest number of counters. Who received the middle number on the first round?

A2. Prove that there is a point D on the side AB of the triangle ABC, such that CD is the geometric mean of AD and DB if and only if sin A sin B $\le \sin^2(C/2)$.

A3. Prove that the sum from k = 0 to n of (2n+1)C(2k+1) 2^{3k} is not divisible by 5 for any non-negative integer n. [rCs denotes the binomial coefficient r!/(s!(r-s)!) .]

B1. An 8 x 8 chessboard is divided into p disjoint rectangles (along the lines between squares), so that each rectangle has the same number of white squares as black squares, and each rectangle has a different number of squares. Find the maximum possible value of p and all possible sets of rectangle sizes.

B2. Determine all possible values of a/(a+b+d) + b/(a+b+c) + c/(b+c+d) + d/(a+c+d) for positive reals a, b, c, d.

B3. Let P(x) be a polynomial with integer coefficients of degree d > 0. Let n be the number of distinct integer roots to P(x) = 1 or -1. Prove that $n \le d + 2$.

17th IMO 1975

A1. Let $x_1 \geq x_2 \geq ... \geq x_n$, and $y_1 \geq y_2 \geq ... \geq y_n$ be real numbers. Prove that if z_i is any permutation of the y_i, then: $\sum_{1\leq i\leq n} (x_i - y_i)^2 \leq \sum_{1\leq i\leq n} (x_i - z_i)^2$.

A2. Let $a_1 < a_2 < a_3 < ...$ be positive integers. Prove that for every $i \geq 1$, there are infinitely many a_n that can be written in the form $a_n = ra_i + sa_j$, with r, s positive integers and $j > i$.

A3. Given any triangle ABC, construct external triangles ABR, BCP, CAQ on the sides, so that $\angle PBC = 45^o$, $\angle PCB = 30^o$, $\angle QAC = 45^o$, $\angle QCA = 30^o$, $\angle RAB = 15^o$, $\angle RBA = 15^o$. Prove that $\angle QRP = 90^o$ and QR = RP.

B1. Let A be the sum of the decimal digits of 4444^{4444}, and B be the sum of the decimal digits of A. Find the sum of the decimal digits of B.

B2. Find 1975 points on the circumference of a unit circle such that the distance between each pair is rational, or prove it impossible.

B3. Find all polynomials P(x, y) in two variables such that:
(1) $P(tx, ty) = t^nP(x, y)$ for some positive integer n and all real t, x, y;
(2) for all real x, y, z: $P(y + z, x) + P(z + x, y) + P(x + y, z) = 0$;
(3) $P(1, 0) = 1$.

18th IMO 1976

A1. A plane convex quadrilateral has area 32, and the sum of two opposite sides and a diagonal is 16. Determine all possible lengths for the other diagonal.

A2. Let $P_1(x) = x^2 - 2$, and $P_{i+1} = P_1(P_i(x))$ for i = 1, 2, 3, ... Show that the roots of $P_n(x) = x$ are real and distinct for all n.

A3. A rectangular box can be completely filled with unit cubes. If one places as many cubes as possible, each with volume 2, in the box, with their edges parallel to the edges of the box, one can fill exactly 40% of the box. Determine the possible dimensions of the box.

B1. Determine the largest number which is the product of positive integers with sum 1976.

B2. n is a positive integer and m = 2n. $a_{ij} = 0$, 1 or -1 for $1 \le i \le n$, $1 \le j \le m$. The m unknowns $x_1, x_2, \ldots, x_m$ satisfy the n equations: $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{im}x_m = 0$, for i = 1, 2, ... , n. Prove that the system has a solution in integers of absolute value at most m, not all zero.

B3. The sequence $u_0, u_1, u_2, \ldots$ is defined by: $u_0 = 2$, $u_1 = 5/2$, $u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$ for n = 1, 2, ... Prove that $[u_n] = 2^{(2n - (-1)n)/3}$, where [x] denotes the greatest integer less than or equal to x.

19th IMO 1977

A1. Construct equilateral triangles ABK, BCL, CDM, DAN on the inside of the square ABCD. Show that the midpoints of KL, LM, MN, NK and the midpoints of AK, BK, BL, CL, CM, DM, DN, AN form a regular dodecahedron.

A2. In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

A3. Given an integer $n > 2$, let V_n be the set of integers $1 + kn$ for k a positive integer. A number m in V_n is called indecomposable if it cannot be expressed as the product of two members of V_n. Prove that there is a number in V_n which can be expressed as the product of indecomposable members of V_n in more than one way (decompositions which differ solely in the order of factors are not regarded as different).

B1. Define $f(x) = 1 - a \cos x - b \sin x - A \cos 2x - B \sin 2x$, where a, b, A, B are real constants. Suppose that $f(x) \geq 0$ for all real x. Prove that $a^2 + b^2 \leq 2$ and $A^2 + B^2 \leq 1$.

B2. Let a and b be positive integers. When $a^2 + b^2$ is divided by $a + b$, the quotient is q and the remainder is r. Find all pairs a, b such that $q^2 + r = 1977$.

B3. The function f is defined on the set of positive integers and its values are positive integers. Given that $f(n+1) > f(f(n))$ for all n, prove that $f(n) = n$ for all n.

20th IMO 1978

A1. m and n are positive integers with $m < n$. The last three decimal digits of 1978^m are the same as the last three decimal digits of 1978^n. Find m and n such that $m + n$ has the least possible value.

A2. P is a point inside a sphere. Three mutually perpendicular rays from P intersect the sphere at points U, V and W. Q denotes the vertex diagonally opposite P in the parallelepiped determined by PU, PV, PW. Find the locus of Q for all possible sets of such rays from P.

A3. The set of all positive integers is the union of two disjoint subsets {f(1), f(2), f(3), ... }, {g(1), g(2), g(3), ... }, where $f(1) < f(2) < f(3) < \ldots$, and $g(1) < g(2) < g(3) < \ldots$, and $g(n) = f(f(n)) + 1$ for n = 1, 2, 3, ... Determine f(240).

B1. In the triangle ABC, AB = AC. A circle is tangent internally to the circumcircle of the triangle and also to AB, AC at P, Q respectively. Prove that the midpoint of PQ is the center of the incircle of the triangle.

B2. $\{a_k\}$ is a sequence of distinct positive integers. Prove that for all positive integers n, $\sum_{1\le k\le n} a_k/k^2 \ge \sum_{1\le k\le n} 1/k$.

B3. An international society has its members from six different countries. The list of members has 1978 names, numbered 1, 2, ... , 1978. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice the number of a member from his own country.

21st IMO 1979

A1. Let m and n be positive integers such that: m/n = 1 - 1/2 + 1/3 - 1/4 + ... - 1/1318 + 1/1319. Prove that m is divisible by 1979.

A2. A prism with pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$ as the top and bottom faces is given. Each side of the two pentagons and each of the 25 segments A_iB_j is colored red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been colored has two sides of a different color. Prove that all 10 sides of the top and bottom faces have the same color.

A3. Two circles in a plane intersect. A is one of the points of intersection. Starting simultaneously from A two points move with constant speed, each traveling along its own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that the two points are always equidistant from P.

B1. Given a plane k, a point P in the plane and a point Q not in the plane, find all points R in k such that the ratio (QP + PR)/QR is a maximum.

B2. Find all real numbers a for which there exist non-negative real numbers x_1, x_2, x_3, x_4, x_5 satisfying:

$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = a$,

$x_1 + 2^3x_2 + 3^3x_3 + 4^3x_4 + 5^3x_5 = a^2$,

$x_1 + 2^5x_2 + 3^5x_3 + 4^5x_4 + 5^5x_5 = a^3$.

B3. Let A and E be opposite vertices of an octagon. A frog starts at vertex A. From any vertex except E it jumps to one of the two adjacent vertices. When it reaches E it stops. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that: $a_{2n-1} = 0$, $a_{2n} = (2 + \sqrt{2})^{n-1}/\sqrt{2} - (2 - \sqrt{2})^{n-1}/\sqrt{2}$.

22nd IMO 1981

A1. P is a point inside the triangle ABC. D, E, F are the feet of the perpendiculars from P to the lines BC, CA, AB respectively. Find all P which minimise: BC/PD + CA/PE + AB/PF.

A2. Take r such that $1 \le r \le n$, and consider all subsets of r elements of the set {1, 2, ... , n}. Each subset has a smallest element. Let F(n,r) be the arithmetic mean of these smallest elements. Prove that: F(n,r) = (n+1)/(r+1).

A3. Determine the maximum value of $m^2 + n^2$, where m and n are integers in the range 1, 2, ... , 1981 satisfying $(n^2 - mn - m^2)^2 = 1$.

B1. (a) For which n > 2 is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining n - 1 numbers?
(b) For which n > 2 is there exactly one set having this property?

B2. Three circles of equal radius have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle are collinear with the point O.

B3. The function f(x,y) satisfies: f(0,y) = y + 1, f(x+1,0) = f(x,1), f(x+1,y+1) = f(x,f(x+1,y)) for all non-negative integers x, y. Find f(4, 1981).

23rd IMO 1982

A1. The function f(n) is defined on the positive integers and takes non-negative integer values. f(2) = 0, f(3) > 0, f(9999) = 3333 and for all m, n: f(m+n) - f(m) - f(n) = 0 or 1. Determine f(1982).

A2. A non-isosceles triangle $A_1A_2A_3$ has sides a_1, a_2, a_3 with a_i opposite A_i. M_i is the midpoint of side a_i and T_i is the point where the incircle touches side a_i. Denote by S_i the reflection of T_i in the interior bisector of angle A_i. Prove that the lines M_1S_1, M_2S_2 and M_3S_3 are concurrent.

A3. Consider infinite sequences $\{x_n\}$ of positive reals such that $x_0 = 1$ and $x_0 \geq x_1 \geq x_2 \geq \ldots$.
(a) Prove that for every such sequence there is an $n \geq 1$ such that: $x_0^2/x_1 + x_1^2/x_2 + \ldots + x_{n-1}^2/x_n \geq 3.999$.
(b) Find such a sequence for which: $x_0^2/x_1 + x_1^2/x_2 + \ldots + x_{n-1}^2/x_n < 4$ for all n.

B1. Prove that if n is a positive integer such that the equation: $x^3 - 3xy^2 + y^3 = n$ has a solution in integers x, y, then it has at least three such solutions. Show that the equation has no solutions in integers for n = 2891.

B2. The diagonals AC and CE of the regular hexagon ABCDEF are divided by inner points M and N respectively, so that: AM/AC = CN/CE = r. Determine r if B, M and N are collinear.

B3. Let S be a square with sides length 100. Let L be a path within S which does not meet itself and which is composed of line segments A_0A_1, A_1A_2, A_2A_3, ... , $A_{n-1}A_n$ with $A_0 = A_n$. Suppose that for every point P on the boundary of S there is a point of L at a distance from P no greater than 1/2. Prove that there are two points X and Y of L such that the distance between X and Y is not greater than 1 and the length of the part of L which lies between X and Y is not smaller than 198.

24th IMO 1983

A1. Find all functions f defined on the set of positive reals which take positive real values and satisfy:
$f(x(f(y))) = yf(x)$ for all x, y; and $f(x) \to 0$ as $x \to \infty$.

A2. Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centers O_1 and O_2 respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1 and M_2 the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.

A3. Let a , b and c be positive integers, no two of which have a common divisor greater than 1. Show that $2abc - ab - bc - ca$ is the largest integer which cannot be expressed in the form $xbc + yca + zab$, where x, y, z are non-negative integers.

B1. Let ABC be an equilateral triangle and E the set of all points contained in the three segments AB, BC and CA (including A, B and C). Determine whether, for every partition of E into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle.

B2. Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression?

B3. Let a, b and c be the lengths of the sides of a triangle. Prove that:$a^2b(a - b) + b^2c(b - c) + c^2a(c - a) \geq 0$.

25th IMO 1984

A1. Prove that $0 \le yz + zx + xy - 2xyz \le 7/27$, where x, y and z are non-negative real numbers satisfying $x + y + z = 1$.

A2. Find one pair of positive integers a, b such that ab(a+b) is not divisible by 7, but $(a+b)^7 - a^7 - b^7$ is divisible by 7^7.

A3. Given points O and A in the plane. Every point in the plane is colored with one of a finite number of colors. Given a point X in the plane, the circle C(X) has center O and radius OX + $\angle$AOX/OX, where $\angle$AOX is measured in radians in the range $[0, 2\pi)$. Prove that we can find a point X, not on OA, such that its color appears on the circumference of the circle C(X).

B1. Let ABCD be a convex quadrilateral with the line CD tangent to the circle on diameter AB. Prove that the line AB is tangent to the circle on diameter CD if and only if BC and AD are parallel.

B2. Let d be the sum of the lengths of all the diagonals of a plane convex polygon with $n > 3$ vertices. Let p be its perimeter. Prove that: $n - 3 < 2d/p < [n/2]\,[(n+1)/2] - 2$, where [x] denotes the greatest integer not exceeding x.

B3. Let a, b, c, d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a + d = 2^k$ and $b + c = 2^m$ for some integers k and m, then $a = 1$.

26th IMO 1985

A1. A circle has center on the side AB of the cyclic quadrilateral ABCD. The other three sides are tangent to the circle. Prove that AD + BC = AB.

A2. Let n and k be relatively prime positive integers with $k < n$. Each number in the set M = {1, 2, 3, ... , n-1} is colored either blue or white. For each i in M, both i and n-i have the same color. For each i in M not equal to k, both i and |i-k| have the same color. Prove that all numbers in M must have the same color.

A3. For any polynomial $P(x) = a_0 + a_1x + ... + a_kx^k$ with integer coefficients, the number of odd coefficients is denoted by o(P). For i = 0, 1, 2, ... let $Q_i(x) = (1 + x)^i$. Prove that if $i_1, i_2, ... , i_n$ are integers satisfying $0 \le i_1 < i_2 < ... < i_n$, then: $o(Q_{i1} + Q_{i2} + ... + Q_{in}) \ge o(Q_{i1})$.

B1. Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 23, prove that M contains a subset of 4 elements whose product is the 4th power of an integer.

B2. A circle center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. The circumcircles of ABC and KBN intersect at exactly two distinct points B and M. Prove that $\angle$OMB is a right angle.

B3. For every real number x_1, construct the sequence $x_1, x_2, ...$ by setting: $x_{n+1} = x_n(x_n + 1/n)$. Prove that there exists exactly one value of x_1 which gives $0 < x_n < x_{n+1} < 1$ for all n.

27th IMO 1986

A1. Let d be any positive integer not equal to 2, 5 or 13. Show that one can find distinct a, b in the set {2, 5, 13, d} such that ab - 1 is not a perfect square.

A2. Given a point P_0 in the plane of the triangle $A_1A_2A_3$. Define $A_s = A_{s-3}$ for all s >= 4. Construct a set of points P_1, P_2, P_3, ... such that P_{k+1} is the image of P_k under a rotation center A_{k+1} through an angle 120^o clockwise for k = 0, 1, 2, ... Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ is equilateral.

A3. To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and y < 0, then the following operation is allowed: x, y, z are replaced by x + y, -y, z + y respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

B1. Let A, B be adjacent vertices of a regular n-gon (n ≥ 5) with center O. A triangle XYZ, which is congruent to and initially coincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, with X remaining inside the polygon. Find the locus of X.

B2. Find all functions f defined on the non-negative reals and taking non-negative real values such that: f(2) = 0, f(x) ≠ 0 for 0 ≤ x < 2, and f(xf(y)) f(y) = f(x + y) for all x, y.

B3. Given a finite set of points in the plane, each with integer coordinates, is it always possible to color the points red or white so that for any straight line L parallel to one of the coordinate axes the difference (in absolute value) between the numbers of white and red points on L is not greater than 1?

28th IMO 1987

A1. Let $p_n(k)$ be the number of permutations of the set {1, 2, 3, ... , n} which have exactly k fixed points. Prove that the sum from k = 0 to n of (k $p_n(k)$) is n!. [A permutation f of a set S is a one-to-one mapping of S onto itself. An element i of S is called a fixed point if f(i) = i.]

A2. In an acute-angled triangle ABC the interior bisector of angle A meets BC at L and meets the circumcircle of ABC again at N. From L perpendiculars are drawn to AB and AC, with feet K and M respectively. Prove that the quadrilateral AKNM and the triangle ABC have equal areas.

A3. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$. Prove that for every integer $k \geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all zero, such that $|a_i| \leq k - 1$ for all i, and $|a_1x_1 + a_2x_2 + \ldots + a_nx_n| \leq (k - 1)\sqrt{n}/(k^n - 1)$.

B1. Prove that there is no function f from the set of non-negative integers into itself such that f(f(n)) = n + 1987 for all n.

B2. Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of 3 points determines a non-degenerate triangle with rational area.

B3. Let n be an integer greater than or equal to 2. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \sqrt{(n/3)}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n-2$.

29th IMO 1988

A1. Consider two coplanar circles of radii $R > r$ with the same center. Let P be a fixed point on the smaller circle and B a variable point on the larger circle. The line BP meets the larger circle again at C. The perpendicular to BP at P meets the smaller circle again at A (if it is tangent to the circle at P, then A = P).

(i) Find the set of values of $AB^2 + BC^2 + CA^2$.

(ii) Find the locus of the midpoint of BC.

A2. Let n be a positive integer and let $A_1, A_2, \ldots, A_{2n+1}$ be subsets of a set B. Suppose that:

(i) Each A_i has exactly 2n elements,

(ii) The intersection of every two distinct A_i contains exactly one element, and

(iii) Every element of B belongs to at least two of the A_i.

For which values of n can one assign to every element of B one of the numbers 0 and 1 in such a way that each A_i has 0 assigned to exactly n of its elements?

A3. A function f is defined on the positive integers by: $f(1) = 1$; $f(3) = 3$; $f(2n) = f(n)$, $f(4n + 1) = 2f(2n + 1) - f(n)$, and $f(4n + 3) = 3f(2n + 1) - 2f(n)$ for all positive integers n. Determine the number of positive integers n less than or equal to 1988 for which $f(n) = n$.

B1. Show that the set of real numbers x which satisfy the inequality: $1/(x - 1) + 2/(x - 2) + 3/(x - 3) + \ldots + 70/(x - 70) \geq 5/4$ is a union of disjoint intervals, the sum of whose lengths is 1988.

B2. ABC is a triangle, right-angled at A, and D is the foot of the altitude from A. The straight line joining the incenters of the triangles ABD and ACD intersects the sides AB, AC at K, L respectively. Show that the area of the triangle ABC is at least twice the area of the triangle AKL.

B3. Let a and b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that $(a^2 + b^2)/(ab + 1)$ is a perfect square.

30th IMO 1989

A1. Prove that the set {1, 2, ... , 1989} can be expressed as the disjoint union of subsets A_1, A_2, ... , A_{117} in such a way that each A_i contains 17 elements and the sum of the elements in each A_i is the same.

A2. In an acute-angled triangle ABC, the internal bisector of angle A meets the circumcircle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that the area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$ and at least four times the area of the triangle ABC.

A3. Let n and k be positive integers, and let S be a set of n points in the plane such that no three points of S are collinear, and for any point P of S there are at least k points of S equidistant from P. Prove that $k < 1/2 + \sqrt{(2n)}$.

B1. Let ABCD be a convex quadrilateral such that the sides AB, AD, BC satisfy AB = AD + BC. There exists a point P inside the quadrilateral at a distance h from the line CD such that AP = h + AD and BP = h + BC. Show that: $1/\sqrt{h} \ge 1/\sqrt{AD} + 1/\sqrt{BC}$.

B2. Prove that for each positive integer n there exist n consecutive positive integers none of which is a prime or a prime power.

B3. A permutation $\{x_1, x_2, ... , x_m\}$ of the set {1, 2, ... , 2n} where n is a positive integer is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in {1, 2, ... , 2n-1}. Show that for each n there are more permutations with property P than without.

31st IMO 1990

A1. Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent at E to the circle through D, E and M intersects the lines BC and AC at F and G respectively. Find EF/EG in terms of t = AM/AB.

A2. Take n ≥ 3 and consider a set E of 2n-1 distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is "good" if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

A3. Determine all integers greater than 1 such that (2n + 1)/n2 is an integer.

B1. Construct a function from the set of positive rational numbers into itself such that f(x f(y)) = f(x)/y for all x, y.

B2. Given an initial integer n0 > 1, two players A and B choose integers n1, n2, n3, ... alternately according to the following rules: Knowing n2k, A chooses any integer n2k+1 such that n2k ≤ n2k+1 ≤ n2k2. Knowing n2k+1, B chooses any integer n2k+2 such that n2k+1/n2k+2 = pr for some prime p and integer r ≥ 1.

Player A wins the game by choosing the number 1990; player B wins by choosing the number 1. For which n0 does

(a) A have a winning strategy?

(b) B have a winning strategy?

(c) Neither player have a winning strategy?

B3. Prove that there exists a convex 1990-gon such that all its angles are equal and the lengths of the sides are the numbers 12, 22, ... , 19902 in some order.

32nd IMO 1991

A1. Given a triangle ABC, let I be the incenter. The internal bisectors of angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that: $1/4 < AI\cdot BI\cdot CI/(AA'\cdot BB'\cdot CC') \le 8/27$.

A2. Let $n > 6$ be an integer and let $a_1, a_2, \ldots, a_k$ be all the positive integers less than n and relatively prime to n. If $a_2 - a_1 = a_3 - a_2 = \ldots = a_k - a_{k-1} > 0$, prove that n must be either a prime number or a power of 2.

A3. Let S = {1, 2, 3, ... 280}. Find the smallest integer n such that each n-element subset of S contains five numbers which are pairwise relatively prime.

B1. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges 1, 2, ... , k in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is 1.
[A graph is a set of points, called vertices, together with a set of edges joining certain pairs of distinct vertices. Each pair of edges belongs to at most one edge. The graph is connected if for each pair of distinct vertices x, y there is some sequence of vertices $x = v_0, v_1, \ldots, v_m = y$, such that each pair v_i, v_{i+1} $(0 \le i < m)$ is joined by an edge.]

B2. Let ABC be a triangle and X an interior point of ABC. Show that at least one of the angles XAB, XBC, XCA is less than or equal to 30^o.

B3. Given any real number $a > 1$ construct a bounded infinite sequence $x_0, x_1, x_2, \ldots$ such that $|x_i - x_j|\,|i - j|^a \ge 1$ for every pair of distinct i, j. [An infinite sequence $x_0, x_1, x_2, \ldots$ of real numbers is bounded if there is a constant C such that $|x_i| < C$ for all i.]

33rd IMO 1992

A1. Find all integers a, b, c satisfying $1 < a < b < c$ such that $(a - 1)(b - 1)(c - 1)$ is a divisor of $abc - 1$.

A2. Find all functions f defined on the set of all real numbers with real values, such that $f(x^2 + f(y)) = y + f(x)^2$ for all x, y.

A3. Consider 9 points in space, no 4 coplanar. Each pair of points is joined by a line segment which is colored either blue or red or left uncolored. Find the smallest value of n such that whenever exactly n edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.

B1. L is a tangent to the circle C and M is a point on L. Find the locus of all points P such that there exist points Q and R on L equidistant from M with C the incircle of the triangle PQR.

B2. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane respectively. Prove that: $|S|^2 \le |S_x|\ |S_y|\ |S_z|$, where |A| denotes the number of points in the set A. [The orthogonal projection of a point onto a plane is the foot of the perpendicular from the point to the plane.]

B3. For each positive integer n, S(n) is defined as the greatest integer such that for every positive integer $k \le S(n)$, n^2 can be written as the sum of k positive squares.

(a) Prove that $S(n) \le n^2 - 14$ for each $n \ge 4$.

(b) Find an integer n such that $S(n) = n^2 - 14$.

(c) Prove that there are infinitely many integers n such that $S(n) = n^2 - 14$.

34th IMO 1993

A1. Let $f(x) = x^n + 5x^{n-1} + 3$, where $n > 1$ is an integer. Prove that f(x) cannot be expressed as the product of two non-constant polynomials with integer coefficients.

A2. Let D be a point inside the acute-angled triangle ABC such that $\angle ADB = \angle ACB + 90^o$, and AC·BD = AD·BC.

(a) Calculate the ratio AB·CD/(AC·BD).

(b) Prove that the tangents at C to the circumcircles of ACD and BCD are perpendicular.

A3. On an infinite chessboard a game is played as follows. At the start n^2 pieces are arranged in an n x n block of adjoining squares, one piece on each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is removed. Find those values of n for which the game can end with only one piece remaining on the board.

B1. For three points P, Q, R in the plane define m(PQR) as the minimum length of the three altitudes of the triangle PQR (or zero if the points are collinear). Prove that for any points A, B, C, X:

$m(ABC) \le m(ABX) + m(AXC) + m(XBC)$.

B2. Does there exist a function f from the positive integers to the positive integers such that f(1) = 2, f(f(n)) = f(n) + n for all n, and f(n) < f(n+1) for all n?

B3. There are n > 1 lamps $L_0, L_1, \dots, L_{n-1}$ in a circle. We use L_{n+k} to mean L_k. A lamp is at all times either on or off. Initially they are all on. Perform steps $s_0, s_1, \dots$ as follows: at step s_i, if L_{i-1} is lit, then switch L_i from on to off or vice versa, otherwise do nothing. Show that:

(a) There is a positive integer M(n) such that after M(n) steps all the lamps are on again;

(b) If $n = 2^k$, then we can take $M(n) = n^2 - 1$.

(c) If $n = 2^k + 1$, then we can take $M(n) = n^2 - n + 1$.

35th IMO 1994

A1. Let m and n be positive integers. Let $a_1, a_2, \dots, a_m$ be distinct elements of {1, 2, ... , n} such that whenever $a_i + a_j \le n$ for some i, j (possibly the same) we have $a_i + a_j = a_k$ for some k. Prove that:

$(a_1 + \dots + a_m)/m \ge (n + 1)/2$.

A2. ABC is an isosceles triangle with AB = AC. M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB. Q is an arbitrary point on BC different from B and C. E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear. Prove that OQ is perpendicular to EF if and only if QE = QF.

A3. For any positive integer k, let f(k) be the number of elements in the set {k+1, k+2, ... , 2k} which have exactly three 1s when written in base 2. Prove that for each positive integer m, there is at least one k with f(k) = m, and determine all m for which there is exactly one k.

B1. Determine all ordered pairs (m, n) of positive integers for which $(n^3 + 1)/(mn - 1)$ is an integer.

B2. Let S be the set of all real numbers greater than -1. Find all functions f from S into S such that $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x and y, and f(x)/x is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$.

B3. Show that there exists a set A of positive integers with the following property: for any infinite set S of primes, there exist two positive integers m in A and n not in A, each of which is a product of k distinct elements of S for some $k \ge 2$.

36th IMO 1995

A1. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameter AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

A2. Let a, b, c be positive real numbers with abc = 1. Prove that: $1/(a^3(b + c)) + 1/(b^3(c + a)) + 1/(c^3(a + b)) \geq 3/2$.

A3. Determine all integers n > 3 for which there exist n points $A_1, \dots , A_n$ in the plane, no three collinear, and real numbers $r_1, \dots , r_n$ such that for any distinct i, j, k, the area of the triangle $A_iA_jA_k$ is $r_i + r_j + r_k$.

B1. Find the maximum value of x_0 for which there exists a sequence $x_0, x_1, \dots , x_{1995}$ of positive reals with $x_0 = x_{1995}$ such that for i = 1, ... , 1995: $x_{i-1} + 2/x_{i-1} = 2x_i + 1/x_i$.

B2. Let ABCDEF be a convex hexagon with AB = BC = CD and DE = EF = FA, such that $\angle BCD = \angle EFA = 60^o$. Suppose that G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120^o$. Prove that AG + GB + GH + DH + HE ≥ CF.

B3. Let p be an odd prime number. How many p-element subsets A of {1, 2, ... , 2p} are there, the sum of whose elements is divisible by p?

37th IMO 1996

A1. We are given a positive integer r and a rectangular board divided into 20 x 12 unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading between two adjacent corners of the board which lie on the long side.

(a) Show that the task cannot be done if r is divisible by 2 or 3.

(b) Prove that the task is possible for r = 73.

(c) Can the task be done for r = 97?

A2. Let P be a point inside the triangle ABC such that $\angle APB - \angle ACB = \angle APC - \angle ABC$. Let D, E be the incenters of triangles APB, APC respectively. Show that AP, BD, CE meet at a point.

A3. Let S be the set of non-negative integers. Find all functions f: S→S such that f(m + f(n)) = f(f(m)) + f(n) for all m, n.

B1. The positive integers a, b are such that 15a + 16b and 16a - 15b are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

B2. Let ABCDEF be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to FA. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF respectively, and let p denote the perimeter of the hexagon. Prove that:

$$R_A + R_C + R_E \geq p/2.$$

B3. Let p, q, n be three positive integers with p + q < n. Let $x_0, x_1, \ldots, x_n$ be integers such that $x_0 = x_n = 0$, and for each $1 \leq i \leq n$, $x_i - x_{i-1} = p$ or $-q$. Show that there exist indices i < j with (i, j) not (0, n) such that $x_i = x_j$.

38th IMO 1997

A1. In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white as on a chessboard. For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along the edges of the squares. Let S_1 be the total area of the black part of the triangle, and S_2 be the total area of the white part. Let $f(m, n) = |S_1 - S_2|$.
(a) Calculate f(m, n) for all positive integers m and n which are either both even or both odd.
(b) Prove that $f(m, n) \le \max(m, n)/2$ for all m, n.
(c) Show that there is no constant C such that f(m, n) < C for all m, n.

A2. The angle at A is the smallest angle in the triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T. Show that AU = TB + TC.

A3. Let $x_1, x_2, \dots, x_n$ be real numbers satisfying $|x_1 + x_2 + \dots + x_n| = 1$ and $|x_i| \le (n+1)/2$ for all i. Show that there exists a permutation y_i of x_i such that $|y_1 + 2y_2 + \dots + ny_n| \le (n+1)/2$.

B1. An n x n matrix whose entries come from the set S = {1, 2, ... , 2n-1} is called a silver matrix if, for each i = 1, 2, ... , n, the ith row and the ith column together contain all elements of S. Show that:

(d) there is no silver matrix for n = 1997;

(e) silver matrices exist for infinitely many values of n.

B2. Find all pairs (a, b) of positive integers that satisfy $a^{b2} = b^a$.

B3. For each positive integer n, let f(n) denote the number of ways of representing n as a sum of powers of 2 with non-negative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For example, f(4) = 4, because 4 can be represented as 4, 2 + 2, 2 + 1 + 1 or 1 + 1 + 1 + 1. Prove that for any integer $n \ge 3$, $2^{n2/4} < f(2^n) < 2^{n2/2}$.

39th IMO 1998

A1. In the convex quadrilateral ABCD, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. The point P, where the perpendicular bisectors of AB and DC meet, is inside ABCD. Prove that ABCD is cyclic if and only if the triangles ABP and CDP have equal areas.

A2. In a competition there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that for any two judges their ratings coincide for at most k contestants. Prove $k/a \geq (b-1)/2b$.

A3. For any positive integer n, let d(n) denote the number of positive divisors of n (including 1 and n). Determine all positive integers k such that $d(n^2) = k\, d(n)$ for some n.

B1. Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

B2. Let I be the incenter of the triangle ABC. Let the incircle of ABC touch the sides BC, CA, AB at K, L, M respectively. The line through B parallel to MK meets the lines LM and LK at R and S respectively. Prove that the angle RIS is acute.

B3. Consider all functions f from the set of all positive integers into itself satisfying $f(t^2f(s)) = s\, f(t)^2$ for all s and t. Determine the least possible value of f(1998).

40th IMO 1999

A1. Find all finite sets S of at least three points in the plane such that for all distinct points A, B in S, the perpendicular bisector of AB is an axis of symmetry for S.

A2. Let $n \geq 2$ be a fixed integer. Find the smallest constant C such that for all non-negative reals $x_1, \dots, x_n$:

$$\sum_{i<j} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum x_i \right)^4.$$

Determine when equality occurs.

A3. Given an n x n square board, with n even. Two distinct squares of the board are said to be adjacent if they share a common side, but a square is not adjacent to itself. Find the minimum number of squares that can be marked so that every square (marked or not) is adjacent to at least one marked square.

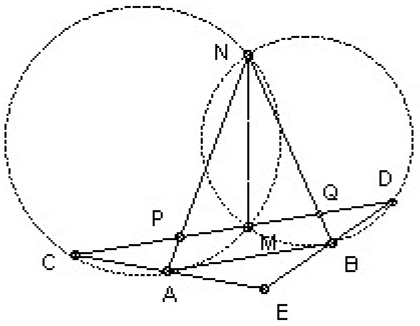
B1. Find all pairs (n, p) of positive integers, such that: p is prime; $n \leq 2p$; and $(p - 1)^n + 1$ is divisible by n^{p-1}.

B2. The circles C_1 and C_2 lie inside the circle C, and are tangent to it at M and N, respectively. C_1 passes through the center of C_2. The common chord of C_1 and C_2, when extended, meets C at A and B. The lines MA and MB meet C_1 again at E and F. Prove that the line EF is tangent to C_2.

B3. Determine all functions f: R→R such that $f(x - f(y)) = f(f(y)) + x f(y) + f(x) - 1$ for all x, y in R. [R is the reals.]

41st IMO 2000

A1. AB is tangent to the circles CAMN and NMBD. M lies between C and D on the line CD, and CD is parallel to AB. The chords NA and CM meet at P; the chords NB and MD meet at Q. The rays CA and DB meet at E. Prove that PE = QE.



A2. A, B, C are positive reals with product 1. Prove that (A - 1 + 1/B)(B - 1 + 1/C)(C - 1 + 1/A) $\le$ 1.

A3. k is a positive real. N is an integer greater than 1. N points are placed on a line, not all coincident. A *move* is carried out as follows. Pick any two points A and B which are not coincident. Suppose that A lies to the right of B. Replace B by another point B' to the right of A such that AB' = k BA. For what values of k can we move the points arbitarily far to the right by repeated moves?

B1. 100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?

B2. Can we find N divisible by just 2000 different primes, so that N divides $2^N + 1$? [N may be divisible by a prime power.]

B3. $A_1A_2A_3$ is an acute-angled triangle. The foot of the altitude from A_i is K_i and the incircle touches the side opposite A_i at L_i. The line K_1K_2 is reflected in the line L_1L_2. Similarly, the line K_2K_3 is reflected in L_2L_3 and K_3K_1 is reflected in L_3L_1. Show that the three new lines form a triangle with vertices on the incircle.

42nd IMO 2001

A1. ABC is acute-angled. O is its circumcenter. X is the foot of the perpendicular from A to BC. Angle C ≥ angle B + 30^o. Prove that angle A + angle COX < 90^o.

A2. a, b, c are positive reals. Let $a' = \sqrt{(a^2 + 8bc)}$, $b' = \sqrt{(b^2 + 8ca)}$, $c' = \sqrt{(c^2 + 8ab)}$. Prove that $a/a' + b/b' + c/c' \geq 1$.

A3. Integers are placed in each of the 441 cells of a 21 x 21 array. Each row and each column has at most 6 different integers in it. Prove that some integer is in at least 3 rows and at least 3 columns.

B1. Let $n_1, n_2, \ldots, n_m$ be integers where m is odd. Let $x = (x_1, \ldots, x_m)$ denote a permutation of the integers 1, 2, ... , m. Let $f(x) = x_1n_1 + x_2n_2 + \ldots + x_mn_m$. Show that for some distinct permutations a, b the difference f(a) - f(b) is a multiple of m!.

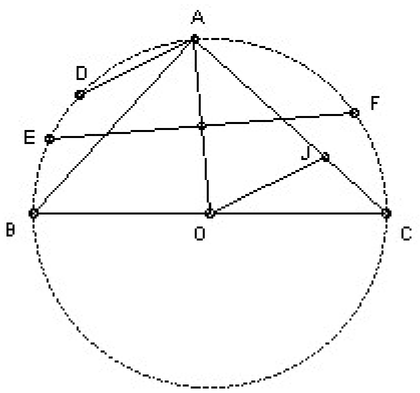
B2. ABC is a triangle. X lies on BC and AX bisects angle A. Y lies on CA and BY bisects angle B. Angle A is 60^o. AB + BX = AY + YB. Find all possible values for angle B.

B3. K > L > M > N are positive integers such that KM + LN = (K + L - M + N)(-K + L + M + N). Prove that KL + MN is composite.

43rd IMO 2002

A1. S is the set of all (h, k) with h, k non-negative integers such that $h + k < n$. Each element of S is colored red or blue, so that if (h, k) is red and $h' \le h$, $k' \le k$, then (h', k') is also red. A type 1 subset of S has n blue elements with different first member and a type 2 subset of S has n blue elements with different second member. Show that there are the same number of type 1 and type 2 subsets.

A2. BC is a diameter of a circle center O. A is any point on the circle with angle AOC > 60^o. EF is the chord which is the perpendicular bisector of AO. D is the midpoint of the minor arc AB. The line through O parallel to AD meets AC at J. Show that J is the incenter of triangle CEF.



A3. Find all pairs of integers $m > 2$, $n > 2$ such that there are infinitely many positive integers k for which $(k^n + k^2 - 1)$ divides $(k^m + k - 1)$.

B1. The positive divisors of the integer $n > 1$ are $d_1 < d_2 < ... < d_k$, so that $d_1 = 1$, $d_k = n$. Let $d = d_1d_2 + d_2d_3 + ... + d_{k-1}d_k$. Show that $d < n^2$ and find all n for which d divides n^2.

B2. Find all real-valued functions f on the reals such that $(f(x) + f(y)) (f(u) + f(v)) = f(xu - yv) + f(xv + yu)$ for all x, y, u, v.

B3. $n > 2$ circles of radius 1 are drawn in the plane so that no line meets more than two of the circles. Their centers are $O_1, O_2, ... , O_n$. Show that $\sum_{i<j} 1/O_iO_j \le (n-1)\pi/4$.

44th IMO 2003

A1. S is the set {1, 2, 3, ... , 1000000}. Show that for any subset A of S with 101 elements we can find 100 distinct elements x_i of S, such that the sets x_i + A are all pairwise disjoint. [Note that x_i + A is the set {a + x_i | a is in A}].

A2. Find all pairs (m, n) of positive integers such that $m^2/(2mn^2 - n^3 + 1)$ is a positive integer.

A3. A convex hexagon has the property that for any pair of opposite sides the distance between their midpoints is $(\sqrt{3})/2$ times the sum of their lengths. Show that all the hexagon's angles are equal.

B1. ABCD is cyclic. The feet of the perpendicular from D to the lines AB, BC, CA are P, Q, R respectively. Show that the angle bisectors of ABC and CDA meet on the line AC iff RP = RQ.

B2. Given n > 2 and reals $x_1 \le x_2 \le ... \le x_n$, show that $(\sum_{i,j} |x_i - x_j|)^2 \le (2/3) (n^2 - 1) \sum_{i,j} (x_i - x_j)^2$. Show that we have equality iff the sequence is an arithmetic progression.

B3. Show that for each prime p, there exists a prime q such that $n^p - p$ is not divisible by q for any positive integer n.

Junior Balkan
(1997 – 2003)

1st Junior Balkan 1997

1. Show that given any 9 points inside a square side 1 we can always find three which form a triangle with area < 1/8.

2. Given reals x, y with $(x^2 + y^2)/(x^2 - y^2) + (x^2 - y^2)/(x^2 + y^2) = k$, find $(x^8 + y^8)/(x^8 - y^8) + (x^8 - y^8)/(x^8 + y^8)$ in terms of k.

3. I is the incenter of ABC. N, M are the midpoints of sides AB, CA. The lines BI, CI meet MN at K, L respectively. Prove that AI + BI + CI > BC + KL.

4. A triangle has circumradius R and sides a, b, c with $R(b+c) = a\sqrt{(bc)}$. Find its angles.

5. $n_1, n_2, \dots, n_{1998}$ are positive integers such that $n_1^2 + n_2^2 + \dots + n_{1997}^2 = n_{1998}^2$. Show that at least two of the numbers are even.

2nd Junior Balkan 1998

1. The number 11...122...25 has 1997 1s and 1998 2s. Show that it is a perfect square.

2. The convex pentagon ABCDE has AB = AE = CD = 1 and angle ABC = angle DEA = 90^o and BC + DE = 1. Find its area.

3. Find all positive integers m, n such that $m^n = n^{m-n}$.

4. Do there exist 16 three digit numbers, using only three different digits in all, so that the numbers are all different mod 16?

3rd Junior Balkan 1999

1. a, b, c are distinct reals and there are reals x, y such that $a^3 + ax + y = 0$, $b^3 + bx + y = 0$ and $c^3 + cx + y = 0$. Show that $a + b + c = 0$.

2. Let $a_n = 2^{3n} + 3^{6n+2} + 5^{6n+2}$. Find $\gcd(a_0, a_1, a_2, \ldots, a_{1999})$.

3. A square has side 20. S is a set of 1999 points inside the square and the 4 vertices. Show that we can find three points in S which form a triangle with area ≤ 1/10.

4. The triangle ABC has AB = AC. D is a point on the side BC. BB' is a diameter of the circumcircle of ABD, and CC' is a diameter of the circumcircle of ACD. M is the midpoint of B'C'. Show that the area of BCM is independent of the position of D.

4th Junior Balkan 2000

1. x and y are positive reals such that $x^3 + y^3 + (x + y)^3 + 30xy = 2000$. Show that $x + y = 10$.

2. Find all positive integers n such that $n^3 + 3^3$ is a perfect square.

3. ABC is a triangle. E, F are points on the side BC such that the semicircle diameter EF touches AB at Q and AC at P. Show that the intersection of EP and FQ lies on the altitude from A.

4. n girls and 2n boys played a tennis tournament. Every player played every other player. The boys won 7/5 times as many matches as the girls (and there were no draws). Find n.

5th Junior Balkan 2001

1. Find all positive integers a, b, c such that $a^3 + b^3 + c^3 = 2001$.

2. ABC is a triangle with $\angle C = 90^o$ and CA ≠ CB. CH is an altitude and CL is an angle bisector. Show that for X ≠ C on the line CL, we have $\angle XAC \neq \angle XBC$. Show that for Y ≠ C on the line CH we have $\angle YAC \neq \angle YBC$.

3. ABC is an equilateral triangle. D, E are points on the sides AB, AC respectively. The angle bisector of $\angle ADE$ meets AE at F, and the angle bisector of $\angle AED$ meets AD at G. Show that area DEF + area DEG ≤ area ABC. When do we have equality?

4. N is a convex polygon with 1415 vertices and perimeter 2001. Prove that we can find three vertices of N which form a triangle of area < 1.

6th Junior Balkan 2002

1. The triangle ABC has CA = CB. P is a point on the circumcircle between A and B (and on the opposite side of the line AB to C). D is the foot of the perpendicular from C to PB. Show that PA + PB = 2·PD.

2. The circles center O_1 and O_2 meet at A and B with the centers on opposite sides of AB. The lines BO_1 and BO_2 meet their respective circles again at B_1 and B_2. M is the midpoint of B_1B_2. M_1, M_2 are points on the circles center O_1 and O_2 respectively such that angle AO_1M_1 = angle AO_2M_2, and B_1 lies on the minor arc AM_1 and B lies on the minor arc AM_2. Show that angle MM_1B = angle MM_2B.

3. Find all positive integers which have exactly 16 positive divisors $1 = d_1 < d_2 < ... < d_{16}$ such that the divisor d_k, where $k = d_5$, equals $(d_2 + d_4) d_6$.

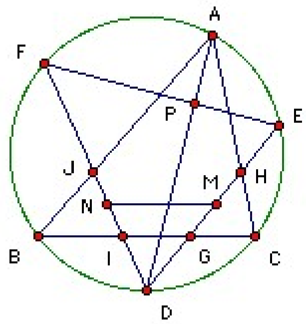
4. Show that $2/(b(a+b)) + 2/(c(b+c)) + 2/(a(c+a)) \geq 27/(a+b+c)^2$ for positive reals a, b, c.

7th Junior Balkan 2003

1. Let A = 44...4 (2n digits) and B = 88...8 (n digits). Show that A + 2B + 4 is a square.

2. $A_1, A_2, \dots, A_n$ are points in the plane, so that if we take the points in any order $B_1, B_2, \dots, B_n$, then the broken line $B_1B_2\dots B_n$ does not intersect itself. What is the largest possible value of n?

3. ABC is a triangle. D is the midpoint of the arc BC not containing A. Similarly E, F. DE meets BC at G and AC at H. M is the midpoint of GH. DF meets BC at I and AB at J, and N is the midpoint of IJ. Find the angles of DMN in terms of the angles of ABC. AD meets EF at P. Show that the circumcenter of DMN lies on the circumcircle of PMN.



4. Show that $(1+x^2)/(1+y+z^2) + (1+y^2)/(1+z+x^2) + (1+z^2)/(1+x+y^2) \geq 2$ for reals x, y, z > -1.

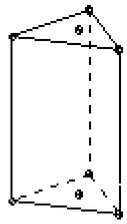
IMO SHORTLIST
(1959 – 2002)

1st - 9th IMO 1959-67 [selected] shortlist problems 1/3

1. Given $n > 3$ points in the plane, no 3 collinear, show that there is a circle through 3 of the points such that none of the points lies inside the circle.

2. Given n positive reals $a_1, a_2, \ldots, a_n$, with product 1, show that $(1 + a_1)(1 + a_2) \ldots (1 + a_n) \geq 2^n$.

3. A right prism with equilateral triangles side k as base and top and height h has small holes in the center of the base and in the center of the top. The inside of the three vertical walls has a mirror surface. Light enters through the small hole in the top and strikes each vertical wall once before leaving through the hole in the bottom. Find the angle at which it enters and the length of each part of the path.



4. Given any 5 points in the plane, no 3 collinear, show that 4 of them form a convex quadrilateral.

5. Show that $\tan\left((\pi \sin x)/(4 \sin y) \right) + \tan\left((\pi \cos x)/(4 \cos y) \right) > 1$ for $0 \leq x \leq \pi/2$ and $\pi/6 < y < \pi/3$.
Alternative version

(a) Show that for some x the expression does not exceed 2.

(b) Prove that $1/z^4 + z^2/1000 \leq 2/100$ and show that for some z, we have $1/z^4 + z^2/1000 \leq 2/100$.

6. Let P be a convex plane polygon with perimeter L and area A. Let S be the set of all points in space which lie a distance R or less from a point of P. Show that the volume of S is $4/3\ \pi R^3 + 1/2\ LR^2 + 2\ AR$
Alternative version

(a) Calculate the area of the figure formed in the plane by the points within a distance R of a polygon P, which is not necessarily convex.

(b) Calculate the volume of the figure formed by points within a distance R from a convex polyhedron, such as a cube or triangular pyramid.

7. Find all the ways in which two infinite circular cylinders can be arranged so that their intersection lies in a plane.
Alternative version

(a) Same but for intersection of cylinder and sphere.

(b) When does the intersection of two cylinders lie in two planes?

8. Given a two-pan balance, a single 1 kg weight, and a large quantity of sugar, show how to obtain 1000 kg of sugar in the smallest possible number of weighings? (You are allowed to put the weight in either pan (or neither), to pour sugar from one pan to the other, and to add sugar to either pan.)

9. Find x such that sin 3x cos(60^o - 4x) + 1 = 0 and sin(60^o - 7x) - cos(30^o + x) + m is non-zero, where m is a fixed real number.

10. How many real solutions are there to the equation x = 1964 sin x - 189?

11. Does there exist an integer k which can be expressed as the sum of two factorials k = m! + n! (with $m \le n$) in two different ways?

12. Find digits a, b, c such that if A is the number made up of 2n digits a, B is the number made up of n digits b, and C is the number made up of n digits c, we have √(A - B) = C for at least two different values of n. Find all n for which the equation holds.

13. Let $x_1, x_2, \ldots, x_n$ be positive reals. Show that ½ n(n-1) $\sum_{i<j} 1/(x_ix_j) \ge 4 (\sum_{i<j} 1/(x_i + x_j))^2$ and find the conditions on x_i for equality.

14. What is the largest number of regions into which we can divide a disk by placing n points on its circumference and joining all the points?

15. The points C, D lie on the circle diameter AB and do not form a diameter. The lines AC and BD meet at L, and the tangents at C and D meet at N. Show that the line LN is perpendicular to AB.

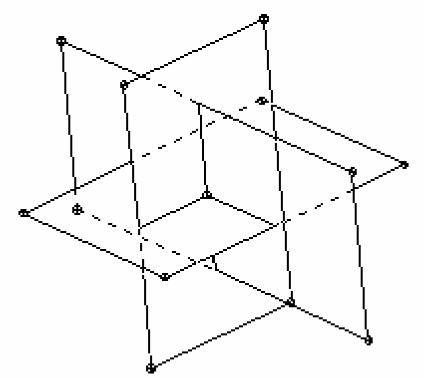
16. C is a circle center O and radius 1. S is a square (in the same plane) with center X and side 2. The variable point X lies inside or on C, and the variable point Y lies inside or on S. Z is situated so that $\angle ZXY = \angle ZYX = 45^o$. Find the locus of Z.

17. ABCD and A'B'C'D' are two parallelograms arbitrarily arranged in space. The points M, N, P, Q lie on the segments AA', BB', CC', DD' respectively, so that the ratios AM/MA', BN/NB', CP/PC', DQ/QD' are all equal. Show that MNPQ is a parallelogram and find the locus of its center as M varies along the segment AA'.

18. Solve the equation 1/sin x + 1/cos x = 1/k, where k is a real parameter. Find for which values of k the equations has a solution, and the number of solutions.

19. Construct a triangle given the three exradii.

20. Three equal rectangles with the same center are mutually perpendicular. The long sides are also mutually perpendicular. P is the polyhedron whose vertices are the vertices of the rectangles. Find the volume of P and find the condition for P to be regular.



22nd IMO 1981 shortlisted problems

1. A 3 x 3 cube is assembled from 27 white unit cubes. The large cube is then painted black on the outside and then disassembled. If it is reassembled at random, what is the probability that the large cube is still completely black on the outside?

2. Let Fn be the Fibonacci sequence defined by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Find all pairs (a, b) of real numbers such that for each n, $aF_n + bF_{n+1}$ is a member of the sequence. Find all pairs (u, v) of real numbers such that for each n, $uF_n^2 + vF_{n+1}^2$ is a member of the sequence.

4. A real sequence u_n is defined by u_1 and $4u_{n+1} = (64u_n + 15)^{1/3}$. Describe the behavior of the sequence as $n \to \infty$.

5. a+b+c+d+e+f+g = 1 and a, b, c, d, e, f, g are non-negative. Find the minimum value of max(a+b+c, b+c+d, c+d+e, d+e+f, e+f+g).

6. p(k) is a polynomial of degree n such that p(k) = (n+1-k)!k!/(n+1)! for k = 0, 1, ... , n. Find p(n+1).

7. AB, BC, CD, DE are consecutive chords on a semicircle of unit radius with lengths a, b, c, d. Prove that $a^2 + b^2 + c^2 + d^2 + abc + bcd < 4$.

8. A convex pentagon ABCDE has equal sides and $\angle A \geq \angle B \geq \angle C \geq \angle D \geq \angle E$. Prove that it is regular.

9. Find the smallest positive integer n such that for every integer $m \geq n$, it is possible to partition a given square into m squares, not necessarily of the same size.

10. A finite set of unit circular disks is given in a plane, such that the area of their union is S. Prove that there exists a subset of mutually disjoint disks whose union has area > 2S/9.

11. Several equal spherical planets are in outer space. On the surface of each planet is a set of points which is invisible from any of the remaining planets. Prove that the sum of the areas of all these sets equals the surface area of one planet.

12. A sphere S is tangent to the edges AB, BC, CD, DA of a tetrahedron ABCD at the points E, F, G, H respectively. If EFGH is a square, prove that the sphere is tangent to the edge AC iff it is tangent to the edge BD.

23rd IMO 1982 shortlisted problems

1. An urn contains w white balls and b black balls. In each move, two balls are drawn at random and removed from the urn, and one ball is added. If the two balls drawn have the same color, then a black ball is added. If they are opposite colors, then a white ball is added. Eventually, only one ball is left. What is the probability that it is white?

2. p(x) is a cubic polynomial with integer coefficients and leading coefficient 1. One of its roots is the product of the other two. Prove that 2p(-1) is a multiple of p(1) + p(-1) - 2(1 + p(0)).

3. Let $a_1, a_2, \ldots, a_n$ be a permutation of 1, 2, ... , n. Find the permutation which maximises $a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n + a_na_1$ and the permutation which minimises it.

4. Let M be the set of real numbers of the form $(m+n)/\sqrt{(m^2+n^2)}$, where m and n are positive integers. Show that if x < y are two elements of M, then there is an element z of M such that x < z < y.

5. Prove that $(1 - s^a)/(1 - s) \le (1 + s)^a/(1 + s)$ for every positive real $s \ne 1$ and every positive rational $a \le 1$.

6. Find all real numbers a such that the equation $16x^4 - ax^3 + (2a + 17)x^2 - ax + 16 = 0$ has exactly four distinct real roots which form a geometric progression.

7. P is a point inside the triangle ABC such that ∠PAC = ∠PBC. The perpendiculars from P meet the lines BC, CA at L, M respectively. Prove that DL = DM where D is the midpoint of AB.

8. The triangles ABC and AB'C' have opposite orientation. ∠BCA = ∠B'C'A = 90^o. BC' and B'C intersect at M. Prove that if the lines AM and CC' are well defined, then they are perpendicular.

9. ABCD is a convex quadrilateral. A_1, B_1, C_1, D_1 are the circumcenters of BCD, CDA, DAB, ABC respectively. Prove that if two of A_1, B_1, C_1, D_1 coincide, then they all coincide. Prove that if they are distinct, then $A_1B_1C_1D_1$ is convex. In this case let A_2, B_2, C_2, D_2 be the circumcenters of $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, $A_1B_1C_1$ respectively. Show that $A_2B_22C_2D_2$ is similar to ABCD.

10. ABCD is a convex quadrilateral. ABM and CDP are equilateral triangles on the outside of the sides AB, CD. BCN and DAQ are equilateral triangles on the inside of sides BC and DA. Prove that MN = AC. What can be said about MNPQ?

11. A convex figure lies inside a circle. The points on the boundary of the figure are considered to be in the figure. For each point on the circle, draw the smallest angle with this point as the vertex which contains the figure. If the angle is always a right angle, prove that the center of the circle is a center of symmetry of the figure.

12. Exactly one quarter of the area of a convex polygon in the coordinate plane lies in each quadrant. If (0,0) is the only lattice point in or on the polygon, prove that its area is less than 4.

13. S is a unit sphere with center at the origin. For any point P on S, the unit sphere with center P intersects the x-axis at O and X, the y-axis at O and Y and the z-axis at O and Z (where X, Y or Z may coincide with O). What is the locus of the centroid of XYZ as P varies over the sphere.

24th IMO 1983 shortlisted problems

1. 1983 cities are served by ten airlines. All services are both ways. There is a direct service between any two cities. Show that at least one of the airlines can offer a round trip with an odd number of landings.

2. Let s(n) be the sum of the positive divisors of n. For example, s(6) = 1 + 2 + 3 + 6 = 12. Show that there are infinitely many n such that s(n)/n > s(m)/m for all m < n.

4. ABC is a triangle with AB not equal to AC. P is taken on the opposite side of AB to C such that PA = PB. Q is taken on the opposite side of AC to B such that QA = QC and $\angle Q = \angle P$. R is taken on the *same* side of BC as A such that RB = RC and $\angle R = \angle P$. Show that APRQ is a parallelogram.

5. Let S_n be the set of all strictly decreasing sequences of n positive integers such that no term divides any other term. Given two sequences A = (a_i) and B = (b_j) we say that A < B if for some k, $a_k < b_k$ and $a_i = b_i$ for i < k. For example, (7, 5, 3) < (9, 7, 2) < (9, 8, 7). Find the sequence A in S_n such that A < B for any other B in S_n.

6. The n positive integers $a_1, a_2, \ldots, a_n$ have sum 2n+2. Show that we can find an integer r such that:
$a_{r+1} \le 3$
$a_{r+1} + a_{r+2} \le 5$
...
$a_{r+1} + a_{r+2} + \ldots + a_n + a_1 + \ldots + a_{r-1} \le 2n-1$
Note that a_r does not appear in the inequalities. Show that there are either one or two r for which the inequalities hold, and that if there is only one, then all the inequalities are strict.
For example, suppose n = 3, $a_1 = 1$, $a_2 = 6$, $a_3 = 1$. Then r = 1 and r = 3 do not work, but r = 2 does work: $a_3 = 1 < 3$, $a_3 + a_1 = 2 < 5$. So r is unique and the inequalities are strict.

7. Let N be a positive integer. Define the sequence $a_0, a_1, a_2, \ldots$ by $a_0 = 0$, $a_{n+1} = N(a_n + 1) + (N + 1)a_n + 2(N(N+1)a_n(a_n+1))^{1/2}$. Show that all terms are positive integers.

8. 3n students are sitting in 3 rows of n. The students leave one at a time. All leaving orders are equally likely. Find the probability that there are never two rows where the number of students remaining differs by 2 or more.

9. Let m be any integer and n any positive integer. Show that there is a polynomial p(x) with integral coefficients such that $| p(x) - m/n | < 1/n^2$ for all x in some interval of length 1/n.

10. c is a positive real constant and b = (1 + c)/(2 + c). f is a real-valued function defined on the interval [0, 1] such that f(2x) = f(x)/b for $0 \le x \le 1/2$ and f(x) = b - (1 - b) f(2x - 1) for $1/2 \le x \le 1$. Show that 0 < f(x) - x < c for all 0 < x < 1.

12. Let S be the set of lattice points (a, b, c) in space such that $0 \le a, b, c \le 1982$. Find the number of ways of coloring each point red or blue so that the number of red vertices of each rectangular parallelepiped (with sides parallel to the axes) is 0, 4 or 8.

14. Does there exist a set S of positive integers such that given any integer n > 1 we can find a, b in S such that a + b = n, and if a, b, c, d are all in S and all < 10 and a + b = c + d, then a = c or d.

15. Let S be the set of polynomials $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0$ with non-negative real coefficients such that $a_0 = a_n \le a_1 = a_{n-1} \le a_2 = a_{n-2} \le \ldots$. For example, $x^3 + 2.1 x^2 + 2.1 x + 1$ or $0.1 x^2 + 15 x + 0.1$. Show that the product of any two members of S belongs to S.

16. Given n distinct points in the plane, let D be the greatest distance between two points and d the shortest distance between two points. Show that $D \ge d \sqrt{3} (\sqrt{n} - 1)/2$.

18. Let $F_1 = 1$, $F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ be the Fibonacci sequence. Let p(x) be a polynomial of degree 990 such that $p(k) = F_k$ for k = 992, 993, ... , 1982. Show that $p(1983) = F_{1983} - 1$.

19. k is a positive real. Solve the equations:

$x_1 | x_1 | = x_2 | x_2| + (x_1 - k) | x_1 - k |$

$x_2 | x_2 | = x_3 | x_3| + (x_2 - k) | x_2 - k |$

...

$x_n | x_n | = x_1 | x_1| + (x_n - k) | x_n - k |$.

20. Find the greatest integer not exceeding $1 + 1/2^k + 1/3^k + ... + 1/N^k$, where k = 1982/1983 and $N = 2^{1983}$.

21. Let n be a positive integer which is not a prime power. Show that there is a permutation $a_1, a_2, ... , a_n$ of 1, 2, ... , n such that $\cos(2\pi a_1/n) + 2 \cos(2\pi a_2/n) + 3 \cos(2\pi a_3/n) + ... + n \cos(2\pi a_n/n) = 0$.

24. d_n is the last non-zero digit of n! . Show that there is no k such that $d_n = d_{n+k}$ for all sufficiently large n.

Note: problems 3, 11, 13, 17, 22 and 23 were used in the Olympiad and are not shown here.

25th IMO 1984 shortlisted problems

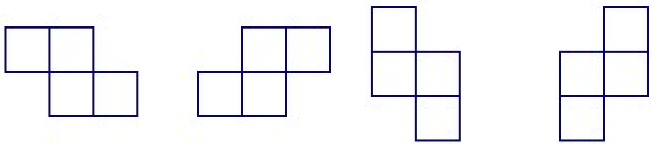
1. Given real A, find all solutions $(x_1, x_2, \dots, x_n)$ to the n equations (i = 1, 2, ... , n): $x_i |x_i| - (x_i - A)|(x_i - A)| = x_{i+1} |x_{i+1}|$, where we take x_{n+1} to mean x_1. *(France 1)*

2. Prove that there are infinitely many triples of positive integers (m, n, p) satisfying $4mn - m - n = p^2 - 1$, but none satisfying $4mn - m - n = p^2$. *(Canada 2)*

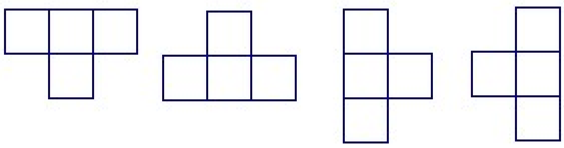
3. Find all positive integers n such that $n = d_6^2 + d_7^2 - 1$, where $1 = d_1 < d_2 < \dots < d_k = n$ are all the positive divisors of n. *(USSR 3)*

6. c is a positive integer. The sequence $f_1, f_2, f_3, \dots$ is defined by $f_1 = 1$, $f_2 = c$, $f_{n+1} = 2 f_n - f_{n-1} + 2$. Show that for each k there is an r such that $f_k f_{k+1} = f_r$. *(Canada 3)*

7. Can we number the squares of an 8 x 8 board with the numbers 1, 2, ... , 64 so that any four squares with any of the following shapes



have sum = 0 mod 4? Can we do it for the following shapes?



(German Federal Republic 5)

9. Let a, b, c be positive reals such that $\sqrt{a} + \sqrt{b} + \sqrt{c} = \sqrt{3}/2$. Show that the equations:

$\sqrt{y - a} + \sqrt{z - a} = 1$

$\sqrt{z - b} + \sqrt{x - b} = 1$

$\sqrt{x - c} + \sqrt{y - c} = 1$

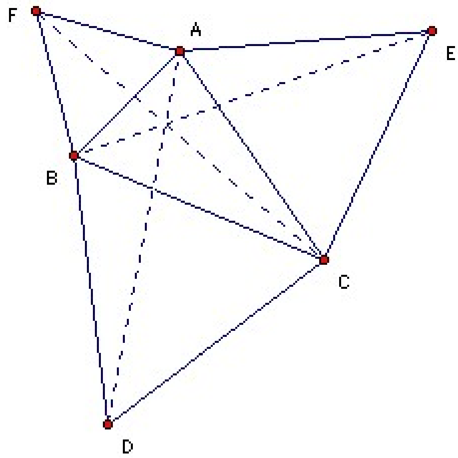
have exactly one solution in reals x, y, z. *(Poland 2)*

10. Prove that the product of five consecutive positive integers cannot be the square of an integer. *(Great Britain 1)*

11. $a_1, a_2, \dots, a_{2n}$ are distinct integers. Find all integers x which satisfy $(x - a_1)(x - a_2) \dots (x - a_{2n}) = (-1)^n(n!)^2$. *(Canada 1)*

13. A tetrahedron is inscribed in a straight circular cylinder of volume 1. Show that its volume cannot exceed $2/(3\pi)$. *(Bulgaria 5)*

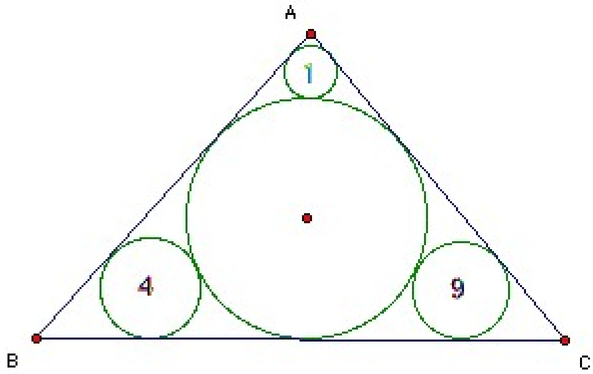
15. The angles of the triangle ABC are all $< 120^o$. Equilateral triangles are constructed on the outside of each side as shown. Show that the three lines AD, BE, CF are concurrent. Suppose they meet at S. Show that SD + SE + SF = 2(SA + SB + SC).



(Luxembourg 2)

17. If $(x_1, x_2, ... , x_n)$ is a permutation of (1, 2, ... , n) we call the pair (x_i, x_j) discordant if $i < j$ and $x_i > x_j$. Let d(n, k) be the number of permutations of (1, 2, ... , n) with just k discordant pairs. Find d(n, 2) and d(n, 3). *(German Federal Republic 3)*

18. ABC is a triangle. A circles with the radii shown are drawn inside the triangle each touching two sides and the incircle. Find the radius of the incircle.



(USA 5)

19. The harmonic table is a triangular array:

1
1/2 1/2

1/3 1/6 1/3

1/4 1/12 1/12 1/4

...
where $a_{n,1} = 1/n$ and $a_{n,k+1} = a_{n-1,k} - a_{n,k}$. Find the harmonic mean of the 1985th row. *(Canada 5)*

20. Find all pairs of positive reals (a, b) with a not 1 such that $\log_a b < \log_{a+1}(b+1)$. *(USA 2)*

Note: This list does not incude the problems used in the Olympiad (4, 5, 8, 12, 14, 16 on the shortlist, which were 5, 1, 3, 2, 4, 6 in the Olympiad).

26th IMO 1985 shortlisted problems

1. Show that if n is a positive integer and a and b are integers, then n! divides a(a + b)(a + 2b) ... (a + (n-1)b) b^{n-1}.

2. A convex quadrilateral ABCD is inscribed in a circle radius 1. Show that 0 < |AB + BC + CD + DA - AC - BD| < 2.

3. Given n > 1, find the maximum value of $\sin^2 x_1 + \sin^2 x_2 + ... + \sin^2 x_n$, where x_i are non-negative and have sum π.

4. Show that $x_1^2/(x_1^2 + x_2x_3) + x_2^2/(x_2^2 + x_3x_4) + ... + x_{n-1}^2/(x_{n-1}^2 + x_nx_1) + x_n^2/(x_n^2 + x_1x_2) \le n-1$ for all positive reals x_i.

5. T is the set of all lattice points in space. Two lattice points are *neighbors* if they have two coordinates the same and the third differs by 1. Show that there is a subset S of T such that if a lattice point x belongs to S then none of its neighbors belong to S, and if x does not belong to S, then exactly one of its neighbors belongs to S.

6. Let A be a set of positive integers such that |m - n| ≥ mn/25 for any m, n in A. Show that A cannot have more than 9 elements. Give an example of such a set with 9 elements.

7. Do there exist 100 distinct lines in the plane having just 1985 distinct points of intersection?

8. Find 8 positive integers $n_1, n_2, ... , n_8$ such that we can express every integer n with |n| < 1986 as $a_1n_1 + ... + a_8n_8$ with each $a_i = 0, \pm 1$.

9. The points A, B, C are not collinear. There are three ellipses, each pair of which intersects. One has foci A and B, the second has foci B and C and the third has foci C and A. Show that the common chords of each pair intersect.

10. The polynomials $p_0(x, y, z)$, $p_1(x, y, z)$, $p_2(x, y, z)$, ... are defined by $p_0(x, y, z) = 1$ and $p_{n+1}(x, y, z) = (x + z)(y + z)\, p_n(x, y, z+1) - z^2p_n(x, y, z)$. Show that each polynomial is symmetric in x, y, z.

11. Show that if there are $a_i = \pm 1$ such that $a_1a_2a_3a_4 + a_2a_3a_4a_5 + ... + a_na_1a_2a_3 = 0$, then n is divisible by 4.

12. Given 1985 points inside a unit cube, show that we can always choose 32 such that any polygon with these points as vertices has perimeter less than 8√3.

13. A die is tossed repeatedly. A wins if it is 1 or 2 on two consecutive tosses. B wins if it is 3 - 6 on two consecutive tosses. Find the probability of each player winning if the die is tossed at most 5 times. Find the probability of each player winning if the die is tossed until a player wins.

14. At time t = 0 a point starts to move clockwise around a regular n-gon from each vertex. Each of the n points moves at constant speed. At time T all the points reach vertex A simultaneously. Show that they will never all be simultaneously at any other vertex. Can they be together again at vertex A?

15. On each edge of a regular tetrahedron of side 1 there is a sphere with that edge as diameter. S be the intersection of the spheres (so it is all points whose distance from the midpoint of every edge is at most 1/2). Show that the distance between any two points of S is at most 1/√6.

16. Let $x_2 = 2^{1/2}$, $x_3 = (2 + 3^{1/3})^{1/2}$, $x_4 = (2 + (3 + 4^{1/4})^{1/3})^{1/2}$, ... , $x_n = (2 + (3 + ... + n^{1/n} ...)^{1/3})^{1/2}$ (where the positive root is taken in every case). Show that $x_{n+1} - x_n < 1/n!$.

17. p is a prime. For which k can the set {1, 2, ... , k} be partitioned into p subsets such that each subset has the same sum?

18. a, b, c, ... , k are positive integers such that a divides $2^b - 1$, b divides $2^c - 1$, ... , k divides $2^a - 1$. Show that a = b = c = ... = k = 1.

19. Show that the sequence $[n \sqrt{2}]$ for n = 1, 2, 3, ... contains infinitely many powers of 2.

20. Two equilateral triangles are inscribed in a circle radius r. Show that the area common to both triangles is at least $r^2(\sqrt{3})/2$.

21. Show that if the real numbers x, y, z satisfy $1/(yz - x^2) + 1/(zx - y^2) + 1/(xy - z^2) = 0$, then $x/(yz - x^2)^2 + y/(zx - y^2)^2 + z/(xy - z^2)^2 = 0$.

22. Show how to construct the triangle ABC given the distance between the circumcenter O and the orthocenter H, the fact that OH is parallel to the side AB, and the length of the side AB.

23. Find all positive integers a, b, c such that 1/a + 1/b + 1/c = 4/5.

24. Factorise $5^{1985} - 1$ as a product of three integers, each greater than 5^{100}.

25. 34 countries each sent a leader and a deputy leader to a meeting. Some of the participants shook hands before the meeting, but no leader shook hands with his deputy. Let S be the set of all 68 participants except the leader of country X. Every member of S shook hands with a different number of people (possibly zero). How many people shook hands with the leader or deputy leader of X?

26. Find the smallest positive integer n such that n has exactly 144 positive divisors including 10 consecutive integers.

27. Find the largest and smallest values of w(w + x)(w + y)(w + z) for reals w, x, y, z such that w + x + y + z = 0 and $w^7 + x^7 + y^7 + z^7 = 0$.

28. X is the set {1, 2, ... , n}. $P_1, P_2, ... , P_n$ are distinct pairs of elements of X. P_i and P_j have an element in common iff {i, j} is one of the pairs. Show that every element of X belongs to exactly two of the pairs.

29. Show that for any point P on the surface of a regular tetrahedron we can find another point Q such that there are at least three different paths of minimal length from P to Q.

30. C is a circle and L a line not meeting it. M and N are variable points on L such that the circle diameter MN touches C but does not contain it. Show that there is a fixed point P such that the ∠ MPN is constant.

Note: This list does not incude the problems used in the Olympiad.

27th IMO 1986 shortlisted problems

7. Let $A_1 = 0.12345678910111213...$, $A_2 = 0.14916253649...$, $A_3 = 0.182764125216...$, $A_4 = 0.11681256625...$, and so on. The decimal expansion of A_n is obtained by writing out the *n*th powers of the integers one after the other. Are any of the A_n rational?

8. A, B, C are three points on the edge of a circular pond and form an equilateral triangle with side 86 and B due west of C. A boy swims from A directly towards B. After a distance x he turns and swims due west a distance y to reach the edge of the pond. Given that x and y are positive integers, find y.

9. For any positive integer n and any prime p > 3, find at least 3(n+1) sets of positive integers $a \le b < c$ such that $abc = p^n(a + b + c)$.

10. Find four positive integers < 70000 each with more than 100 positive divisors.

11. The real numbers $x_0, x_1, ... , x_{n+1}$ satisfy $0 = x_0 < x_1 < ... < x_n < x_{n+1} = 1$ and $1/(x_i - x_0) + 1/(x_i - x_1) + ... + 1/(x_i - x_{i-1}) + 1/(x_i - x_{i+1}) + 1/(x_i - x_{i+2}) + ... + 1/(x_i - x_{n+1}) = 0$ for i = 1, 2, ... , n. Show that $x_i + x_{n+1-i} = 1$ for all i.

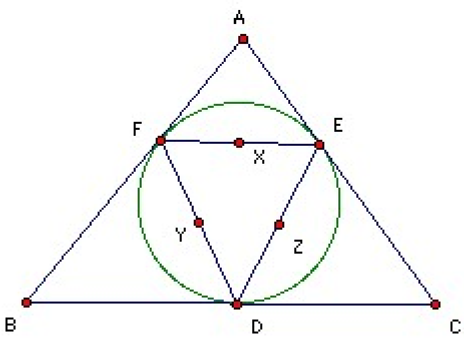
12. A graph has n points and q edges. The edges are labeled 1, 2, ... , q. Show that we can always find a sequence of at least 2q/n edges such that the labels increase monotonically and adjacent edges have a common vertex.

13. A k-element subset is chosen at random from {1, 2, ... , 1986}. For which k is there an equal chance that the sum of the elements in the subset will be 0, 1 or 2 mod 3.

14. Find an explicit expression for f(n), the least number of distinct points in the plane such that for each k = 1, 2, ... , n there is a straight line containing just k points.

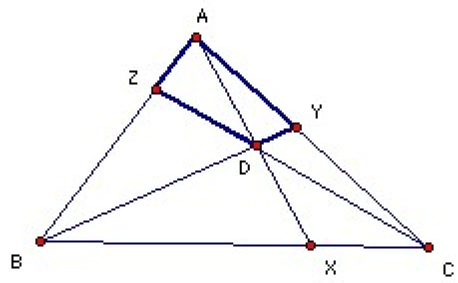
15. A particle starts at (0, 0). A fair coin is tossed repeatedly. For each head the x-coordinate is increased by 1 unless x = n, when it does not move. For each tail the y-coordinate is increased by 1 unless y = n, when it does not move. Find the probability that just 2n+k tosses are needed to reach (n, n).

16. The incircle of ABC touches the sides at DEF and the midpoints of the sides of DEF are X, Y, Z. Show that the incenter of ABC, and the circumcenters of ABC and XYZ are collinear.



17. Q is a convex quadrilateral which is not cyclic. f(Q) is the quadrilateral formed by the circumcenters of the four triangles whose vertices are vertices of Q. Show that f(f(Q)) is similar to Q and find the ratio of similitude in terms of the angles of Q.

18. X, Y, Z are points on the sides of the triangle ABC, such that AX, BY, CZ meet at D inside the triangle. Show that if AYDZ and BZDX are cyclic, so is CXDY.



19. The tetrahedron ABCD has AD = BC = a, AC = BD = b and AB·CD = c^2. Find the smallest value of AP + BP + CP + DP for any point P in space.

20. Show that the sum of the face angles at each vertex of a tetrahedron is 180^o iff the faces are all congruent triangles.

21. A tetrahedron has each sum of opposite edges equal to 1. Show that the sum of the four inradii of the faces is at most $1/\sqrt{3}$ with equality iff the tetrahedron is regular.

Note: This list does not incude problems 1-6 which were used in the Olympiad.

28th IMO 1987 shortlisted problems

1. f is a real-valued function on the reals such that:

(1) if $x \geq y$ and $f(y) - y \geq v \geq f(x) - x$, then $f(z) = v + z$ for some z between x and y;

(2) for some k, $f(k) = 0$ and if $f(h) = 0$, then $h <= k$;

(3) $f(0) = 1$;

(4) $f(1987) \leq 1988$;

(5) $f(x) f(y) = f(x f(y) + y f(x) - xy)$ for all x, y.

Find f(1987).

(Australia 6)

2. $S = \{a_1, a_2, ... , a_n, b_1, b_2, ... , b_n\}$. There are subsets $C_1, C_2, ... , C_k$, such that (1) for no i, j do both a_i and b_i both belong to C_j, (2) for any pair of distinct elements of S, not of the form a_i, b_i, there is just one C_j containing both elements. Show that if $n > 3$, then $k \geq 2n$. *(USA 3)*

3. Does there exist a polynomial p(x, y) of degree 2 such that, for each non-negative integer n, we have $n = p(a, b)$ for just one pair (a, b) of non-negative integers? *(Finland 3)*

4. ABCDA'B'C'D' is any parallelepiped (with ABCD, A'B'C'D' faces and AA', BB', CC', DD' edges). Show that $AC + AB' + AD' <= AB + AD + AA' + AC'$ (the sum of the thre short diagonals from A is less than the sum of the three edges from A plus the long diagonal from A). *(France 5)*

5. Find the smallest real c such that $x_1^{1/2} + x_2^{1/2} + ... + x_n^{1/2} \leq c\,(x_1 + x_2 + ... + x_n)^{1/2}$ for all n and all real sequences $x_1, x_2, x_3, ...$ which satisfy $x_1 + x_2 + ... + x_n \leq x_{n+1}$. *(United Kingdom 6)*

6. Show that $a^n/(b+c) + b^n/(c+a) + c^n/(a+b) \geq (2/3)^{n-2} s^{n-1}$ for all $n \geq 1$, where a, b, c are the sides of a triangle and $s = (a + b + c)/2$. *(Greece 4)*

7. Given any 5 real numbers u_0, u_1, u_2, u_3, u_4, show that we can always find 5 real numbers v_0, v_1, v_2, v_3, v_4 such that each $u_i - v_i$ is integral and $\sum_{i<j}(v_i - v_j)^2 < 4$. *(Netherlands 1)*

8. Does there exist a subset M of Euclidean space such that any plane meets M in a finite non-empty set? *(Hungary 1)*

9. Show that for any relatively prime positive integers m, n we can find integers $a_1, a_2, ... , a_m$ and $b_1, b_2, ... , b_n$ such that each product a_ib_j gives a different residue mod mn. *(Hungary 2)*

10. Two spheres S and S' touch externally and lie inside a cone C. Each sphere touches the cone in a full circle. n solid spheres are arranged in the cone in a ring so that each touches S and S' externally, touches the cone, and touches its two neighbouring solid spheres. What are the possible values of n? *(Iceland 3)*

11. Find the number of ways of partitioning $\{1, 2, 3, ... , n\}$ into three (possibly empty) subsets A, B, C such that (1) for each subset, if the elements are written in ascending order, then they alternate in parity, and (2) if all three subsets are non-empty, then just one of them has its smallest element even. *(Poland 1)*

12. ABC is a non-equilateral triangle. Find the locus of the centroid of all equilateral triangles A'B'C' such that A, B', C' are collinear, A', B, C' are collinear, A', B', C are collinear, and both ABC and A'B'C' have their vertices anti-clockwise. *(Poland 5)*

14. How many n-digit words can be formed from the alphabet $\{0, 1, 2, 3, 4\}$ if neighboring digits must differ by exactly 1? *(German Federal Republic 1)*

17. Show that we can color the elements of the set {1, 2, ... , 1987} with 4 colors so that any arithmetic progression with 10 terms in the set is not monochromatic. *(Romania 1)*

18. For any positive integer r find the smallest positive integer h(r) such that for any partition of {1, 2, ... , h(r) } into r parts, there are integers $a \geq 0$ and $1 \leq x \leq y$ such that a + x, a + y and a + x + y all belong to the same part. *(Romania 4)*

19. Given angles A, B, C such that $A + B + C < 180^o$, show that there is a triangle with sides sin A, sin B, sin C and that its area is less than (sin 2A + sin 2B + sin 2C)/8. *(USSR 2)*

23. Show that for any integer k > 1, there is an irrational number r such that $[r^m] = -1 \bmod k$ for every natural number m. *(Yugoslavia 2)*

Note: 13 (German Democratic Republic 2), 15 (German Federal Republic 2), 16 (German Federal Republic 3), 20 (USSR 2), 21 (USSR 4), and 22 (Vietnam 4) do not appear here, because they were used in the Olympiad.

29th IMO 1988 shortlisted problems

1. The sequence $a_0, a_1, a_2, \ldots$ is defined by $a_0 = 0$, $a_1 = 1$, $a_{n+2} = 2a_{n+1} + a_n$. Show that 2^k divides a_n iff 2^k divides n.

2. Find the number of odd coefficients of the polynomial $(x^2 + x + 1)^n$.

3. The angle bisectors of the triangle ABC meet the circumcircle again at A', B', C'. Show that area A'B'C' ≥ area ABC.

4. The squares of an n x n chessboard are numbered in an arbitrary manner from 1 to n^2 (every square has a different number). Show that there are two squares with a common side whose numbers differ by at least n.

6. ABCD is a tetrahedron. Show that any plane through the midpoints of AB and CD divides the tetrahedron into two parts of equal volume.

7. c is the largest positive root of $x^3 - 3x^2 + 1$. Show that $[c^{1788}]$ and $[c^{1988}]$ are multiples of 17.

8. $u_1, u_2, \ldots, u_n$ are vectors in the plane, each with length at most 1, and with sum zero. Show that one can rearrange them as $v_1, v_2, \ldots, v_n$ so that all the partial sums v_1, $v_1 + v_2$, $v_1 + v_2 + v_3$, ... , $v_1 + v_2 + \ldots + v_n$ have length at most $\sqrt{5}$.

10. Let X = {1, 2, ... , n}. Find the smallest number of subsets f(n) of X with union X such that given any distinct a, b in X, one of the subsets contains just one of a, b.

11. The lock on a safe has three wheels, each of which has 8 possible positions. It is known that the lock is defective and will open if any two wheels are in the correct position. How many combinations must be tried to guarantee opening the safe?

12. ABC is a triangle. K, L, M are points on the sides BC, CA, AB respectively. D, E, F are points on the sides LM, MK, KL respectively. Show that (area AME)(area CKE)(area BKF)(area ALF)(area BDM)(area CLD) ≤ $(1/8 \text{ area ABC})^6$.

14. For what values of n, does there exist an n x n array of entries 0, ±1 such that all rows and columns have different sums?

15. ABC is an acute-angled triangle. H is the foot of the perpendicular from A to BC. M and N are the feet of the perpendiculars from H to AB and AC. L_A is the line through A perpendicular to MN. L_B and L_C are defined similarly. Show that L_A, L_B and L_C are concurrent.

17. ABCDE is a convex pentagon such that BC, CD and DE are equal and each diagonal is parallel to a side (AC is parallel to DE, BD is parallel to AE etc). Show that the pentagon is regular.

19. f has positive integer values and is defined on the positive integers. It satisfies f(f(m) + f(n)) = m + n for all m, n. Find all possible values for f(1988).

20. Find the smallest n such that if {1, 2, ... , n} is divided into two disjoint subsets then we can always find three distinct numbers a, b, c in the same subset with ab = c.

21. 49 students solve a set of 3 problems. Each problem is marked from 0 to 7. Show that there are two students A and B such that A scores at least as many as B for each problem.

22. Show that there are only two values of N for which $(4N+1)(x_1^2 + x_2^2 + \ldots + x_N^2) = 4(x_1 + x_2 + \ldots + x_N)^2 + 4N + 1$ has an integer solution x_i.

23. I is the incenter of the triangle ABC. Show that for any point P, $BC\cdot PA^2 + CA\cdot PB^2 + AB\cdot PC^2 = BC\cdot IA^2 + CA\cdot IB^2 + AB\cdot IC^2 + (AB + BC + CA)\cdot IP^2$.

24. $x_1, x_2, x_3, \ldots$ is a sequence of non-negative reals such that $x_{n+2} = 2x_{n+1} - x_n$ and $x_1 + x_2 + \ldots + x_n \le 1$ for all $n > 0$. Show that $x_{n+1} \le x_n$ and $x_{n+1} > x_n - 2/n^2$ for all $n > 0$.

25. A *double* number has an even number of digits and the first half of its digits are the same as the second half. For example, 360360 is a double number, but 36036 is not. Show that infinitely many double numbers are squares.

27. ABC is an acute-angled triangle area S. L is a line. The lengths of the perpendiculars from A, B, C to L are u, v, w respectively. Show that $u^2\tan A + v^2\tan B + w^2\tan C \ge 2S$. For which L does equality occur?

28. The sequence of integers $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 2$, $a_2 = 7$, and $-1/2 < a_{n+2} - a_{n+1}^2/a_n \le 1/2$. Show that a_n is odd for $n > 1$.

29. n signals are equally spaced along a rail track. No train is allowed to leave a signal whilst there is a moving train between that signal and the next. Any number of trains can wait at a signal. At time 0, k trains are waiting at the first signal. Except when waiting at a signal each train travels at a constant speed, but each train has a different speed. Show that the last train reaches signal n at the same time irrespective of the order in which the trains are arranged at the first signal.

30. ABC is a triangle. M is a point on the side AC such that the inradii of ABM and CBM are the same. Show that $BM^2 = \cot(B/2)$ area ABC.

31. An even number of people have a discussion sitting at a circular table. After a break they sit down again in a different order. Show that there must be two people with the same number of people sitting between them before and after the break.

Note: problems 5, 9, 13, 16, 18 and 26 were used in the Olympiad and are not shown here.

30th IMO 1989 shortlisted problems

1. Ali Baba has a rectangular piece of carpet. He finds that if he lays if flat on the floor of either of his storerooms then each corner of the carpet touches a different wall of the room. He knows that the storerooms have widths 38 and 50 feet and the same (unknown but integral) length. What are the dimensions of the carpet?

2. Prove that for $n > 1$ the polynomial $x^n/n! + x^{n-1}/(n-1)! + ... + x/1! + 1$ has no rational roots.

3. The polynomial $x^n + n x^{n-1} + a_2x^{n-2} + ... + a_0$ has n roots whose 16th powers have sum n. Find the roots.

4. The angle bisectors of the triangle ABC meet the circumcircle again at A'B'C'. Show that 16 (area A'B'C')$^3 \geq 27$ area ABC R^4, where R is the circumradius of ABC.

5. Show that any two points P and Q inside a regular n-gon can be joined by two circular arcs PQ which lie inside the n-gon and meet at an angle at least $(1 - 2/n)\pi$.

6. The rectangle R is covered by a finite number of rectangles $R_1, ... , R_n$ such that (1) each R_i is a subset of R, (2) the sides of each R_i are parallel to the sides of R, (3) the rectangles R_i have disjoint interiors, and (4) each R_i has a side of integral length. Show that R has a side of integral length.

7. For each $n > 0$ we write $(1 + 2^{1/3}4 - 4^{1/3}4)^n$ as $a_n + b_n 2^{1/3} + c_n 4^{1/3}$, where a_n, b_n, c_n are integers. Show that c_n is non-zero.

8. Let C represent the complex numbers. Let g: C → C be an arbitrary function. Let w be a cube root of 1 other than 1 and let v be any complex number. Find a function f: C → C such that $f(z) + f(wz + v) = g(z)$ for all z and show that it is unique.

9. Define the sequence $a_1, a_2, a_3, ...$ by 2^n = the sum of a_d such that d divides n. Show that a_n is divisible by n. [For example, $a_1 = 2$, $a_2 = 2$, $a_3 = 6$.]

10. There are n cars waiting at distinct points of a circular race track. At the starting signal each car starts. Each car may choose arbitrarily which of the two possible directions to go. Each car has the same constant speed. Whenever two cars meet they both change direction (but not speed). Show that at some time each car is back at its starting point.

11. A quadrilateral has both a circumcircle and an incircle. Show that intersection point of the diagonals lies on the line joining the centers of the two circles.

12. a, b, c, d, m, n are positive integers such that $a^2 + b^2 + c^2 + d^2 = 1989$, $a + b + c + d = m^2$ and the largest of a, b, c, d is n^2. Find m and n.

13. The real numbers $a_0, a_1, ... , a_n$ satisfy $a_0 = a_n = 0$, $a_k = c + \sum_{i=k}^{n-1} a_{i-k}(a_i + a_{i+1})$. Show that $c \leq 1/(4n)$.

14. Given 7 points in the plane, how many segments (each joining two points) are needed so that given any three points at least two have a segment joining them?

15. Given a convex n-gon $A_1A_2 ... A_n$ with area A and a point P, we rotate P through an angle x about A_i to get the point P_i. Find the area of the polygon $P_1P_2 ... P_n$.

16. A positive integer is written in each square of an m x n board. The allowed move is to add an integer k to each of two adjacent numbers (whose squares have a common side). Find a necessary and sufficient condition for it to be possible to get all numbers zero by a finite sequence of moves.

17. Show that the intersection of a plane and a regular tetrahedron can be an obtuse-angled triangle, but that the obtuse angle cannot exceed 120^o.

18. Five points are placed on a sphere of radius 1. That is the largest possible value for the shortest distance between two of the points? Find all configurations for which the maximum is attained.

19. a and b are non-square integers. Show that $x^2 - ay^2 - bz^2 + abw^2 = 0$ has a solution in integers not all zero iff $x^2 - ay^2 - bz^2 = 0$ has a solution in integers not all zero.

20. b > 0 and a are fixed real numbers and n is a fixed positive integer. The real numbers x_0, $x_1, \ldots, x_n$ satisfy $x_0 + x_1 + \ldots + x_n = a$ and $x_0^2 + \ldots + x_n^2 = b$. Find the range of x_0.

21. Let m > 1 be a positive odd integer. Find the smallest positive integer n such that 2^{1989} divides $m^n - 1$.

22. The points O, A, B, C, D in the plane are such that the six triangles OAB, OAC, OAD, OBC, OBD, OCD all have area at least 1. Show that one of the triangles must have area at least √2.

23. 155 birds sit on a circle center O. Birds at A and B are mutually visible iff $\angle AOB \le 10^o$. More than one bird may sit at the same point. What is the smallest possible number of mutually visible pairs?

24. Given positive integers a ≥ b ≥ c, let N(a, b, c) be the number of solutions in positive integers x, y, z to a/x + b/y + c/z = 1. Show that N(a, b, c) ≤ 6ab(3 + ln(2a)).

25. ABC is an acute-angled triangle with circumcenter O and orthocenter H. AO = AH. Find all possible values for the $\angle A$.

There were 109 problems proposed in total, although this included one problems which was merely a variant. 32 were shortlisted (from which 6 were chosen and are not shown above, nor is the variant shown).

31st IMO 1990 shortlisted problems

1. Is there a positive integer which can be written as the sum of 1990 consecutive positive integers and which can be written as a sum of two or more consecutive positive integers in just 1990 ways?

2. 11 countries each have 3 representatives. Is it possible to find 1990 committees $C_1, C_2, \ldots, C_{1990}$ such that each committee has just one representative from each country, no two committees have the same members, and every two committees have at least one member in common except for the pairs (C_1, C_2), (C_2, C_3), (C_3, C_4), ... , (C_{1991}, C_{1992}), (C_{1992}, C_1)?

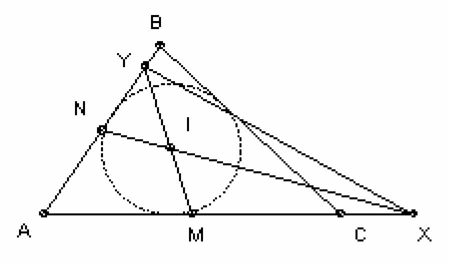
4. The set of all positive integers is divided into r disjoint subsets. Show that for one of them we can find a positive integer m such that for any k there are numbers $a_1 < a_2 < \ldots < a_k$ in the subset with the difference between consecutive numbers in the sequence at most m.

5. The triangle ABC has unequal sides, centroid G, incenter I and orthocenter H. Show that angle GIH > 90^o.

7. Define f(0) = 0, f(1) = 0, and $f(n+2) = 4^{n+2}f(n+1) - 16^{n+1}f(n) + n\,2^{n2}$. Show that f(1989), f(1990) and f(1991) are all divisible by 13.

8. For a positive integer k, let $f_1(k)$ be the square of the sum of its digits. Let $f_{n+1}(k) = f_1(f_n(k))$. Find the value of $f_{1991}(2^{1990})$.

9. ABC is a triangle with incenter I. M is the midpoint of AC and N is the midpoint of AB. The lines NI and AC meet at X, and the lines MI and AB meet at Y. The triangles AXY and ABC have the same area. Find $\angle$ A.



10. A plane cuts a right circular cone of volume V into two parts. The plane is tangent to the circumference of the base of the cone and passes through the midpoint of the altitude. Find the volume of the smaller part.

12. ABC is a triangle with angle bisectors AD and BF. The lines AD, BF meet the line through C parallel to AB at E and G respectively, and FG = DE. Show that CA = CB.

13. A gymnast ascends a ladder of n steps A steps at a time and descends B steps at a time. Find the smallest n such that starting from the bottom, he can get to the top and back again. Note that he does not have to go directly to the top. For example, if A = 3, B = 2, then n = 4 works: 0 3 1 4 2 0.

13. A gymnast ascends a ladder of n steps A steps at a time and descends B steps at a time. Find the smallest n such that starting from the bottom, he can get to the top and back again. Note that he does not have to go directly to the top. For example, if A = 3, B = 2, then n = 4 works: 0 3 1 4 2 0.

14. R is the rectangle with vertices (0, 0), (m, 0), (0, n), (m, n), where m and n are odd integers. R is divided into triangles. Each triangle has at least one *good* side which lies on a line of the form x = i or y = j, where i and j are integers, and has the altitude to this side of length 1. Any side which is not a good side is a common side to two triangles. Show that there are at least two triangles each with two good sides.

15. For which k can the set {1990, 1991, 1992, ... , 1990 + k} be divided into two disjoint subsets with equal sums?

17. Holes are drilled through a long diagonal of each of pqr unit cubes and the cubes are threaded are put onto a string. For which p, q, r can the cubes be arranged to form a p x q x r cube (whilst still on the string and with neighbouring cubes on the string continuing to touch at their adjacent vertices)? Suppose that the ends of the string are tied to form a loop so that every cube touches two neighbouring cubes. For which p, q, r can a cube be made now?

18. a <= b are positive integers, m = (a + b)/2. Define the function f on the integers by f(n) = n + a if n < m, n - b if $n \geq m$. Let $f_1(n) = f(n)$, $f_2(n) = f(f_1(n))$, $f_3(n) = f(f_2(n))$ etc. Find the smallest k such that $f_k(0) = 0$.

19. P is a point inside a regular tetrahedron of unit volume. The four planes through P parallel to the faces of the tetrahedron partition it into 14 pieces. Let v(P) be the total volume of the pieces which are neither a tetrahedron nor a parallelepiped (in other words, the pieces which are adjacent to an edge, but not to a vertex). Find the smallest and largest possible values for v(P).

20. Show that every positive integer n > 1 has a positive multiple less than n^4 which uses at most 4 different digits.

21. Ten cities are served by two airlines. All services are both ways. There is a direct service between any two cities. Show that at least one of the airlines can offer two disjoint round trips, each with an odd number of landings.

23. w, x, y, z are non-negative reals such that wx + xy + yz + zw = 1. Show that $w^3/(x + y + z) + x^3/(w + y + z) + y^3/(w + x + z) + z^3/(w + x + y) \geq 1/3$.

25. Let p(x) be a cubic polynomial with rational coefficients. q_1, q_2, q_3, ... is a sequence of rationals such that $q_n = p(q_{n+1})$ for all positive n. Show that for some k, we have $q_{n+k} = q_n$ for all positive n.

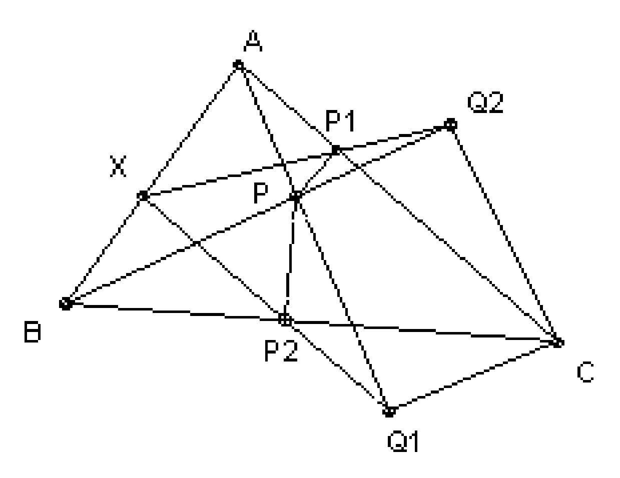
26. Find all positive integers n such that every positive integer with n digits, one of which is 7 and the others 1, is prime.

27. Show that it is not possible to find a finite number of points P_1, P_2, ... , P_n in the plane such that each point has rational coordinates, each edge P_1P_2, P_2P_3, P_3P_4, ... , $P_{n-1}P_n$, P_nP_1 has length 1, and n is odd.

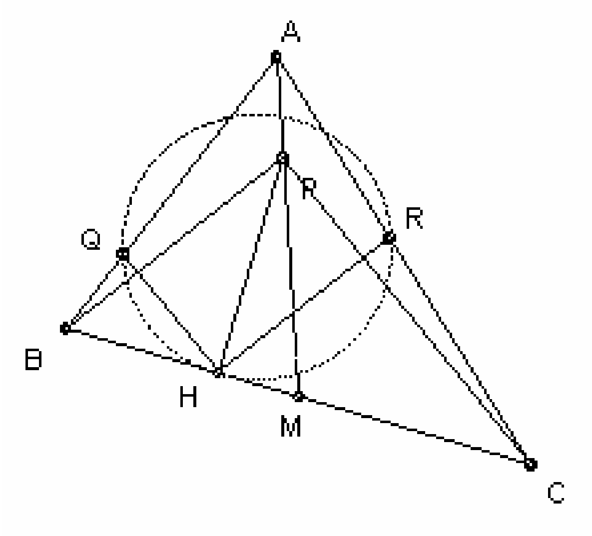
Note: problems 3, 6, 11, 16, 22 and 24 were used in the Olympiad and are not shown here.

32nd IMO 1991 shortlisted problems

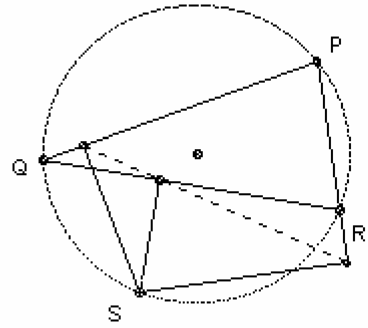
1. ABC is a triangle and P an interior point. Let the feet of the perpendiculars from P to AC, BC be P_1, P_2 respectively, and let the feet of the perpendiculars from C to AP, BP be Q_1, Q_2 respectively. Show that P_1Q_2 and P_2Q_1 meet on the line AB.

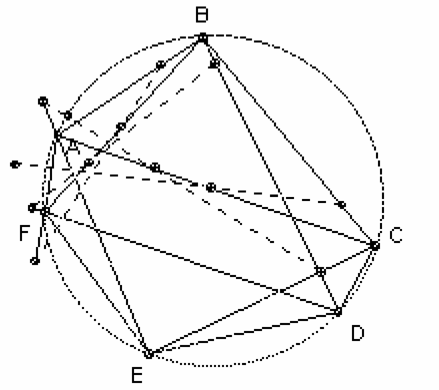


2. ABC is an acute-angled triangle. M is the midpoint of BC and P is the point on AM such that MB = MP. H is the foot of the perpendicular from P to BC. The lines through H perpendicular to PB, PC meet AB, AC respectively at Q, R. Show that BC is tangent to the circle through Q, H, R at H.

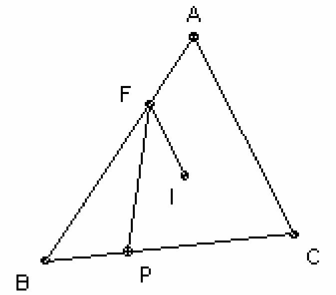


3. If S is a point on the circumcircle of the triangle PQR, show that the feet of the perpendiculars from S to the lines PQ, QR, RP are collinear. Denote the line by [S, PQR]. If ABCDEF is a hexagon whose vertices lie on a circle, show that the lines [A, BDF], [B, ACE], [D, ABF], [E, ABC] are concurrent iff CDEF is a rectangle.





5. ABC is a triangle with ∠A = 60^o and incenter I. P lies on BC with 3 BP = BC. The point F lies on AB and IF is parallel to AC. Show that ∠BFP = ∠FBI.



7. O is the circumcenter of the tetrahedron ABCD. The midpoints of BC, CA, AB are L, M, N respectively. AB + BC = AD + CD, CB + CA = BD + AD and CA + AB = CD + BD. Show that ∠LOM = ∠MON = ∠NOL.

8. Given a set A of n points in the plane, no three collinear, show that we can find a set B of 2n - 5 points such that a point of B lies in the interior of every triangle whose vertices belong to A.

9. A graph has 1991 points and every point has degree at least 1593. Show that there are six points, each of which is joined to the others. Is 1593 the best possible?

11. Let mCn represent the binomial coefficient m!/(n! (m-n)!). Prove that (1991C0)/1991 - (1990C1)/1990 + (1989C2)/1989 - (1988C3)/1988 + ... - (996C995)/996 = 1/1991.

13. No term of the sequence $a_1, a_2, \dots, a_n$ is divisible by n and the sum of all n terms is not divisible by n. Show that for n > 1 there are at least n different subsequences whose sum is divisible by n.

14. The quadratic $ax^2 + bx + c$ has integer coefficients. For some prime p its values are squares for 2p-1 consecutive integers x. Show that p divides $b^2 - 4ac$.

15. Let a_n be the last non-zero digit of n! Does the sequence a_n become periodic after a finite number of terms?

17. Find all positive integer solutions to $3^m + 4^n = 5^k$.

18. What is the largest power of 1991 which divides $1990^m + 1992^n$, where $m = 1991^{1992}$ and $n = 1991^{1990}$?

19. Show that the only rational solution q in the range 0 < q < 1 to the equation cos(3πq) + 2 cos(2πq) = 0 is 2/3.

20. Let k be the positive root of the equation $x^2 = 1991x + 1$. For positive integers m, n define m # n = mn + [km] [kn]. Show that the operation # is associative.

21. The polynomial $p(x) = x^{1991} + a_{1990}x^{1990} + ... + a_0$ has integer coefficients. Show that the equation $p(x)^2 = 9$ has at most 1995 distinct integer solutions.

22. There is exactly one square with its vertices on the curve $y = x^3 + ax^2 + bx + c$, where a, b, c are real. Show that the square must have side $72^{1/4}$.

23. f(n) is an integer-valued function defined on the integers which satisfies f(m + f(f(n))) = - f(f(m+1)) - n for all m, n. The polynomial g(n) has integer coefficients and g(n) = g(f(n)) for all n. Find f(1991) and the most general form for g.

24. Find all odd integers n > 1 for which there is at least one permutation $a_1, a_2, ... , a_n$ of 1, 2, 3, ... , n such that the sums $a_1 - a_2 + a_3 - ... + a_n$, $a_2 - a_3 + a_4 - ... - a_n + a_1$, $a_3 - a_4 + ... + a_n - a_1 + a_2$, ...
$a_n - a_1 + a_2 - ... + a_{n-1}$ are all positive.

25. Given n > 1 real numbers $0 \le x_i \le 1$, show that for some i < n we have $x_i(1 - x_{i+1}) \ge x_1(1 - x_n)/4$.

26. $a_1, a_2, ... , a_n$ and $b_1, b_2, ... , b_n$ are non-negative reals such that $a_1 + a_2 + ... + a_n = b_1 + b_2 + ... + b_n = 1$. Let p_i be the product $a_1a_2 ... a_{i-1}b_ia_{i+1} ... a_n$. If none of the b_i exceed k, where $½ \le k \le 1$, show that $p_1 + p_2 + ... + p_n \le k/(n - 1)^{n-1}$

27. Find the maximum possible value of $x_1x_2(x_1 + x_2) + x_1x_3(x_1 + x_3)\ x_1x_2(x_1 + x_2) + x_1x_4(x_1 + x_4) + ... + x_1x_n(x_1 + x_n)$ for non-negative real numbers $x_1, x_2, ... , x_n$ with sum 1.

29. Find all subsets X of the real line such that for any stretch S there is a translation T such that S(X) = T(X). A stretch is a transformation $x \to h + k(x - h)$, where h and $k \neq 0$ are constants, and a translation is a transformation $x \to x + t$, where t is a constant.

30. Two people A and B and an umpire U play the following game. A chooses an arbitrary positive integer m, and B chooses an arbitrary positive integer n. Each gives his integer to U but not to the other player. U then writes two integers on a blackboard, one of which is m + n. U asks A if he knows n. If he does not, then U asks B if he knows m. If he does not, then U asks A if he knows n, and so on. Show that one player will be able to deduce the other's number after a finite number of questions.

Note: problems 4, 6, 10, 12, 16, and 28 were used in the Olympiad and are not shown here.

33rd IMO 1992 shortlisted problems

1. m is a positive integer. If there are two coprime integers a, b such that a divides $m + b^2$ and b divides $m + a^2$, show that we can also find two such coprime integers with the additional restriction that a and b are positive and their sum does not exceed m + 1.

2. a and b are positive reals. Let X be the set of non-negative reals. Show that there is a unique function $f:X \to X$ such that $f(f(x)) = b(a + b)x - af(x)$ for all x.

3. The quadrilateral ABCD has perpendicular diagonals. Squares ABEF, BCGH, CDIJ and DAKL are constructed on the outside of the sides. The lines CL and DF meet at P, the lines DF and AH meet at Q, the lines AH and BJ meet at R, and the lines BJ and CL meet at S. The lines AI and BK meet at P', the lines BK and CE meet at Q', the lines CE and DG meet at R', and the lines DG and AI meet at S'. Show that the quadrilaterals PQRS and P'Q'R'S' are congruent.

5. A convex quadrilateral has equal diagonals. Equilateral triangles are constructed on the outside of each side. The centers of the triangles on opposite sides are joined. Show that the two joining lines are perpendicular.

7. Two circles touch externally at I. The two circles lie inside a large circle and both touch it. The chord BC of the large circle touches both smaller circles (not at I). The common tangent to the two smaller circles at I meets the large circle at A, where A and I are on the same side of the chord BC. Show that I is the incenter of ABC.

8. Show that there is a convex 1992-gon with sides 1, 2, 3, ... , 1992 in some order, and an inscribed circle (touching every side).

9. p(x) is a polyonomial with rational coefficients such that $k^3 - k = 33^{1992} = p(k)^3 - p(k)$ for some real number k. Let $p_n(x)$ be p(p(... p(x) ...)) (iterated n times). Show that $p_n(k)^3 - p_n(k) = 33^{1992}$.

11. BD and CE are angle bisectors of the triangle ABC. $\angle BDE = 24^o$ and $\angle CED = 18^o$. Find angles A, B, C.

12. The polynomials f(x), g(x) and a(x, y) have real coefficients. They satisfy f(x) - f(y) = a(x, y) (g(x) - g(y)) for all x, y. Show that there is a polynomial h(x) such that f(x) = h(g(x)) for all x.

14. For each positive integer n, let d(n) be the largest odd divisor of n and define f(n) to be n/2 + n/d(n) for n even and $2^{(n+1)/2}$ for n odd. Define the sequence $a_1, a_2, a_3, ...$ by $a_1 = 1$, $a_{n+1} = f(a_n)$. Show that 1992 occurs in the sequence and find the first time it occurs. Does it occur more than once?

15. Does there exist a set of 1992 positive integers such that each subset has a sum which is a square, cube or higher power?

16. Show that $(5^{125} - 1)/(5^{25} - 1)$ is composite.

17. Let b(n) be the number of 1s in the binary representation of a positive integer n. Show that $b(n^2) \le b(n) (b(n) + 1)/2$ with equality for infinitely many positive integers n. Show that there is a sequence of positive integers $a_1 < a_2 < a_3 < ...$ such that $b(a_n^2)/b(a_n)$ tends to zero.

18. Let x_1 be a real number such that $0 < x_1 < 1$. Define $x_{n+1} = 1/x_n - [1/x_n]$ for x_n non-zero and 0 for x_n zero. Show that $x_1 + x_2 + ... + x_n < F_1/F_2 + F_2/F_3 + ... + F_n/F_{n+1}$, where F_n is the Fibonacci sequence: $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$.

Note: problems 4, 6, 10, 13, 19 and 20 were used in the Olympiad and are not shown here.

34th IMO 1993 shortlisted problems

1. Show that there is a finite set of points in the plane such that for any point P in the set we can find 1993 points in the set a distance 1 from P.

2. ABC is a triangle with circumradius R and inradius r. If p is the inradius of the orthic triangle, show that $p/R \le 1 - (1/3)(1 + r/R)^2)$. [The *orthic* triangle has as vertices the feet of the altitudes of ABC.]

3. The triangle ABC has incenter I. A circle lies inside the circumcircle and touches it. It also touches the sides CA, BC at D, E respectively. Show that I is the midpoint of DE.

4. ABC is a triangle. D and E are points on the side BC such that $\angle BAD = \angle CAE$. The incircles of ABD and ACE touch BC at M and N respectively. Show that 1/MB + 1/MD = 1/NC + 1/NE.

7. $a > 0$ and b, c are integers such that $ac - b^2$ is a square-free positive integer P. [For example P could be $3\cdot5$, but not $3^2 5$.] Let f(n) be the number of pairs of integers d, e such that $ad^2 + 2bde + ce^2 = n$. Show that f(n) is finite and that $f(n) = f(P^k n)$ for every positive integer k.

8. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 1$, $a_n = a_{n-1} - n$ if $a_{n-1} > n$, $a_{n-1} + n$ if $a_{n-1} \le n$. Let S be the set of n such that $a_n = 1993$. Show that S is infinite. Find the smallest member of S. If the element of S are written in ascending order show that the ratio of consecutive terms tends to 3.

9. Show that the set of positive rationals can be partitioned into three disjoint sets A, B, C such that BA = B, BB = C and BC = A, where HK denotes the set {hk: h is in H and k is in K}. Show that all positive rational cubes must lie in A. Find such a partition with the additional property that for each of the sets {1, 2}, {2, 3}, {3, 4}, ... {34, 35} at least one member is not in A.

10. A positive integer n has *property P* if, for all a, n^2 divides $a^n - 1$ whenever n divides $a^n - 1$. Show that every prime number has property P. Show that infinitely many other numbers have property P.

12. Let $k \le n$ be positive integers. Let S be any set of n distinct real numbers. Let T be the set of all sums of k distinct elements of S. Show that T has at least $k(n - k) + 1$ distinct elements.

13. If m and n are relatively prime positive integers, denote the pair (m, n) by s and define f(s) to be the pair (k, m+n-k), where k is the largest odd integer dividing n. Show that k and m+n-k are relatively prime. Show that f(f(... f(s)...)) = s where f is iterated t times for some t $\le (m + n + 1)/4$. Show that if m + n is prime and does not divide $2^h - 1$ for h = 1, 2, ... , m+n-2, then the smallest t is [(m+n-1)/4].

14. The triangle ABC has side lengths BC = a, CA = b, AB = c, as usual. Points D, E, F on the sides BC, CA, AB respectively are such that DEF is an equilateral triangle. Show that DE $\sqrt{(a^2 + b^2 + c^2 + k\, 4\sqrt{3})} \ge k\, 2\sqrt{2}$, where k is the area of ABC.

16. A is an n-tuple of non-negative integers $(a_1, a_2, \ldots , a_n)$ such that $a_i \le i-1$. Given any such n-tuple, we define the successor $A' = (b_1, \ldots , b_n)$, where $b_1 = 0$, b_{i+1} is the number of earlier members of A which are at least a_i. Let A_k be the sequence defined by $A_0 = A$, A_{k+1} is the successor of A_k. Show that $A_{k+2} = A_k$ for some k.

18. Let S_n be the number of sequences of n 0s and 1s such that the sequence does not contain six consecutive identical blocks of numbers. [For example, 1000100100100100100110 is not allowed because it has six consecutive blocks 001.] Show that S_n tends to infinity.

19. $b > 1$, a and n are positive integers such that $b^n - 1$ divides a. Show that in base b, the number a has at least n non-zero digits.

20. The $n > 1$ real numbers $x_1, x_2, \ldots, x_n$ satisfy $0 \le x_1 + \ldots + x_n \le n$. Show that there are integers k_i with sum 0 such that $1 - n \le x_i + nk_i \le n$ for each i.

21. A circle S *bisects* a circle S' if it cuts S' at opposite ends of a diameter. S_A, S_B, S_C are circles with distinct centers A, B, C (respectively). Show that A, B, C are collinear iff there is no unique circle S which bisects each of S_A, S_B, S_C. Show that if there is more than one circle S which bisects each of S_A, S_B, S_C, then all such circles pass through two fixed points. Find these points.

23. Show that for any finite set S of distinct positive integers, we can find a set T □ S such that every member of T divides the sum of all the members of T.

24. Show that $a/(b + 2c + 3d) + b/(c + 2d + 3a) + c/(d + 2a + 3b) + d/(a + 2b + 3c) \ge 2/3$ for any positive reals.

25. a is a real number such that $|a| > 1$. Solve the equations:

$x_1^2 = ax^2 + 1$

$x_2^2 = ax_3 + 1$

...

$x_{999}^2 = ax_{1000} + 1$

$x_{1000}^2 = ax_1 + 1$.

26. a, b, c, d are non-negative reals with sum 1. Show that $abc + bcd + cda + dab \le 1/27 + 176abcd/27$.

Note: Problems 5, 6, 11, 15, 17 and 22 were used in the Olympiad and are not shown here.

35th IMO 1994 shortlisted problems

Algebra

A1. The sequence x_0, x_1, x_2, ... is defined by $x_0 = 1994$, $x_{n+1} = x_n^2/(x_n + 1)$. Show that $[x_n] = 1994 - n$ for $0 \le n \le 998$.

A4. h and k are reals. Find all real-valued functions f defined on the positive reals such that $f(x) f(y) = y^h f(x/2) + x^k f(y/2)$ for all x, y.

A5. Let $f(x) = (x^2 + 1)/(2x)$ for x non-zero. Define $f_0(x) = x$ and $f_{n+1}(x) = f(f_n(x))$. Show that for x not -1, 0 or 1 we have $f_n(x)/f_{n+1}(x) = 1 + 1/f(y)$, where $y = (x+1)^N/(x-1)^N$ and $N = 2^n$.

Combinatorics

C1. Two players play alternately on a 5 x 5 board. The first player always enters a 1 into an empty square and the second player always enters a 0 into an empty square. When the board is full, the sum of the numbers in each of the nine 3 x 3 squares is calculated and the first player's score is the largest such sum. What is the largest score the first player can make (if the second player plays so as to minimise this score)?

C2. A finite graph is connected. A positive real number is assigned to each point. Each point is colored red or blue and at least one point is colored red. Show that, given a knowledge of (1) the points and edges of the graph, (2) the number assigned to each red point, and (3) for each blue point the average of the numbers for the points joined to it, one can find the number assigned to each point.

C3. Peter has three accounts at a bank, each with an integral number of dollars. He is only allowed to transfer money from one account to another if the effect of the transfer is to double the amount of money in one of the accounts. Show that by a series of transfers he can always empty one account, but that he cannot always get all his money into one account.

C4. There are n+1 cells in a row labeled from 0 to n and n+1 cards labeled from 0 to n. The cards are arbitrarily placed in the cells, one per cell. The objective is to get card i into cell i for each i. The allowed move is to find the smallest h such that cell h has a card with a label k > h, pick up that card, slide the cards in cells h+1, h+2, ... , k one cell to the left and to place card k in cell k. Show that at most $2^n - 1$ moves are required to get every card into the correct cell and that there is a unique starting position which requires $2^n - 1$ moves. [For example, if n = 2 and the initial position is 210, then we get 102, then 012, a total of 2 moves.]

C5. 1994 girls are seated at a round table. Initially one girl holds n tokens. Each turn a girl who is holding more than one token passes one token to each of her neighbours. Show that if n < 1994, the game must terminate. Show that if n = 1994 it cannot terminate. [Note: Sweden proposed the problem with 1991 girls (in which case you must show the game terminates for n ≤ 1991), but the PIG (see below) decided that was too difficult.]

C6. Two players play alternatively on an infinite square grid. The first player puts a X in an empty cell and the second player puts a O in an empty cell. The first player wins if he gets 11 adjacent Xs in a line, horizontally, vertically or diagonally. Show that the second player can always prevent the first player from winning.

C7. n > 2. Show that there is a set of 2^{n-1} points in the plane, no three collinear such that no 2n form a convex 2n-gon.

Geometry

G1. C and D are points on a semicircle. The tangent at C meets the extended diameter of the semicircle at B, and the tangent at D meets it at A, so that A and B are on opposite sides of the center. The lines AC and BD meet at E. F is the foot of the perpendicular from E to AB. Show that EF bisects angle CFD.

G2. ABCD is a quadrilateral with BC parallel to AD. M is the midpoint of CD, P is the midpoint of MA and Q is the midpoint of MB. The lines DP and CQ meet at N. Prove that N is inside the quadrilateral ABCD.

G3. A circle C has two parallel tangents L' and L". A circle C' touches L' at A and C at X. A circle C" touches L" at B, C at Y and C' at Z. The lines AY and BX meet at Q. Show that Q is the circumcenter of XYZ.

G5. L is a line not meeting a circle center O. E is the foot of the perpendicular from O to L and M is a variable point on L (not E). The tangents from O to the circle meet it at A and B. The feet of the perpendiculars from E to MA and MB are C and D respectively. The lines CD and OE meet at F. Show that F is fixed.

Number theory

N1. Find the largest possible subset of {1, 2, ... , 15} such that the product of any three distinct elements of the subset is not a square.

N4. Define the sequences a_n, b_n, c_n as follows. $a_0 = k$, $b_0 = 4$, $c_0 = 1$. If a_n is even then $a_{n+1} = a_n/2$, $b_{n+1} = 2b_n$, $c_{n+1} = c_n$. If a_n is odd, then $a_{n+1} = a_n - b_n/2 - c_n$, $b_{n+1} = b_n$, $c_{n+1} = b_n + c_n$. Find the number of positive integers $k < 1995$ such that some $a_n = 0$.

N6. Define the sequence $a_1, a_2, a_3, \ldots$ as follows. a_1 and a_2 are coprime positive integers and $a_{n+2} = a_{n+1}a_n + 1$. Show that for every $m > 1$ there is an $n > m$ such that a_m^m divides a_n^n. Is it true that a_1 must divide a_n^n for some $n > 1$?

N7. A *wobbly* number is a positive integer whose digits are alternately zero and non-zero with the last digit non-zero (for example, 201). Find all positive integers which do not divide any wobbly numbers.

Note: problems A2, A3, G4, N2, N3, and N5 were used in the Olympiad and are not shown here.

Apparently the problems committee for this year was known as the "Problem Interpolation Group (P.I.G.)" ! The previous year it seems to have been simply the Problems Committee, and in 1995 it became the Problems Selection Committee.

36th IMO 1995 shortlisted problems

Algebra

A2. a, b, c are integers such that $a \ge 0$, $b \ge 0$, $ab \ge c^2$. Show that for some n we can find integers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ such that $x_1^2 + x_2^2 + \ldots + x_n^2 = a$, $y_1^2 + y_2^2 + \ldots + y_n^2 = b$, $x_1y_1 + x_2y_2 + \ldots + x_ny_n = c$.

A3. $n > 2$. $x_1, x_2, \ldots, x_n$ are real numbers such that $2 \le x_i \le 3$. Show that $(x_1^2 + x_2^2 - x_3^2)/(x_1 + x_2 - x_3) + (x_2^2 + x_3^2 - x_4^2)/(x_2 + x_3 - x_4) + \ldots + (x_{n-1}^2 + x_n^2 - x_1^2)/(x_{n-1} + x_n - x_1) + (x_n^2 + x_1^2 - x_2^2)/(x_n + x_1 - x_2) \le 2(x_1 + x_2 + \ldots + x_n) - 2n$.

A4. a, b, c are fixed positive reals. Find all positive real solutions x, y, z to: $x + y + z = a + b + c$ and $4xyz - (a^2x + b^2y + c^2z) = abc$.

A5. Does there exist a real-valued function f on the reals such that f(x) is bounded, $f(1) = 1$ and $f(x + 1/x^2) = f(x) + f(1/x)^2$ for all non-zero x?

A6. $x_1 < x_2 < \ldots < x_n$ are real numbers, where $n > 2$. Show that $n(n-1)/2 \sum_{i<j} x_ix_j > ((n-1)x_1 + (n-2)x_2 + \ldots + 2x_{n-2} + x_{n-1}) (x_2 + 2x_3 + \ldots + (n-1)x_n)$.

Geometry

G2. ABC is a triangle. Show that there is a unique point P such that $PA^2 + PB^2 + AB^2 = PB^2 + PC^2 + BC^2 = PC^2 + PA^2 + CA^2$.

G3. ABC is a triangle. The incircle touches BC, CA, AB at D, E, F respectively. X is a point inside the triangle such that the incircle of XBC touches BC at D. It touches CX at Y and XB at Z. Show that EFZY is cyclic.

G4. ABC is an acute-angled triangle. There are points A_1, A_2 on the side BC, B_1 and B_2 on the side CA, and C_1, C_2 on the side AB such that the points are in the order: A, C_1, C_2, B; B, A_1, A_2, C; and C, B_1, B_2, A. Also $\angle AA_1A_2 = \angle AA_2A_1 = \angle BB_1B_2 = \angle BB_2B_1 = \angle CC_1C_2 = \angle CC_2C_1$. The three lines AA_1, BB_1 and CC_1 meet in three points and the three lines AA_2, BB_2, CC_2 meet in three points. Show that all six points lie on a circle.

G6. ABCD is a tetrahedron with centroid G. The line AG meets the circumsphere again at A'. The points B', C' and D' are defined similarly. Show that $GA.GB.GC.GD \le GA'.GB'.GC'.GD'$ and $1/GA + 1/GB + 1/GC + 1/GC \ge 1/GA' + 1/GB' + 1/GC' + 1/GD'$.

G7. O is a point inside the convex quadrilateral ABCD. The line through O parallel to AB meets the side BC at L and the line through O parallel to BC meets the side AB at K. The line through O parallel to AD meets the side CD at M and the line through O parallel to CD meets the side DA at N. The area of ABCD is k the area of AKON is k_1 and the area of LOMC is k_2. Show that $k^{1/2} \ge k_1^{1/2} + k_2^{1/2}$.

G8. ABC is a triangle. A circle through B and C meets the side AB again at C' and meets the side AC again at B'. Let H be the orthocenter of ABC and H' the orthocenter of AB'C'. Show that the lines BB', CC' and HH' are concurrent.

Number theory and combinatorics

N1. k is a positive integer. Show that there are infinitely many squares of the form $2^kn - 7$.

N2. Show that for any integers a, b one can find an integer c such that there are no integers m, n with $m^2 + am + b = 2n^2 + 2n + c$.

N4. Find all positive integers m, n such that $m + n^2 + d^3 = mnd$, where d is the greatest common divisor of m and n.

N5. A graph has 12k points. Each point has 3k+6 edges. For any two points the number of points joined to both is the same. Find k.

N7. Does there exist n > 1 such that the set of positive integers may be partitioned into n non-empty subsets so that if we take an arbitrary element from every set but one then their sum belongs to the remaining set?

N8. For each odd prime p, find positive integers m, n such that $m \le n$ and $(2p)^{1/2} - m^{1/2} - n^{1/2}$ is non-negative and as small as possible.

Sequences

S1. Find a sequence f(1), f(2), f(3), ... of non-negative integers such that 0 occurs in the sequence, all positive integers occur in the sequence infinitely often, and $f(f(n^{163})) = f(f(n)) + f(f(361))$.

S3. For any integer n > 1, let p(n) be the smallest prime which does not divide n and let q(n) = the product of all primes less than p(n), or 1 if p(n) = 2. Define the sequence $a_0, a_1, a_2, \ldots$ by $a_0 = 1$ and $a_{n+1} = a_n p(a_n)/q(a_n)$. Find all n such that $a_n = 1995$.

S4. x is a positive real such that $1 + x + x^2 + \ldots x^{n-1} = x^n$. Show that $2 - 1/2^{n-1} \le x < 2 - 1/2^n$.

S5. The function f(n) is defined on the positive integers as follows. f(1) = 1. f(n+1) is the largest positive integer m such that there is a strictly increasing arithmetic progression of m positive integers ending with n such that f(k) = f(n) for each k in the arithmetic progression. Show that there are positive integers a and b such that f(an + b) = n + 2 for all positive integers n.

S6. Show that there is a unique function f on the positive integers with positive integer values such that $f(m + f(n)) = n + f(m + 95)$ for all m, n. Find $f(1) + f(2) + \ldots + f(19)$.

Note: problems A1, G1, G5, N3, N6, S2 were used in the Olympiad and are not shown here.

37th IMO 1996 shortlisted problems

1. x, y, z are positive real numbers with product 1. Show that $xy/(x^5 + xy + y^5) + yz/(y^5 + yz + z^5) + zx/(z^5 + zx + x^5) \le 1$. When does equality occur?

2. $x_1 \ge x_2 \ge ... \ge x_n$ are real numbers such that $x_1^k + x_2^k + ... + x_n^k \ge 0$ for all positive integers k. Let $d = \max\{ |x_1|, ... , |x_n| \}$. Show that $x_1 = d$ and that $(x - x_1)(x - x_2) ... (x - x_n) \ge x^n - d^n$ for all real $x > d$.

3. Given $a > 2$, define the sequence $a_0, a_1, a_2, ...$ by $a_0 = 1$, $a_1 = a$, $a_{n+2} = a_{n+1}(a_{n+1}^2/a_n^2 - 2)$. Show that $1/a_0 + 1/a_1 + 1/a_2 + ... + 1/a_n < 2 + a - (a^2 - 4)^{1/2}$.

4. Show that the polynomial $x^n - a_1x^{n-1} - ... - a_n = 0$, where a_i are non-negative reals, not all zero, has just one positive real root. Let this root be k. Put $s = a_1 + a_2 + ... + a_n$ and $s' = a_1 + 2a_2 + 3a_3 + ... + na_n$. Show that $s^s \le k^{s'}$.

5. The real polynomial $p(x) = ax^3 + bx^2 + cx + d$ is such that $|p(x)| \le 1$ for all x such that $|x| \le 1$. Show that $|a| + |b| + |c| + |d| \le 7$.

6. Show that there are polynomials p(x), q(x) with integer coefficients such that $p(x)\,(x + 1)^{2n} + q(x)\,(x^{2n} + 1) = k$, for some positive integer k. Find the smallest such k (for each n).

7. f is a real-valued function on the reals such that $| f(x) | <= 1$ and $f(x + 13/42) + f(x) = f(x + 1/6) + f(x + 1/7)$ for all x. Show that there is a real number $c > 0$ such that $f(x + c) = f(x)$ for all x.

9. The sequence $a_1, a_2, a_3, ...$ is defined by $a_1 = 0$ and $a_{4n} = a_{2n} + 1$, $a_{4n+1} = a_{2n} - 1$, $a_{4n+2} = a_{2n+1} - 1$, $a_{4n+3} = a_{2n+1} + 1$. Find the maximum and minimum values of a_n for n = 1, 2, ... , 1996 and the values of n at which they are attained. How many terms a_n for n = 1, 2, ... , 1996 are 0?

10. ABC is a triangle with orthocenter H. P is a point on the circumcircle (distinct from the vertices). BE is an altitude. Q is the intersection of the line through A parallel to PB and the line through B parallel to AP. R is the intersection of the line through A parallel to PC and the line through C parallel to PA. The lines HR and AQ meet at X. Show that EX is parallel to AP.

12. ABC is an acute-angled triangle with BC longer than AC. O is the circumcenter and H is the orthocenter. CF is an altitude. The line through F perpendicular to OF meets the side AC at P. Show that $\angle FHP = \angle A$.

13. P is a point inside the equilateral triangle ABC. The lines AP, BP, CP meet the opposite sides at A', B', C' respectively. Show that $A'B'\cdot B'C'\cdot C'A' \ge A'B\cdot B'C\cdot C'A$.

15. One rectangle has sides a, b and another has sides c, d, where $a < c \le d < b$ and $ab < cd$. Show that the rectangle with smaller area can be placed inside the other rectangle iff $(b^2 - a^2)^2 <= (bc - ad)^2 + (bd - ac)^2$.

16. ABC is an acute-angled triangle with circumcenter O and circumradius R. The line AO meets the circumcircle of BOC again at A'. B' and C' are defined similarly. Show that $OA'.OB'.OC' \ge 8R^3$. When does equality occur?

17. ABCD is a convex quadrilateral. R_A is the circumradius of the three vertices other than A. R_B, R_C and R_D are defined similarly. Show that $R_A + R_C > R_B + R_D$ iff $\angle A + \angle C > \angle B + \angle D$.

18. On the plane are given a point O and a polygon U (not necessarily convex). Let P denote the perimeter of U, D the sum of the distances from O to the vertices of U, and H the sum of the distances from O to the lines containing the sides of U. Prove that $D^2 - H^2 \ge P^2/4$.

19. We start with the numbers a, b, c, d. We then replace them with a' = a - b, b' = b - c, c' = c - d, d' = d - a. We carry out this process 1996 times. Is it possible to end up with numbers A, B, C, D such that |BC - AD|, |AC - BD|, |AB - CD| are all primes?

21. A finite sequence of integers $a_0, a_1, \ldots, a_n$ is called *quadratic* if $|a_1 - a_0| = 1^2$, $|a_2 - a_1| = 2^2$, ... , $|a_n - a_{n-1}| = n^2$. Show that any two integers h, k can be linked by a quadratic sequence (in other words for some n we can find a quadratic sequence a_i with $a_0 = h$, $a_n = k$). Find the shortest quadratic sequence linking 0 and 1996.

22. Find all positive integers m and n such that $[m^2/n] + [n^2/m] = [m/n + n/m] + mn$.

23. Let X be the set of non-negative integers. Find all bijections f on X such that f(3mn + m + n) = 4 f(m) f(n) + f(m) + f(n) for all m, n.

24. An (n-1) x (n-1) square is divided into $(n-1)^2$ unit squares. Each of the n^2 vertices is colored red or blue. Find the number of possible colorings such that every small square has two vertices of each color.

26. k, m, n are positive integers such that 1 < n < m < k+2. What is the largest subset of {1, 2, 3, ... , k} which has no n distinct elements with sum m.

27. Do there exist two disjoint infinite sets A and B of points in the plane such that: (1) no three points in the union of A and B are collinear; (2) the distance between any two points in the union of A and B is at least 1; (3) given any three points in B, there is a point of A in the triangle formed by the three points; (4) given any three points in A, there is a point of B in the triangle formed by the three points? A triangle means all points in or on the triangle (the vertices, all points on the sides of the triangle and all points in the interior of the triangle).

28. A finite number of beans are placed in an infinite row of baskets. A move is to select a basket with at least two beans and to remove two beans, putting one in the basket on the left and the other in the basket on the right. Moves are made until no basket contains more than one bean. Show that the number of moves and the final configuration depend only on the initial configuration.

30. X is a finite set and f, g are bijections on X such that for any point x in X either f(f(x)) = g(g(x)) or f(g(x)) = g(f(x)) or both. Show that for any x, f(f(f(x))) = g(g(f(x))) iff f(f(g(x))) = g(g(g(x))).

Note: problems 8, 11, 14, 20, 25, 29 were used in the Olympiad and are not shown here.

38th IMO 1997 shortlisted problems

2. The sequences R_n are defined as follows. $R_1 = (1)$. If $R_n = (a_1, a_2, ... , a_m)$, then $R_{n+1} = (1, 2, ... , a_1, 1, 2, ... , a_2, 1, 2, ... , 1, 2, ... , a_m, n+1)$. For example, $R_2 = (1, 2)$, $R_3 = (1, 1, 2, 3)$, $R_4 = (1, 1, 1, 2, 1, 2, 3, 4)$. Show that for $n > 1$, the kth term from the left in R_n is 1 iff the kth term from the right is not 1.

3. If S is a finite set of non-zero vectors in the plane, then a maximal subset is a subset whose vector sum has the largest possible magnitude. Show that if S has n vectors, then there are at most 2n maximal subsets of S. Give a set of 4 vectors with 8 maximal subsets and a set of 5 vectors with 10 maximal subsets.

5. Let ABCD be a regular tetrahedron. Let M be a point in the plane ABC and N a point different from M in the plane ADC. Show that the segments MN, BN and MD can be used to form a triangle.

6. Let a, b, c be positive integers such that a and b are relatively prime and c is relatively prime to a or b. Show that there are infinitely many solutions to $m^a + n^b = k^c$, where m, n, k are distinct positive integers.

7. ABCDEF is a convex hexagon with AB = BC, CD = DE, EF = FA. Show that $BC/BE + DE/DA + FA/FC \geq 3/2$. When does equality occur?

9. ABC is a non-isosceles triangle with incenter I. The smaller circle through I tangent to CA and CB meets the smaller circle through I tangent to BC and BA at A' (and I). B' and C' are defined similarly. Show that the circumcenters of AIA', BIB' and CIC' are collinear.

10. Find all positive integers n such that if p(x) is a polynomial with integer coefficients such that $0 \leq p(k) \leq n$ for $k = 0, 1, 2, ... , n+1$ then $p(0) = p(1) = ... = p(n+1)$.

11. p(x) is a polynomial with real coefficients such that $p(x) > 0$ for $x \geq 0$. Show that $(1 + x)^n p(x)$ has non-negative coefficients for some positive integer n.

12. p is prime. q(x) is a polynomial with integer coefficients such that q(k) = 0 or 1 mod p for every positive integer k, and q(0) = 0, q(1) = 1. Show that the degree of q(x) is at least p-1.

13. In town A there are n girls and n boys and every girl knows every boy. Let a(n,r) be the number of ways in which r girls can dance with r boys, so that each girl knows her partner. In town B there are n girls and 2n-1 boys such that girl i knows boys 1, 2, ... , 2i-1 (and no others). Let b(n,r) be the number of ways in which r girls from town B can dance with r boys from town B so that each girl knows her partner. Show that a(n,r) = b(n,r).

14. $b > 1$ and $m > n$. Show that if $b^m - 1$ and $b^n - 1$ have the same prime divisors then $b + 1$ is a power of 2. [For example, $7 - 1 = 2.3$, $7^2 - 1 = 2^4.3$.]

15. If an infinite arithmetic progression of positive integers contains a square and a cube, show that it must contain a sixth power.

16. ABC is an acute-angled triangle with incenter I and circumcenter O. AD and BE are altitudes, and AP and BQ are angle bisectors. Show that D, I, E are collinear iff P, O, Q are collinear.

18. ABC is an acute-angled triangle. The altitudes are AD, BE and CF. The line through D parallel to EF meets AC at Q and AB at R. The line EF meets BC at P. Show that the midpoint of BC lies on the circumcircle of PQR.

19. Let $x_1 \geq x_2 \geq x_3 ... \geq x_{n+1} = 0$. Show that $\sqrt{(x_1 + x_2 + ... + x_n)} \leq (\sqrt{x_1} - \sqrt{x_2}) + (\sqrt{2}) (\sqrt{x_2} - \sqrt{x_3}) + ... + (\sqrt{n}) (\sqrt{x_n} - \sqrt{x_{n+1}})$.

20. ABC is a triangle. D is a point on the side BC (not at either vertex). The line AD meets the circumcircle again at X. P is the foot of the perpendicular from X to AB, and Q is the foot of the perpendicular from X to AC. Show that the line PQ is a tangent to the circle on diameter XD iff AB = AC.

22. Do there exist real-valued functions f and g on the reals such that f(g(x)) = x^2 and g(f(x)) = x^3? Do there exist real-valued functions f and g on the reals such that f(g(x)) = x^2 and g(f(x)) = x^4?

23. ABCD is a convex quadrilateral and X is the point where its diagonals meet. XA sin A + XC sin C = XB sin B + XD sin D. Show that ABCD must be cyclic.

25. ABC is a triangle. The bisectors of A, B, C meet the circumcircle again at K, L, M respectively. X is a point on the side AB (not one of the vertices). P is the intersection of the line through X parallel to AK and the line through B perpendicular to BL. Q is the intersection of the line through X parallel to BL and the line through A perpendicular to AK. Show that KP, LQ and MX are concurrent.

26. Find the minimum value of $x_0 + x_1 + ... + x_n$ for non-negative real numbers x_i such that $x_0 = 1$ and $x_i \le x_{i+1} + x_{i+2}$.

Note: problems 1, 4, 8, 17, 21, 24 were used in the Olympiad and are not shown here.

39th IMO 1998 shortlisted problems

Algebra

A1. $x_1, x_2, \dots, x_n$ are positive reals with sum less than 1. Show that $n^{n+1}x_1x_2 \dots x_n(1 - x_1 - \dots - x_n) \le (x_1 + x_2 + \dots + x_n)(1 - x_1)(1 - x_2) \dots (1 - x_n)$.

A2. $x_1, x_2, \dots, x_n$ are reals not less than 1. Show that $1/(1 + x_1) + 1/(1 + x_2) + \dots + 1/(1 + x_n) \ge n/(1 + (x_1x_2 \dots x_n)^{1/n})$.

A3. x, y, z are positive reals with product 1. Show that $x^3/((1 + y)(1 + z)) + y^3/((1 + z)(1 + x)) + z^3/((1 + x)(1 + y)) \ge 3/4$.

A4. Define the numbers c(n, k) for k = 0, 1, 2, ... , n by c(n, 0) = c(n, n) = 1, $c(n+1, k) = 2^k c(n, k) + c(n, k-1)$. Show that c(n, k) = c(n-k, k).

Combinatorics

C1. An m x n array of real numbers has the sum of each row and column integral. Show that each non-integral element x can be changed to [x] or [x] + 1, so that the row and column sums are unchanged.

C2. n is a fixed positive integer. An *odd n-admissible* sequence $a_1, a_2, a_3, \dots$ satisfies the following conditions: (1) $a_1 = 1$; (2) $a_{2k} = a_{2k-1} + 2$ or $a_{2k-1} + n$; (3) $a_{2k+1} = 2a_{2k}$ or $n\, a_{2k}$. An *even n-admissible* sequence satisfies (1) $a_1 = 1$; (2) $a_{2k} = 2a_{2k-1}$ or $n\, a_{2k-1}$; (3) $a_{2k+1} = a_{2k} + 2$ or $a_{2k} + n$. An integer m > 1 is *n-attainable* if it belongs to an odd n-admissible sequence or an even n-admissible sequence. Show that for n > 8 there are infinitely many positive integers which are not n-attainable. Show that all positive integers except 7 are 3-attainable.

C3. The numbers from 1 to 9 are written in an arbitrary order. A *move* is to reverse the order of any block of consecutive increasing or decreasing numbers. For example, a move changes 91**6532**748 to 91**3562**748. Show that at most 12 moves are needed to arrange the numbers in increasing order.

C4. If A is a permutation of 1, 2, 3, ... , n and B is a subset of {1, 2, ... , n}, then we say that A splits B if we can find three elements of A such that the middle element does not belong to B, but the outer two do. For example, the permutation 13542 of 12345 splits {1, 2, 3} (because, for example, 4 appears between 3 and 2) but it does not split {3, 4, 5}. Show that if we take any n-2 subsets of {1, 2, ... , n}, each with at least 2 and at most n-1 elements, then there is a permutation of 1, 2, ... , n which splits all of them.

C6. G is the complete graph on 10 points (so that there is an edge between each pair of points). Show that we can color each edge with one of 5 colors so that for any 5 points we can find 5 differently colored edges all of whose endpoints are amongst the 5 points. Show that we cannot color each edge with one of 4 colors so that for any 4 points we can find 4 differently colored edges with all endpoints amongst the 4 points.

C7. Some cards have one side white and the other side black. A card is placed on each square of an m x n board. All cards are white side up, except for the card in the top left corner. A move is to remove any black card and to turn over any cards on squares which share an edge with the square from which the card was taken. For which m, n can one remove all the cards from the board by a sequence of moves.

Geometry

G2. ABCD is a cyclic quadrilateral. E and F are variable points on the sides AB and CD respectively, such that AE/EB = CF/FD. P is a point on the segment EF such that EP/PF = AB/CD. Show that area APD/area BPC does not depend on the positions of E and F.

G4. M and N are points inside the triangle ABC such that $\angle MAB = \angle NAC$ and $\angle MBA = \angle NBC$. Show that $AM \cdot AN/(AB \cdot AC) + BM \cdot BN/(BA \cdot BC) + CM \cdot CN/(CA \cdot CB) = 1$.

G5. ABC is a triangle with circumcenter O, orthocenter H and circumradius R. The points D, E, F are the reflections of A, B, C respectively, in the opposite sides. Show that they are collinear iff OH = 2R.

G6. ABCDEF is a convex hexagon with $\angle B + \angle D + \angle F = 360^o$ and $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$. Show that $BC \cdot DF \cdot EA = EF \cdot AC \cdot BD$.

G7. ABC is a triangle with angle $C = 2\angle B$. D is a point on the side BC such that DC = 2 BD. E is a point on the line AD such that D is the midpoint of AE. Show that $\angle ECB + 180^o = 2\angle EBC$.

G8. ABC is a triangle with angle $A = 90^o$. The tangent at A to the circumcircle meets the line BC at D. E is the reflection of A in BC. X is the foot of the perpendicular from A to BE. Y is the midpoint of AX. The line BY meets the circumcircle again at Z. Show that BD is tangent to the circumcircle of ADZ.

Number theory

N2. Find all pairs of real numbers (x, y) such that $x [ny] = y [nx]$ for all positive integers n.

N3. Find the smallest n such that given any n distinct integers one can always find 4 different integers a, b, c, d such that $a + b = c + d \mod 20$.

N4. The sequence $a_1, a_2, a_3, \ldots$ is defined as follows. $a_1 = 1$. a_n is the smallest integer greater than a_{n-1} such that we cannot find $1 \le i, j, k \le n$ (not necessarily distinct) such that $a_i + a_j = 3a_k$. Find a_{1998}.

N5. Find all positive integers n for which there is an integer m such that $m^2 + 9$ is a multiple of $2^n - 1$.

N7. Show that for any n > 1 there is an n digit number with all digits non-zero which is divisible by the sum of its digits.

N8. The sequence $0 \le a_0 < a_1 < a_2 < \ldots$ is such that every non-negative integer can be uniquely expressed as $a_i + 2a_j + 4a_k$ (where i, j, k are not necessarily distinct). Find a_{1998}.

Note: problems A5, C6, G1, G3, N1 and N6 were used in the Olympiad and are not shown here.

40th IMO 1999 shortlisted problems

Algebra

A2. The numbers 1 to n^2 are arranged in the squares of an n x n board (1 per square). There are n^2 (n-1) pairs of numbers in the same row or column. For each such pair take the larger number divided by the smaller. Then take the smallest such ratio and call it the *minrat* of the arrangement. So, for example, if n^2 and n^2 - 1 were in the same row, then the minrat would be $n^2/(n^2 - 1)$. What is the largest possible minrat?

A3. Each of n > 1 girls has a doll with her name on it. Each pair of girls swaps dolls in some order. For which values of n is it possible for each girl to end up (1) with her own doll, (2) with another girl's doll? [For example, if n = 4, denote girls by A, B, C, D and dolls by a, b, c, d. So the initial position is Aa Bb Cc Dd. If the swaps are in the order AB, CD, AC, BD, AD, BC, then the successive positions are Ab Ba Cc Dd, Ab Ba Cd Dc, Ad Ba Cb Dc, Ad Bc Cb Da, Aa Bc Cb Dd and Aa Bb Cc Dd. So for n = 4, (1) is possible.]

A4. Show that we cannot partition the positive integers into three non-empty parts, so that if a and b belong to different parts, then $a^2 - ab + b^2$ belongs to the third part.

A6. Given real numbers $x_1 \leq x_2$ <= ... <= x_n (with n > 2), carry out the following procedure:

(2) arrange the numbers in a circle;

(3) delete one of the numbers;

(4) if just two numbers are left, then take their sum. Otherwise replace each number by the sum of it and the number on its right. Go to step 2. Show that the largest sum that can be achieved by this procedure is (n-2)C0 x_2 + (n-2)C0 x_3 + (n-2)C1 x_4 + (n-2)C1 x_5 + (n-2)C2 x_6 + ... + (n-2)C([n/2]-1) x_n, where mCk represents the binomial coefficient.

Combinatorics

C1. A *low path* from the lattice point (0, 0) to the lattice point (n, n) is a sequence of 2n moves, each one point up or one point to the right, starting at (0, 0) and ending at (n, n) such that all points of the path satisfy y <= x. A *step* is two consecutive moves, the first right and the second up. Show that the number of low paths from (0, 0) to (n, n) with just k steps is 1/k (n-1)C(k-1) nC(k-1), where mCk is the binomial coefficient.

C2. A *tile* is made of 5 unit squares as shown. Show that if a 5 x n rectangle can be covered with n tiles, then n is even. [Tiles can be turned over and rotated.] Show that a 5 x 2n rectangle can be tiled in at least $3^{n-1}2$ ways.

C3. A chameleon repeatedly rests and then catches a fly. The first rest is for a period of 1 minute. The rest before catching the fly 2n is the same as the rest before catching fly n. The rest before catching fly 2n+1 is 1 minute more than the rest before catching fly 2n. How many flies does the chameleon catch before his first rest of 9 minutes? How many minutes (in total) does the chameleon rest before catching fly 98? How many flies has the chameleon caught after 1999 total minutes of rest?

C4. Let A be any set of n residues mod n^2. Show that there is a set B of n residues mod n^2 such that at least half of the residues mod n^2 can be written as a + b with a in A and b in B.

C6. Every integer is colored red, blue, green or yellow. m and n are distinct odd integers such that m + n is not zero. Show that we can find two integers a and b with the same color such that a - b = m, n m + n or m - n.

C7. Let $p > 3$ be a prime. Let h be the number of sequences $a_1, a_2, \ldots, a_{p-1}$ such that $a_1 + 2a_2 + 3a_3 + \ldots + (p-1)a_{p-1}$ is divisible by p and each a_i is 0, 1 or 2. Let k be defined similarly except that each a_i is 0, 1 or 3. Show that $h \le k$ with equality iff $p = 5$.

Geometry

G1. P is a point inside the triangle ABC. k = min(PA, PB, PC). Show that k + PA + PB + PC ≤ AB + BC + CA.

G2. Given any 5 points, no three collinear and no four concyclic, show that just 4 of the 10 circles through 3 points contain just one of the other two points.

G4. ABC is a triangle. The point X on the side AB is such that AX/XB = 4/5. The point Y on the segment CX is such that CY = 2 XY, and the point Z on the ray CA is such that $\angle CXZ + \angle B = 180^o$. Also $\angle XYZ = 45^o$. Show that the angles of ABC are determined and find the smallest angle.

G5. ABC is a triangle with inradius r. The circle through A and C orthogonal to the incircle meets the circle through A and B orthogonal to the incircle at A and A'. The points B' and C' are defined similarly. Show that the circumradius of A'B'C' is r/2.

G7. P is a point inside the convex quadrilateral ABCD such that PA = PC, $\angle APB = \angle PAD + \angle PCD$, and $\angle CPD = \angle PCB + \angle PAB$. Show that AB·PC = BC·PD and CD·PA = DA·PB.

G8. X is a variable point on the arc AB of the circumcircle of ABC which does not contain C. I' is the incenter of AXC and I" is the incenter of BXC. Show that the circumcircle of XI'I" passes through a fixed point of the circumcircle of ABC.

Number theory

N2. Prove that every positive rational number can be written as $(a^3 + b^3)/(c^3 + d^3)$ for some positive integers a, b, c, d.

N3. Show that there exist two strictly increasing sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ such that $a_n(a_n + 1)$ divides $b_n^2 + 1$ for each n.

N4. Show that there are infinitely many primes p such that 1/p has its shortest period divisible by 3: $1/p = 0.a_1a_2 \ldots a_{3k}a_1a_2\ldots$. Amongst such primes find the largest possible value for $a_i + a_{i+k} + a_{i+2k}$. For example, 1/7 = 0.142857... So the possible values for p = 7 are 1+2+5 = 8 and 4+8+7 = 19.

N5. Show that for any n not divisible by 3 and k >= n, there is a multiple of n which has sum of digits k.

N6. Show that for any k > 0 we can find an infinite arithmetic progression of positive integers each of which has sum of digits greater than k and where the common difference is not a multiple of 10.

Note: problems A1, A5, C5, G3, G6 and N1 were used in the Olympiad and are not shown here.

41st IMO 2000 shortlisted problems

Algebra

A2. a, b, c are positive integers such that $b > 2a$, $c > 2b$. Show that there is a real k such that the fractional parts of ka, kb, kc all exceed 1/3 and do not exceed 2/3.

A3. Find all pairs of real-valued functions f, g on the reals such that $f(x + g(y)) = x f(y) - y f(x) + g(x)$ for all real x, y.

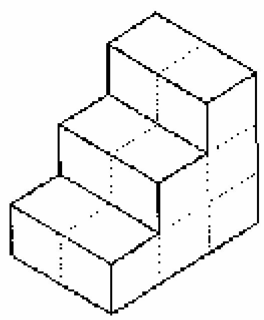
A4. The function f on the non-negative integers takes non-negative integer values and satisfies $f(4n) = f(2n) + f(n)$, $f(4n+2) = f(4n) + 1$, $f(2n+1) = f(2n) + 1$ for all n. Show that the number of non-negative integers n such that $f(4n) = f(3n)$ and $n < 2^m$ is $f(2^{m+1})$.

A6. $0 = a_0 < a_1 < a_2 < ...$ and $0 = b_0 < b_1 < b_2 < ...$ are sequences of real numbers such that: (1) if $a_i + a_j + a_k = a_r + a_s + a_t$, then (i, j, k) is a permutation of (r, s, t); and (2) a positive real x can be represented as $x = a_j - a_i$ iff it can be represented as $b_m - b_n$. Prove that $a_k = b_k$ for all k.

A7. A polynomial p(x) of degree 2000 with distinct real coefficients satisfies condition n if (1) $p(n) = 0$ and (2) if q(x) is obtained from p(x) by permuting its coefficients, then either $q(n) = 0$, or we can obtain a polynomial r(x) by transposing two coefficients of q(x) such that $r(n) = 0$. Find all integers n for which there is a polynomial satisfying condition n.

Combinatorics

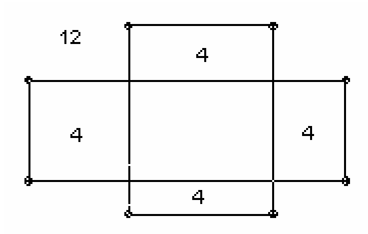
C2. A piece is made of 12 unit cubes. It looks like a staircase of 3 steps, each of width 2. Thus the bottom layer is 2 x 3, the second layer is 2 x 2 and the top layer is 1 x 2. For which n can we make an n x n x n cube with such pieces?



C3. $n > 3$. $S = \{P_1, ... , P_n\}$ is a set of n points in the plane, no three collinear and no four concyclic. Let a_i be the number of circles through three points of S which have P_i as an interior point. Let $m(S) = a_1 + ... + a_n$. Show that the points of S are the vertices of a convex polygon iff $m(S) = f(n)$, where f(n) is a value that depends only on n.

C4. Find the smallest number of pawns that can be placed on an n x n board such that no row or column contains k adjacent unoccupied squares. Assume that $n/2 < k \le 2n/3$.

C5. n rectangles are drawn in the plane. Each rectangle has parallel sides and the sides of distinct rectangles lie on distinct lines. The rectangles divide the plane into a number of regions. For each region R let v(R) be the number of vertices. Take the sum $\sum v(R)$ over the regions which have one or more vertices of the rectangles in their boundary. Show that this sum is less than 40n.



For example, for the two rectangles illustrated the sum is 28 < 80. Note that the unbounded outer region has 12 vertices, and we do not count the central region because it does not contain any vertices of the two rectangles.

C6. a and b are positive coprime integers. A subset S of the non-negative integers is called *admissible* if 0 belongs to S and whenever k belongs to S, so do k + a and k + b. Find f(a, b), the number of admissible sets.

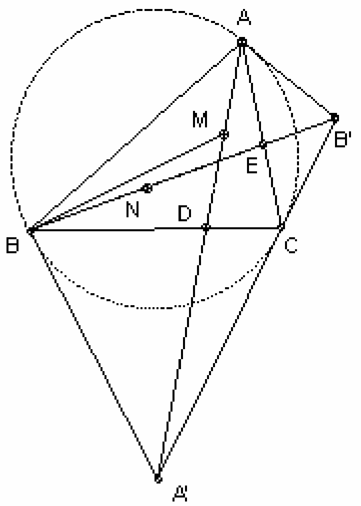
Geometry

G1. Two circles C and C' meet at X and Y. Find four points such that if a circle touches C and C' at P and Q and meets the line XY at R and S, then each of the lines PR, PS, QR, QS passes through one of the four points.

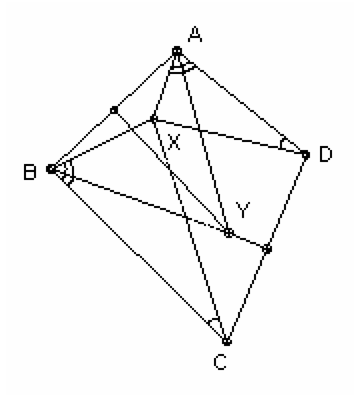
G3. ABC is an acute-angled triangle with orthocenter H and circumcenter O. Show that there are points D, E, F on BC, CA, AB respectively such that OD + DH = OE + EH = OF + FH and AD, BE, CF are concurrent.

G4. Show that a convex n-gon can be inscribed in a circle iff there is a pair of real numbers a_i, b_i associated with each vertex P_i such that the distance between each pair of vertices P_iP_j with $i < j$ is $a_ib_j - a_jb_i$.

G5. ABC is an acute angled triangle. The tangent at A to the circumcircle meets the tangent at C at the point B'. BB' meets AC at E, and N is the midpoint of BE. Similarly, the tangent at B meets the tangent at C at the point A'. AA' meets BC at D, and M is the midpoint of AD. Show that $\angle$ ABM = $\angle$ BAN. If AB = 1, find BC and AC for the triangles which maximise $\angle$ ABM.



G6. ABCD is a convex quadrilateral. The perpendicular bisectors of AB and CD meet at Y. X is a point inside ABCD such that $\angle$ ADX = $\angle$ BCX < 90° and $\angle$ DAX = $\angle$ CBX < 90°. Show that $\angle$ AYB = 2 $\angle$ ADX.



G7. Ten gangsters are standing in a field. The distance between each pair of gangsters is different. When the clock strikes, each gangster shoots the nearest gangster dead. What is the largest number of gangsters that can survive?

Number theory

N1. Find all positive integers $n > 1$ such that if a and b are relatively prime then $a = b \mod n$ iff $ab = 1 \mod n$.

N2. Find all positive integers n such that the number of positive divisors of n is $(4n)^{1/3}$.

N4. Find all positive integers a, m, n such that $a^m + 1$ divides $(a + 1)^n$.

N5. Show that for infinitely many positive integers n, we can find a triangle with integral sides whose semiperimeter divided by its inradius is n.

N6. Show that all but finitely many positive integers can be represented as a sum of distinct squares.

Note: problems A1, A5, C1, G2, G8 and N3 were used in the Olympiad and are not shown here.

42nd IMO 2001 shortlisted problems

Algebra

A1. Let T be the set of all triples (a, b, c) where a, b, c are non-negative integers. Find all real-valued functions f on T such that f(a, b, c) = 0 if any of a, b, c are zero and f(a, b, c) = 1 + 1/6 f(a+1, b-1, c) + 1/6 f(a-1, b+1, c) + 1/6 f(a+1, b, c-1) + 1/6 f(a-1, b, c+1) + 1/6 f(a, b+1, c-1) + 1/6 f(a, b-1, c+1) for a, b, c all positive.

A2. Show that any infinite sequence a_n of positive numbers satisfies $a_n > a_{n-1}2^{1/n} - 1$ for infinitely many n.

A3. Show that $x_1/(1 + x_1^2) + x_2/(1 + x_1^2 + x_2^2) + ... + x_n/(1 + x_2^2 + ... + x_n^2) < \sqrt{n}$ for all real numbers x_i.

A4. Find all real-valued functions f on the reals such that f(xy) (f(x) - f(y)) = (x - y) f(x) f(y) for all x, y.

A5. Find all positive integers $a_0 = 1, a_1, a_2, ... , a_n$ such that $a_0/a_1 + a_1/a_2 + ... + a_{n-1}/a_n = 99/100$ and $(a_{k+1} - 1)a_{k-1} \geq a_k^3 - a_k^2$ for k = 1, 2, ... , n-1.

Combinatorics

C1. What is the largest number of subsequences of the form n, n+1, n+2 that a sequence of 2001 positive integers can have? For example, the sequence 1, 2, 2, 3, 3, of 5 terms has 4 such subsequences.

C3. A finite graph is such that there are no subsets of 5 points with all 15 edges and every two triangles have at least one point in common. Show that there are at most two points X such that removing X leaves no triangles.

C4. Let P be the set of all sets of three integers of the form {n, n+1776, n + 3777} or {n, n+2001, n+3777}. Show that we can write the set {0, 1, 2, 3, ... } as a union of disjoint members of P.

C5. Find all finite sequences $a_0, a_1, a_2, ... , a_n$ such that a_m equals the number of times that m appears in the sequence.

C6. Show that there is a set P of (2n)!/(n! (n+1)!) sequences of 2n terms, half 1s and half 0s, such that any sequence of 2n terms, half 1s and half 0s, is either in P or can be derived from a member of P by *moving* one term. Moving a term means changing $a_1, a_2, ... , a_{2n}$ to $a_1, a_2, ... , a_{i-1}, a_{i+1}, ... , a_ja_ia_{j+1} ... a_n$ for some i, j. For example, by moving the initial 0 we can change **0**110 to 1**0**10 or 11**0**0, or by moving the first 1 we can change 0**1**10 to **1**010 or 010**1**.

C7. There are n piles of stones in a line. If a pile contains at least two stones more than the pile on its right, then a stone can be moved to the pile on its right. Initially, all the piles are empty except the leftmost pile which has n stones. If more than one move is possible, then a possible move is chosen arbitrarily. Show that after a finite number of moves, no more moves are possible and that the final configuration is independent of the moves made. Describe the final configuration. For example, we n = 6, one might get successively: 5 1, 4 2, 4 1 1, 3 2 1.

Geometry

G1. ABC is an acute-angled triangle. A' is the center of the square with two vertices on BC, one on AB and one on AC. Similarly, B' is the center of the square with two vertices on CA, one on AB and one on BC, and C' is the center of the square with two vertices on AB, one on BC and one on CA. Show that AA', BB' and CC' are concurrent.

G3. ABC is a triangle with centroid G and side lengths a = BC, b = CA, c = AB. Find the point P in the plane which minimises AP.AG + BP.BG + CP.CG and find the minimum in terms of a, b, c.

G4. P is a point inside the triangle ABC. The feet of the perpendiculars to the sides are D, E, F. Find the point P which maximises PD.PE.PF/(PA.PB.PC). Which triangles give the largest maximum value?

G5. ABC is an acute-angled triangle. B' is a point on the perpendicular bisector of AC on the opposite side of AC to B such that angle AB'C = 2A. A' and C' are defined similarly (with angle CA'B = 2C, angle BC'A = 2B). The lines AA' and B'C' meet at A". The points B" and C" are defined similarly. Find AA'/A"A' + BB'/B"B' + CC'/C"C'.

G6. P is a point outside the triangle ABC. The perpendiculars from P meet the lines BC, CA, AB at D, E, F, respectively. The triangles PAF, PBD, PCE all have equal area. Show that their area must equal that of ABC.

G7. P is a point inside acute-angled the triangle ABC. The perpendiculars from P meet the sides BC, CA, AB at D, E, F, respectively. Show that P is the circumcenter iff each of the triangles AEF, BDF, CDE has perimeter not exceeding that of DEF.

Number theory

N1. Prove that we cannot find consecutive factorials with first digits 1, 2, ... , 9.

N2. Find the largest real k such that if a, b, c, d are positive integers such that a + b = c + d, 2ab = cd and a ≥ b, then a/b ≥ k.

N3. The sequence a_n is defined by $a_1 = 11^{11}$, $a_2 = 12^{12}$, $a_3 = 13^{13}$, and $a_{n+3} = |a_{n+2} - a_{n+1}| + |a_{n+1} - a_n|$. Find a_n, where $n = 14^{14}$.

N4. Let p > 3 be a prime. Show that there is a positive integer n < p-1 such that $n^{p-1} - 1$ and $(n+1)^{p-1} - 1$ are not divisible by p^2.

N6. Do there exist 100 positive integers not exceeding 25000 such that the sum of every pair is distinct?

Note: problems A6, C2, C8, G2, G8 and N5 were used in the Olympiad and are not shown here.

43rd IMO 2002 shortlisted problems

Number theory

N1. Express 2002^{2002} as the smallest possible number of (positive or negative) cubes.

N3. If N is the product of n distinct primes, each greater than 3, show that $2^N + 1$ has at least 4^n divisors.

N4. Does the equation 1/a + 1/b + 1/c + 1/(abc) = m/(a + b + c) have infinitely many solutions in positive integers a, b, c for any positive integer m?

N5. m, n are integers > 1. $a_1, a_2, \ldots, a_n$ are integers, and none is a multiple of m^{n-1}. Show that there are integers e_i, not all zero, with $|e_i| < m$, such that $e_1a_1 + e_2a_2 + \ldots + e_na_n$ is a multiple of m^n.

Geometry

G1. B is a point on the circle S. A is a point (distinct from B) on the tangent to S at B. C is a point not on S such that the line segment AC meets the circle S at two distinct points. S' is a circle which touches AC at C and S at D, where B and D are on opposite sides of the line AC. Show that the circumcenter of BCD lies on the circumcircle of ABC.

G2. The sides of the triangle ABC subtend the same angles at the point F inside the triangle. The lines BF, CF meet the sides AC, AB at D, E respectively. Show that AB + AC ≥ 4 DE.

G4. S and S' are circles intersecting at P and Q. A, B are distinct variable points on the circle S not at P or Q. The lines AP, BP meet the circle S' again at A', B' respectively. The lines AB, A'B' meet at C. Show that the circumcenter of AA'C lies on a fixed circle (as A, B vary).

G5. S is a set of 5 coplanar points, no 3 collinear. M(S) is the largest area of a triangle with vertices in S. Similarly, m(S) is the smallest area. What is the smallest possible value of M(S)/m(S) as S varies?

G7. ABC is an acute-angled triangle. The incircle touches BC at K. The altitude AD has midpoint M. The line KM meets the incircle again at N. Show that the circumcircle of BCN touches the incircle of ABC at N.

G8. The circles S and S' meet at A and B. A line through A meets S, S' again at C, D respectively. M is a point on CD. The line through M parallel to BC meets BD at K, and the line through M parallel to BD meets BC at N. The perpendicular to BC at N meets S at the point E on the opposite side of BC to A. The perpendicular to BD at K meets S' at F on the opposite side of BD to A. Show that $\angle EMF = 90^o$.

Algebra

A1. Find all real-valued functions f(x) on the reals such that f(f(x) + y) = 2x + f(f(y) - x) for all x, y.

A2. The infinite real sequence $x_1, x_2, x_3, \ldots$ satisfies $|x_i - x_j| \geq 1/(i + j)$ for all unequal i, j. Show that if all x_i lie in the interval [0, c], then c ≥ 1.

A3. $p(x) = ax^3 + bx^2 + cx + d$ is a polynomial with integer coefficients and a non-zero. We have x p(x) = y p(y) for infinitely many pairs (x, y) of unequal integers. Show that p(x) has an integer root.

A5. Given a positive integer n which is not a cube, define $a = n^{1/3}$, b = 1/(a - [a]), c = 1/(b - [b]). Show that there are infinitely many such n for which we can find integers r, s, t, not all zero, such that ra + sb + tc = 0.

A6. A is a non-empty set of positive integers. There are positive integers $b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n$ such that each set $b_iA + c_i = \{b_ia + c_i$: a belongs to A$\}$ is a subset of A, and the n sets $b_iA + c_i$ are pairwise disjoint. Prove that $1/b_1 + 1/b_2 + \ldots + 1/b_n \le 1$.

Combinatorics

C2. n is an odd integer. The squares of an n x n chessboard are colored alternately black and white, with the four corner squares black. A tromino is an L shape formed from three squares. For which n can the black squares all be covered by non-overlapping trominoes. What is the minimum number required?

C3. A sequence of n positive integers is *full* if for each k > 1, k only occurs if k-1 occurs before the last occurrence of k. How many full sequences are there for each n?

C4. T is the set of all triples (x, y, z) with x, y, z non-negative integers < 10. A chooses a member (X, Y, Z) of T. B seeks to identify it. He is allowed to name a triple (a, b, c) in T. A must then reply with $|X + Y - a - b| + |Y + Z - b - c| + |Z + X - c - a|$. How many triples does B need to name to be sure of determining A's triple?

C5. r > 1 is a fixed positive integers. F is an infinite family of sets, each of size r, no two of which are disjoint. Show that some set of size r - 1 has non-empty intersection with every element of F.

C6. n is even. Show that there is a permutation $x_1, x_2, \ldots, x_n$ of 1, 2, ... n such that for each i, x_{i+1} belongs to $\{2x_i, 2x_i - 1, 2x_i - n, 2x_i - n - 1\}$, where, as usual we take x_{n+1} to mean x_1.

C7. A graph has 120 points. A weak quartet is a set of four points with just one edge. What is the maximum possible number of weak quartets?

Note: problems N2, N6, G3, G6, A4, C1 were used in the Olympiad and are not shown here.

OMCC (1999 – 2003)

1st OMCC 1999

A1. A, B, C, D, E each has a unique piece of news. They make a series of phone calls to each other. In each call, the caller tells the other party all the news he knows, but is not told anything by the other party. What is the minimum number of calls needed for all five people to know all five items of news? What is the minimum for n people?

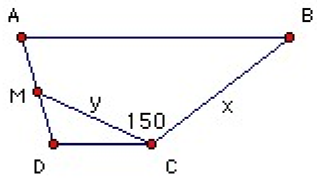
A2. Find a positive integer n with 1000 digits, none 0, such that we can group the digits into 500 pairs so that the sum of the products of the numbers in each pair divides n.

A3. A and B play a game as follows. Starting with A, they alternately choose a number from 1 to 9. The first to take the total over 30 loses. After the first choice each choice must be one of the four numbers in the same row or column as the last number (but not equal to the last number):

7	8	9
4	5	6
1	2	3

Find a winning strategy for one of the players.

B1. ABCD is a trapezoid with AB parallel to CD. M is the midpoint of AD, $\angle MCB = 150^o$, $BC = x$ and $MC = y$. Find area ABCD in terms of x and y.



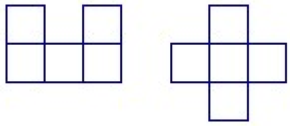
B2. $a > 17$ is odd and $3a-2$ is a square. Show that there are positive integers $b \neq c$ such that $a+b$, $a+c$, $b+c$ and $a+b+c$ are all squares.

B3. $S \subseteq \{1, 2, 3, \ldots, 1000\}$ is such that if m and n are distinct elements of S, then m+n does not belong to S. What is the largest possible number of elements in S?

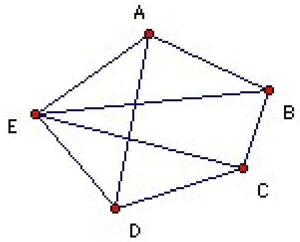
2nd OMCC 2000

A1. Find all three digit numbers abc (with a ≠ 0) such that $a^2 + b^2 + c^2$ divides 26.

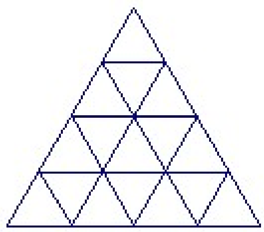
A2. The diagram shows two pentominos made from unit squares. For which n > 1 can we tile a 15 x n rectangle with these pentominos?



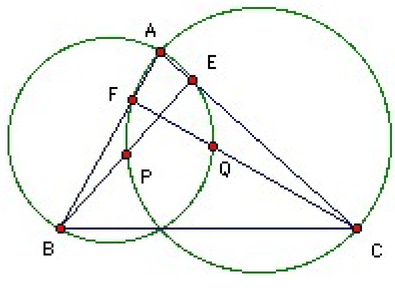
A3. ABCDE is a convex pentagon. Show that the centroids of the 4 triangles ABE, BCE, CDE, DAE from a parallelogram with whose area is 2/9 area ABCD.



B1. Write an integer in each small triangles so that every triangle with at least two neighbors has a number equal to the difference between the numbers in two of its neighbors.



B2. ABC is acute-angled. The circle diameter AC meets AB again at F, and the circle diameter AB meets AC again at E. BE meets the circle diameter AC at P, and CF meets the circle diameter AB at Q. Show that AP = AQ.

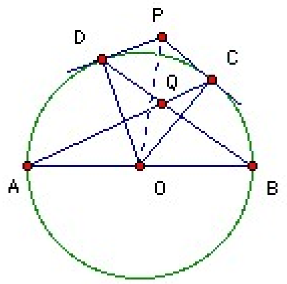


B3. A nice representation of a positive integer n is a representation of n as sum of powers of 2 with each power appearing at most twice. For example, 5 = 4 + 1 = 2 + 2 + 1. Which positive integers have an even number of nice representations?

3rd OMCC 2001

A1. A and B stand in a circle with 2001 other people. A and B are not adjacent. Starting with A they take turns in touching one of their neighbors. Each person who is touched must immediately leave the circle. The winner is the player who manages to touch his opponent. Show that one player has a winning strategy and find it.

A2. C and D are points on the circle diameter AB such that $\angle AQB = 2 \angle COD$. The tangents at C and D meet at P. The circle has radius 1. Find the distance of P from its center.



A3. Find all squares which have only two non-zero digits, one of them 3.

B1. Find the smallest n such that the sequence of positive integers $a_1, a_2, \dots, a_n$ has each term ≤ 15 and $a_1! + a_2! + \dots + a_n!$ has last four digits 2001.

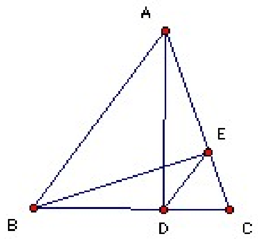
B2. a, b, c are reals such that if p_1, p_2 are the roots of $ax^2 + bx + c = 0$ and q_1, q_2 are the roots of $cx^2 + bx + a = 0$, then p_1, q_1, p_2, q_2 is an arithmetic progression of distinct terms. Show that $a + c = 0$.

B3. 10000 points are marked on a circle and numbered clockwise from 1 to 10000. The points are divided into 5000 pairs and the points of each pair are joined by a segment, so that each segment intersects just one other segment. Each of the 5000 segments is labeled with the product of the numbers at its endpoints. Show that the sum of the segment labels is a multiple of 4.

4th OMCC 2002

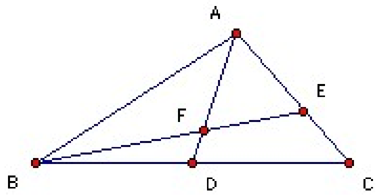
A1. For which n > 2 can the numbers 1, 2, ... , n be arranged in a circle so that each number divides the sum of the next two numbers (in a clockwise direction)?

A2. ABC is acute-angled. AD and BE are altitudes. area BDE ≤ area DEA ≤ area EAB ≤ ABD. Show that the triangle is isosceles.



A3. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = A$, $a_{n+1} = a_n + d(an)$, where d(m) is the largest factor of m which is < m. For which integers A > 1 is 2002 a member of the sequence?

B1. ABC is a triangle. D is the midpoint of BC. E is a point on the side AC such that BE = 2AD. BE and AD meet at F and $\angle FAE = 60^o$. Find $\angle FEA$.



B2. Find an infinite set of positive integers such that the sum of any finite number of distinct elements of the set is not a square.

B3. A path from (0,0) to (n,n) on the lattice is made up of unit moves upward or rightward. It is *balanced* if the sum of the x-coordinates of its 2n+1 vertices equals the sum of their y-coordinates. Show that a balanced path divides the square with vertices (0,0), (n,0), (n,n), (0,n) into two parts with equal area.

5th OMCC 2003

A1. There are 2003 stones in a pile. Two players alternately select a positive divisor of the number of stones currently in the pile and remove that number of stones. The player who removes the last stone loses. Find a winning strategy for one of the players.

A2. AB is a diameter of a circle. C and D are points on the tangent at B on opposite sides of B. AC, AD meet the circle again at E, F respectively. CF, DE meet the circle again at G, H respectively. Show that AG = AH.

A3. Given integers $a > 1$, $b > 2$, show that $a^b + 1 \geq b(a+1)$. When do we have equality?

B1. Two circles meet at P and Q. A line through P meets the circles again at A and A'. A parallel line through Q meets the circles again at B and B'. Show that PBB' and QAA' have equal perimeters.

B2. An 8 x 8 board is divided into unit squares. Each unit square is painted red or blue. Find the number of ways of doing this so that each 2 x 2 square (of four unit squares) has two red squares and two blue squares.

B3. Call a positive integer a *tico* if the sum of its digits (in base 10) is a multiple of 2003. Show that there is an integer N such that N, 2N, 3N, ... , 2003N are all ticos. Does there exist a positive integer such that all its multiples are ticos?

PUTNAM (1938 – 2003)

1st Putnam 1938

A1. A solid in Euclidean 3-space extends from $z = -h/2$ to $z = +h/2$ and the area of the section $z = k$ is a polynomial in k of degree at most 3. Show that the volume of the solid is $h(B + 4M + T)/6$, where B is the area of the bottom ($z = -h/2$), M is the area of the middle section ($z = 0$), and T is the area of the top ($z = h/2$). Derive the formulae for the volumes of a cone and a sphere.

A2. A solid has a cylindrical middle with a conical cap at each end. The height of each cap equals the length of the middle. For a given surface area, what shape maximizes the volume?

A3. A particle moves in the Euclidean plane. At time t (taking all real values) its coordinates are $x = t^3 - t$ and $y = t^4 + t$. Show that its velocity has a maximum at $t = 0$, and that its path has an inflection at $t = 0$.

A4. A notch is cut in a cylindrical vertical tree trunk. The notch penetrates to the axis of the cylinder and is bounded by two half-planes. Each half-plane is bounded by a horizontal line passing through the axis of the cylinder. The angle between the two half-planes is θ. Prove that the volume of the notch is minimized (for given tree and θ) by taking the bounding planes at equal angles to the horizontal plane.

A5. (1) Find $\lim_{x\to\inf} x^2/e^x$.
(2) Find $\lim_{k\to 0} 1/k \int_0^k (1 + \sin 2x)^{1/x}\, dx$.

A6. A swimmer is standing at a corner of a square swimming pool. She swims at a fixed speed and runs at a fixed speed (possibly different). No time is taken entering or leaving the pool. What path should she follow to reach the opposite corner of the pool in the shortest possible time?

A7. *Do either (1) or (2)*

(1) S is a thin spherical shell of constant thickness and density with total mass M and center O. P is a point outside S. Prove that the gravitational attraction of S at P is the same as the gravitational attraction of a point mass M at O.

(2) K is the surface $z = xy$ in Euclidean 3-space. Find all straight lines lying in S. Draw a diagram to illustrate them.

B1. *Do either (1) or (2)*

(1) Let A be matrix (a_{ij}), $1 \le i,j \le 4$. Let $d = \det(A)$, and let A_{ij} be the cofactor of a_{ij}, that is, the determinant of the 3 x 3 matrix formed from A by deleting a_{ij} and other elements in the same row and column. Let B be the 4 x 4 matrix (A_{ij}) and let D be det B. Prove $D = d^3$.

(2) Let P(x) be the quadratic $Ax^2 + Bx + C$. Suppose that $P(x) = x$ has unequal real roots. Show that the roots are also roots of $P(P(x)) = x$. Find a quadratic equation for the other two roots of this equation. Hence solve $(y^2 - 3y + 2)^2 - 3(y^2 - 3y + 2) + 2 - y = 0$.

B2. Find all solutions of the differential equation $zz'' - 2z'z' = 0$ which pass through the point $x=1$, $z=1$.

B3. A horizontal disk diameter 3 inches rotates once every 15 seconds. An insect starts at the southernmost point of the disk facing due north. Always facing due north

B4. The parabola P has focus a distance m from the directrix. The chord AB is normal to P at A. What is the minimum length for AB?

B5. Find the locus of the foot of the perpendicular from the center of a rectangular hyperbola to a tangent. Obtain its equation in polar coordinates and sketch it.

B6. What is the shortest distance between the plane $Ax + By + Cz + 1 = 0$ and the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. You may find it convenient to use the notation $h = (A^2 + B^2 + C^2)^{-1/2}$, $m = (a^2A^2 + b^2B^2 + c^2C^2)^{1/2}$. What is the algebraic condition for the plane not to intersect the ellipsoid?

2nd Putnam 1939

A1. Let C be the curve $y^2 = x^3$ (where x takes all non-negative real values). Let O be the origin, and A be the point where the gradient is 1. Find the length of the curve from O to A.

A2. Let C be the curve $y = x^3$ (where x takes all real values). The tangent at A meets the curve again at B. Prove that the gradient at B is 4 times the gradient at A.

A3. The roots of $x^3 + a x^2 + b x + c = 0$ are α, β and γ. Find the cubic whose roots are α^3, β^3, γ^3.

A4. Given 4 lines in Euclidean 3-space:

L_1: x = 1, y = 0;
L_2: y = 1, z = 0;
L_3: x = 0, z = 1;
L_4: x = y, y = -6z.

Find the equations of the two lines which both meet all of the L_i.

A5. *Do either (1) or (2)*

(1) x and y are functions of t. Solve x' = x + y - 3, y' = -2x + 3y + 1, given that x(0) = y(0) = 0.

(2) A weightless rod is hinged at O so that it can rotate without friction in a vertical plane. A mass m is attached to the end of the rod A, which is balanced vertically above O. At time t = 0, the rod moves away from the vertical with negligible initial angular velocity. Prove that the mass first reaches the position under O at $t = \sqrt{(OA/g)} \ln (1 + \sqrt{2})$.

A6. *Do either (1) or (2):*

(1) A circle radius r rolls around the inside of a circle radius 3r, so that a point on its circumference traces out a curvilinear triangle. Find the area inside this figure.

(2) A frictionless shell is fired from the ground with speed *v* at an unknown angle to the vertical. It hits a plane at a height *h*. Show that the gun must be sited within a radius $v/g\ (v^2 - 2gh)^{1/2}$ of the point directly below the point of impact.

A7. *Do either (1) or (2):*

(1) Let C_a be the curve $(y - a^2)^2 = x^2(a^2 - x^2)$. Find the curve which touches all C_a for a > 0. Sketch the solution and at least two of the C_a.

(2) Given that $(1 - hx)^{-1}(1 - kx)^{-1} = \sum_{i\geq 0} a_i x^i$, prove that $(1 + hkx)(1 - hkx)^{-1}(1 - h^2x)^{-1}(1 - k^2x)^{-1} = \sum_{i\geq 0} a_i^2 x^i$.

B1. The points P(a,b) and Q(0,c) are on the curve y/c = cosh (x/c). The line through Q parallel to the normal at P cuts the x-axis at R. Prove that QR = b.

B2. Evaluate $\int_1^3 ((x - 1)(3 - x))^{-1/2} dx$ and $\int_1^\infty (e^{x+1} + e^{3-x})^{-1} dx$.

B3. Given $a_n = (n^2 + 1) 3^n$, find a recurrence relation $a_n + p a_{n+1} + q a_{n+2} + r a_{n+3} = 0$. Hence evaluate $\sum_{n\geq 0} a_n x^n$.

B4. The *axis* of a parabola is its axis of symmetry and its *vertex* is its point of intersection with its axis. Find: the equation of the parabola which touches y = 0 at (1,0) and x = 0 at (0,2); the equation of its axis; and its vertex.

B5. *Do either (1) or (2):*

(1) Prove that $\int_1^k [x] f'(x)\,dx = [k] f(k) - \sum_1^{[k]} f(n)$, where k > 1, and [z] denotes the greatest integer ≤ z. Find a similar expression for: $\int_1^k [x^2] f'(x)\,dx$.

(2) A particle moves freely in a straight line except for a resistive force proportional to its speed. Its speed falls from 1,000 ft/s to 900 ft/s over 1,200 ft. Find the time taken to the nearest 0.01 s. [No calculators or log tables allowed!]

B6. *Do either (1) or (2):*

(1) f is continuous on the closed interval [a, b] and twice differentiable on the open interval (a, b). Given $x_0 \in (a, b)$, prove that we can find $\xi \in (a, b)$ such that $((f(x_0) - f(a))/(x_0 - a) - (f(b) - f(a))/(b - a))/(x_0 - b) = f''(\xi)/2$.

(2) AB and CD are identical uniform rods, each with mass *m* and length 2*a*. They are placed a distance *b* apart, so that ABCD is a rectangle. Calculate the gravitational attraction between them. What is the limiting value as *a* tends to zero?

B7. *Do either (1) or (2):*

(1) Let $a_i = \sum_{n=0}^{\infty} x^{3n+i}/(3n+i)!$ Prove that $a_0^3 + a_1^3 + a_2^3 - 3a_0a_1a_2 = 1$.

(2) Let O be the origin, λ a positive real number, C be the conic $ax^2 + by^2 + cx + dy + e = 0$, and C_λ the conic $ax^2 + by^2 + \lambda cx + \lambda dy + \lambda^2 e = 0$. Given a point P and a non-zero real number k, define the transformation D(P,k) as follows. Take coordinates (x',y') with P as the origin. Then D(P,k) takes (x',y') to (kx',ky'). Show that D(O,λ) and D(A,-λ) both take C into C_λ, where A is the point $(-c\lambda/(a(1 + \lambda)), -d\lambda/(b(1 + \lambda)))$. Comment on the case λ = 1.

3rd Putnam 1940

A1. p(x) is a polynomial with integer coefficients. For some positive integer c, none of p(1), p(2), ... , p(c) are divisible by c. Prove that p(b) is not zero for any integer b.

A2. y = f(x) is continuous with continuous derivative. The arc PQ is concave to the chord PQ. X is chosen on the arc PQ to maximize PX + XQ. Prove that XP and XQ are equally inclined to the tangent at X.

A3. α is a fixed real number. Find all functions f: R → R (where R is the reals) which are continuous, have a continuous derivative, and satisfy $\int_b^y f^\alpha(x)\,dx = \left(\int_b^y f(x)\,dx\right)^\alpha$ for all y and some b.

A4. p is a positive constant. Let R is the curve $y^2 = 4px$. Let S be the mirror image of R in the y-axis ($y^2 = -4px$). R remains fixed and S rolls around it without slipping. O is the point of S initially at the origin. Find the equation for the locus of O as S rolls?

A5. Prove that the set of points satisfying $x^4 - x^2 = y^4 - y^2 = z^4 - z^2$ is the union of 4 straight lines and 6 ellipses.

A6. p(x) is a polynomial with real coefficients and derivative r(x) = p'(x). For some positive integers a, b, $r^a(x)$ divides $p^b(x)$. Prove that for some real numbers A and α and for some integer n, we have $p(x) = A(x - \alpha)^n$.

A7. a_i and b_i are real, and $\sum_1^\infty a_i^2$ and $\sum_1^\infty b_i^2$ converge. Prove that $\sum_1^\infty (a_i - b_i)^p$ converges for $p \geq 2$.

A8. Show that the area of the triangle bounded by the lines $a_ix + b_iy + c_i = 0$ (i = 1, 2, 3) is $\Delta^2/|2(a_2b_3 - a_3b_2)(a_3b_1 - a_1b_3)(a_1b_2 - a_2b_1)|$, where Δ is the 3 x 3 determinant with columns a_i, b_i, c_i.

B1. A stone is thrown from the ground with speed v at an angle θ to the horizontal. There is no friction and the ground is flat. Find the total distance it travels before hitting the ground. Show that the distance is greatest when $\sin\theta \ln(\sec\theta + \tan\theta) = 1$.

B2. C_1, C_2 are cylindrical surfaces with radii r_1, r_2 respectively. The axes of the two surfaces intersect at right angles and $r_1 > r_2$. Let S be the area of C_1 which is enclosed within C_2. Prove that $S = 8r_2^2A = 8r_1^2C - 8(r_1^2 - r_2^2)B$, where $A = \int_0^1 (1 - x^2)^{1/2}(1 - k^2x^2)^{-1/2}\,dx$, $B = \int_0^1 (1 - x^2)^{-1/2}(1 - k^2x^2)^{-1/2}\,dx$, and $C = \int_0^1 (1 - x^2)^{-1/2}(1 - k^2x^2)^{1/2}\,dx$, and $k = r_2/r_1$.

B3. Let p be a positive real, let S be the parabola $y^2 = 4px$, and let P be a point with coordinates (a, b). Show that there are 1, 2 or 3 normals from P to S according as $4(2p - a)^2 + 27pb^2 >, = \text{or} < 0$.

B4. Let S be the surface $ax^2 + by^2 + cz^2 = 1$ (a, b, c all non-zero), and let K be the sphere $x^2 + y^2 + z^2 = 1/a + 1/b + 1/c$ (known as the *director* sphere). Prove that if a point P lies on 3 mutually perpendicular planes, each of which is tangent to S, then P lies on K.
Comment. The original question also asked, apparently in error, for a proof that every point of K had this property, which is (a) false unless we allow planes which are asymptotes (or tangents at infinity), and (b) unreasonably hard - at least, I cannot see a neat proof.

B5. Find all rational triples (a, b, c) for which a, b, c are the roots of $x^3 + ax^2 + bx + c = 0$.

B6. The n x n matrix (m_{ij}) is defined as $m_{ij} = a_ia_j$ for $i \neq j$, and $a_i^2 + k$ for i = j. Show that $\det(m_{ij})$ is divisible by k^{n-1} and find its other factor.

B7. Given n > 8, let $a = \sqrt{n}$ and $b = \sqrt{(n+1)}$. Which is greater a^b or b^a?

4th Putnam 1941

A1. Prove that $(a - x)^6 - 3a(a - x)^5 + 5/2\ a^2(a - x)^4 - 1/2\ a^4(a - x)^2 < 0$ for $0 < x < a$.

A2. Define $f(x) = \int_0^x \sum_{i=0}^{n-1} (x - t)^i / i!\ dt$. Find the nth derivative $f^{(n)}(x)$.

A3. A circle radius a rolls in the plane along the x-axis the envelope of a diameter is the curve C. Show that we can find a point on the circumference of a circle radius a/2, also rolling along the x-axis, which traces out the curve C.

A4. The real polynomial $x^3 + px^2 + qx + r$ has real roots $a \leq b \leq c$. Prove that f ' has a root in the interval $[b/2 + c/2, b/3 + 2c/3]$. What can we say about f if the root is at one of the endpoints?

A5. The line L is parallel to the plane $y = z$ and meets the parabola $y^2 = 2x$, $z = 0$ and the parabola $3x = z^2$, $y = 0$. Prove that if L moves freely subject to these constraints then it generates the surface $x = (y - z)(y/2 - z/3)$.

A6. f is defined for the non-negative reals and takes positive real values. The centroid of the area lying under the curve $y = f(x)$ between $x = 0$ and $x = a$ has x-coordinate $g(a)$. Prove that for some positive constant k, $f(x) = k\ g'(x)/(x - g(x))^2\ e^{\int 1/(t - g(t))\ dt}$.

A7. *Do either (1) or (2):*

(1) Let A be the 3 x 3 matrix

$1+x^2-y^2-z^2$	$2(xy+z)$	$2(zx-y)$
$2(xy-z)$	$1+y^2-z^2-x^2$	$2(yz+x)$
$2(zx+y)$	$2(yz-x)$	$1+z^2-x^2-y^2$

Show that $\det A = (1 + x^2 + y^2 + z^2)^3$.

(2) A solid is formed by rotating about the x-axis the first quadrant of the ellipse $x^2/a^2 + y^2b^2 = 1$. Prove that this solid can rest in stable equilibrium on its vertex (corresponding to $x = a$, $y = 0$ on the ellipse) iff $a/b \leq \sqrt{(8/5)}$.

B1. A particle moves in the plane so that its angular velocity about the point (1, 0) equals minus its angular velocity about the point (-1, 0). Show that its trajectory satisfies the differential equation $y'\ x(x^2 + y^2 - 1) = y(x^2 + y^2 + 1)$. Verify that this has as solutions the rectangular hyperbolae with center at the origin and passing through $(\pm 1, 0)$.

B2. Find:

a. $\lim_{n\to\infty} \sum_{1\leq i\leq n} 1/\sqrt{(n^2 + i^2)}$;

b. $\lim_{n\to\infty} \sum_{1\leq i\leq n} 1/\sqrt{(n^2 + i)}$;

c. $\lim_{n\to\infty} \sum_{1\leq i\leq n}{}^{2}\ 1/\sqrt{(n^2 + i)}$;

B3. Let y_1 and y_2 be any two linearly independent solutions of the differential equation $y'' + p(x)\ y' + q(x)\ y = 0$. Let $z = y_1y_2$. Find the differential equation satisfied by z.

B4. Given an ellipse center O, take two perpendicular diameters AOB and COD. Take the diameter A'OB' parallel to the tangents to the ellipse at A and B (this is said to be *conjugate* to the diameter AOB). Similarly, take C'OD' conjugate to COD. Prove that the rectangular hyperbola through A'B'C'D' passes through the foci of the ellipse.

B5. A wheel radius r is traveling along a road without slipping with angular velocity $\omega > \sqrt{(g/r)}$. A particle is thrown off the rim of the wheel. Show that it can reach a maximum height above the road of $(r\omega + g/\omega)^2/(2g)$. [Ignore air resistance.]

B6. f is a real valued function on [0, 1], continuous on (0, 1). Prove that $\int_{x=0}^{x=1} \int_{y=x}^{y=1} \int_{z=x}^{z=y} f(x) f(y) f(z)\, dz\, dy\, dx = 1/6 \left(\int_{x=0}^{x=1} f(x)\, dx \right)^3$.

B7. *Do either (1) or (2):*

(1) f is a real-valued function defined on the reals with a continuous second derivative and satisfies $f(x + y) f(x - y) = f(x)^2 + f(y)^2 - 1$ for all x, y. Show that for some constant k we have $f''(x) = \pm k^2 f(x)$. Deduce that f(x) is one of ±cos kx, ±cosh kx.

(2) a_i and b_i are constants. Let A be the (n+1) x (n+1) matrix A_{ij}, defined as follows: $A_{i1} = 1$; $A_{1j} = x^{j-1}$ for $j \le n$; $A_{1\,(n+1)} = p(x)$; $A_{ij} = a_{i-1}^{j-1}$ for $i > 1$, $j \le n$; $A_{i\,(n+1)} = b_{i-1}$ for $i > 1$. We use the identity det A = 0 to define the polynomial p(x). Now given any polynomial f(x), replace b_i by $f(b_i)$ and p(x) by q(x), so that det A = 0 now defines a polynomial q(x). Prove that f(p(x)) is a multiple of $\prod (x - a_i)$ plus q(x).

5th Putnam 1942

A1. ABCD is a square side 2a with vertices in that order. It rotates in the first quadrant with A remaining on the positive x-axis and B on the positive y-axis. Find the locus of its center.

A2. a and b are unequal reals. What is the remainder when the polynomial p(x) is divided $(x - a)^2(x - b)$.

A3. Does $\sum_{n\geq 0} n!\, k^n/(n + 1)^n$ converge or diverge for k = 19/7?

A4. Let C be the family of conics $(2y + x)^2 = a(y + x)$. Find C', the family of conics which are orthogonal to C. At what angle do the curves of the two families meet at the origin?

A5. C is a circle radius a whose center lies a distance b from the coplanar line L. C is rotated through π about L to form a solid whose center of gravity lies on its surface. Find b/a.

A6. P is a plane and H is the half-space on one side of P. K is a fixed circle in P. C is a circle in P which cuts K at an angle α. Let C have center O and radius r. f(C) is the point in H on the normal to P through O and a distance r from O. Show that the locus of f(C) is a one-sheet hyperboloid and that it has two families of rulings in it.

B1. S is a solid square side 2a. It lies in the quadrant $x \geq 0$, $y \geq 0$, and it is free to move around provided a vertex remains on the x-axis and an adjacent vertex on the y-axis. P is a point of S. Show that the locus of P is part of a conic. For what P does the locus degenerate?

B2. Let P_a be the parabola $y = a^3x^2/3 + a^2x/2 - 2a$. Find the locus of the vertices of P_a, and the envelope of P_a. Sketch the envelope and two P_a.

B3. f(x, y) and g(x, y) satisfy the differential equation $f_1(x, y)\, g_2(x, y) - f_2(x, y)\, g_1(x, y) = 1$ (*). Taking r = f(x, y) and y as independent variables, and x = h(r, y), g(x, y) = k(r, y), show that $k_2(r, y) = h_1(r, y)$. Integrate and hence obtain a solution to (*). What other solutions does (*) have?

B4. A particle moves in a circle through the origin under the influence of a force a/r^k towards the origin (where r is its distance from the origin). Find k.

B5. Let $f(x) = x/(1 + x^6\sin^2x)$. Sketch the curve y = f(x) and show that $\int_0^\infty f(x)\, dx$ exists.

6th Putnam 1946

A1. p(x) is a real polynomial of degree less than 3 and satisfies $|p(x)| \le 1$ for $x \in [-1, 1]$. Show that $|p'(x)| \le 4$ for $x \in [-1, 1]$.

A2. R is the reals. For functions f, g : R → R and x ∈ R define $I(fg) = \int_1^x f(t)\, g(t)\, dt$. If a(x), b(x), c(x), d(x) are real polynomials, show that I(ac) I(bd) - I(ad) I(bc) is divisible by $(x - 1)^4$.

A3. ABCD are the vertices of a square with A opposite C and side AB = s. The distances of a point P in space from A, B, C, D are a, b, c, d respectively. Show that $a^2 + c^2 = b^2 + d^2$, and that the perpendicular distance k of P from the plane ABCD is given by $8k^2 = 2(a^2 + b^2 + c^2 + d^2) - 4s^2 - (a^4 + b^4 + c^4 + d^4 - 2a^2c^2 - 2b^2d^2)/s^2$.

A4. R is the reals. f : R → R has a continuous derivative, f(0) = 0, and $|f '(x)| \le |f(x)|$ for all x. Show that f is constant.

A5. Let T be a tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. What is the smallest possible volume for the tetrahedral volume bounded by T and the planes x = 0, y = 0, z = 0?

A6. A particle moves in one dimension. Its distance x from the origin at time t is $at + bt^2 + ct^3$. Find an expression for the particle's acceleration in terms of a, b, c and its speed v.

B1. Two circles C_1 and C_2 intersect at A and B. C_1 has radius 1. L denotes the arc AB of C_2 which lies inside C_1. L divides C_1 into two parts of equal area. Show L has length > 2.

B2. P_0 is the parabola $y^2 = mx$, vertex K (0, 0). If A and B points on P_0 whose tangents are at right angles, let C be the centroid of the triangle ABK. Show that the locus of C is a parabola P_1. Repeat the process to define P_n. Find the equation of P_n.

B3. The density of a solid sphere depends solely on the radial distance. The gravitational force at any point inside the sphere, a distance r from the center, is kr^2 (where k is a constant). Find the density (in terms of G, k and r), and the gravitational force at a point outside the sphere. [You may assume the usual results about the gravitational attraction of a spherical shell.]

B4. Define $a_n = 2(1 + 1/n)^{2n+1}/((1 + 1/n)^n + (1 + 1/n)^{n+1})$. Prove that a_n is strictly monotonic increasing.

B5. Let m be the smallest integer greater than $(\sqrt{3} + 1)^{2n}$. Show that m is divisible by 2^{n+1}.

B6. The particle P moves in the plane. At t = 0 it starts from the point A with velocity zero. It is next at rest at t = T, when its position is the point B. Its path from A to B is the arc of a circle center O. Prove that its acceleration at each point in the time interval [0, T] is non-zero, and that at some point in the interval its acceleration is directly towards the center O.

7th Putnam 1947

A1. The sequence a_n of real numbers satisfies $a_{n+1} = 1/(2 - a_n)$. Show that $\lim_{n\to\infty} a_n = 1$.

A2. R is the reals. f **:** R → R is continuous and satisfies f(r) = f(x) f(y) for all x, y, where $r = \sqrt{(x^2 + y^2)}$. Show that f(x) = f(1) to the power of x^2.

A3. ABC is a triangle and P an interior point. Show that we cannot find a piecewise linear path $K = K_1K_2 \ldots K_n$ (where each K_iK_{i+1} is a straight line segment) such that: (1) none of the K_i do not lie on any of the lines AB, BC, CA, AP, BP, CP; (2) none of the points A, B, C, P lie on K; (3) K crosses each of AB, BC, CA, AP, BP, CP just once; (4) K does not cross itself.

A4. Take the x-axis as horizontal and the y-axis as vertical. A gun at the origin can fire at any angle into the first quadrant ($x, y \geq 0$) with a fixed muzzle velocity v. Assuming the only force on the pellet after firing is gravity (acceleration g), which points in the first quadrant can the gun hit?

A5. The sequences a_n, b_n, c_n of positive reals satisfy: (1) $a_1 + b_1 + c_1 = 1$; (2) $a_{n+1} = a_n^2 + 2b_nc_n$, $b_{n+1} = b_n^2 + 2c_na_n$, $c_{n+1} = c_n^2 + 2a_nb_n$. Show that each of the sequences converges and find their limits.

A6. A is the matrix

a b c

d e f

g h i

det A = 0 and the cofactor of each element is its square (for example the cofactor of b is $fg - di = b^2$). Show that all elements of A are zero.

B1. Let R be the reals. f **:** [1, ∞) → R is differentiable and satisfies $f'(x) = 1/(x^2 + f(x)^2)$ and f(1) = 1. Show that as x → ∞, f(x) tends to a limit which is less than $1 + \pi/4$.

B2. R is the reals. f **:**(0, 1) → R is differentiable and has a bounded derivative: $|f'(x)| <= k$. Prove that : $|\int_0^1 f(x)\,dx - \sum_1^n f(i/n)/n| \leq k/n$.

B3. Let O be the origin (0, 0) and C the line segment { (x, y) **:** x ∈ [1, 3], y = 1 }. Let K be the curve { P **:** for some Q ∈ C, P lies on OQ and PQ = 0.01 }. Let k be the length of the curve K. Is k greater or less than 2?

B4. $p(z) = z^2 + az + b$ has complex coefficients. $|p(z)| = 1$ on the unit circle $|z| = 1$. Show that a = b = 0.

B5. Let p(x) be the polynomial (x - a)(x - b)(x - c)(x - d). Assume p(x) = 0 has four distinct integral roots and that p(x) = 4 has an integral root k. Show that k is the mean of a, b, c, d.

B6. P is a variable point in space. Q is a fixed point on the z-axis. The plane normal to PQ through P cuts the x-axis at R and the y-axis at S. Find the locus of P such that PR and PS are at right angles.

8th Putnam 1948

A1. C is the complex numbers. f : C → R is defined by $f(z) = |z^3 - z + 2|$. What is the maximum value of f on the unit circle $|z| = 1$?

A2. K is a cone. s is a sphere radius r, and S is a sphere radius R. s is inside K touches it along all points of a circle. S is also inside K and touches it along all points of a circle. s and S also touch each other. What is the volume of the finite region between the two spheres and inside K?

A3. a_n is a sequence of positive reals decreasing monotonically to zero. b_n is defined by $b_n = a_n - 2a_{n+1} + a_{n+2}$ and all b_n are non-negative. Prove that $b_1 + 2b_2 + 3b_3 + ... = a_1$.

A4. Let D be a disk radius r. Given $(x, y) \in D$, and $R > 0$, let $a(x, y, R)$ be the length of the arc of the circle center (x, y), radius R, which is outside D. Evaluate $\lim_{R\to0} R^{-2} \int_D a(x, y, R)\, dx\, dy$.

A5. Let $\alpha_1, \alpha_2, ... , \alpha_n$ be the nth roots of unity. Find $\prod_{i<j} (\alpha_i - \alpha_j)^2$.

A6. *Do either (1) or (2):*

(1) On each element ds of a closed plane curve there is a force 1/R ds, where R is the radius of curvature. The force is towards the center of curvature at each point. Show that the curve is in equilibrium.
(2) Prove that $x + 2/3\, x^3 + 2.4/3.5\, x^5 + ... + 2.4. ... .2n/(3.5. ... .2n+1)\, x^{2n+1} + ... = (1 - x^2)^{-1/2} \sin^{-1}x$

B1. p(x) is a cubic polynomial with roots α, β, γ and p'(x) divides p(2x). Find the ratios α : β : γ.

B2. A circle radius r is tangent to the three coordinate planes (x =0, y =0, z = 0) in space. Find the locus of its center.

B3. Show that $[\sqrt{n} + \sqrt{(n + 1)}] = [\sqrt{(4n + 2)}]$ for positive integers n.

B4. R is the reals. For what λ can we find a continuous function f : (0, 1) → R, not identically zero, such that $\int_0^1 \min(x, y) f(y)\, dy = \lambda f(x)$ for all $x \in (0, 1)$?

B5. Find the area of the region $\{ (x, y) : |x + yt + t^2| \le 1$ for all $t \in [0, 1] \}$.

B6. *Do either (1) or (2):*

(1) Take the origin O of the complex plane to be the vertex of a cube, so that OA, OB, OC are edges of he cube. Let the feet of the perpendiculars from A, B, C to the complex plane be the complex numbers u, v, w. Show that $u^2 + v^2 + w^2 = 0$.

(2) Let (a_{ij}) be an n x n matrix. Suppose that for each i, $2 |a_{ii}| > \sum_1^n |a_{ij}|$. By considering the corresponding system of linear equations or otherwise, show that $\det a_{ij} \ne 0$.

9th Putnam 1949

A1. *Do either (1) or (2)*

(1) Let L be the line through (0, -a, a) parallel to the x-axis, M the line through (a, 0, -a) parallel to the y-axis, and N the line through (-a, a, 0) parallel to the z-axis. Find the equation of S, the surface formed from the union of all lines K which intersect each of L, M and N.

(2) Let S be the surface $xy + yz + zx = 0$. Which planes cut S in circles? In parabolas?

A2. Take points O, P, Q, R in space. Let the volume of the parallelepiped with edges OP, OQ, OR be V. Let V' be the volume of the parallelepiped which has O as one vertex and which has OP, OQ, OR as altitudes to three faces. Show that $V V' = OP^2 OQ^2 OR^2$. Generalize to n dimensions.

A3. All the complex numbers z_n are non-zero and $|z_m - z_n| > 1$ (for any $m \neq n$). Show that $\sum 1/z_n^3$ converges.

A4. Take P inside the tetrahedron ABCD to minimize PA + PB + PC + PD. Show that ∠APB = ∠CPD and that the bisector of APB also bisects CPD.

A5. Let $p(z) = z^6 + 6z + 10$. How many roots lie in each quadrant of the complex plane?

A6. Show that $\prod_1^\infty (1 + 2\cos(2z/3^n))/3 = (\sin z)/z$ for all complex z.

B1. Show that for any rational $a/b \in (0, 1)$, we have $|a/b - 1/\sqrt{2}| > 1/(4b^2)$.

B2. *Do either (1) or (2)*

(1) Prove that $\sum_2^\infty \cos(\ln \ln n) / \ln n$ diverges.

(2) Let k, a, b, c be real numbers such that $a, k > 0$ and $b^2 < ac$. Show that $\int_U (k + ax^2 + 2bxy + cy^2)^{-2}\, dx\, dy = \pi/(k \sqrt{(ac - b^2)})$, where U is the entire plane.

B3. C is a closed plane curve. If $P, Q \in C$, then $|PQ| < 1$. Show that we can find a disk radius $1/\sqrt{3}$ which contains C.

B4. Let $(1 + x - \sqrt{(x^2 - 6x + 1)})/4 = \sum_1^\infty a_n x^n$. Show that all a_n are positive integers.

B5. a_n is a sequence of positive reals. Show that $\limsup_{n\to\infty}((a_1 + a_{n+1})/a_n)^n \geq e$.

B6. C is a closed convex curve. If P lies on C and T_P is the tangent at P, then T_P varies continuously with P. Let O be a point inside C. Given a point P on C, define f(P) to be the point where the perpendicular from O to T_P intersects C. Given P_1, define the sequence P_n by $P_{n+1} = f(P_n)$. Assume that f is continuous and that, for each P, C lies entirely on one side of T_P. Show that P_n converges. Find $S = \{ P : P = \lim_{n\to\infty} P_n \text{ for some } P_1\}$.

10th Putnam 1950

A1. a and b are positive reals and $a > b$. Let C be the plane curve $r = a - b \cos \theta$. For what values of b/a is C convex?

A2. Does the series $\sum_2^\infty 1/\ln n!$ converge? Does the series $1/3 + 1/(3\ 3^{1/2}) + 1/(3\ 3^{1/2}\ 3^{1/3}) + ... + 1/(3\ 3^{1/2}\ 3^{1/3} ... 3^{1/n}) + ...$ converge?

A3. The sequence a_n is defined by $a_0 = \alpha$, $a_1 = \beta$, $a_{n+1} = a_n + (a_{n-1} - a_n)/(2n)$. Find $\lim a_n$.

A4. *Do either (1) or (2)*

(1) P is a prism with triangular base. A is a vertex. The total area of the three faces containing A is 3k. Show that if the volume of P is maximized, then each of the three faces has area k and the two lateral faces are perpendicular to each other.

(2) Let $f(x) = x + x^3/(1\cdot3) + x^5/(1\cdot3\cdot5) + x^7/(1\cdot3\cdot5\cdot7) + ...$, and $g(x) = 1 + x^2/2 + x^4/(2\cdot4) + x^6/(2\cdot4\cdot6) + ...$. Show that $\int_0^x \exp(-t^2/2)\,dt = f(x)/g(x)$.

A5. Let N be the set of natural numbers {1, 2, 3, ... }. Let Z be the integers. Define $d : N \to Z$ by $d(1) = 0$, $d(p) = 1$ for p prime, and $d(mn) = m\,d(n) + n\,d(m)$ for any integers m, n. Determine d(n) in terms of the prime factors of n. Find all n such that $d(n) = n$. Define $d_1(m) = d(m)$ and $d_{n+1}(m) = d(d_n(m))$. Find $\lim_{n\to\infty} d_n(63)$.

A6. Let $f(x) = \sum_0^\infty a_n x^n$ and suppose that each $a_n = 0$ or 1. *Do either (1) or (2):*

(1) Show that if f(1/2) is rational, then f(x) has the form p(x)/q(x) for some integer polynomials p(x) and q(x).

(2) Show that if f(1/2) is not rational, then f(x) does not have the form p(x)/q(x) for any integer polynomials p(x) and q(x).

B1. Given n, not necessarily distinct, points $P_1, P_2, ... , P_n$ on a line. Find the point P on the line to minimize $\sum |PP_i|$.

B2. An ellipse with semi-axes a and b has perimeter length p(a, b). For b/a near 1, is $\pi(a + b)$ or $2\pi\sqrt{(ab)}$ the better approximation to p(a, b)?

B3. Leap years have 366 days; other years have 365 days. Year $n > 0$ is a leap year iff (1) 4 divides n, but 100 does not divide n, or (2) 400 divides n. n is chosen at random from the natural numbers. Show that the probability that December 25 in year n is a Wednesday is not 1/7.

B4. A long, light cylinder has elliptical cross-section with semi-axes $a > b$. It lies on the ground with its main axis horizontal and the major axes horizontal. A thin heavy wire of the same length as the cylinder is attached to the line along the top of the cylinder. [We could take the cylinder to be the surface $|x| \le L$, $y^2/a^2 + z^2/b^2 = 1$. Contact with the ground is along $|x| \le L$, $y = 0$, $z = -b$. The wire is along $|x| \le L$, $y = 0$, $z = b$.] For what values of b/a is the cylinder in stable equilibrium?

B5. *Do either (1) or (2):*

(1) Show that if $\sum(a_n + 2a_{n+1})$ converges, then so does $\sum a_n$.

(2) Let S be the surface $2xy = z^2$. The surface S and the variable plane P enclose a cone with volume $\pi a^3/3$, where a is a positive real constant. Find the equation of the envelope of P. What is the envelope in the case of a general cone?

B6.

(1) The convex polygon C' lies inside the polygon C. Is it true that the perimeter of C' is no longer than the perimeter of C?

(2) C is the convex polygon with shortest perimeter enclosing the polygon C'. Is it true that the perimeter of C is no longer than the perimeter of C' ?

(3) The closed convex surface S' lies inside the closed surface S. Is it true that area S' ≤ area S?

(4) S is the smallest convex surface containing the closed surface S'. Is it true that area S ≤ area S'?

11th Putnam 1951

A1. A is a skew-symmetric real 4 x 4 matrix. Show that det A $\geq$ 0.

A2. k is a positive real and P_1, P_2, ... , P_n are points in the plane. What is the locus of P such that $\sum PP_i^2 = k$? State in geometric terms the conditions on k for such points P to exist.

A3. Find $\sum_0^\infty (-1)^n/(3n + 1)$.

A4. Sketch the curve $y^4 - x^4 - 96y^2 + 100x^2 = 0$.

A5. Show that a line in the plane with rational slope contains either no lattice points or an infinite number. Show that given any line L of rational slope we can find $\delta > 0$, such that no lattice point is a distance k from L where $0 < k < \delta$.

A6. Let C be a parabola. Take points P, Q on C such that (1) PQ is perpendicular to the tangent at P, (2) the area enclosed by the parabola and PQ is as small as possible. What is the position of the chord PQ?

A7. Show that if $\sum a_n$ converges, then so does $\sum a_n/n$.

B1. R is the reals. f, g **:** $R^2 \to R$ have continuous partial derivatives of all orders. What conditions must they satisfy for the differential equation f(x, y) dx + g(x, y) dy = 0 to have an integrating factor h(xy)?

B2. R is the reals. Find an example of functions f, g **:** R → R, which are differentiable, not identically zero, and satisfy (f/g)' = f '/g' .

B3. Show that $\ln(1 + 1/x) > 1/(1 + x)$ for $x > 0$.

B4. Can we find four distinct concentric circles all touching an ellipse?

B5. T is a torus, center O. The plane P contains O and touches T. Prove that P $\in$ T is two circles.

B6. The real polynomial $p(x) = x^3 + ax^2 + bx + c$ has three real roots $\alpha < \beta < \gamma$. Show that $\sqrt{(a^2 - 3b)} < (\gamma - \alpha) \leq 2\sqrt{(a^2/3 - b)}$.

B7. In 4-space let S be the 3-sphere radius r: $w^2 + x^2 + y^2 + z^2 = r^2$. What is the 3-dimensional volume of S? What is the 4-dimensional volume of its interior?

12th Putnam 1952

A1. p(x) is a polynomial with integral coefficients. The leading coefficient, the constant term, and p(1) are all odd. Show that p(x) has no rational roots.

A2. Show that the solutions of the differential equation $(9 - x^2) (y')^2 = 9 - y^2$ are conics touching the sides of a square.

A3. Let the roots of the cubic $p(x) = x^3 + ax^2 + bx + c$ be α, β, γ. Find all (a, b, c) so that $p(\alpha^2) = p(\beta^2) = p(\gamma^2) = 0$.

A4. A map represents the polar cap from latitudes -45^o to 90^o. The pole (latitude 90^o) is at the center of the map and lines of latitude on the globe are represented as concentric circles with radii proportional to (90^o - latitude). How are east-west distances exaggerated compared to north-south distances on the map at a latitude of -30^o?

A5. a_i are reals ≠ 1. Let $b_n = 1 - a_n$. Show that $a_1 + a_2b_1 + a_3b_1b_2 + a_4b_1b_2b_3 + ... + a_nb_1b_2 ... b_{n-1} = 1 - b_1b_2 ... b_n$.

A6. Prove that there are only finitely many cuboidal blocks with integer sides a x b x c, such that if the block is painted on the outside and then cut into unit cubes, exactly half the cubes have no face painted.

A7. Let O be the center of a circle C and P_0 a point on the circle. Take points P_n on the circle such that angle $P_nOP_{n-1} = +1$ for all integers n. Given that π is irrational, show that given any two distinct points P, Q on C, the (shorter) arc PQ contains a point P_n.

B1. ABC is a triangle with, as usual, AB = c, CA = b. Find necessary and sufficient conditions for $b^2c^2/(2bc \cos A) = b^2 + c^2 - 2bc \cos A$.

B2. Find the surface comprising the curves which satisfy dx/(yz) = dy/(zx) = dz/(xy) and which meet the circle $x = 0, y^2 + z^2 = 1$.

B3. Let A(x) = be the matrix

```
 0     a-x   b-x

-a-x    0     c-x

-b-x   -c-x    0
```

For which (a, b, c) does det A(x) = 0 have a repeated root in x?

B4. The solid S consists of a circular cylinder radius r, height h, with a hemispherical cap at one end. S is placed with the center of the cap on the table and the axis of the cylinder vertical. For some k, equilibrium is stable if r/h > k, unstable if r/h < k and neutral if r/h = k. Find k and show that if r/h = k, then the body is in equilibrium if any point of the cap is in contact with the table.

B5. The sequence a_n is monotonic and $\sum a_n$ converges. Show that $\sum n(a_n - a_{n+1})$ converges.

B6. A, B, C are points of a fixed ellipse E. Show that the area of ABC is a maximum iff the centroid of ABC is at the center of E.

B7. Let R be the reals. Define a_n by $a_1 = \alpha \in R$, $a_{n+1} = \cos a_n$. Show that a_n converges to a limit independent of α.

13th Putnam 1953

A1. Show that $(2/3)\ n^{3/2} < \sum_1^n \sqrt{r} < (2/3)\ n^{3/2} + (1/2)\ \sqrt{n}$.

A2. The complete graph with 6 points and 15 edges has each edge colored red or blue. Show that we can find 3 points such that the 3 edges joining them are the same color.

A3. a, b, c are real, and the sum of any two is greater than the third. Show that $2(a + b + c)(a^2 + b^2 + c^2)/3 > a^3 + b^3 + c^3 + abc$.

A4. Using $\sin x = 2 \sin x/2 \cos x/2$ or otherwise, find $\int_0^{\pi/2} \ln \sin x\ dx$.

A5. S is a parabola with focus F and axis L. Three distinct normals to S pass through P. Show that the sum of the angles which these make with L less the angle which PF makes with L is a multiple of π.

A6. Show that $\sqrt{7}, \sqrt{7 - \sqrt{7}}, \sqrt{7 - \sqrt{7 + \sqrt{7}}}, \sqrt{7 - \sqrt{7 + \sqrt{7 - \sqrt{7}}}}, \ldots$ converges and find its limit.

A7. $p(x) = x^3 + ax^2 + bx + c$ has three positive real roots. Find a necessary and sufficient condition on a, b, c for the roots to be cos A, cos B, cos C for some triangle ABC.

B1. Does $\sum_1^\infty 1/n^{1 + 1/n}$ converge?

B2. p(x) is a real polynomial of degree n such that p(m) is integral for all integers m. Show that if k is a coefficient of p(x), then n! k is an integer.

B3. k is real. Solve the differential equations $y' = z(y + z)^k$, $z' = y(y + z)^k$ subject to $y(0) = 1$, $z(0) = 0$.

B4. R is the reals. S is a surface in R^3 containing the point (1, 1, 1) such that the tangent plane at any point $P \in S$ cuts the axes at three points whose orthocenter is P. Find the equation of S.

B5. The coefficients of the complex polynomial $z^4 + az^3 + bz^2 + cz + d$ satisfy $a^2d = c^2 \neq 0$. Show that the ratio of two of the roots equals the ratio of the other two.

B6. A and B are equidistant from O. Given k > OA, find the point P in the plane OAB such that OP = k and PA + PB is a minimum.

B7. Show that we can express any irrational number $\alpha \in (0, 1)$ uniquely in the form $\sum_1^\infty (-1)^{n+1}\ 1/(a_1a_2 \ldots a_n)$, where a_i is a strictly monotonic increasing sequence of positive integers. Find a_1, a_2, a_3 for $\alpha = 1/\sqrt{2}$.

14th Putnam 1954

A1. Let N be the set {1, 2, ... , n}, where n is an odd integer. Let $f : N \times N \to N$ satisfy: (1) $f(r, s) = f(s, r)$ for all r, s; (2) $\{f(r, s) : s \in N\} = N$ for each r. Show that $\{f(r, r) : r \in N\} = N$.

A2. Given any five points in the interior of a square side 1, show that two of the points are a distance apart less than $k = 1/\sqrt{2}$. Is this result true for a smaller k?

A3. Let S be the set of all curves satisfying $y' + a(x) y = b(x)$, where a(x) and b(x) are never zero. Show that if $C \in S$, then the tangent at the point $x = k$ on C passes through a point P_k which is independent of C.

A4. A uniform rod length 2a is suspended in midair with one end resting against a smooth vertical wall at X and the other end attached by a string length 2b to a point on the wall above X. For what angles between the rod and the string is equilibrium possible?

A5. R is the reals. $f : (0, 1) \to R$ satisfies $\lim_{x\to 0} f(x) = 0$, and $f(x) - f(x/2) = o(x)$ as $x\to 0$. Show that $f(x) = o(x)$ as $x\to 0$.

A6. The real sequence a_n satisfies $a_n = \sum_{n+1}^{\infty} a_k^2$. Show $\sum a_n$ does not converge unless all a_n are zero.

A7. Prove that the equation $m^2 + 3mn - 2n^2 = 122$ has no integral solutions.

B1. Show that for any positive integer r, we can find integers m, n such that $m^2 - n^2 = r^3$.

B2. Let $a_n = \sum_1^n (-1)^{i+1}/i$. Assume that $\lim_{n\to\infty} a_n = k$. Rearrange the terms by taking two positive terms, then one negative term, then another two positive terms, then another negative term and so on. Let b_n be the sum of the first n terms of the rearranged series. Assume that $\lim_{n\to\infty} b_n = h$. Show that $b_{3n} = a_{4n} + a_{2n}/2$, and hence that $h \neq k$.

B3. Let S be a finite collection of closed intervals on the real line such that any two have a point in common. Prove that the intersection of all the intervals is non-empty.

B4. Let F be a point, and L and D lines, in the plane. Show how to construct the point of intersection (if any) between L and the parabola with focus F and directrix D.

B5. Let R be the reals. Let $f : (-1, 1) \to R$ be a function with a derivative at 0. Let a_n be a sequence in (-1, 0) tending to zero and b_n a sequence in (0, 1) tending to zero. Show that $\lim_{n\to\infty} (f(b_n) - f(a_n))/(b_n - a_n) = f'(0)$.

B6. If x is a positive rational, show that we can find distinct positive integers $a_1, a_2, \ldots , a_n$ such that $x = \sum 1/a_i$.

B7. Let α be a positive real. Let $a_n = \Sigma_1^n (\alpha/n + i/n)^n$. Show that $\lim a_n \in (e^{\alpha}, e^{\alpha+1})$.

15th Putnam 1955

A1. Prove that if a, b, c are integers and $a\sqrt{2} + b\sqrt{3} + c = 0$, then $a = b = c = 0$.

A2. O is the center of a regular n-gon $P_1P_2 \ldots P_n$ and X is a point outside the n-gon on the line OP_1. Show that $XP_1\, XP_2 \ldots XP_n + OP_1^n = OX^n$.

A3. a_n is a sequence of monotonically decreasing positive terms such that $\sum a_n$ converges. S is the set of all $\sum b_n$, where b_n is a subsequence of a_n. Show that S is an interval iff $a_{n-1} \le \sum_n^\infty a_i$ for all n.

A4. n vertices are taken on a circle and all possible chords are drawn. No three chords are concurrent (except at a vertex). How many points of intersection are there (excluding vertices)?

A5. Given a parabola, construct the focus (with ruler and compass).

A6. For what positive integers n does the polynomial $p(x) = x^n + (2 + x)^n + (2 - x)^n$ have a rational root.

A7. k is a real constant. y satisfies $y'' = (x^3 + kx)\, y$ with initial conditions $y = 1$, $y' = 0$ at $x = 0$. Show that the solutions of $y = 0$ are bounded above but not below.

B1. The lines L and M are horizontal and intersect at O. A sphere rolls along supported by L and M. What is the locus of its center?

B2. Let R be the reals. $f : R \to R$ is twice differentiable, f'' is continuous and $f(0) = 0$. Define $g : R \to R$ by $g(x) = f(x)/x$ for $x \ne 0$, $g(0) = f'(0)$. Show that g is differentiable and that g' is continuous.

B3. Let S be a spherical cap with distance taken along great circles. Show that we cannot find a distance preserving map from S to the plane.

B4. Can we find n such that $\mu(n) = \mu(n + 1) = \ldots = \mu(n + 1000000) = 0$? [The Möbius function $\mu(r) = 0$ iff r has a square factor > 1.]

B5. n is a positive integer. An infinite sequence of 0s and 1s is such that it only contains n different blocks of n consecutive terms. Show that it is eventually periodic.

B6. Let N be the set of positive integers and R^+ the positive reals. $f : N \to R^+$ satisfies $f(n) \to 0$ as $n \to \infty$. Show that there are only finitely many solutions to $f(a) + f(b) + f(c) = 1$.

B7. A three-dimensional solid acted on by four constant forces is in equilibrium. No two lines of force are in the same plane. Show that the four lines of force are rulings on a hyperboloid.

16th Putnam 1956

A1. $\alpha \neq 1$ is a positive real. Find $\lim_{x\to\infty} \left((\alpha^x - 1)/(\alpha x - x) \right)^{1/x}$.

A2. Given any positive integer n, show that we can find a positive integer m such that mn uses all ten digits when written in the usual base 10.

A3. Find the trajectory of a particle which moves from rest in a vertical plane under (constant) gravity and a force kv perpendicular to its velocity v.

A4. Let p(x) be a real polynomial of degree n with leading coefficient 1 and all roots real. Let R be the reals and $f : [a, b] \to R$ be an n times differentiable function with at least n + 1 distinct zeros. Show that p(D) f(x) has at least one zero on [a, b], where D denotes d/dx.

A5. Show that there are just (n-k+1)Ck subsets of {1, 2, ... , n} with k elements and not containing both i and i+1 for any i.

A6. Let R be the reals. Find $f : R \to R$ which preserves all rational distances but not all distances. Show that if $f : R^2 \to R^2$ preserves all rational distances then it preserves all distances.

A7. Show that for any given positive integer n, the number of odd nCm with $0 \le m \le n$ is a power of 2.

B1. The differential equation a(x, y) dx + b(x, y) dy = 0 is homogeneous and exact (meaning that a(x, y) and b(x, y) are homogeneous polynomials of the same degree and that $\partial a/\partial y = \partial b/\partial x$). Show that the solution y = y(x) satisfies x a(x, y) + y b(x, y) = c, for some constant c.

B2. Let P be the set of all subsets of the plane. $f : P \to P$ satisfies $f(X \in Y)$ □ $f(f(X)) \in f(Y) \in Y$ for all $X, Y \in P$ (*). Show that (1) f(X) □ X, (2) f(f(X)) = f(X) , (3) if X □ Y, then f(X) □ f(Y), for all $X, Y \in P$. Show conversely that if $f : P \to P$ satisfies (1), (2), (3), then f satisfies (*).

B3. ABCD is an arbitrary tetrahedron. The inscribed sphere touches ABC at S, ABD at R, ACD at Q and BCD at P. Show that the four sets of angles {ASB, BSC, CSA}, {ARB, BRD, DRA}, {AQC, CQD, DQA}, {BPC, CPD, DPB} are the same.

B4. Show that for any triangle ABC, we have sin A cos C + A cos B > 0.

B5. Show that a graph with 2n points and $n^2 + 1$ edges necessarily contains a 3-cycle, but that we can find a graph with 2n points and n^2 edges without a 3-cycle.

B6. The sequence a_n is defined by $a_1 = 2$, $a_{n+1} = a_n^2 - a_n + 1$. Show that any pair of values in the sequence are relatively prime and that $\sum 1/a_n = 1$.

B7. p(z) and q(z) are complex polynomials with the same set of roots (but possibly different multiplicities). p(z) + 1 and q(z) + 1 also have the same set of roots. Show that p(z) = q(z).

17th Putnam 1957

A1. A surface S in 3-space is such that every normal intersects a fixed line L. Show that we can find a surface of revolution containing S.

A2. k is a real number greater than 1. A uniform wire consists of the curve $y = e^x$ between x = 0 and x = k, and the horizontal line $y = e^k$ between x = k - 1 and x = k. The wire is suspended from $(k - 1, e^k)$ and a horizontal force applied at the other end, (0, 1) to keep it in equilibrium. Show that the force is directed towards increasing x.

A3. A and B are real numbers such that cos A ≠ cos B. Show that for any integer n > 1, $|\cos nA \cos B - \cos A \cos nB| < (n^2 - 1) |\cos A - \cos B|$.

A4. p(z) is a polynomial of degree n with complex coefficients. Its roots (in the complex plane) can be covered by a disk radius r. Show that for any complex k, the roots of n p(z) - k p'(z) can be covered by a disk radius r + |k|.

A5. Let S be a set of n points in the plane such that the greatest distance between two points of S is 1. Show that at most n pairs of points of S are a distance 1 apart.

A6. Define a_n by $a_1 = \ln \alpha$, $a_2 = \ln(\alpha - a_1)$, $a_{n+1} = a_n + \ln(\alpha - a_n)$. Show that $\lim_{n\to\infty} a_n = \alpha - 1$.

A7. Show that we can find a set of disjoint circles such that given any rational point on the x-axis, there is a circle touching the x-axis at that point. Show that we cannot find such a set for the irrational points.

B1. Let A be the 100 x 100 matrix with $a_{mn} = mn$. Show that the absolute value of each of the 100! products in the expansion of det A is congruent to 1 mod 101.

B2. The sequence a_n is defined by its initial value a_1, and $a_{n+1} = a_n(2 - k\, a_n)$. For what real a_1 does the sequence converge to 1/k?

B3. R^+ is the positive reals, $f : [0, 1] \to R^+$ is monotonic decreasing. Show that $\int_0^1 f(x)\,dx \int_0^1 x f(x)^2\,dx \le \int_0^1 x f(x)\,dx \int_0^1 f(x)^2\,dx$.

B4. Show that the number of ways of representing n as an ordered sum of 1s and 2s equals the number of ways of representing n + 2 as an ordered sum of integers > 1. For example: 4 = 1 + 1 + 1 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 (5 ways) and 6 = 4 + 2 = 2 + 4 = 3 + 3 = 2 + 2 + 2 (5 ways).

B5. Let S be a set and P the set of all subsets of S. f : P → P is such that if X □ Y, then f(X) □ f(Y). Show that for some K, f(K) = K.

B6. y is the solution of the differential equation $(x^2 + 9) y'' + (x^2 + 4)y = 0$, y(0) = 0, y'(0) = 1. Show that y(x) = 0 for some $x \in [\sqrt{(63/53)}\pi, 3\pi/2]$.

B7. Let P be a regular polygon and its interior. Show that for any n > 1, we can find a subset S_n of the plane such that we cannot translate and rotate P to cover S_n but we can translate and rotate P to cover any n points of S_n.

18th Putnam 1958

A1. Show that the real polynomial $\sum_0^n a_i x^i$ has at least one real root if $\sum a_i/(i + 1) = 0$.

A2. A rough sphere radius R rests on top of a fixed rough sphere radius R. It is displaced slightly and starts to roll off. At what point does it lose contact?

A3. A sequence of numbers $\alpha_i \in [0, 1]$ is chosen at random. Show that the expected value of n, where $\sum_1^n \alpha_i > 1$, $\sum_1^{n-1} \alpha_i \le 1$ is e.

A4. $z_1, z_2, \dots, z_n$ are complex numbers with modulus $a > 0$. Let $f(n, m)$ denote the sum of all products of m of the numbers. For example, $f(3, 2) = z_1z_2 + z_2z_3 + z_3z_1$. Show that $|f(n, m)|/a^m = |f(n, n-m)|/a^{n-m}$.

A5. Let R be the reals. Show that there is at most one continuous function $f : [0, 1]^2 \to R$ satisfying $f(x, y) = 1 + \int_0^x\int_0^y f(s, t)\, dt\, ds$.

A6. Assume that the interest rate is r, so that capital of k becomes $k(1 + r)^n$ after n years. How much do we need to invest to be able to withdraw 1 at the end of year 1, 4 at the end of year 2, 9 at the end of year 3, 16 at the end of year 4 and so on (in perpetuity)?

A7. Show that we cannot place 10 unit squares in the plane so that no two have an interior point in common and one has a point in common with each of the others.

B1. *Do both (1) and (2):*

(1) Given real numbers a, b, c, d with $a > b, c, d$, show how to construct a quadrilateral with sides a, b, c, d and the side length a parallel to that length b. What conditions must a, b, c, d satisfy?

(2) H is the foot of the altitude from A in the acute-angled triangle ABC. D is any point on the segment AH. BD meets AC at E, and CD meets AB at F. Show that $\angle AHE = \angle AHF$.

B2. Let n be a positive integer. Prove that $n(n + 1)(n + 2)(n + 3)$ cannot be a square or a cube.

B3. In a tournament of n players, every pair of players plays once. There are no draws. Player i wins w_i games. Prove that we can find three players i, j, k such that i beats j, j beats k and k beats i iff $\sum w_i^2 < (n - 1)n(2n - 1)/6$.

B4. Let S be a spherical shell radius 1. Find the average straight line distance between two points of S. [In other words S is the set of points (x, y, z) with $x^2 + y^2 + z^2 = 1$).

B5. S is an infinite set of points in the plane. The distance between any two points of S is integral. Prove that S is a subset of a straight line.

B6. A particle of unit mass moves in a vertical plane under the influence of constant gravitational force g and a resistive force which is in the opposite direction to its velocity and with magnitude a function of its speed. The particle starts at time t = 0 and has coordinates (x, y) at time t. Given that x = x(t) and is not constant, show that $y(t) = - g\, x(t) \int_0^t ds/x'(s) + g \int_0^t x(s) / x'(s)\, ds + a\, x(t) + b$, where a and b are constants.

B7. R is the reals. $f : [a, b] \to R$ is continuous and $\int_a^b x^n f(x)\, dx = 0$ for all non-negative integers n. Show that $f(x) = 0$ for all x.

19th Putnam 1958

A1. Define f(i, j) as follows: f(i, 1) = f(1, j) = 1, f(i+1, j+1) = f(i, j+1) + f(i+1, j) + f(i, j). Let d(n) = f(1, n-1) + f(2, n-2) + ... + f(n-1, 1). Show that d(n + 2) = d(n) + 2 d(n + 1).

A2. Define $a_1 = 1$, $a_{n+1} = 1 + n/a_n$. Show that $\sqrt{n} \le a_n < 1 + \sqrt{n}$.

A3. Assuming that there is a unique function f(x) satisfying f(0) = 1, $f'(x) = f(x) + \int_0^1 f(t)\,dt$, find it.

A4. Find the general solution in real numbers a, b, c, d to the inequalities 2a > a + b > a + c > 2b > b + c > a + d > 2c > b + d > c + d > d + d. Find the smallest solution in positive integers.

A5. Let $A = (a_{ij})$ be the n x n matrix with $a_{ij} = 1$ if $i \neq j$, and $a_{ii} = 0$. Show that the number of non-zero terms in the expansion of det A is $n! \sum_o^n (-1)^i/i!$.

A6. R is the reals. $\alpha \in [0, 1)$. f **:** [0, 1] → [0, α] and g **:** [0, 1] → R are continuous. β satisfies β = max (g(x) + f(x) β). When do we have β = max g(x)/(1 - f(x))?

A7. m, n are relatively prime positive integers with n even. Given any positive integer r, define f(r) to be the integer which minimizes |f(r)/r - m/n|. Show that $\lim_{k\to\infty} \sum_1^k |f(r)/r - m/n|\, r/k = 1/4$.

B1. Let $a_n = \sum_0^n 1/nCr$. Show that $a_n = 1 + a_{n-1} (n + 1)/(2n)$. Deduce that $\lim a_n = 2$.

B2. Let X be the set {1, 2, 3, ... , 2n}, take Y □ X with |Y| = n + 1. Show that we can find a, b ∈ Y with a dividing b.

B3. Show that if X is a square side 1 and X = A ∈ B, then A or B has diameter at least $\sqrt{5}/2$. Show that we can find A and B both having diameter $\le \sqrt{5}/2$.

B4. Let R be the reals. f **:** R → R is three times differentiable. As x → ∞, f(x) tends to a finite limit, and f '''(x) tends to zero. Show that f '(x) and f ''(x) also tend to zero.

B5. A sequence of points P_n in the plane is defined by: P_0 is at the origin; P_1 is at (1, 0); and $P_{n-1}P_n$ is length 1/n and at an angle θ to the previous segment. Find the coordinates of lim P_n.

B6. A graph has n vertices {1, 2, ... , n} and a complete set of edges. Each edge is oriented, as either i → j or j → i. Show that we can find a permutation of the vertices a_i so that $a_1 \to a_2 \to a_3 \to \ldots \to a_n$.

B7. Let N = {1, 2, ... , n}. Given a permutation f **:** N → N, define d(f) = no. of i such that f(i) > f(j) for all j > i. Find the mean of d(f) over all permutations f on N.

20th Putnam 1959

A1. Prove that we can find a real polynomial p(y) such that $p(x - 1/x) = x^n - 1/x^n$ (where n is a positive integer) iff n is odd.

A2. Let $\omega^3 = 1$, $\omega \neq 1$. Show that z_1, z_2, $-\omega z_1 - \omega^2 z_2$ are the vertices of an equilateral triangle.

A3. Let C be the complex numbers. $f : C \to C$ satisfies $f(z) + z f(1 - z) = 1 + z$ for all z. Find f.

A4. R is the reals. $f, g : [0, 1] \to R$ are arbitary functions. Show that we can find x, y such that $|xy - f(x) - g(y)| \geq 1/4$.

A5. At a particular moment, A, T and B are in a vertical line, with A 50 feet above T, and T 100 feet above B. T flies in a horizontal line at a fixed speed. A flies at a fixed speed directly towards B, B flies at twice T's speed, also directly towards T. A and B reach T simultaneously. Find the distance traveled by each of A, B and T, and A's speed.

A6. Given any real numbers $\alpha_1, \alpha_2, \dots , \alpha_m, \beta$, show that for m, n > 1 we can find m real n x n matrices $A_1, \dots , A_m$ such that $\det A_i = \alpha_i$, and $\det(\sum A_i) = \beta$.

A7. Let R be the reals. Let $f : [a, b] \to R$ have a continuous derivative, and suppose that if $f(x) = 0$, then $f'(x) \neq 0$. Show that we can find $g : [a, b] \to R$ with a continuous derivative, such that $f(x)g'(x) > f'(x)g(x)$ for all $x \in [a, b]$.

B1. Join each of m points on the positive x-axis to each of n points on the positive y-axis. Assume that no three of the resulting segments are concurrent (except at an endpoint). How many points of intersection are there (excluding endpoints)?

B2. Show that any positive real can be expressed in infinitely many ways as a sum $\sum 1/(10^{a_n})$, where $a_1 < a_2 < a_3 < \dots$ are positive integers.

B3. Find a continuous function $f : [0, 1] \to [0, 1]$ such that given any $\beta \in [0, 1]$, we can find infinitely many α such that $f(\alpha) = \beta$.

B4. A is the 5 x 5 array:

11	17	25	19	16
24	10	13	15	3
12	5	14	2	18
23	4	1	8	22
6	20	7	21	9

Pick 5 elements, one from each row and column, whose minimum is as large as possible (and prove it so).

B5. L_1 is the line $\{ (t + 1, 2t - 4, -3t + 5) : t \text{ real} \}$ and L_2 is the line $\{ (4t - 12, -t + 8, t + 17) : t \text{ real} \}$. Find the smallest sphere touching L_1 and L_2.

B6. α and β are positive irrational numbers satisfying $1/\alpha + 1/\beta = 1$. Let $a_n = [n \alpha]$ and $b_n = [n \beta]$, for n = 1, 2, 3, ... Show that the sequences a_n and b_n are disjoint and that every positive integer belongs to one or the other.

B7. Given any finite ordered tuple of real numbers X, define a real number [X], so that for all x_i, α:

(1) [X] is unchanged if we permute the order of the numbers in the tuple X;

(2) $[(x_1 + \alpha, x_2 + \alpha, \dots, x_n + \alpha)] = [(x_1, x_2, \dots, x_n)] + \alpha$;

(3) $[(-x_1, -x_2, \dots, -x_n)] = -[(x_1, x_2, \dots, x_n)]$;

(4) for $y_1 = y_2 = \dots = y_n = [(x_1, x_2, \dots, x_n)]$, we have $[(y_1, y_2, \dots, y_n, x_{n+1})] = [(x_1, x_2, \dots, x_{n+1})]$.

Show that $[(x_1, x_2, \dots, x_n)] = (x_1 + x_2 + \dots + x_n)/n$.

21st Putnam 1960

A1. For n a positive integer find f(n), the number of pairs of positive integers (a, b) such that ab/(a + b) = n.

A2. Let S be the set consisting of a square with side 1 and its interior. Show that given any three points of S, we can find two whose distance apart is at most $\sqrt{6} - \sqrt{2}$.

A3. Let α, β, γ, δ, ε be arbitary reals. Show that $(1 - \alpha) e^{\alpha} + (1 - \beta) e^{\alpha+\beta} + (1 - \gamma) e^{\alpha+\beta+\gamma} + (1 - \delta) e^{\alpha+\beta+\gamma+\delta} + (1 - \varepsilon) e^{\alpha+\beta+\gamma+\delta+\varepsilon} \le k_4$, where $k_1 = e$, $k_2 = k_1^e$, $k_3 = k_2^e$, $k_4 = k_3^e$ (so k_4 is 10^k with k approx 1.66 million).

A4. Given two points P, Q on the same side of a line *l*, find the point X which minimises the sum of the distances from X to P, Q and *l*.

A5. The real polynomial p(x) is such that for any real polynomial q(x), we have p(q(x)) = q(p(x)). Find p(x).

A6. A player throws a fair die (prob 1/6 for each of 1, 2, 3, 4, 5, 6 and each throw independent) repeatedly until his total score ≥ n. Let p(n) be the probability that his final score is n. Find lim p(n).

A7. Let f(n) be the smallest integer such that any permutation on n elements, repeated f(n) times, gives the identity. Show that f(n) = p f(n - 1) if n is a power of p, and f(n) = f(n - 1) if n is not a prime power.

B1. Find all pairs of unequal integers m, n such that $m^n = n^m$.

B2. Let $f(m, n) = 3m+n+(m+n)^2$. Find $\sum_0^{\infty}\sum_0^{\infty} 2^{-f(m, n)}$.

B3. Fluid flowing in the plane has the velocity $(y + 2x - 2x^3 - 2xy^2, - x)$ at (x, y). Sketch the flow lines near the origin. What happens to an individual particle as $t \to \infty$?

B4. Show that if an (infinite) arithmetic progression of positive integers contains an nth power, then it contains infinitely many nth powers.

B5. Define a_n by $a_0 = 0$, $a_{n+1} = 1 + \sin(a_n - 1)$. Find $\lim (\sum_0^n a_i)/n$.

B6. Let $2^{f(n)}$ be the highest power of 2 dividing n. Let g(n) = f(1) + f(2) + ... + f(n). Prove that $\sum \exp(-g(n))$ converges.

B7. Let R' be the non-negative reals. Let f, g **:** R' → R be continuous. Let a **:** R' → R be the solution of the differential equation: a' + f a = g, a(0) = c. Show that if b **:** R' → R satisfies b' + f b ≥ g for all x and b(0) = c, then b(x) ≥ a(x) for all x. Show that for sufficiently small x the solution of $y' + f y = y^2$, y(0) = d, is $y(x) = \max (d e^{-h(x)} - \int_0^x e^{-(h(x)-h(t))} u(t)^2 dt)$, where the maximum is taken over all continuous u(t), and $h(t) = \int_0^t (f(s) - 2u(s)) ds$.

22nd Putnam 1961

A1. The set of pairs of positive reals (x, y) such that $x^y = y^x$ form the straight line y = x and a curve. Find the point at which the curve cuts the line.

A2. For which real numbers α, β can we find a constant k such that $x^\alpha y^\beta < k(x + y)$ for all positive x, y?

A3. Find $\lim_{n\to\infty} \sum_1^N n/(N + i^2)$, where $N = n^2$.

A4. If $n = \prod p^r$ be the prime factorization of n, let $f(n) = (-1)^{\sum r}$ and let $F(n) = \sum_{d|n} f(d)$. Show that F(n) = 0 or 1. For which n is F(n) = 1?

A5. Let X be a set of n points. Let P be a set of subsets of X, such that if A, B ∈ P, then X - A, A ∈ B, A ∩ B ∈ P. What are the possible values for the number of elements of P?

A6. Consider polynomials in one variable over the finite field F_2 with 2 elements. Show that if n + 1 is not prime, then $1 + x + x^2 + ... + x^n$ is reducible. Can it be reducible if n + 1 is prime?

A7. S is a non-empty closed subset of the plane. The disk (a circle and its interior) D □ S and if any disk D' □ S, then D' □ D. Show that if P belongs to the interior of D, then we can find two distinct points Q, R ∈ S such that P is the midpoint of QR.

B1. a_n is a sequence of positive reals. $h = \lim (a_1 + a_2 + ... + a_n)/n$ and $k = \lim (1/a_1 + 1/a_2 + ... + 1/a_n)/n$ exist. Show that $h k \geq 1$.

B2. Two points are selected independently and at random from a segment length β. What is the probability that they are at least a distance α (< β) apart?

B3. A, B, C, D lie in a plane. No three are collinear and the four points do not lie on a circle. Show that one point lies inside the circle through the other three.

B4. Given $x_1, x_2, ... , x_n \in [0, 1]$, let $s = \sum_{1\leq i<j\leq n} |x_i - x_j|$. Find f(n), the maximum value of s over all possible $\{x_i\}$.

B5. Let n be an integer greater than 2. Define the sequence a_m by $a_1 = n$, a_{m+1} = n to the power of a_m. *Either* show that a_m < n!! ... ! (where the factorial is taken m times), *or* show that a_m > n!! ... ! (where the factorial is taken m-1 times).

B6. Let y be the solution of the differential equation $y'' = - (1 + \sqrt{x})\, y$ such that y(0) = 1, y'(0) = 0. Show that y has exactly one zero for x ∈ (0, π/2) and find a positive lower bound for it.

B7. The sequence of non-negative reals satisfies $a_{n+m} \leq a_n a_m$ for all m, n. Show that $\lim a_n^{1/n}$ exists.

23rd Putnam 1962

A1. 5 points lie in a plane, no 3 collinear. Show that 4 of the points form a convex quadrilateral.

A2. Let R be the reals. Find all f **:** K → R, where K is [0, ∞) or a finite interval [0, a), such that $(1/k \int_0^k f(x)\,dx)^2 = f(0) f(k)$ for all k in K.

A3. ABC is a triangle and k > 0. Take A' on BC, B' on CA, C' on AB so that AB' = k B'C, CA' = k A'B, BC' = k C'A. Let the three points of intersection of AA', BB', CC' be P, Q, R. Show that the area PQR $(k^2 + k + 1)$ = area ABC $(k - 1)^2$.

A4. R is the reals. [a, b] is an interval with b ≥ a + 2. f **:** [a, b] → R is twice differentiable and |f(x)| ≤ 1 and |f "(x)| ≤ 1. Show that |f '(x)| ≤ 2.

A5. Find nC1 1^2 + nC2 2^2 + nC3 3^2 + ... + nCn n^2 (where nCr is the binomial coefficient).

A6. X is a subset of the rationals which is closed under addition and multiplication. 0 □ X. For any rational x ≠ 0, just one of x, -x ∈ X. Show that X is the set of all positive rationals.

B1. Define $x^{(n)} = x(x - 1)(x - 2) \ldots (x - n + 1)$ and $x^{(0)} = 1$. Show that $(x + y)^{(n)}$ = nC0 $x^{(0)}y^{(n)}$ + nC1 $x^{(1)}y^{(n-1)}$ + nC2 $x^{(2)}y^{(n-2)}$ + ... + nCn $x^{(n)}y^{(0)}$.

B2. Let R be the reals, let N be the set of positive integers, and let P = {X **:** X □ N}. Find f **:** R → P such that f(a) □ f(b) (and f(a) ≠ f(b)) if a < b.

B3. Show that a convex open set in the plane containing the point P, but not containing any ray from P, must be bounded. Is this true for any convex set in the plane?

B4. A finite set of circles divides the plane into regions. Show that we can color the plane with two colors so that no two adjacent regions (with a common arc of non-zero length forming part of each region's boundary) have the same color.

B5. Show that for n > 1, $(3n + 1)/(2n + 2) < \sum_1^n r^n/n^n < 2$.

B6. f **:** [0, 2π) → [-1, 1] satisfies $f(x) = \sum_0^n (a_j \sin jx + b_j \cos jx)$ for some real constants a_j, b_j. Also |f(x)| = 1 for just 2n distinct values in the interval. Show that f(x) = cos(nx + k) for some k.

24th Putnam 1963

A1. Dissect a regular 12-gon into a regular hexagon, 6 squares and 6 equilateral triangles. Let the regular 12-gon have vertices $P_1, P_2, \dots, P_{12}$ (in that order). Show that the diagonals P_1P_9, $P_{12}P_4$ and P_2P_{11} are concurrent.

A2. The sequence $a_1, a_2, a_3, \dots$ of positive integers is strictly monotonic increasing, $a_2 = 2$ and $a_{mn} = a_m a_n$ for m, n relatively prime. Show that $a_n = n$.

A3. Let D be the differential operator d/dx and E the differential operator $xD(xD - 1)(xD - 2) \dots (xD - n)$. Find an expression of the form $y = \int_1^x g(t)\, dt$ for the solution of the differential equation $Ey = f(x)$, with initial conditions $y(1) = y'(1) = \dots = y^{(n)}(1) = 0$, where f(x) is a continuous real-valued function on the reals.

A4. Show that for any sequence of positive reals, a_n, we have $\limsup_{n\to\infty} n\left((a_{n+1} + 1)/a_n - 1\right) \geq 1$. Show that we can find a sequence where equality holds.

A5. R is the reals. $f : [0, \pi] \to R$ is continuous and $\int_0^\pi f(x) \sin x\, dx = \int_0^\pi f(x) \cos x\, dx = 0$. Show that f is zero for at least two points in $(0, \pi)$. Hence or otherwise, show that the centroid of any bounded convex open region of the plane is the midpoint of at least three distinct chords of its boundary.

A6. M is the midpoint of a chord PQ of an ellipse. A, B, C, D are four points on the ellipse such that AC and BD intersect at M. The lines AB and PQ meet at R, and the lines CD and PQ meet at S. Show that M is also the midpoint of RS.

B1. Find all integers n for which $x^2 - x + n$ divides $x^{13} + x + 90$.

B2. Is the set $\{ 2^m 3^n : m, n \text{ are integers} \}$ dense in the positive reals?

B3. R is the reals. Find all $f : R \to R$ which are twice differentiable and satisfy: $f(x)^2 - f(y)^2 = f(x + y) f(x - y)$.

B4. Γ is a closed plane curve enclosing a convex region and having a continuously turning tangent. A, B, C are points of Γ such that ABC has the maximum possible perimeter p. Show that the normals to Γ at A, B, C are the angle bisectors of ABC. If A, B, C have this property, does ABC necessarily have perimeter p? What happens if Γ is a circle?

B5. The series $\sum a_n$ of non-negative terms converges and $a_i \leq 100a_n$ for $i = n, n + 1, n + 2, \dots, 2n$. Show that $\lim_{n\to\infty} na_n = 0$.

B6. Let $S = S_0$ be a set of points in space. Let $S_n = \{ P : P$ belongs to the closed segment AB, for some $A, B \in S_{n-1}\}$. Show that $S_2 = S_3$.

25th Putnam 1964

A1. Let A_1, A_2, A_3, A_4, A_5, A_6 be distinct points in the plane. Let D be the longest distance between any pair, and d the shortest distance. Show that $D/d \geq \sqrt{3}$.

A2. α is a real number. Find all continuous real-valued functions $f :[0, 1] \to (0, \infty)$ such that $\int_0^1 f(x)\,dx = 1$, $\int_0^1 x f(x)\,dx = \alpha$, $\int_0^1 x^2 f(x)\,dx = \alpha^2$.

A3. The distinct points x_n are dense in the interval (0, 1). $x_1, x_2, \ldots, x_{n-1}$ divide (0, 1) into n sub-intervals, one of which must contain x_n. This part is divided by x_n into two sub-intervals, lengths a_n and b_n. Prove that $\sum a_n b_n(a_n + b_n) = 1/3$.

A4. The sequence of integers u_n is bounded and satisfies $u_n = (u_{n-1} + u_{n-2} + u_{n-3}u_{n-4})/(u_{n-1}u_{n-2} + u_{n-3} + u_{n-4})$. Show that it is periodic for sufficiently large n.

A5. Find a constant k such that for any positive a_i, $\sum_1^\infty n/(a_1 + a_2 + \ldots + a_n) \leq k \sum_1^\infty 1/a_n$.

A6. S is a finite set of collinear points. Let k be the maximum distance between any two points of S. Given a pair of points of S a distance d < k apart, we can find another pair of points of S also a distance d apart. Prove that if two pairs of points of S are distances a and b apart, then a/b is rational.

B1. a_n are positive integers such that $\sum 1/a_n$ converges. b_n is the number of a_n which are $\leq n$. Prove $\lim b_n/n = 0$.

B2. S is a finite set. A set P of subsets of S has the property that any two members of P have at least one element in common and that P cannot be extended (whilst keeping this property). Prove that P contains just half of the subsets of S.

B3. R is the reals. $f : R \to R$ is continuous and for any $\alpha > 0$, $\lim_{n\to\infty} f(n\alpha) = 0$. Prove $\lim_{x\to\infty} f(x) = 0$.

B4. n great circles on the sphere are in general position (in other words at most two circles pass through any two points on the sphere). How many regions do they divide the sphere into?

B5. Let a_n be a strictly monotonic increasing sequence of positive integers. Let b_n be the least common multiple of $a_1, a_2, \ldots, a_n$. Prove that $\sum 1/b_n$ converges.

B6. D is a disk. Show that we cannot find congruent sets A, B with $A \cap B = \square$, $A \in B = D$.

26th Putnam 1965

A1. The triangle ABC has an obtuse angle at B, and ∠A < ∠C. The external angle bisector at A meets the line BC at D, and the external angle bisector at B meets the line AC at E. Also, BA = AD = BE. Find ∠A.

A2. Let k = [(n - 1)/2]. Prove that $\sum_0^k$ ((n - 2r)/n $)^2$ $(nCr)^2$ = 1/n (2n-2)C(n-1) (where nCr is the binomial coefficient).

A3. $\{a_r\}$ is an infinite sequence of real numbers. Let b_n = 1/n $\sum_1^n$ exp(i a_r). Prove that b_1, b_2, b_3, b_4, ... converges to k iff b_1, b_4, b_9, b_{16}, ... converges to k.

A4. S and T and finite sets. U is a collection of ordered pairs (s, t) with s ∈ S and t ∈ T. There is no element s ∈ S such that all possible pairs (s, t) ∈ U. Every element t ∈ T appears in at least one element of U. Prove that we can find distinct s_1, s_2 ∈ S and distinct t_1, t_2 ∈ T such that (s_1, t_1), (s_2, t_2) ∈ U, but (s_1, t_2), (s_2, t_1) □ U.

A5. How many possible bijections f on {1, 2, ... , n} are there such that for each i = 2, 3, ... , n we can find j < i with f(i) - f(j) = ±1?

A6. α and β are positive real numbers such that 1/α + 1/β = 1. Prove that the line mx + ny = 1 with m, n positive reals is tangent to the curve $x^\alpha + y^\alpha = 1$ in the first quadrant (x, y ≥ 0) iff $m^\beta + n^\beta = 1$.

B1. X is the unit n-cube, $[0, 1]^n$. Let $k_n = \int_X \cos^2(\pi(x_1 + x_2 + ... + x_n)/(2n))\, dx_1 ... dx_n$. What is $\lim_{n\to\infty} k_n$?

B2. Every two players play each other once. The outcome of each game is a win for one of the players. Player n wins a_n games and loses b_n games. Prove that $\sum a_n^2 = \sum b_n^2$.

B3. Show that there are just three right angled triangles with integral side lengths a < b < c such that ab = 4(a + b + c).

B4. Define f_n **:**R → R by $f_n(x)$ = (nC0 + nC2 x + nC4 x^2 + ...) / (nC1 + nC3 x + nC5 x^2 + ...), where nCm is the binomial coefficient. Find a formula for $f_{n+1}(x)$ in terms of $f_n(x)$ and x, and determine $\lim_{n\to\infty} f_n(x)$ for all real x.

B5. Let S be a set with n > 3 elements. Prove that we can find a collection of $[n^2/4]$ 2-subsets of S such that for any three distinct elements A, B, C of the collection, A ∈ B ∈ C has at least 4 elements.

B6. Four distinct points A_1, A_2, B_1, B_2 have the property that any circle through A_1 and A_2 has at least one point in common with any circle through B_1 and B_2. Show that the four points are collinear or lie on a circle.

27th Putnam 1966

A1. Let $f(n) = \sum_1^n [r/2]$. Show that $f(m + n) - f(m - n) = mn$ for $m > n > 0$.

A2. A triangle has sides a, b, c. The radius of the inscribed circle is r and $s = (a + b + c)/2$. Show that $1/(s - a)^2 + 1/(s - b)^2 + 1/(s - c)^2 \geq 1/r^2$.

A3. Define the sequence $\{a_n\}$ by $a_1 \in (0, 1)$, and $a_{n+1} = a_n(1 - a_n)$. Show that $\lim_{n\to\infty} n\, a_n = 1$.

A4. Delete all the squares from the sequence 1, 2, 3, ... Show that the nth number remaining is n + m, where m is the nearest integer to $\sqrt{n}$.

A5. Let S be the set of continuous real-valued functions on the reals. $\varphi : S \to S$ is a linear map such that if $f, g \in S$ and $f(x) = g(x)$ on an open interval (a, b), then $\varphi f = \varphi g$ on (a, b). Prove that for some $h \in S$, $(\varphi f)(x) = h(x)f(x)$ for all f and x.

A6. Let $a_n = \sqrt{(1 + 2 \sqrt{(1 + 3 \sqrt{(1 + 4 \sqrt{(1 + 5 \sqrt{(... + (n - 1) \sqrt{(1 + n)} ...)})})})})}$. Prove $\lim a_n = 3$.

B1. A convex polygon does not extend outside a square side 1. Prove that the sum of the squares of its sides is at most 4.

B2. Prove that at least one integer in any set of ten consecutive integers is relatively prime to the others in the set.

B3. a_n is a sequence of positive reals such that $\sum 1/a_n$ converges. Let $s_n = \sum_1^n a_i$. Prove that $\sum n^2 a_n/s_n^2$ converges.

B4. Given a set of (mn + 1) unequal positive integers, prove that we can either (1) find m + 1 integers b_i in the set such that b_i does not divide b_j for any unequal i, j, or (2) find n+1 integers a_i in the set such that a_i divides a_{i+1} for i = 1, 2, ... , n.

B5. Given n points in the plane, no three collinear, prove that we can label them P_i so that $P_1P_2P_3 ... P_n$ is a simple closed polygon (with no edge intersecting any other edge except at its endpoints).

B6. $y = f(x)$ is a solution of $y'' + e^x y = 0$. Prove that f(x) is bounded.

28th Putnam 1967

A1. We are given a positive integer n and real numbers a_i such that $|\sum_1^n a_k \sin kx| \le |\sin x|$ for all real x. Prove $|\sum_1^n k\, a_k| \le 1$.

A2. Let u_n be the number of symmetric n x n matrices whose elements are all 0 or 1, with exactly one 1 in each row. Take $u_0 = 1$. Prove $u_{n+1} = u_n + n\, u_{n-1}$ and $\sum_0^\infty u_n x^n/n! = e^{f(x)}$, where $f(x) = x + (1/2)\, x^2$.

A3. Find the smallest positive integer n such that we can find a polynomial $nx^2 + ax + b$ with integer coefficients and two distinct roots in the interval (0, 1).

A4. Let $1/2 < \alpha \in R$, the reals. Show that there is no function $f : [0, 1] \to R$ such that $f(x) = 1 + \alpha \int_x^1 f(t)\, f(t - x)\, dt$ for all $x \in [0, 1]$.

A5. K is a convex, finite or infinite, region of the plane, whose boundary is a union of a finite number of straight line segments. Its area is at least π/4. Show that we can find points P, Q in K such that PQ = 1.

A6. a_i and b_i are reals such that $a_1b_2 \ne a_2b_1$. What is the maximum number of possible 4-tuples (sign x_1, sign x_2, sign x_3, sign x_4) for which all x_i are non-zero and x_i is a simultaneous solution of $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$ and $b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 = 0$. Find necessary and sufficient conditions on a_i and b_i for this maximum to be achieved.

B1. A hexagon is inscribed in a circle radius 1. Alternate sides have length 1. Show that the midpoints of the other three sides form an equilateral triangle.

B2. $\alpha, \beta \in [0, 1]$ and we have $ax^2 + bxy + cy^2 \equiv (\alpha x + (1 - \alpha)y)^2$, $(\alpha x + (1 - \alpha)y)(\beta x + (1 - \beta)y) \equiv dx^2 + exy + fy^2$. Show that at least one of $a, b, c \ge 4/9$ and at least one of $d, e, f \ge 4/9$.

B3. R is the reals. f, g are continuous functions R → R with period 1. Show that $\lim_{n\to\infty} \int_0^1 f(x)\, g(nx)\, dx = (\int_0^1 f(x)\, dx)\, (\int_0^1 g(x)\, dx)$.

B4. We are given a sequence $a_1, a_2, \dots, a_n$. Each a_i can take the values 0 or 1. Initially, all $a_i = 0$. We now successively carry out steps 1, 2, ... , n. At step m we change the value of a_i for those i which are a multiple of m. Show that after step n, $a_i = 1$ iff i is a square. Devise a similar scheme to give $a_i = 1$ iff i is twice a square.

B5. The first n terms of the exansion of $(2 - 1)^{-n}$ are $2^{-n} (1 + n/1!\, (1/2) + n(n + 1)/2!\, (1/2)^2 + \dots + n(n + 1) \dots (2n -2)/(n - 1)!\, (1/2)^{n-1})$. Show that they sum to 1/2.

B6. R is the reals. D is the closed unit disk $x^2 + y^2 \le 1$ in R^2. The function $f : D \to [-1, 1]$ has partial derivatives $f_1(x, y)$ and $f_2(x, y)$. Show that there is a point (a, b) in the interior of D such that $f_1(a, b)^2 + f_2(a, b)^2 \le 16$.

29th Putnam 1968

A1. Prove that $\int_0^1 x^4(1 - x)^4/(1 + x^2)\, dx = 22/7 - \pi$.

A2. Given integers a, b, c, d such that $ad - bc \neq 0$, integers m, n and a real $\varepsilon > 0$, show that we can find rationals x, y, such that $0 < |ax + by - m| < \varepsilon$ and $0 < |cx + dy - n| < \varepsilon$.

A3. S is a finite set. P is the set of all subsets of S. Show that we can label the elements of P as A_i, such that $A_1 =$ □ and for each $n \geq 1$, either A_{n-1} □ A_n and $|A_n - A_{n-1}| = 1$, or A_{n-1} □ A_n and $|A_{n-1} - A_n| = 1$.

A4. Let S_2 be the 2-sphere $\{ (x, y, z) : x^2 + y^2 + z^2 = 1\}$. Show that for any n points on S_2, the sum of the squares of the $n(n - 1)/2$ distances between them (measured in space, not in S_2) is at most n^2.

A5. Find the smallest possible α such that if $p(x) \equiv ax^2 + bx + c$ satisfies $|p(x)| \leq 1$ on [0, 1], then $|p'(0)| \leq \alpha$.

A6. Find all finite polynomials whose coefficients are all ±1 and whose roots are all real.

B1. The random variables X, Y can each take a finite number of integer values. They are not necessarily independent. Express prob(min(X, Y) = k) in terms of p_1 = prob(X = k), p_2 = prob(Y = k) and p_3 = prob(max(X, Y) = k).

B2. (G, *) is a finite group with n elements. K is a subset of G with more than n/2 elements. Prove that for every $g \in G$, we can find $h, k \in K$ such that $g = h * k$.

B3. Given that a 60^o angle cannot be trisected with ruler and compass, prove that a $120^o/n$ angle cannot be trisected with ruler and compass for n = 1, 2, 3, ...

B4. R is the reals. $f : R \to R$ is continuous and $L = \int_{-\infty}^{\infty} f(x)\, dx$ exists. Show that $\int_{-\infty}^{\infty} f(x - 1/x)\, dx = L$.

B5. Let F be the field with p elements. Let S be the set of 2 x 2 matrices over F with trace 1 and determinant 0. Find |S|.

B6. A *compact* set of real numbers is closed and bounded. Show that we cannot find compact sets A_1, A_2, A_3, ... such that (1) all elements of A_n are rational and (2) given any compact set K whose members are all rationals, K □ some A_n.

30th Putnam 1969

A1. R^2 represents the usual plane (x, y) with $-\infty < x, y < \infty$. p: $R^2 \to R$ is a polynomial with real coefficients. What are the possibilities for the image $p(R^2)$?

A2. A is an n x n matrix with elements $a_{ij} = |i - j|$. Show that the determinant $|A| = (-1)^{n-1} (n - 1) 2^{n-2}$.

A3. An n-gon (which is not self-intersecting) is triangulated using m interior vertices. In other words, there is a set of N triangles such that: (1) their union is the original n-gon; (2) the union of their vertices is the set consisting of the n vertices of the n-gon and the m interior vertices; (3) the intersection of any two distinct triangles in the set is either empty, a vertex of both triangles, or a side of both triangles. What is N?

A4. Prove that $\int_0^1 x^x \, dx = 1 - 1/2^2 + 1/3^3 - 1/4^4 + \dots$.

A5. u(t) is a continuous function. x(t), y(t) is the solution of $x' = -2y + u(t)$, $y' = -2x + u(t)$ satisfying the initial condition $x(0) = x_0$, $y(0) = y_0$. Show that if $x_0 \neq y_0$, then we do not have $x(t) = y(t) = 0$ for any t, but that given any $x_0 = y_0$ and any $T > 0$, we can always find some u(t) such that $x(T) = y(T) = 0$.

A6. The sequence $a_1 + 2a_2, a_2 + 2a_3, a_3 + 2a_4, \dots$ converges. Prove that the sequence $a_1, a_2, a_3, \dots$ also converges.

B1. The positive integer n is divisible by 24. Show that the sum of all the positive divisors of n - 1 (including 1 and n - 1) is also divisible by 24.

B2. G is a finite group with identity 1. Show that we cannot find two proper subgroups A and B ($\neq \{1\}$ or G) such that $A \in B = G$. Can we find three proper subgroups A, B, C such that $A \in B \in C = G$?

B3. The sequence $a_1, a_2, a_3, \dots$ satisfies $a_1a_2 = 1$, $a_2a_3 = 2$, $a_3a_4 = 3$, $a_4a_5 = 4$, ... Also, $\lim_{n\to\infty} a_n/a_{n+1} = 1$. Prove that $a_1 = \sqrt{(2/\pi)}$.

B4. Γ is a plane curve of length 1. Show that we can find a closed rectangle area 1/4 which covers Γ.

B5. The sequence a_i, i = 1, 2, 3, ... is strictly monotonic increasing and the sum of its inverses converges. Let f(x) = the largest i such that $a_i < x$. Prove that $f(x)/x \to 0$ as $x \to \infty$.

B6. M is a 3 x 2 matrix, N is a 2 x 3 matrix. MN =

8	2	-2
2	5	4
-2	4	5

Show that NM =

9	0
0	9

31st Putnam 1970

A1. e^{bx} cos cx is expanded in a Taylor series $\sum a_n x^n$. b and c are positive reals. Show that either all a_n are non-zero, or infinitely many a_n are zero.

A2. $p(x, y) = a x^2 + b x y + c y^2$ is a homogeneous real polynomial of degree 2 such that $b^2 < 4ac$, and q(x, y) is a homogeneous real polynomial of degree 3. Show that we can find k > 0 such that p(x, y) = q(x, y) has no roots in the disk $x^2 + y^2 < k$ except (0, 0).

A3. A perfect square has *length* n if its last n digits (in base 10) are the same and non-zero. What is the longest possible length? What is the smallest square achieving this length?

A4. The real sequence $a_1, a_2, a_3, \ldots$ has the property that $\lim_{n\to\infty} (a_{n+2} - a_n) = 0$. Prove that $\lim_{n\to\infty} (a_{n+1} - a_n)/n = 0$.

A5. Find the radius of the largest circle on an ellipsoid with semi-axes a > b > c.

A6. x is chosen at random from the interval [0, a] (with the uniform distribution). y is chosen similarly from [0, b], and z from [0, c]. The three numbers are chosen independently, and a ≥ b ≥ c. Find the expected value of min(x, y, z).

B1. Let $f(n) = (n^2 + 1)(n^2 + 4)(n^2 + 9) \ldots (n^2 + (2n)^2)$. Find $\lim_{n\to\infty} f(n)^{1/n}/n^4$.

B2. A weather station measures the temperature T continuously. It is found that on any given day T = p(t), where p is a polynomial of degree <= 3, and t is the time. Show that we can find times $t_1 < t_2$, which are independent of p, such that the average temperature over the period 9am to 3pm is $(p(t_1) + p(t_2)) / 2$. Show that $t_1 \approx$ 10:16am, $t_2 \approx$ 1:44pm.

B3. S is a closed subset of the real plane. Its projection onto the x-axis is bounded. Show that its projection onto the y-axis is closed.

B4. A vehicle covers a mile (= 5280 ft) in a minute, starting and ending at rest and never exceeding 90 miles/hour. Show that its acceleration or deceleration exceeded 6.6 ft/sec^2.

B5. $k_n(x) = -n$ on $(-\infty, -n]$, x on $[-n, n]$, and n on $[n, \infty)$. Prove that the (real valued) function f(x) is continuous iff all $k_n(f(x))$ are continuous.

B6. The quadrilateral Q contains a circle which touches each side. It has side lengths a, b, c, d and area $\sqrt{(abcd)}$. Prove it is cyclic.

32nd Putnam 1971

A1. Given any 9 lattice points in space, show that we can find two which have a lattice point on the interior of the segment joining them.

A2. Find all possible polynomials f(x) such that f(0) = 0 and $f(x^2 + 1) = f(x)^2 + 1$.

A3. The vertices of a triangle are lattice points in the plane. Show that the diameter of its circumcircle does not exceed the product of its side lengths.

A4. α lies in the open interval (1, 2). Show that the polynomial formed by expanding $(x + y)^n(x^2 - \alpha xy + y^2)$ has positive coefficients for sufficiently large n. Find the smallest such n for α = 1.998.

A5. A player scores either A or B at each turn, where A and B are unequal positive integers. He notices that his cumulative score can take any positive integer value except for those in a finite set S, where |S| =35, and 58 ∈ S. Find A and B.

A6. α is a real number such that $1^\alpha, 2^\alpha, 3^\alpha$, ... are all integers. Show that α ≥ 0 and that α is an integer.

B1. S is a set with a binary operation * such that (1) a * a = a for all a ∈ S, and (2) (a * b) * c = (b * c) * a for all a, b, c ∈ S. Show that * is associative and commutative.

B2. Let X be the set of all reals except 0 and 1. Find all real valued functions f(x) on X which satisfy f(x) + f(1 - 1/x) = 1 + x for all x in X.

B3. Car A starts at time t = 0 and, traveling at a constant speed, completes 1 lap every hour. Car B starts at time t = α > 0 and also completes 1 lap every hour, traveling at a constant speed. Let a(t) be the number of laps completed by A at time t, so that a(t) = 0 for t < 1, a(t) = 1 for 1 ≤ t < 2 and so on. Similarly, let b(t) be the number of laps completed by B at time t. Let S = {t : a(t) = 2 b(t) }. Show that S is made up of intervals of total length 1.

B4. A and B are two points on a sphere. S(A, B, k) is defined to be the set {P : $\underline{AP}$ + $\underline{BP}$ = k}, where $\underline{XY}$ denotes the great-circle distance between points X and Y on the sphere. Determine all sets S(A, B, k) which are circles.

B5. A *hypocycloid* is the path traced out by a point on the circumference of a circle rolling around the inside circumference of a larger fixed circle. Show that the plots in the (x, y) plane of the solutions (x(t), y(t)) of the differential equations x'' + y' + 6x = 0, y'' - x' + 6y = 0 with initial conditions x'(0) = y'(0) = 0 are hypocycloids. Find the possible radii of the circles.

B6. Prove that: |f(1)/1 + f(2)/2 + ... + f(n)/n - 2n/3| < 1, where f(n) is the largest odd divisor of n.

33rd Putnam 1972

A1. Show that we cannot have 4 binomial coefficients nCm, nC(m+1), nC(m+2), nC(m+3) with n, m > 0 (and $m + 3 \le n$) in arithmetic progression.

A2. Let S be a set with a binary operation * such that (1) a * (a * b) = b for all a, b $\in$ S, (2) (a * b) * b = a for all a, b $\in$ S. Show that * is commutative. Give an example for which S is not associative.

A3. A sequence { x_i } is said to have a Cesaro limit iff $\lim_{n\to\infty}(x_1 + x_2 + ... + x_n)/n$ exists. Find all (real-valued) functions f on the closed interval [0, 1] such that { $f(x_i)$ } has a Cesaro limit iff { x_i } has a Cesaro limit.

A4. Show that a circle inscribed in a square has a larger perimeter than any other ellipse inscribed in the square.

A5. Show that n does not divide $2^n - 1$ for n > 1.

A6. f is an integrable real-valued function on the closed interval [0, 1] such that $\int_0^1 x^m f(x)\,dx = 0$ for m = 0, 1, 2, ... , n - 1, and 1 for m = n. Show that $|f(x)| \ge 2^n(n + 1)$ on a set of positive measure.

B1. Let $\sum_0^\infty x^n(x - 1)^{2n} / n! = \sum_0^\infty a_n x^n$. Show that no three consecutive a_n are zero.

B2. A particle moves in a straight line with monotonically decreasing acceleration. It starts from rest and has velocity v a distance d from the start. What is the maximum time it could have taken to travel the distance d?

B3. A group has elements g, h satisfying: $ghg = hg^2h$, $g^3 = 1$, $h^n = 1$ for some odd n. Prove h = 1.

B4. Show that for n > 1 we can find a polynomial p(a, b, c) with integer coefficients such that $p(x^n, x^{n+1}, x + x^{n+2}) \equiv x$.

B5. A, B, C and D are non-coplanar points. $\angle ABC = \angle ADC$ and $\angle BAD = \angle BCD$. Show that AB = CD and BC = AD.

B6. The polynomial p(x) has all coefficients 0 or 1, and p(0) = 1. Show that if the complex number z is a root, then $|z| \ge (\sqrt{5} - 1)/2$.

34th Putnam 1973

A1. ABC is a triangle. P, Q, R are points on the sides BC, CA, AB. Show that one of the triangles AQR, BRP, CPQ has area no greater than PQR. If BP ≤ PC, CQ ≤ QA, AR ≤ RB, show that the area of PQR is at least 1/4 of the area of ABC.

A2. $a_n = \pm 1/n$ and $a_{n+8} > 0$ iff $a_n > 0$. Show that if four of $a_1, a_2, \ldots, a_8$ are positive, then $\sum a_n$ converges. Is the converse true?

A3. n is a positive integer. Prove that $[\sqrt{(4n + 1)}\,] = [\,\min (k + n/k)\,]$, where the minimum is taken over all positive integers k.

A4. How many real roots does $2^x = 1 + x^2$ have?

A5. An object's equations of motion are: $x' = yz$, $y' = zx$, $z' = xy$. Its coordinates at time $t = 0$ are (x_0, y_0, z_0). If two of these coordinates are zero, show that the object is stationary for all t. If $(x_0, y_0, z_0) = (1, 1, 0)$, show that at time t, $(x, y, z) = (\sec t, \sec t, \tan t)$. If $(x_0, y_0, z_0) = (1, 1, -1)$, show that at time t, $(x, y, z) = (1/(1 + t), 1/(1 + t), -1/(1 + t))$. If two of the coordinates x_0, y_0, z_0 are non-zero, show that the object's distance from the origin $d \to \infty$ at some finite time in the past or future.

A6. Show that there are no seven lines in the plane such that there are at least six points on just three lines and at least four points on just two lines.

B1. S is a finite collection of integers, not necessarily distinct. If any element of S is removed, then the remaining integers can be divided into two collections with the same size and the same sum. Show that all elements of S are equal.

B2. The real and imaginary parts of z are rational, and z has unit modulus. Show that $|z^{2n} - 1|$ is rational for any integer n.

B3. The prime p has the property that $n^2 - n + p$ is prime for all positive integers less than p. Show that there is exactly one integer triple (a, b, c) such that $b^2 - 4ac = 1 - 4p$, $0 < a \le c$, $-a \le b < a$.

B4. f is defined on the closed interval [0, 1], f(0) = 0, and f has a continuous derivative with values in (0, 1]. By considering the inverse f^{-1} or otherwise, show that $\left(\int_0^1 f(x)\,dx \right)^2 \ge \int_0^1 f(x)^3\,dx$. Give an example where we have equality.

B5. If x is a solution of the quadratic $ax^2 + bx + c = 0$, show that, for any n, we can find polynomials p and q with rational coefficients such that $x = p(x^n, a, b, c) / q(x^n, a, b, c)$. Hence or otherwise find polynomials r, s with rational coefficients so that $x = r(x^3, x + 1/x) / s(x^3, x + 1/x)$.

B6. Show that $\sin^2 x \sin 2x$ has two maxima in the interval $[0, 2\pi]$, at $\pi/3$ and $4\pi/3$. Let f(x) = the absolute value of $\sin^2 x \sin^3 4x \sin^3 8x \ldots \sin^3 2^{n-1}x \sin 2^n x$. Show that $f(\pi/3) \ge f(x)$. Let $g(x) = \sin^2 x \sin^2 4x \sin^2 8x \ldots \sin^2 2^n x$. Show that $g(x) \le 3^n/4^n$.

35th Putnam 1974

A1. S is a subset of {1, 2, 3, ... , 16} which does not contain three integers which are relatively prime in pairs. How many elements can S have?

A2. C is a vertical circle fixed to a horizontal line. P is a fixed point outside the circle and above the horizontal line. For a point Q on the circle, $f(Q) \in (0, \infty]$ is the time taken for a particle to slide down the straight line from P to Q (under the influence of gravity). What point Q minimizes f(Q)?

A3. Which odd primes p can be written in the form $m^2 + 16n^2$? In the form $4m^2 + 4mn + 5n^2$,where m and n may be negative? [You may assume that p can be written in the form $m^2 + n^2$ iff p = 1 (mod 4).]

A4. Find $1/2^{n-1} \sum_1^{[n/2]} (n - 2i)$ nCi, where nCi is the binomial coefficient.

A5. The parabola $y = x^2$ rolls around the fixed parabola $y = -x^2$. Find the locus of its focus (initially at x = 0, y = 1/4).

A6. Let f(n) be the degree of the lowest order polynomial p(x) with integer coefficients and leading coefficient 1, such that n divides p(m) for all integral m. Describe f(n). Evaluate f(1000000).

B1. P, Q, R, S, T are points on a circle radius 1. How should they be placed to maximise the sum of the perimeter and the five diagonals?

B2. f(x) is a real valued function on the reals, and has a continuous derivative. $f'(x)^2 + f(x)^3 \to 0$ as $x \to \infty$. Show that f(x) and $f'(x) \to 0$ as $x \to \infty$.

B3. Prove that $(\cos^{-1}(1/3))/\pi$ is irrational.

B4. R is the reals. $f : R^2 \to R$ is such that $f_{x0} : R \to R$ defined by $f_{x0}(x) = f(x_0, x)$ is continuous for every x_0 and $g_{y0} : R \to R$ defined by $g_{y0}(x) = f(x, y_0)$ is continuous for every y_0. Show that there is a sequence of continuous functions $h_n : R^2 \to R$ which tend to f pointwise.

B5. Let $f_n(x) = \sum_0^n x^i/i!$. Show that $f(n) > e^n/2$. [Assume $e^x - f(x) = 1/n! \int_0^x (x - t)^n e^t\, dt$, and $\int_0^\infty t^n e^{-t}\, dt = n!$]

B6. Let S be a set with 1000 elements. Find a, b, c, the number of subsets R of S such that |R| = 0, 1, 2 (mod 3) respectively. Find a, b, c if |S| = 1001.

36th Putnam 1975

A1. A triangular number is a positive integer of the form n(n + 1)/2. Show that m is a sum of two triangular numbers iff 4m + 1 is a sum of two squares.

A2. For what region of the real (a, b) plane, do both (possibly complex) roots of the polynomial $z^2 + az + b = 0$ satisfy $|z| < 1$?

A3. Let $0 < \alpha < \beta < \gamma \in R$, the reals. Let $K = \{ (x, y, z) : x^\beta + y^\beta + z^\beta = 1$, and $x, y, z \geq 0\} \in R^3$. Define $f : K \to R$ by $f(x, y, z) = x^\alpha + y^\beta + z^\gamma$. At what points of K does f assume its maximum and minimum values?

A4. m > 1 is odd. Let n = 2m and $\theta = e^{2\pi i/n}$. Find a finite set of integers $\{a_i\}$ such that $\sum a_i \theta^i = 1/(1 - \theta)$.

A5. Let I be an interval and f(x) a continuous real-valued function on I. Let y_1 and y_2 be linearly independent solutions of y" = f(x) y, which take positive values on I. Show that from some positive constant k, $k \sqrt{(y_1 y_2)}$ is a solution of $y'' + 1/y^3 = f(x) y$.

A6. Given three points in space forming an acute-angled triangle, show that we can find two further points such that no three of the five points are collinear and the line through any two is normal to the plane through the other three.

B1. Let G be the group { (m, n) : m, n are integers } with the operation (a, b) + (c, d) = (a + c, b + d). Let H be the smallest subgroup containing (3, 8), (4, -1) and (5, 4). Let H_{ab} be the smallest subgroup containing (0, a) and (1, b). Find a > 0 such that $H_{ab} = H$.

B2. A *slab* is the set of points strictly between two parallel planes. Prove that a countable sequence of slabs, the sum of whose thicknesses converges, cannot fill space.

B3. Let n be a fixed positive integer. Let S be any finite collection of at least n positive reals (not necessarily all distinct). Let $f(S) = (\sum_{a \in S} a)^n$, and let g(S) = the sum of all n-fold products of the elements of S (in other words, the nth symmetric function). Find $\sup_S g(S)/f(S)$.

B4. Does a circle have a subset which is topologically closed and which contains just one of each pair of diametrically opposite points?

B5. Define $f_0(x) = e^x$, $f_{n+1}(x) = x f_n'(x)$. Show that $\sum_0^\infty f_n(1)/n! = e^e$.

B6. Let $h_n = \sum_1^n 1/r$. Show that $n - (n - 1) n^{-1/(n-1)} > h_n > n(n + 1)^{1/n} - n$ for n > 2.

37th Putnam 1976

A1. Given two rays OA and OB and a point P between them. Which point X on the ray OA has the property that if XP is extended to meet the ray OB at Y, then XP.PY has the smallest possible value.

A2. Let $a(x, y)$ be the polynomial $x^2y + xy^2$, and $b(x, y)$ the polynomial $x^2 + xy + y^2$. Prove that we can find a polynomial $p_n(a, b)$ which is identically equal to $(x + y)^n + (-1)^n (x^n + y^n)$. For example, $p_4(a, b) = 2b^2$.

A3. Find all solutions to $p^n = q^m \pm 1$, where p and q are primes and $m, n \geq 2$.

A4. Let $p(x) \equiv x^3 + ax^2 + bx - 1$, and $q(x) \equiv x^3 + cx^2 + dx + 1$ be polynomials with integer coefficients. Let α be a root of $p(x) = 0$. $p(x)$ is irreducible over the rationals. $\alpha + 1$ is a root of $q(x) = 0$. Find an expression for another root of $p(x) = 0$ in terms of α, but not involving a, b, c, or d.

A5. Let P be a convex polygon. Let Q be the interior of P and $S = P \in Q$. Let p be the perimeter of P and A its area. Given any point (x, y) let $d(x, y)$ be the distance from (x, y) to the nearest point of S. Find constants α, β, γ such that $\int_U e^{-d(x,y)}\, dx\, dy = \alpha + \beta p + \gamma A$, where U is the whole plane.

A6. Let R be the real line. $f : R \to [-1, 1]$ is twice differentiable and $f(0)^2 + f'(0)^2 = 4$. Show that $f(x_0) + f''(x_0) = 0$ for some x_0.

B1. Show that $\lim_{n\to\infty} 1/n \sum_1^n ([2n/i] - 2[n/i]) = \ln a - b$ for some positive integers a and b.

B2. G is a group generated by the two elements g, h, which satisfy $g^4 = 1$, $g^2 \neq 1$, $h^7 = 1$, $h \neq 1$, $ghg^{-1}h = 1$. The only subgroup containing g and h is G itself. Write down all elements of G which are squares.

B3. Let $0 < \alpha < 1/4$. Define the sequence p_n by $p_0 = 1$, $p_1 = 1 - \alpha$, $p_{n+1} = p_n - \alpha p_{n-1}$. Show that if each of the events $A_1, A_2, \dots, A_n$ has probability at least $1 - \alpha$, and A_i and A_j are independent for $| i - j | > 1$, then the probability of all A_i occurring is at least p_n. You may assume that all p_n are positive.

B4. Let an ellipse have center O and foci A and B. For a point P on the ellipse let d be the distance from O to the tangent at P. Show that $PA\cdot PB\cdot d^2$ is independent of the position of P.

B5. Find $\sum_0^n (-1)^i\, nCi\, (x - i)^n$, where nCi is the binomial coefficient.

B6. Let $\sigma(n)$ be the sum of all positive divisors of n, including 1 and n. Show that if $\sigma(n) = 2n + 1$, then n is the square of an odd integer.

38th Putnam 1977

A1. Show that if four distinct points of the curve $y = 2x^4 + 7x^3 + 3x - 5$ are collinear, then their average x-coordinate is some constant k. Find k.

A2. Find all real solutions (a, b, c, d) to the equations $a + b + c = d$, $1/a + 1/b + 1/c = 1/d$.

A3. R is the reals. f, g, h are functions $R \to R$. $f(x) = (h(x + 1) + h(x - 1))/2$, $g(x) = (h(x + 4) + h(x - 4))/2$. Express h(x) in terms of f and g.

A4. Find polynomials p(x) and q(x) with integer coefficients such that $p(x)/q(x) = \sum_0^\infty x^{2n}/(1 - x^{2n+1})$ for $x \in (0, 1)$.

A5. p is a prime and $m \geq n$ are non-negative integers. Show that (pm)C(pn) = mCn (mod p), where mCn is the binomial coefficient.

A6. R is the reals. X is the square [0, 1] x [0, 1]. $f : X \to R$ is continuous. If $\int_Y f(x, y)\, dx\, dy = 0$ for all squares Y such that (1) Y □ X, (2) Y has sides parallel to those of X, (3) at least one of Y's sides is contained in the boundary of X, is it true that f(x, y) = 0 for all x, y?

B1. Find $\prod_2^\infty (n^3 - 1)/(n^3 + 1)$.

B2. P is a plane containing a convex quadrilateral ABCD. X is a point not in P. Find points A', B', C', D' on the lines XA, XB, XC, XD respectively so that A'B'C'D' is a parallelogram.

B3. Let S be the set of all collections of 3 (not necessarily distinct) positive irrational numbers with sum 1. If $A = \{x, y, z\} \in S$ and $x > 1/2$, define $A' = \{2x - 1, 2y, 2z\}$. Does repeated application of this operation necessarily give a collection with all elements < 1/2?

B4. Let P be a point inside a continuous closed curve in the plane which does not intersect itself. Show that we can find two points on the curve whose midpoint is P.

B5. $a_1, a_2, \ldots, a_n$ are real and $b < (\sum a_i)^2/(n - 1) - \sum a_i^2$. Show that $b < 2a_ia_j$ for all distinct i, j.

B6. G is a group. H is a finite subgroup with n elements. For some element $g \in G$, $(gh)^3 = 1$ for all elements $h \in H$. Show that there are at most $3n^2$ distinct elements which can be written as a product of a finite number of elements of the coset Hg.

39th Putnam 1978

A1. Let S = {1, 4, 7, 10, 13, 16, ... , 100}. Let T be a subset of 20 elements of S. Show that we can find two distinct elements of T with sum 104.

A2. Let A be the real n x n matrix (a_{ij}) where a_{ij} = a for i < j, b (≠ a) for i > j, and c_i for i = j. Show that det A = (b p(a) - a p(b))/(b - a), where p(x) = ∏ (c_i - x).

A3. Let $p(x) = 2(x^6 + 1) + 4(x^5 + x) + 3(x^4 + x^2) + 5x^3$. Let $a = \int_0^\infty x/p(x)\,dx$, $b = \int_0^\infty x^2/p(x)\,dx$, $c = \int_0^\infty x^3/p(x)\,dx$, $d = \int_0^\infty x^4/p(x)\,dx$. Which of a, b, c, d is the smallest?

A4. A binary operation (represented by multiplication) on S has the property that (ab)(cd) = ad for all a, b, c, d. Show that: (1) if ab = c, then cc = c; (2) if ab = c, then ad = cd for all d. Find a set S, and such a binary operation, which also satisfies: (A) a a = a for all a; (B) ab = a ≠ b for some a, b; (C) ab ≠ a for some a, b.

A5. Let $a_1, a_2, ... , a_n$ be reals in the interval (0, π) with arithmetic mean μ. Show that ∏ (sin a_i)/a_i ≤ ((sin μ)/μ $)^n$.

A6. Given n points in the plane, prove that less than $2n^{3/2}$ pairs of points are a distance 1 apart.

B1. A convex octagon inscribed in a circle has 4 consecutive sides length 3 and the remaining sides length 2. Find its area.

B2. Find $\sum_1^\infty \sum_1^\infty 1/(i^2j + 2ij + ij^2)$.

B3. The polynomials $p_n(x)$ are defined by $p_1(x) = 1 + x$, $p_2(x) = 1 + 2x$, $p_{2n+1}(x) = p_{2n}(x) + (n + 1)\, x\, p_{2n-1}(x)$, $p_{2n+2}(x) = p_{2n+1}(x) + (n + 1)\, x\, p_{2n}(x)$. Let a_n be the largest real root of $p_n(x)$. Prove that a_n is monotonic increasing and tends to zero.

B4. Show that we can find integers a, b, c, d such that $a^2 + b^2 + c^2 + d^2 = abc + abd + acd + bcd$, and the smallest of a, b, c, d is arbitarily large.

B5. Find the real polynomial p(x) of degree 4 with largest possible coefficient of x^4 such that p([-1, 1]) □ [0, 1].

B6. a_{ij} are reals in [0, 1]. Show that $(\sum_{i=1}^n \sum_{j=1}^{mi} a_{ij}/i)^2 \le 2m \sum_{i=1}^n \sum_{j=1}^{mi} a_{ij}$.

40th Putnam 1979

A1. Find the set of positive integers with sum 1979 and maximum possible product.

A2. R is the reals. For what real k can we find a continuous function f : R → R such that $f(f(x)) = k x^9$ for all x.

A3. a_n are defined by $a_1 = \alpha$, $a_2 = \beta$, $a_{n+2} = a_n a_{n+1}/(2a_n - a_{n+1})$. α, β are chosen so that $a_{n+1} \neq 2a_n$. For what α, β are infinitely many a_n integral?

A4. A and B are disjoint sets of n points in the plane. No three points of A ∈ B are collinear. Can we always label the points of A as $A_1, A_2, \ldots, A_n$, and the points of B as $B_1, B_2, \ldots, B_n$ so that no two of the n segments A_iB_i intersect?

A5. Show that we can find two distinct real roots α, β of $x^3 - 10x^2 + 29x - 25$ such that we can find infinitely many positive integers n which can be written as $n = [r\alpha] = [s\beta]$ for some integers r, s.

A6. Given n reals $\alpha_i \in [0, 1]$ show that we can find $\beta \in [0, 1]$ such that $\sum 1/|\beta - \alpha_i| \le 8n \sum_1^n 1/(2i - 1)$.

B1. Can we find a line normal to the curves y = cosh x and y = sinh x?

B2. Given $0 < \alpha < \beta$, find $\lim_{\lambda\to 0} \left(\int_0^1 (\beta x + \alpha(1 - x))^\lambda \, dx \right)^{1/\lambda}$.

B3. F is a finite field with n elements. n is odd. $x^2 + bx + c$ is an irreducible polynomial over F. For how many elements d ∈ F is $x^2 + bx + c + d$ irreducible?

B4. Find a non-trivial solution of the differential equation $F(y) \equiv (3x^2 + x - 1)y'' - (9x^2 + 9x - 2)y' + (18x + 3)y = 0$. The solution of $F(y) = 6(6x + 1)$ such that $f(0) = 1$, and $(f(-1) - 2)(f(1) - 6) = 1$ is $y=f(x)$. Find a relation of the form $(f(-2) - a)(f(2) - b) = c$.

B5. A convex set S in the plane contains (0, 0) but no other lattice points. The intersections of S with each of the four quadrants have the same area. Show that the area of S is at most 4.

B6. z_i are complex numbers. Show that $|\text{Re}[(z_1^2 + z_2^2 + \ldots + z_n^2)^{1/2}]| \le |\text{Re } z_1| + |\text{Re } z_2| + \ldots + |\text{Re } z_n|$.

41st Putnam 1980

A1. Let $f(x) = x^2 + bx + c$. Let C be the curve $y = f(x)$ and let P_i be the point $(i, f(i))$ on C. Let A_i be the point of intersection of the tangents at P_i and P_{i+1}. Find the polynomial of smallest degree passing through $A_1, A_2, \ldots, A_9$.

A2. Find $f(m, n)$, the number of 4-tuples (a, b, c, d) of positive integers such that the lowest common multiple of any three integers in the 4-tuple is $3^m 7^n$.

A3. Find $\int_0^{\pi/2} f(x)\,dx$, where $f(x) = 1/(1 + \tan^{\sqrt{2}} x)$.

A4. Show that for any integers a, b, c, not all zero, and such that $|a|, |b|, |c| < 10^6$, we have $|a + b\sqrt{2} + c\sqrt{3}| > 10^{-21}$. But show that we can find such a, b, c with $|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}$.

A5. Let $p(x)$ be a polynomial with real coefficients of degree 1 or more. Show that there are only finitely many values α such that $\int_0^\alpha p(x) \sin x\,dx = 0$ and $\int_0^\alpha p(x) \cos x\,dx = 0$.

A6. Let R be the reals and C the set of all functions $f : [0, 1] \to R$ with a continuous derivative and satisfying $f(0) = 0$, $f(1) = 1$. Find $\inf_C \int_0^1 | f'(x) - f(x) |\,dx$.

B1. For which real k do we have $\cosh x \le \exp(k x^2)$ for all real x?

B2. S is the region of space defined by $x, y, z \ge 0$, $x + y + z \le 11$, $2x + 4y + 3z \le 36$, $2x + 3z \le 24$. Find the number of vertices and edges of S. For which a, b is $ax + by + z \le 2a + 5b + 4$ for all points of S?

B3. Define a_n by $a_0 = \alpha$, $a_{n+1} = 2a_n - n^2$. For which α are all a_n positive?

B4. S is a finite set with subsets $S_1, S_2, \ldots, S_{1066}$ each containing more than half the elements of S. Show that we can find $T \subseteq S$ with $|T| \le 10$, such that all $T \cap S_i$ are non-empty.

B5. R^{0+} is the non-negative reals. For $\alpha \ge 0$, C_α is the set of continuous functions $f : [0, 1] \to R^{0+}$ such that: (1) f is convex [$f(\lambda x + \mu y) \le \lambda f(x) + \mu f(y)$ for $\lambda, \mu \ge 0$ with $\lambda + \mu = 1$]; (2) f is increasing; (3) $f(1) - 2 f(2/3) + f(1/3) \ge \alpha (f(2/3) - 2 f(1/3) + f(0))$. For which α is C_α is closed under pointwise multiplication?

B6. An array of rationals $f(n, i)$ where n and i are positive integers with $i > n$ is defined by $f(1, i) = 1/i$, $f(n+1, i) = (n+1)/i \, (f(n, n) + f(n, n+1) + \ldots + f(n, i - 1))$. If p is prime, show that $f(n, p)$ has denominator (when in lowest terms) not a multiple of p (for $n > 1$).

42nd Putnam 1981

A1. Let the largest power of 5 dividing $1^1 2^2 3^3 \dots n^n$ be $5^{f(n)}$. What is $\lim_{n\to\infty} f(n)/n^2$?

A2. We can label the squares of an 8 x 8 chess board from from 1 to 64 in 64! different ways. For each way we find D, the largest difference between the labels of two squares which are adjacent (orthogonally or diagonally). What is the smallest possible D?

A3. Evaluate: $\lim_{k\to\infty} e^{-k} \int_R (e^x - e^y) / (x - y)\, dx\, dy$, where R is the rectangle $0 \le x, y, \le k$.

A4. A particle moves in a straight line inside a square side 1. It is reflected from the sides, but absorbed by the four corners. It starts from an arbitrary point P inside the square. Let c(k) be the number of possible starting directions from which it reaches a corner after traveling a distance k or less. Find the smallest constant a_2, such that from some constants a_1 and a_0, $c(k) \le a_2 k^2 + a_1 k + a_0$ for all P and all k.

A5. p(x) is a real polynomial with at least n distinct real roots greater than 1. [To be precise we can find at least n distinct values $a_i > 1$ such that $p(a_i) = 0$. It is possible that one or more of the a_i is a multiple root, and it is possible that there are other roots.] Put $q(x) = (x^2 + 1) p(x) p'(x) + x p(x)^2 + x p'(x)^2$. Must q(x) have at least 2n - 1 distinct real roots?

A6. A, B, C are lattice points in the plane. The triangle ABC contains exactly one lattice point, X, in its interior. The line AX meets BC at E. What is the largest possible value of AX/XE?

B1. Evaluate $\lim_{n\to\infty} 1/n^5 \sum (5 r^4 - 18 r^2 s^2 + 5 s^4)$, where the sum is over all r, s satisfying $0 < r, s \le n$.

B2. What is the minimum value of $(a - 1)^2 + (b/a - 1)^2 + (c/b - 1)^2 + (4/c - 1)^2$, over all real numbers a, b, c satisfying $1 \le a \le b \le c \le 4$.

B3. Prove that infinitely many positive integers n have the property that for any prime p dividing $n^2 + 3$, we can find an integer m such that (1) p divides $m^2 + 3$, and (2) $m^2 < n$.

B4. A is a set of 5 x 7 real matrices closed under scalar multiplication and addition. It contains matrices of ranks 0, 1, 2, 4 and 5. Does it necessarily contain a matrix of rank 3?

B5. f(n) is the number of 1s in the base 2 representation of n. Let $k = \sum f(n) / (n + n^2)$, where the sum is taken over all positive integers. Is e^k rational?

B6. Let P be a convex polygon each of whose sides touches a circle C of radius 1. Let A be the set of points which are a distance 1 or less from P. If (x, y) is a point of A, let f(x, y) be the number of points in which a unit circle center (x, y) intersects P (so certainly $f(x, y) \ge 1$). What is $\sup 1/|A| \int_A f(x, y)\, dx\, dy$, where the sup is taken over all possible polygons P?

43rd Putnam 1982

A1. Let S be the set of points (x, y) in the plane such that $|x| \le y \le |x| + 3$, and $y \le 4$. Find the position of the centroid of S.

A2. Let $B_n(x) = 1^x + 2^x + ... + n^x$ and let $f(n) = B_n(\log_n 2) / (n \log_2 n)^2$. Does $f(2) + f(3) + f(4) + ...$ converge?

A3. Evaluate $\int_0^\infty (\tan^{-1}(\pi x) - \tan^{-1}x) / x \, dx$.

A4. Given that the equations $y' = -z^3$, $z' = y^3$ with initial conditions $y(0) = 1$, $z(0) = 0$ have the unique solution $y = f(x)$, $z = g(x)$ for all real x, prove $f(x)$ and $g(x)$ are both periodic with the same period.

A5. a, b, c, d are positive integers satisfying $a + c \le 1982$ and $a/b + c/d < 1$. Prove that $1 - a/b - c/d > 1/1983^3$.

A6. a_i are real numbers and $\sum_1^\infty a_i = 1$. Also $|a_1| > |a_2| > |a_3| > ...$. $f(i)$ is a bijection of the positive integers onto itself, and $|f(i) - i|\,|a_i| \to 0$ as $i \to \infty$. Prove or disprove that $\sum_1^\infty a_{f(i)} = 1$.

B1. ABC is an arbitrary triangle, and M is the midpoint of BC. How many pieces are needed to dissect AMB into triangles which can be reassembled to give AMC?

B2. Let $a(r)$ be the number of lattice points inside the circle center the origin, radius r. Let $k = 1 + e^{-1} + e^{-4} + ... + \exp(-n^2) + ...$. Express $\int_U a(\sqrt{(x^2 + y^2)}) \exp(-(x^2+y^2))\, dx\, dy$ as a polynomial in k, where U represents the entire plane.

B3. Let p_n be the probability that two numbers selected independently and randomly from $\{1, 2, 3, ... , n\}$ have a sum which is a square. Find $\lim_{n\to\infty} p_n \sqrt{n}$.

B4. A set S of k distinct integers n_i is such that $\prod n_i$ divides $\prod (n_i + m)$ for all integers m. Must 1 or -1 belong to S? If all members of S are positive, is S necessarily just $\{1, 2, ... , k\}$?

B5. Given $x > e^e$, define the sequence $f(n)$ as follows: $f(0) = e$, $f(n+1) = (\ln x)/(\ln f(n))$. Prove that the sequence converges. Let the limit be $g(x)$. Prove that g is continuous.

B6. Let $A(a, b, c)$ be the area of a triangle with sides a, b, c. Let $f(a, b, c) = \sqrt{A(a, b, c)}$. Prove that for any two triangles with sides a, b, c and a', b', c' we have $f(a, b, c) + f(a',b',c') \le f(a + a', b + b', c + c')$. When do we have equality?

44th Putnam 1983

A1. How many positive integers divide at least one of 10^{40} and 20^{30}?

A2. A clock's minute hand has length 4 and its hour hand length 3. What is the distance between the tips at the moment when it is increasing most rapidly?

A3. Let $f(n) = 1 + 2n + 3n^2 + ... + (p - 1)n^{p-2}$, where p is an odd prime. Prove that if $f(m) = f(n) \pmod p$, then $m = n \pmod p$.

A4. Prove that for $m = 5 \pmod 6$, $mC2 - mC5 + mC8 - mC11 + ... - mC(m-6) + mC(m-3) \neq 0$.

A5. Does there exist a positive real number α such that $[\alpha^n] - n$ is even for all integers $n > 0$?

A6. Let T be the triangle with vertices (0, 0), (a, 0), and (0, a). Find $\lim_{a\to\infty} a^4\exp(-a^3) \int_T \exp(x^3+y^3)\, dx\, dy$.

B1. Let C be a cube side 4, center O. Let S be the sphere center O radius 2. Let A be one of the vertices of the cube. Let R be the set of points in C but not S, which are closer to A than to any other vertex of C. Find the volume of R.

B2. Let f(n) be the number of ways of representing n as a sum of powers of 2 with no power being used more than 3 times. For example, f(7) = 4 (the representations are 4 + 2 + 1, 4 + 1 + 1 + 1, 2 + 2 + 2 + 1, 2 + 2 + 1 + 1 + 1). Can we find a real polynomial p(x) such that f(n) = [p(n)] ?

B3. y_1, y_2, y_3 are solutions of $y''' + a(x) y'' + b(x) y' + c(x) y = 0$ such that $y_1^2 + y_2^2 + y_3^2 = 1$ for all x. Find constants α, β such that $y_1'(x)^2 + y_2'(x)^2 + y_3'(x)^2$ is a solution of $y' + \alpha a(x) y + \beta c(x) = 0$.

B4. Let $f(n) = n + [\sqrt{n}]$. Define the sequence a_i by $a_0 = m$, $a_{n+1} = f(a_n)$. Prove that it contains at least one square.

B5. Define $||x||$ as the distance from x to the nearest integer. Find $\lim_{n\to\infty} 1/n \int_1^n ||n/x||\, dx$. You may assume that $\prod_1^\infty 2n/(2n-1)\, 2n/(2n+1) = \pi/2$.

B6. Let α be a complex $(2^n + 1)$th root of unity. Prove that there always exist polynomials p(x), q(x) with integer coefficients, such that $p(\alpha)^2 + q(\alpha)^2 = -1$.

45th Putnam 1984

A1. S is an a x b x c brick. T is the set of points a distance 1 or less from S. Find the volume of T.

A2. Evaluate $6/((9 - 4)(3 - 2)) + 36/((27 - 8)(9 - 4)) + ... + 6^n/((3^{n+1} - 2^{n+1})(3^n - 2^n)) + ...$.

A3. Let A be the 2n x 2n matrix whose diagonal elements are all x and whose off-diagonal elements a_{ij} = a for i + j even, and b for i + j odd. Find $\lim_{x\to a}\det A/(x - a)^{2n-2}$.

A4. A convex pentagon inscribed in a circle radius 1 has two perpendicular diagonals which intersect inside the pentagon. What is the maximum area the pentagon can have?

A5. V is the pyramidal region x, y, z ≥ 0, x + y + z ≤ 1. Evaluate $\int_V x\, y^9 z^8 (1 - x - y - z)^4\, dx\, dy\, dz$.

A6. Let f(n) be the last non-zero digit in the decimal representation of n! . Show that for distinct integers $a_i \ge 0$, $f(5^{a_1} + 5^{a_2} + ... + 5^{a_r})$ depends only on $a_1 + ... + a_r = a$. Write the value as g(a). Find the smallest period for g, or show that it is not periodic.

B1. Define f(n) = 1! + 2! + ... + n! . Find a recurrence relation f(n + 2) = a(n) f(n + 1) + b(n) f(n), where a(x) and b(x) are polynomials.

B2. Find the minimum of $f(x, y) = (x - y)^2 + (\sqrt{2 - x^2} - 9/y)^2$ in the half-infinite strip $0 < x < \sqrt{2}$, $y > 0$.

B3. Let S_n be a set with n elements. Can we find a binary operation * on S which satisfies (1) right cancellation: a*c = b*c implies a = b (for all a, b, c), and (2) total non-associativity: a * (b * c) ≠ (a * b) * c for all a, b, c? Note that we are not just requiring that * is not associative, but that it is *never* associative.

B4. Find all real valued functions f(x) defined on [0, ∞), such that (1) f is continuous on [0, ∞), (2) f(x) > 0 for x > 0, (3) for all $x_0 > 0$, the centroid of the region under the curve y = f(x) between 0 and x_0 has y-coordinate equal to the average value of f(x) on $[0, x_0]$.

B5. Let f(n) be the number of 1s in the binary expression for n. Let $g(m) = \pm 0^m \pm 1^m \pm 2^m ... \pm(2^m - 1)^m$, where we take the + sign for k^m iff f(k) is even. Show that g(m) can be written in the form $(-1)^m a^{p(m)} (q(m))!$ where a is an integer and p(x) and q(x) are polynomials.

B6. Define a sequence of convex polygons P_n as follows. P_0 is an equilateral triangle side 1. P_{n+1} is obtained from P_n by cutting off the corners one-third of the way along each side (for example P_1 is a regular hexagon side 1/3). Find $\lim_{n\to\infty} \text{area}(P_n)$.

46th Putnam 1985

A1. How many triples (A, B, C) are there of sets with $A \in B \in C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A \cap B \cap C =$ □?

A2. ABC is an acute-angled triangle with area 1. A rectangle R has its vertices R_i on the sides of the triangle, R_1 and R_2 on BC, R_3 on AC and R_4 on AB. Another rectangle S has it vertices on the sides of the triangle AR_3R_4, two on R_3R_4 and one one each of the other two sides. What is the maximum total area of R and S over all possible choices of triangle and rectangles?

A3. x is a real. Define $a_{i\,0} = x/2^i$, $a_{i\,j+1} = a_{i\,j}^2 + 2\,a_{i\,j}$. What is $\lim_{n\to\infty} a_{n\,n}$?

A4. Let a_n be the sequence defined by $a_1 = 3$, $a_{n+1} = 3^k$, where $k = a_n$. Let b_n be the remainder when a_n is divided by 100. Which values b_n occur for infinitely many n?

A5. Let $f_n(x) = \cos x \cos 2x \ldots \cos nx$. For which n in the range 1, 2, ... , 10 is $\int_0^{2\pi} f_n(x)\,dx$ non-zero?

A6. Find a polynomial f(x) with real coefficients and f(0) = 1, such that the sums of the squares of the coefficients of $f(x)^n$ and $(3x^2 + 7x + 2)^n$ are the same for all positive integers n.

B1. p(x) is a polynomial of degree 5 with 5 distinct integral roots. What is the smallest number of non-zero coefficients it can have? Give a possible set of roots for a polynomial achieving this minimum.

B2. The polynomials $p_n(x)$ are defined as follows: $p_0(x) = 1$; $p_{n+1}'(x) = (n + 1)\,p_n(x + 1)$, $p_{n+1}(0) = 0$ for $n \geq 0$. Factorize $p_{100}(1)$ into distinct primes.

B3. $a_{i\,j}$ is a positive integer for i, j = 1, 2, 3, ... and for each positive integer we can find exactly eight $a_{i\,j}$ equal to it. Prove that $a_{i\,j} > ij$ for some i, j.

B4. Let C be the circle radius 1, center the origin. A point P is chosen at random on the circumference of C, and another point Q is chosen at random in the interior of C. What is the probability that the rectangle with diagonal PQ, and sides parallel to the x-axis and y-axis, lies entirely inside (or on) C?

B5. Assuming that $\int_{-\infty}^{\infty} e^{-x2}\,dx = \sqrt{\pi}$, find $\int_0^{\infty} x^{-1/2}\,e^{-1985(x + 1/x)}\,dx$.

B6. G is a finite group consisting of real n x n matrices with the operation of matrix multiplication. The sum of the traces of the elements of G is zero. Prove that the sum of the elements of G is the zero matrix.

47th Putnam 1986

A1. S is the set {x real st $x^4 - 13x^2 + 36 \le 0$}. Find the maximum value of $f(x) = x^3 - 3x$ on S.

A2. What is the remainder when the integral part of $10^{20000}/(10^{100} + 3)$ is divided by 10?

A3. Find $\cot^{-1}(1) + \cot^{-1}(3) + ... + \cot^{-1}(n^2+n+1) + ...$, where the $\cot^{-1}(m)$ is taken to be the value in the range $(0, \pi/2]$.

A4. Let r(n) be the number of n x n matrices $A = (a_{ij})$ such that: (1) each $a_{ij} = -1, 0$, or 1; and (2) if we take any n elements a_{ij}, no two in the same row or column, then their sum is the same. Find rational numbers a, b, c, d, u, v, w such that $r(n) = a u^n + b v^n + c w^n + d$.

A5. $\mathbf{f}:R^n \to R^n$ is defined by $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), ... , f_n(\mathbf{x}))$, where $\mathbf{x} = (x_1, x_2, ... , x_n)$ and the n functions $f_i:R^n \to R$ have continuous 2nd order partial derivatives and satisfy $\partial f_i/\partial x_j - \partial f_j/\partial x_i = c_{ij}$ (for all $1 \le i, j \le n$) for some constants c_{ij}. Prove that there is a function $g:R^n \to R$ such that $f_i + \partial g/\partial x_i$ is linear (for all $1 \le i \le n$).

A6. Let p(x) be the real polynomial $1 + \alpha_1 x^{m_1} + \alpha_2 x^{m_2} + ... + \alpha_n x^{m_n}$, where $0 < m_1 < m_2 < ... < m_n$. For some real polynomial q(x) we have $p(x) = (1 - x)^n q(x)$. Find q(1) in terms of $m_1, ... , m_n$.

B1. ABCD is a rectangle. AEB is isosceles with E on the opposite side of AB to C and D and lies on the circle through A, B, C, D. This circle has radius 1. For what values of |AD| do the rectangle and triangle have the same area?

B2. x, y, z are complex numbers satisfying $x(x - 1) + 2yz = y(y - 1) + 2zx = z(z - 1) + 2xy$. Prove that there are only finitely many possible values for the triple (x - y, y - z, z - x) and enumerate them.

B3. We use the congruence notation for polynomials (in one variable) with integer coefficients to mean that the corresponding coefficients are congruent. Thus if $f(x) = a_k x^k + ... + a_0$, and $g(x) = b_k x^k + ... + b_0$, then f = g (mod m) means that $a_i = b_i$ (mod m) for all i. Let p be a prime and f(x), g(x), h(x), r(x), s(x) be polynomials with integer coefficients such that r(x)f(x) + s(x)g(x) = 1 (mod p) and f(x)g(x) = h(x) (mod p). Prove that for any positive integer n we can find F(x) and G(x) with integer coefficients such that F(x) = f(x) (mod p), G(x) = g(x) (mod p) and F(x)G(x) = h(x) (mod p^n).

B4. For real r > 0, define m(r) =min{ $|r - \sqrt{(m^2+2n^2)}|$ for m, n integers}. Prove or disprove: (1) $\lim_{r\to\infty} m(r)$ exists; and (2) is zero.

B5. Let $f(x, y, z) = x^2 + y^2 + z^2 + xyz$. a(x, y, z), b(x, y, z), b(x, y, z) are polynomials with real coefficients such that f(a(x, y, z), b(x, y, z), c(x, y, z)) = f(x, y, z). Prove or disprove that a(x, y, z), b(x, y, z), c(x, y, z) must be ±x, ±y, ±z in some order (with an even number of minus signs).

B6. A, B, C, D are n x n matrices with entries in some field F. The transpose of a matrix X is denoted as X' (defined as $X'_{ij} = X_{ji}$). Given that A B' and C D' are symmetric, and A D' - B C' = 1, prove that A'D - C'B = 1.

48th Putnam 1987

A1. Four planar curves are defined as follows: $C_1 = \{(x, y): x^2 - y^2 = x/(x^2 + y^2)\}$, $C_2 = \{(x, y): 2xy + y/(x^2 + y^2) = 3\}$, $C_3 = \{(x, y): x^3 - 3xy^2 + 3y = 1\}$, $C_4 = \{(x, y): 3yx^2 - 3x - y^3 = 0\}$. Prove that $C_1 \cap C_2 = C_3 \cap C_4$.

A2. An infinite sequence of decimal digits is obtained by writing the positive integers in order: 1234567891011 ... . Define f(n) = m if the 10^n th digit forms part of an m-digit number. For example, f(1) = 2, because the 10th digit is part of 10, and f(2) = 2, because the 100th digit is part of 55. Find f(1987).

A3. y = f(x) is a real-valued solution (for all real x) of the differential equation $y'' - 2y' + y = 2e^x$ which is positive for all x. Is f '(x) necessarily positive for all x? y = g(x) is another real valued solution, which satisfies g'(x) > 0 for all real x. Is g(x) necessarily positive for all x?

A4. p(x, y, z) is a polynomial with real coefficients such that: (1) $p(tx, ty, tz) = t^2f(y - x, z - x)$ for all real x, y, z, t (and some function f); (2) p(1, 0, 0) = 4, p(0 ,1, 0) = 5, and p(0, 0, 1) = 6; and (3) $p(\alpha, \beta, \gamma) = 0$ for some complex numbers α, β, γ such that $|\beta - \alpha| = 10$. Find $|\gamma - \alpha|$.

A5. **f**: $R^2 \to R^3$ (where R is the real line) is defined by $\mathbf{f}(x, y) = (-y/(x^2 + 4y^2), x/(x^2 + 4y^2), 0)$. Can we find **F**: $R^3 \to R^3$, such that:

(1) if $\mathbf{F} = (F_1, F_2, F_3)$, then F_i all have continuous partial derivatives for all (x, y, z) ≠ (0, 0, 0);

(2) □ **x F** = **0** for all (x, y, z) ≠ 0;

(3) **F**(x, y, 0) = **f**(x, y)?

A6. Define f(n) as the number of zeros in the base 3 representation of the positive integer n. For which positive real x does $F(x) = x^{f(1)}/1^3 + x^{f(2)}/2^3 + ... + x^{f(n)}/n^3 + ...$ converge?

B1. Evaluate $\int_2^4 \ln^{1/2}(9 - x)/(\ln^{1/2}(9 - x) + \ln^{1/2}(x + 3))\ dx$.

B2. Let n, r, s be non-negative integers with n ≥ r + s, prove that $\sum_{i=0}^s$ sCi / nC(r+i) = (n+1)/((n+1-s) (n-s)Cr), where mCn denotes the binomial coefficient.

B3. F is a field in which 1 + 1 ≠ 0. Define $P_\alpha = ((\alpha^2 - 1)/(\alpha^2 + 1), 2\alpha/(\alpha^2 + 1))$. Let A = { (β, γ) : β, γ ∈ F, and $\beta^2 + \gamma^2 = 1$}, and let B= { (1, 0) } ∈ { P_α: α ∈ F, and $\alpha^2 \neq -1$}. Prove that A = B.

B4. Define the sequences x_i and y_i as follows. Let $(x_1, y_1) = (0.8, 0.6)$ and let $(x_{n+1}, y_{n+1}) = (x_n\cos y_n - y_n\sin y_n, x_n\sin y_n + y_n \cos y_n)$ for n ≥ 1. Find $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$.

B5. **A** is a complex 2n x n matrix such that if **z** is a real 1 x 2n row vector then **z A** ≠ **0** unless **z** = **0**. Prove that given any real 2n x 1 column vector **x** we can always find an n x 1 column vector **z** such that the real part of **A z** = **x**.

B6. F is a finite field with p^2 elements, where p is an odd prime. S is a set of $(p^2 - 1)/2$ distinct non-zero elements of F such that for each a ∈ F, just one of a and -a is in S. Prove that the number of elements in S ∩ {2a: a ∈ S} is even.

49th Putnam 1988

A1. Let $S = \{ (x, y) : |x| - |y| \le 1 \text{ and } |y| \le 1 \}$. Sketch S and find its area.

A2. Let $f(x) = \exp(x^2)$. Find an open interval I and a non-zero function $g(x)$ on I such that $(fg)' = f\,'g'$ on I, or prove that they do not exist.

A3. For what real numbers α does $(1/\ 1 \operatorname{cosec}(1) - 1)^\alpha + (1/2 \operatorname{cosec}(1/2) - 1)^\alpha + \ldots + (1/n \operatorname{cosec}(1/n) - 1)^\alpha + \ldots$ converge?

A4. The plane is divided into 3 disjoint sets. Can we always find two points in the same set a distance 1 apart? What about 9 disjoint sets?

A5. R^+ denotes the positive reals. Prove that there is a unique function $f : R^+ \to R^+$ satisfying $f(f(x)) = 6x - f(x)$ for all x.

A6. V is an n-dimensional vector space. Can we find a linear map $A : V \to V$ with n+1 eigenvectors, any n of which are linearly independent, which is not a scalar multiple of the identity?

B1. If $n > 3$ is not prime, show that we can find positive integers a, b, c, such that $n = ab + bc + ca + 1$.

B2. α is a non-negative real. x is a real satisfying $(x + 1)^2 \ge \alpha(\alpha + 1)$. Is $x^2 \ge \alpha(\alpha - 1)$?

B3. α_n is the smallest element of the set $\{ |a - b\sqrt{3}| :$ a, b non-negative integers with sum n$\}$. Find sup α_n.

B4. α_n are positive reals, and $\beta_n = \alpha_n^{n/(n+1)}$. Show that if $\sum \alpha_n$ converges, then so does $\sum \beta_n$.

B5. Find the rank of the 2n+1 x 2n+1 skew-symmetric matrix with entries given by $a_{ij} = 1$ for $(i - j) = -2n, -(2n-1), \ldots, -(n+1)$; -1 for $(i - j) = -n, -(n-1), \ldots, -1$; 1 for $(i - j) = 1, 2, \ldots, n$; -1 for $(i - j) = n+1, n+2, \ldots, 2n+1$. In other words, the main diagonal is 0s, the n diagonals immediately below the main diagonal are 1s, the n diagonals below that are -1s, the n diagonals immediately above the main diagonal are -1s, and the n diagonals above that are 1s.

B6. The triangular numbers are 1, 3, 6, 10, ... , $n(n+1)/2$, ... Prove that there are an infinite number of pairs a_i, b_i such that m is triangular iff $a_i m + b_i$ is triangular.

50th Putnam 1989

A1. Which members of the sequence 101, 10101, 1010101, ... are prime?

A2. Let R be the rectangle $0 \le x \le a$, $0 \le y, \le b$. Evaluate $\int_R e^{f(x, y)}\, dx\, dy$, where $f(x, y) = \max(b^2x^2, a^2y^2)$.

A3. Prove that all the roots of $11z^{10} + 10i z^9 + 10i z - 11 = 0$ have unit modulus.

A4. A player plays the following game. At each turn a fair coin is tossed (probability 1/2 of heads, and all tosses independent), and, depending on the results of the tosses to date, (1) the game ends and the player wins, (2) the game ends and the player loses, or (3) the coin is tossed again. Given an irrational p in the interval (0, 1), can we find a rule such that (A) the player wins with probability p, and (B) the game ends after a finite number of tosses with probability 1?

A5. Show that we can find $\alpha > 0$ such that, given any point P inside a regular polygon with an odd number of sides which is inscribed in a circle radius 1, we can find two vertices of the polygon whose distance from P differs by less than $1/n - \alpha/n^3$, where the polygon has $2n + 1$ sides.

A6. Let $\alpha = 1 + a_1x + a_2x^2 + \dots$ where $a_n = 1$ if every block of zeros in the binary expansion of n has even length, 0 otherwise. Prove that, if we calculate in the field of two elements, then $\alpha^3 + x\alpha + 1 = 0$. [For example, calculating in this field, $(1 + x)^2 = 1 + x + x + x^2 = 1 + x^2$.]

B1. S is a square side 1. R is the subset of points closer to its center than to any side. Find the area of R.

B2. Let S be a non-empty set with a binary operation (written like multiplication) such that: (1) it is associative; (2) ab = ac implies b = c; (3) ba = ca implies b = c; (4) for each element, the set of its powers is finite. Is S necessarily a group?

B3. Let R be the reals and R* the non-negative reals. f: R* → R satisfies the following conditions: (1) it is differentiable and $f'(x) = -3 f(x) + 6 f(2x)$ for $x > 0$; (2) $|f(x)| \le e^{-\sqrt{x}}$ for $x \ge 0$. Define $u_n = \int_0^\infty x^n f(x)\, dx$ for $n \ge 0$. Express u_n in terms of u_0, prove that the sequence $u_n 3^n/n!$ converges, and show that the limit is 0 iff $u_0 = 0$.

B4. Does there exist an uncountable set of subsets of the positive integers such that any two distinct subsets have finite intersection?

B5. A quadrilateral is inscribed in a circle radius 1. Two opposite sides are parallel. The difference between their lengths is $d > 0$. The distance from the intersection of the diagonals to the center of the circle is h. Find sup d/h and describe the cases in which it is attained.

B6. f is a continuous real-valued function on the closed interval [0, 1] such that $f(1) = 0$. A point $(a_1, a_2, \dots, a_n)$ is chosen at random from the n-dimensional region $0 < x_1 < x_2 < \dots < x_n < 1$. Define $a_0 = 0$, $a_{n+1} = 1$. Show that the expected value of $\sum_0^n (a_{i+1} - a_i) f(a_{i+1})$ is $\int_0^1 f(x)\, p(x)\, dx$, where p(x) is a polynomial of degree n which maps the interval [0, 1] into itself (and is independent of f).

51st Putnam 1990

A1. Prove that the sequence $a_0 = 2, 3, 6, 14, 40, 152, 784, \ldots$ with general term $a_n = (n+4)\, a_{n-1} - 4n\, a_{n-2} + (4n-8)\, a_{n-3}$ is the sum of two well-known sequences.

A2. Can we find a subsequence of $\{ n^{1/3} - m^{1/3} : n, m = 0, 1, 2, \ldots \}$ which converges to $\sqrt{2}$?

A3. A convex pentagon has all its vertices lattice points in the plane (and no three collinear). Prove that its area is at least 5/2.

A4. Given a point P in the plane, let S_P be the set of points whose distance from P is irrational. What is the smallest number of such sets whose union is the entire plane?

A5. M and N are n x n matrices such that $(MN)^2 = 0$. Must $(NM)^2 = 0$?

A6. How many ordered pairs (A, B) of subsets of {1, 2, ... , 10} can we find such that each element of A is larger than |B| and each element of B is larger than |A|.

B1. R is the real line. Find all possible functions f: R → R with continuous derivative such that $f(\alpha)^2 = 1990 + \int_0^\alpha (f(x)^2 + f'(x)^2)\, dx$ for all α.

B2. Let $P_n(x, z) = \prod_1^n (1 - z x^{i-1}) / (z - x^i)$. Prove that $1 + \sum_1^\infty (1 + x^n) P_n(x, z) = 0$ for $|z| > 1$ and $|x| < 1$.

B3. Let S be the set of 2 x 2 matrices each of whose elements is one of the 15 squares 0, 1, 4, ... , 196. Prove that if we select more than $15^4 - 15^2 - 15 + 2$ matrices from S, then two of those selected must commute.

B4. A finite group with n elements is generated by g and h. Can we arrange two copies of the elements of the group in a sequence (total length 2n) so that each element is g or h times the previous element and the first element is g or h times the last?

B5. Can we find a sequence of reals $\alpha_i \neq 0$ such that each polynomial $\alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n$ has all its roots real and distinct?

B6. C is a non-empty, closed, bounded, convex subset of the plane. Given a support line L of C and a real number $0 \le \alpha \le 1$, let B_α be the band parallel to L, situated midway between L and the parallel support line on the other side of C, and of width α times the distance between the two support lines. What is the smallest α such that $\cap B_\alpha$ contains a point of C, where the intersection is taken over all possible directions for the support line L?

52nd Putnam 1991

A1. The rectangle with vertices (0, 0), (0, 3), (2, 0) and (2, 3) is rotated clockwise through a right angle about the point (2, 0), then about (5, 0), then about (7, 0), and finally about (10, 0). The net effect is to translate it a distance 10 along the x-axis. The point initially at (1, 1) traces out a curve. Find the area under this curve (in other words, the area of the region bounded by the curve, the x-axis and the lines parallel to the y-axis through (1, 0) and (11, 0)).

A2. M and N are real unequal n x n matrices satisfying $M^3 = N^3$ and $M^2N = N^2M$. Can we choose M and N so that $M^2 + N^2$ is invertible?

A3. P(x) of is a polynomial of degree $n \geq 2$ with real coefficients, such that (1) it has n unequal real roots, (2) for each pair of adjacent roots a, b the derivative P'(x) is zero halfway between the roots (at x = (a + b)/2). Find all possible P(x).

A4. Can we find an (infinite) sequence of disks in the Euclidean plane such that: (1) their centers have no (finite) limit point in the plane; (2) the total area of the disks is finite; and (3) every line in the plane intersects at least one of the disks?

A5. Let $f(z) = \int_0^z \sqrt{(x^4 + (z - z^2)^2)}\, dx$. Find the maximum value of f(z) in the range $0 \leq z \leq 1$.

A6. An n-sum of *type 1* is a finite sequence of positive integers $a_1, a_2, \ldots, a_r$, such that: (1) $a_1 + a_2 + \ldots a_r = n$; and (2) $a_1 > a_2 + a_3$, $a_2 > a_3 + a_4$, ... , $a_{r-2} > a_{r-1} + a_r$, and $a_{r-1} > a_r$. For example, there are five 7-sums of type 1, namely: 7; 6, 1; 5, 2; 4, 3; 4, 2, 1. An n-sum of *type 2* is a finite sequence of positive integers $b_1, b_2, \ldots, b_s$ such that: (1) $b_1 + b_2 + \ldots + b_s = n$; (2) $b_1 \geq b_2 \geq \ldots \geq b_s$; (3) each b_i is in the sequence 1, 2, 4, ... , g_j, ... defined by $g_1 = 1$, $g_2 = 2$, $g_j = g_{j-1} + g_{j-2} + 1$; and (4) if $b_1 = g_k$, then 1, 2, 4, ... , g_k is a subsequence. For example, there are five 7-sums of type 2, namely: 4, 2, 1; 2, 2, 2, 1; 2, 2, 1, 1, 1; 2, 1, 1, 1, 1, 1; 1, 1, 1, 1, 1, 1, 1. Prove that for $n \geq 1$ the number of type 1 and type 2 n-sums is the same.

B1. For positive integers n define $d(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Given a positive integer b_0, define a sequence b_i by taking $b_{k+1} = b_k + d(b_k)$. For what b_0 do we have b_i constant for sufficiently large i?

B2. R is the real line. f, g: R -> R are non-constant, differentiable functions satisfying: (1) f(x + y) = f(x)f(y) - g(x)g(y) for all x, y; (2) g(x + y) = f(x)g(y) + g(x)f(y) for all x, y; and (3) f '(0) = 0. Prove that $f(x)^2 + g(x)^2 = 1$ for all x.

B3. Can we find N such that all m x n rectangles with m, n > N can be tiled with 4 x 6 and 5 x 7 rectangles?

B4. p is a prime > 2. Prove that $\sum_{0 \leq n \leq p}$ pCn (p+n)Cn = $2^p + 1$ (mod p^2). [aCb is the binomial coefficient a!/(b! (a-b)!).]

B5. p a prime > 2. How many residues mod p are both squares and squares plus one?

B6. Let a and b be positive numbers. Find the largest number c, in terms of a and b, such that for all x with $0 < |x| \leq c$ and for all α with $0 < \alpha < 1$, we have: $a^\alpha b^{1-\alpha} \leq a \sinh \alpha x/\sinh x + b \sinh x(1 - \alpha)/\sinh x$.

53rd Putnam 1992

A1. Let Z be the integers. Prove that if f **:** Z → Z satisifies f(f(n)) = f(f(n+2) + 2) = n for all n, and f(0) = 1, then f(n) = 1 - n.

A2. Let the coefficient of x^{1992} in the power series $(1 + x)^{\alpha} = 1 + \alpha x + ...$ be C(α). Find $\int_0^1 C(-y-1) \sum_{k=1}^{1992} 1/(y+k)\, dy$.

A3. Find all positive integers a, b, m, n with m relatively prime to n such that $(a^2 + b^2)^m = (ab)^n$.

A4. Let R be the reals. Let f **:** R → R be an infinitely differentiable function such that f(1/n) = $n^2/(n^2+1)$ for n = 1, 2, 3, ... Find the value of the derivatives of f at zero: $f^{(k)}(0)$ for k = 1, 2, 3, ...

A5. Let N be the positive integers. Define f **:** N → {0, 1} by f(n) = 1 if the number of 1s in the binary representation of n is odd and 0 otherwise. Show that there do not exist positive integers k and m such that f(k + j) = f(k + m + j) = f(k + 2m + j) for 0 ≤ j < m.

A6. Four points are chosen independently and at random on the surface of a sphere (using the uniform distribution). What is the probability that the center of the sphere lies inside the resulting tetrahedron?

B1. Let R be the reals. Let S □ R have n ≥ 2 elements. Let A_S = { x ∈ ∈ R **:** x = (s + t)/2 for some s, t ∈ S with s ≠t}. What is the smallest possible $|A_S|$?

B2. Show that the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$ is $\sum_{j=0}^{k}$ nCj nC(k-2j).

B3. Let S be the set of points (x, y) in the plane such that the sequence a_n defined by $a_0 = x$, $a_{n+1} = (a_n^2 + y^2)/2$ converges. What is the area of S?

B4. p(x) is a polynomial of degree < 1992 such that p(0), p(1), p(-1) are all non-zero. The 1992th derivative of $p(x)/(x^3 - x)$ = f(x)/g(x) for polynomials f(x) and g(x). Find the smallest possible degree of f(x).

B5. Let A_n denote the n-1 x n-1 matrix (a_{ij}) with a_{ij} = i + 2 for i = j, and 1 otherwise. Is the sequence (det A_n)/n! bounded?

B6. Let M be a set of real n x n matrices such that: (1) 1 ∈ M; (2) if A, B ∈ M, then just one of AB, - AB is in M; (3) if A, B ∈ M, then either AB = BA or AB = -BA; (4) if 1 ≠ A ∈ M, then there is at least one B ∈ M such that AB = - BA. Prove that M contains at most n^2 matrices.

54th Putnam 1993

A1. Let O be the origin. y = c intersects the curve $y = 2x - 3x^3$ at P and Q in the first quadrant and cuts the y-axis at R. Find c so that the region OPR bounded by the y-axis, the line y = c and the curve has the same area as the region between P and Q under the curve and above the line y = c.

A2. The sequence a_n of non-zero reals satisfies $a_n^2 - a_{n-1}a_{n+1} = 1$ for $n \geq 1$. Prove that there exists a real number α such that $a_{n+1} = \alpha\, a_n - a_{n-1}$ for $n \geq 1$.

A3. Let P be the set of all subsets of {1, 2, ... , n}. Show that there are $1^n + 2^n + ... + m^n$ functions $f : P \rightarrow \{1, 2, ... , m\}$ such that $f(A \cap B) = \min(f(A), f(B))$ for all A, B.

A4. Given a sequence of 19 positive (not necessarily distinct) integers not greater than 93, and a set of 93 positive (not necessarily distinct) integers not greater than 19. Show that we can find non-empty subsequences of the two sequences with equal sum.

A5. Let U be the set formed as the union of three open intervals, U = (-100, -10) ∈ (1/101, 1/11) ∈ (101/100, 11/10). Show that $\int_U (x^2 - x)^2/(x^3 - 3x + 1)^2\, dx$ is rational.

A6. Let $a_0, a_1, a_2, ...$ be a sequence such that: $a_0 = 2$; each $a_n = 2$ or 3; a_n = the number of 3s between the nth and n+1th 2 in the sequence. So the sequence starts: 233233323332332 ... . Show that we can find α such that $a_n = 2$ iff $n = [\alpha m]$ for some integer $m \geq 0$.

B1. What is the smallest integer n > 0 such that for any integer m in the range 1, 2, 3, ... , 1992 we can always find an integral multiple of 1/n in the open interval (m/1993, (m + 1)/1994)?

B2. A deck of 2n cards numbered from 1 to 2n is shuffled and n cards are dealt to A and B. A and B alternately discard a card face up, starting with A. The game when the sum of the discards is first divisible by 2n + 1, and the last person to discard wins. What is the probability that A wins if neither player makes a mistake?

B3. x and y are chosen at random (with uniform density) from the interval (0, 1). What is the probability that the closest integer to x/y is even?

B4. K(x, y), f(x) and g(x) are positive and continuous for x, y ∈ [0, 1]. $\int_0^1 f(y) K(x, y)\, dy = g(x)$ and $\int_0^1 g(y) K(x, y)\, dy = f(x)$ for all x ∈ [0, 1]. Show that f = g on [0, 1].

B5. Show that given any 4 points in the plane we can find two whose distance apart is not an odd integer.

B6. Given a triple of positive integers $x \leq y \leq z$, derive a new triple as follows. Replace x and y by 2x and y - x (and reorder), or replace x and z by 2x and z - x (and reorder), or replace y and z by 2y and z - y (and reorder). Show that by finitely many tranformations of this type we can always derive a triple with smallest element zero.

55th Putnam 1994

A1. a_n is a sequence of positive reals satisfying $a_n \le a_{2n} + a_{2n+1}$ for all n. Prove that $\sum a_n$ diverges.

A2. Let R be the region in the first quadrant bounded by the x-axis, the line 2y = x, and the ellipse $x^2/9 + y^2 = 1$. Let R' be the region in the first quadrant bounded by the y-axis, the line y = mx, and the ellipse. Find m such that R and R' have the same area.

A3. X is the set of points on one or more sides of a triangle with sides length 1, 1 and √2. Show that if X is 4-colored, then there must be two points of the same color a distance 2 - √2 or more apart.

A4. A and B are 2 x 2 matrices with integral values. A, A + B, A + 2B, A + 3B, and A + 4B all have inverses with integral values. Show that A + 5B does also.

A5. Given a sequence of positive real numbers which tends to zero, show that every non-empty interval (a, b) contains a non-empty subinterval (c, d) that does not contain any numbers equal to a sum of 1994 distinct elements of the sequence.

A6. Let Z be the integers. Let $f_1, f_2, \dots, f_{10} : Z \to Z$ be bijections. Given any $n \in Z$ we can find some composition of the f_i (allowing repetitions) which maps 0 to n. Consider the set of 1024 functions S = { $g_1g_2 \dots g_{10}$ }, where g_i = the identity or f_i. Show that at most half the functions in S map a finite (non-empty) subset of Z onto itself.

B1. For a positive integer n, let f(n) be the number of perfect squares d such that $|n - d| \le 250$. Find all n such that f(n) = 15. [The perfect squares are 0, 1, 4, 9, 16, ...]

B2. For which real α does the curve $y = x^4 + 9x^3 + \alpha x^2 + 9x + 4$ contain four collinear points?

B3. Let R be the reals and R^+ the positive reals. $f : R \to R^+$ is differentiable and $f'(x) > f(x)$ for all x. For what k must f(x) exceed e^{kx} for all sufficiently large x?

B4. A is the 2 x 2 matrix (a_{ij}) with $a_{11} = a_{22} = 3$, $a_{12} = 2$, $a_{21} = 4$ and I is the 2 x 2 unit matrix. Show that the greatest common divisor of the entries of $A^n - I$ tends to infinity.

B5. For any real α define $f_\alpha(x) = [\alpha x]$. Let n be a positive integer. Show that there exists an α such that for $1 \le k \le n$, $f_\alpha^k(n^2) = n^2 - k = f_\alpha^k(n^2)$, where f_α^k denotes the k-fold composition of f_α.

B6. a, b, c, d are integers in the range 0 - 99. Show that if $101a - 100 \cdot 2^a + 101b - 100 \cdot 2^b = 101c - 100 \cdot 2^c + 101d - 100 \cdot 2^d \pmod{10100}$ then {a, b} = {c, d}.

56th Putnam 1995

A1. S is a set of real numbers which is closed under multiplication. $S = A \in B$, and $A \cap B =$ □. If $a, b, c \in A$, then $abc \in A$. Similarly, if $a, b, c \in B$, then $abc \in B$. Show that at least one of the A, B is closed under multiplication.

A2. For what positive reals α, β does $\int_\beta^\infty \sqrt{(\sqrt{(x + \alpha)} - \sqrt{x})} - \sqrt{(\sqrt{x} - \sqrt{(x - \beta)})}\, dx$ converge?

A3. d, e and f each have nine digits when written in base 10. Each of the nine numbers formed from d by replacing one of its digits by the corresponding digit of e is divisible by 7. Similarly, each of the nine numbers formed from e by replacing one of its digits by the corresponding digit of f is divisible by 7. Show that each of the nine differences between corresponding digits of d and f is divisible by 7.

A4. n integers totalling n - 1 are arranged in a circle. Prove that we choose one of the integers x_1, so that the other integers going around the circle are, in order, $x_2, \ldots, x_n$ and we have $\sum_1^k x_i \le k - 1$ for $k = 1, 2, \ldots, n$.

A5. R is the reals. $x_i : R \to R$ are differentiable for $i = 1, 2, \ldots, n$ and satisfy $x_i' = a_{i1}x_1 + \ldots + a_{in}x_n$ for some constants $a_{ij} \ge 0$. Also $\lim_{t\to\infty} x_i(t) = 0$. Can the functions x_i be linearly independent?

A6. Each of the n triples (r_i, s_i, t_i) is a randomly chosen permutation of (1, 2, 3) and each triple is chosen independently. Let p be the probability that each of the three sums $\sum r_i$, $\sum s_i$, $\sum t_i$ equals 2n, and let q be the probability that they are 2n - 1, 2n, 2n + 1 in some order. Show that for some $n \ge 1995$, $4p \le q$.

B1. Let X be a set with 9 elements. Given a partition π of X, let $\pi(h)$ be the number of elements in the part containing h. Given any two partitions π_1 and π_2 of X, show that we can find $h \ne k$ such that $\pi_1(h) = \pi_1(k)$ and $\pi_2(h) = \pi_2(k)$.

B2. An ellipse with semi-axes a and b rolls without slipping on the curve $y = c \sin (x/a)$ and completes one revolution in one period of the sine curve. What conditions do a, b, c satisfy?

B3. For each positive integer k with n^2 decimal digits (and leading digit non-zero), let d(k) be the determinant of the matrix formed by writing the digits in order across the rows (so if k has decimal form $a_1a_2 \ldots a_n$, then the matrix has elements $b_{ij} = a_{n(i-1)+j}$). Find $f(n) = \sum d(k)$, where the sum is taken over all $9\cdot 10^{n2-1}$ such integers.

B4. Express $(2207-1/(2207-1/(2207-1/(2207- \ldots))))^{1/8}$ in the form $(a + b\sqrt{c})/d$, where a, b, c, d are integers.

B5. A game starts with four heaps, containing 3, 4, 5 and 6 items respectively. The two players move alternately. A player may take a complete heap of two or three items or take one item from a heap provided that leaves more than one item in that heap. The player who takes the last item wins. Give a winning strategy for the first or second player.

B6. Let N be the positive integers. For any $\alpha > 0$, define $S_\alpha = \{ [n\alpha]: n \in N\}$. Prove that we cannot find α, β, γ such that $N = S_\alpha \in S_\beta \in S_\gamma$ and $S_\alpha, S_\beta, S_\gamma$ are (pairwise) disjoint.

57th Putnam 1996

A1. What is the smallest α such that two squares with total area 1 can always be placed inside a rectangle area α with sides parallel to those of the rectangle and with no overlap (of their interiors)?

A2. Two circles have radii 1 and 3 and centers a distance 10 apart. Find the locus of all points which are the midpoint of a segment with one end on each circle.

A3. There are six courses on offer. Each of 20 students chooses some all or none of the courses. Is it true that we can find two courses C and C' and five students S_1, S_2, S_3, S_4, S_5 such that each S_i has chosen C and C', or such that each S_i has chosen neither C nor C'?

A4. A is a finite set. S is a set of ordered triples (a, b, c) of distinct elements of A, such that:

(a, b, c) $\in$ S iff (b, c, a) $\in$ S;

(a, b, c) $\in$ $\in$ S iff (c, b, a) $\in$ S;

(a, b, c) and (c, d, a) $\in$ S iff (b, c, d) and (d, a, b) $\in$ S.

Prove that there exists an injection g from A to the reals, such that g(a) < g(b) < g(c) implies (a, b, c) $\in$ S.

A5. Let p be a prime ≥ 5. Prove that p^2 divides $\sum_0^{[2p/3]}$ pCr.

A6. Let R be the reals and k a non-negative real. Find all continuous functions f **:** R $\rightarrow$ R such that $f(x) = f(x^2 + k)$ for all x.

B1. Let N be the set {1, 2, 3, ... , n}. X is *selfish* if |X| $\in$ X. How many subsets of N are selfish and have no proper selfish subsets.

B2. Let $f(n) = ((2n+1)/e)^{(2n+1)/2}$. Show that for n > 0: $f(n-1) < 1\cdot3\cdot5 \ldots (2n-1) < f(n)$.

B3. $(x_1, x_2, \ldots , x_n)$ is a permutation of $(1, 2, \ldots , n)$. What is the maximum of $x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n + x_nx_1$?

B4. Let B be the 2 x 2 matrix (b_{ij}) with $b_{11} = b_{22} = 1$, $b_{12} = 1996$, $b_{21} = 0$. Can we find a 2 x 2 matrix A such that sin A = B? [We define sin A by the usual power series: $A - A^3/3! + A^5/5! - \ldots$.]

B5. We call a finite string of the symbols X and O *balanced* iff every substring of consecutive symbols has a difference of at most 2 between the number of Xs and the number of Os. For example, XOOXOOX is not balanced, because the substring OOXOO has a difference of 3. Find the number of balanced strings of length n.

B6. The origin lies inside a convex polygon whose vertices have coordinates (a_i, b_i) for i = 1, 2, ... , n. Show that we can find x, y > 0 such that $a_1x^{a_1}y^{b_1} + a_2x^{a_2}y^{b_2} + \ldots + a_nx^{a_n}y^{b_n} = 0$ and $b_1x^{a_1}y^{b_1} + b_2x^{a_2}y^{b_2} + \ldots + b_nx^{a_n}y^{b_n} = 0$.

58th Putnam 1997

A1. ROMN is a rectangle with vertices in that order and RO = 11, OM = 5. The triangle ABC has circumcenter O and its altitudes intersect at R. M is the midpoint of BC, and AN is the altitude from A to BC. What is the length of BC?

A2. Players 1, 2, ... , n are seated in a circle. Each has one penny. Starting with player 1 passing one penny to player 2, players alternately pass one or two pennies to the next player still seated. A player leaves as soon as he runs out of pennies. So player 1 leaves after move 1, and player 2 leaves after move 2. Find an infinite set of numbers n for which some player ends up with all n pennies.

A3. Let $f(x) = (x - x^3/2 + x^5/(2.4) - x^7/(2.4.6) + \dots)$, and $g(x) = (1 + x^2/2^2 + x^4/(2^2 4^2) + x^6/(2^2 4^2 6^2) + \dots)$. Find $\int_0^\infty f(x)\, g(x)\, dx$.

A4. G is a group, not necessarily abelian. We write the operation as juxtapostion and the identity as 1. There is a function $\varphi : G \to G$ such that if $abc = def = 1$, then $\varphi(a)\varphi(b)\varphi(c) = \varphi(d)\varphi(e)\varphi(f)$. Prove that there exists an element $k \in G$ such that $k\varphi(x)$ is a homomorphism.

A5. Is the number of ordered 10-tuples of positive integers $(a_1, a_2, \dots, a_{10})$ such that $1/a_1 + 1/a_2 + \dots + 1/a_{10} = 1$ even or odd?

A6. Let N be a fixed positive integer. For real α, define the sequence x_k by: $x_0 = 0$, $x_1 = 1$, $x_{k+2} = (\alpha x_{k+1} - (N - k)x_k)/(k + 1)$. Find the largest α such that $x_{N+1} = 0$ and the resulting x_k.

B1. Find $\sum_1^{6N-1} \min(\{r/3N\}, \{r/3N\})$, where $\{\alpha\} = \min(\alpha - [\alpha], [\alpha] + 1 - \alpha)$, the distance to the nearest integer.

B2. Let R be the reals. $f : R \to R$ is twice-differentiable and we can find $g : R \to R$ such that $g(x) \geq 0$ and $f(x) + f''(x) = -xg(x)f'(x)$ for all x. Prove that f(x) is bounded.

B3. Let $(1 + 1/2 + 1/3 + \dots + 1/n) = p_n/q_n$, where p_n and q_n are relatively prime positive integers. For which n is q_n not divisible by 5?

B4. Let $(1 + x + x^2)^m = \sum_0^{2m} a_{m,n}x^n$. Prove that for all $k \geq 0$, $\sum_0^{[2k/3]} (-1)^i a_{k-i,i} \in [0,1]$.

B5. Define the sequence a_n by $a_1 = 2$, $a_{n+1} = 2^{a_n}$. Prove that $a_n = a_{n-1} \pmod{n}$ for $n \geq 2$.

B6. For a plane set S define d(S), the diameter of S, to be sup {PQ : P, Q ∈ S}. Let K be a triangle with sides 3, 4, 5 and its interior. If $K = H_1 \in H_2 \in H_3 \in H_4$, what is the smallest possible value of $\max(d(H_1), d(H_2), d(H_3), d(H_4))$?

59th Putnam 1998

A1. A cone has circular base radius 1, and vertex a height 3 directly above the center of the circle. A cube has four vertices in the base and four on the sloping sides. What length is a side of the cube?

A2. Let C be the circle center (0, 0), radius 1. Let X, Y be two points on C with positive x and y coordinates. Let X_1, Y_1 be the points on the x-axis with the same x-coordinates as X and Y respectively, and let X_2, Y_2 be the points on the y-axis with the same y-coordinates. Show that the area of the region XYY_1X_1 plus the area of the region XYY_2X_2 depends only on the length of the arc XY, and not on its position.

A3. Let R be the reals. Let $f : R \to R$ have a continuous third derivative. Show that there is a point a with $f(a)\, f'(a)\, f''(a)\, f'''(a) \geq 0$.

A4. Define the sequence of decimal integers a_n as follows: $a_1 = 0$; $a_2 = 1$; a_{n+2} is obtained by writing the digits of a_{n+1} immediately followed by those of a_n. When is a_n a multiple of 11?

A5. A finite collection of disks covers a subset X of the plane. Show that we can find a pairwise disjoint subcollection S, such that X □ $\in \{3D : D \in S\}$, where 3D denotes the disk with the same center as D and 3 times the radius.

B1. Find the minimum of $\{ (x + 1/x)^6 - (x^6 + 1/x^6) - 2 \}/\{ (x + 1/x)^3 + (x^3 + 1/x^3) \}$, for $x > 0$.

B2. Let P be the point (a, b) with $0 < b < a$. Find Q on the x-axis and R on $y = x$, so that PQ + QR + RP is as small as possible.

B3. Let S be the sphere center the origin and radius 1. Let P be a regular pentagon in the plane $z = 0$ with vertices on S. Find the surface area of the part of the sphere which lies above ($z > 0$) P or its interior.

B4. For what $m, n > 0$ is $\sum_0^{mn-1} (-1)^{[i/m] + [i/n]} = 0$?

B5. Let n be the decimal integer 11...1 (with 1998 digits). What is the 1000th digit after the decimal point of $\sqrt{n}$?

B6. Show that for any integers a, b, c we can find a positive integer n such that $n^3 + a n^2 + b n + c$ is not a perfect square.

60th Putnam 1999

A1. Find polynomials a(x), b(x), c(x) such that $|a(x)| - |b(x)| + c(x) = -1$ for $x < -1$, $3x + 2$ for $-1 \le x \le 0$, $-2x + 2$ for $x > 0$.

A2. Show that for some fixed positive n we can always express a polynomial with real coefficients which is nowhere negative as a sum of the squares of n polynomials.

A3. Let $1/(1 - 2x - x^2) = s_0 + s_1x + s_2x^2 + \dots$. Prove that for some f(n) we have $s_n^2 + s_{n+1}^2 = s_{f(n)}$.

A4. Let $a_{ij} = i^2j/(3^i(j\,3^i + i\,3^j))$. Find $\sum a_{ij}$ where the sum is taken over all pairs of integers (i, j) with i, j > 0.

A5. Find a constant k such that for any polynomial f(x) of degree 1999, we have $|f(0)| \le k \int_{-1}^{1} |f(x)|\,dx$.

A6. $u_1 = 1$, $u_2 = 2$, $u_3 = 24$, $u_n = (6\,u_{n-1}^2u_{n-3} - 8\,u_{n-1}u_{n-2}^2)/(u_{n-2}u_{n-3})$. Show that u_n is always a multiple of n.

B1. The triangle ABC has AC = 1, $\angle ACB = 90^o$, and $\angle BAC = \varphi$. D is the point between A and B such that AD = 1. E is the point between B and C such that $\angle EDC = \varphi$. The perpendicular to BC at E meets AB at F. Find $\lim_{\varphi \to 0} EF$.

B2. p(x) is a polynomial of degree n. q(x) is a polynomial of degree 2. $p(x) = p''(x)q(x)$ and the roots of p(x) are not all equal. Show that the roots of p(x) are all distinct.

B3. Let R be the reals. Define $f: [0, 1) \times [0, 1) \to R$ by $f(x, y) = \sum x^my^n$, where the sum is taken over all pairs of positive integers (m, n) satisfying $m \ge n/2$, $n \ge m/2$. Find $\lim_{(x, y)\to(1, 1)} (1 - xy^2)(1 - x^2y)f(x, y)$.

B4. Let R be the reals. $f: R \to R$ is three times differentiable, and $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$ are all positive for all x. Also $f(x) \ge f'''(x)$ for all x. Show that $f'(x) < 2 f(x)$ for all x.

B5. n is an integer greater than 2 and $\varphi = 2\pi/n$. A is the n x n matrix (a_{ij}), where $a_{ij} = \cos((i + j)\varphi)$ for $i \ne j$, $1 + \cos(2j\varphi)$ for $i = j$. Find det A.

B6. X is a finite set of integers greater than 1 such that for any positive integer n, we can find $m \in X$ such that m divides n or is relatively prime to n. Show that X contains a prime or two elements whose greatest common divisor is prime.

61st Putnam 2000

A1. k is a positive constant. The sequence x_i of positive reals has sum k. What are the possible values for the sum of x_i^2 ?

A2. Show that we can find infinitely many triples N, N + 1, N + 2 such that each member of the triple is a sum of one or two squares.

A3. An octagon is incribed in a circle. One set of alternate vertices forms a square area 5. The other set forms a rectangle area 4. What is the maximum possible area for the octagon?

A4. Show that $\lim_{k\to\infty}$ &inf;$_0^k$ sin x sin x^2 dx converges.

A5. A, B, C each have integral coordinates and lie on a circle radius R. Show that at least one of the distances AB, BC, CA exceeds $R^{1/3}$.

A6. p(x) is a polynomial with integer coefficients. A sequence x_0, x_1, x_2, ... is defined by x_0 = 0, $x_{n+1} = p(x_n)$. Prove that if $x_n = 0$ for some n > 0, then $x_1 = 0$ or $x_2 = 0$.

B1. We are given N triples of integers (a_1, b_1, c_1), (a_2, b_2, c_2), ... (a_N, b_N, c_N). At least one member of each triple is odd. Show that we can find integers A, B, C such that at least 4N/7 of the N values A a_i + B b_i + C c_i are odd.

B2. m and n are positive integers with m ≤ n. d is their greatest common divisor. nCm is the binomial coefficient. Show that d/n nCm is integral.

B3. a_1, a_2, ... , a_N are real and a_N is non-zero. f(x) = a_1 sin 2πx + a_2 sin 4πx + a_3 sin 6πx + ... + a_N sin 2Nπx. Show that the number of zeros of $f^{(i)}(x) = 0$ in the interval [0, 1) is a non-decreasing function of i and tends to 2N (as i tends to infinity).

B4. f(x) is a continuous real function satisfying $f(2x^2 - 1) = 2x f(x)$. Show that f(x) is zero on the interval [-1, 1].

B5. S_0 is an arbitrary finite set of positive integers. Define S_{n+1} as the set of integers k such that just one of k - 1, k is in S_n. Show that for infinitely many n, S_n is the union of S_0 and a translate of S_0.

B6. Let X be the set of 2^n points (±1, ±1, ... , ±1) in Euclidean n-space. Show that any subset of X with at least $2^{n+1}/n$ points contains an equilateral triangle.

62nd Putnam 2001

A1. Given a set X with a binary operation *, not necessarily associative or commutative, but such that $(x * y) * x = y$ for all x, y in X. Show that $x * (y * x) = y$ for all x, y in X.

A2. You have a set of n biased coins. The mth coin has probability $1/(2m+1)$ of landing heads ($m = 1, 2, \ldots, n$) and the results for each coin are independent. What is the probability that if each coin is tossed once, you get an odd number of heads?

A3. For what integers n is the polynomial $x^4 - (2n + 4) x^2 + (n - 2)^2$ the product of two non-trivial polynomials with integer coefficients?

A4. Points X, Y, Z lie on the sides BC, CA, AB (respectively) of the triangle ABC. AX bisects BY at K, BY bisects CZ at L, CZ bisects AX at M. Find the area of the triangle KLM as a fraction of that of ABC.

A5. Find all solutions to $x^{n+1} - (x + 1)^n = 2001$ in positive integers x, n.

A6. A parabola intersects a disk radius 1. Can the arc length of the parabola inside the disk exceed 4?

B1. Number the cells in a 2n x 2n grid from 1 to $4n^2$, starting at the top left and moving left to right along each row, then continuing at the left of the next row down and so on. Colour half the cells black and half white, so that just half the cells in each row and half the cells in each column are black. Show that however this is done the sum of the black squares equals the sum of the white squares.

B2. Find all real solutions (x, y) to the simultaneous equations:

$1/x - 1/(2y) = 2 y^4 - 2 x^4$

$1/x + 1/(2y) = (3 x^2 + y^2)(x^2 + 3 y^2)$.

B3. Let @n denote the closest integer to √n. Find $\sum (2^{@n} + 2^{-@n})/2^n$, where the sum is taken from n = 1 to n = infinity.

B4. Let X be the set of rationals excluding 0, ±1. Let f: X → X be defined as $f(x) = x - 1/x$. Let $f^1(x) = f(x)$, $f^2(x) = f(f(x))$, $f^3(x) = f(f^2(x))$ etc. Does there exist a value x in X such that for any positive integer n, we can find y in X with $f^n(y) = x$?

B5. f is a continuous real-valued function on the reals such that for some $0 < a, b < 1/2$, we have $f(f(x)) = a f(x) + b x$ for all x. Show that for some constant k, $f(x) = k x$ for all x.

B6. $x_1 < x_2 < x_3 < \ldots$ is a sequence of positive reals such that $\lim x_n/n = 0$. Is it true that we can find arbitrarily large N such that all of $(x_1 + x_{2N-1}), (x_2 + x_{2N-2}), (x_3 + x_{2N-3}), \ldots, (x_{N-1} + x_{N+1})$ are less than $2 x_N$?

63rd Putnam 2002

A1. k and n are positive integers. Let $f(x) = 1/(x^k - 1)$. Let $p(x) = (x^k - 1)^{n+1} f^n(x)$, where f^n is the nth derivative. Find p(1).

A2. Given any 5 distinct points on the surface of a sphere, show that we can find a closed hemisphere which contains at least 4 of them.

A3. A subset of {1, 2, ... , n} is *round* if it is non-empty and the average (arithmetic mean) of its elements is an integer. Show that the number of round subsets of {1, 2, ... , n} has the same parity as n.

A4. Two players play a game on a 3 x 3 board. The first player places a 1 on an empty square and the second player places a 0 on an empty square. Play continues until all squares are occupied. The second player wins if the resulting determinant is 0 and the first player wins if it has any other value. Who wins?

A5. The sequence u_n is defined by $u_0 = 1$, $u_{2n} = u_n + u_{n-1}$, $u_{2n+1} = u_n$. Show that for any positive rational k we can find n such that $u_n/u_{n+1} = k$.

A6. b > 1 is an integer. For any positive integer n, let d(n) be the number of digits in n when it is written in base b. Define the sequence f(n) by f(1) = 1, f(2) = 2, f(n) = n f(d(n)). For which values of b does 1/f(1) + 1/f(2) + 1/f(3) + ... converge?

B1. An event is a hit or a miss. The first event is a hit, the second is a miss. Thereafter the probability of a hit equals the proportion of hits in the previous trials (so, for example, the probability of a hit in the third trial is 1/2). What is the probability of exactly 50 hits in the first 100 trials?

B2. A polyhedron has at least 5 faces, and it has exactly 3 edges at each vertex. Two players play a game. Each in turn selects a face not previously selected. The winner is the first to get three faces with a common vertex. Show that the first player can always win.

B3. Show that for any integer n > 1, we have $1/e - 1/(ne) < (1 - 1/n)^n < 1/e - 1/(2ne)$.

B4. One of the integers in {1, 2, ... , 2002} is selected at random (with each integer having a chance 1/2002 of being selected). You wish to guess the correct integer in an odd number of guesses. After each guess you are told whether the actual integer is less than, equal to, or greater than your guess. You are not allowed to guess an integer which has already been ruled out by the earlier answers. Show that you can guess in such a way that you have a chance > 2/3 of guessing the correct integer in an odd number of guesses.

B5. A base b palindrome is an integer which is the same when read backwards in base b. For example, 200 is not a palindrome in base 10, but it is a palindrome in base 9 (242) and base 7 (404). Show that there is an integer which for at least 2002 values of b has three digits and is a palindrome.

B6. Let p be prime and $q = p^2$. Let D be the determinant:

$$\begin{matrix} x & y & z \\ x^p & y^p & z^p \\ x^q & y^q & z^q \end{matrix}$$

Show that we can find a set of linear polynomials ax + by + cz (with a, b, c integers) whose product Q equals D mod p. (In other words, after expansion the corresponding coefficients of Q and D are equal mod p.)

64th Putnam 2003

A1. Given n, how many ways can we write n as a sum of one or more positive integers $a_1 \leq a_2 \leq ... \leq a_k$ with $a_k - a_1 = 0$ or 1.

A2. $a_1, a_2, ... , a_n, b_1, ... , b_n$ are non-negative reals. Show that $(\prod a_i)^{1/n} + (\prod b_i)^{1/n} \leq (\prod (a_i+b_i))^{1/n}$.

A3. Find the minimum of |sin x + cos x + tan x + cot x + sec x + cosec x| for real x.

A4. a, b, c, A, B, C are reals with a, A non-zero such that $|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$ for all real x. Show that $|b^2 - 4ac| \leq |B^2 - 4AC|$.

A5. An *n-path* is a lattice path starting at (0,0) made up of n upsteps (x,y) → (x+1,y+1) and n downsteps (x,y) → (x-1,y-1). A *downramp* of length m is an upstep followed by m downsteps ending on the line y = 0. Find a bijection between the (n-1)-paths and the n-paths which have no downramps of even length.

A6. Is it possible to partition {0, 1, 2, 3, ... } into two parts such that n = x + y with x ≠ y has the same number of solutions in each part for each n?

B1. Do their exist polynomials a(x), b(x), c(y), d(y) such that $1 + xy + x^2y^2 \equiv a(x)c(y) + b(x)d(y)$?

B2. Given a sequence of n terms, $a_1, a_2, ... , a_n$ the derived sequence is the sequence $(a_1+a_2)/2, (a_2+a_3)/2, ... , (a_{n-1}+a_n)/2$ of n-1 terms. Thus the (n-1)th derivative has a single term. Show that if the original sequence is 1, 1/2, 1/3, ... , 1/n and the (n-1)th derivative is x, then x < 2/n.

B3. Show that $\prod_{i=1}^{n} \text{lcm}(1, 2, 3, ... , [n/i]) = n!$.

B4. $az^4 + bz^3 + cz^2 + dz + e$ has integer coefficients (with a ≠ 0) and roots r_1, r_2, r_3, r_4 with r_1+r_2 rational and $r_3+r_4 \neq r_1+r_2$. Show that r_1r_2 is rational.

B5. ABC is an equilateral triangle with circumcenter O. P is a point inside the circumcircle. Show that there is a triangle with side lengths |PA|, |PB|, |PC| and that its area depends only on |PO|.

B6. Show that $\int_0^1 \int_0^1 |f(x) + f(y)|\, dx\, dy \geq \int_0^1 |f(x)|\, dx$ for any continuous real-valued function on [0,1].

SEMINAR 1 - 109

Seminar 1 – 109

1. Given a field F with operations + (identity 0) and x (identity 1). Express x using the binary operation - (minus) and the unary operation a^{-1} (multiplicative inverse).

2. Find a single operation from which +, -, x and / can all be derived.

3. If additions and subtractions are free, how many real multiplications are needed to multiply two complex numbers?

4. A bug either splits into two perfect copies of itself or dies. If the probability of splitting is p (and is independent of the bug's ancestry), what is the probability that a bug's descendants die out?

5. Given any n distinct points in the plane, show that one of the angles determined by them falls in the closed interval $[0, \pi/n]$.

6. Prove that every sequence of real numbers contains a monotone subsequence.

7. Define $f_{n,1}(x) = x + x^2/n$, $f_{n,r+1} = f_{n,1}(f_{n,r}(x))$ for $1 \le r \le n$. What is the limit of $f_{n,n}(x)$ as n tends to infinity?

8. Given any p in the closed interval [0, 1], devise an experiment using a fair coin which has success probability p.

9. Two players alternately choose a binary digit b_i. The digits are used to construct a real number $b = 0.b_1b_2b_3...$. The first player wins iff b is transcendental. Who wins?

10. A traffic light is green for 30 seconds, then red for 30 seconds, then green for 30 seconds and so on. What is your expected delay?

11. Prove no three lattice points in the plane form an equilateral triangle. In three dimensions?

12. All members of the sequence $a_1, a_2, a_3, ...$ are positive and $a_n < (a_{n-2} + a_{n-1})/2$ for $n > 2$. Prove the sequence converges.

13. $x_0, x_1, x_2, ...$ is a sequence of complex numbers satisfying. $x_{n+1} = (x_n + 1/x_n)/2$ for $n > 0$. Does it converge?

14. $x_0, x_1, x_2, ...$ is a real sequence satisfying: $x_{n+1} = (x_n + x_{n-1})/2$ for $n > 1$. What is the limit of the sequence (in terms of x_0 and x_1)?

15. $x_1, x_2, ... , x_m$ is a sequence of positive integers satisfying $x_1 + ... + x_m = n$. What is the maximum value of $\prod x_i$?

16. $\sum a_n$ is a convergent series of positive reals. Prove that $\sum (a_1a_2... a_n)^{1/n}$ converges.

17. n is a given positive integer. $p(x) \equiv x^m + a_{m-1}x^{m-1} + ... + a_1x + a_0$ is a polynomial with real coefficients a_i such that p(r) is an integral multiple of n for all integers r. What is the smallest possible value of m (in terms of n)?

18. Evaluate $\sqrt{(1 + 2\sqrt{(1 + 3\sqrt{(1 + ...)})})}$.

19. Prove that at any party two people have the same number of friends present.

20. If n is any integer greater than 1, then n does not divide $2^n - 1$.

21. Let n be a positive integer. Prove that if m = 0 (mod 3), then $m = 2^r \pmod{3^n}$ for some positive integer r.

22. How many square residues are there mod 2^n?

23. What is the maximum value of $1/2^x + 1/2^{1/x}$ for x > 0?

24. Prove that n distinct points in the plane, but not all on a single line, determine at least n distinct lines.

25. [0, 1] is the closed real interval. f:[0, 1] → [0, 1] is monotonic increasing, and f(0) = 0, f(1) = 1. Prove that the graph of f can be covered by n rectangles with sides parallel to the x-axis and y-axis and with area $1/n^2$.

26. Prove that any finite set of closed squares with total area 3 can be arranged to cover the unit square.

27. Prove that any finite set of closed squares with total area 1/2 can be fitted inside a unit square without overlapping.

28. Find the smallest subset X of the plane such that no point of the plane is at a rational distance from all points of X.

29. $x_1, x_2, x_3, ...$ are distinct points in the plane such that the distance between any two points is an integer. Prove that all the points are collinear.

30. A and B are positive integers. $x_i = (A + 1/2)^i + (B + 1/2)^i$. Prove that only finitely many of $x_1, x_2, x_3, ...$ are integers.

31. Let R be the reals. $f:R^2 \to R$ is such that f(x, y) is a polynomial in y for each fixed x, and a polynomial in x for each fixed y. Is f(x, y) a polynomial in x and y? What if we replace R by the rationals Q?

32. n is a positive integer. Prove that we cannot find an integer m and an integer r > 1 such that $n(n + 1)(n + 2) = m^r$.

33. K is a polygon in the plane with perimeter length P and enclosing an area A. Prove that we can find a disk radius > A/P which lies inside K.

34. A chooses an integer from {0, 1, 2, ... , 15} and must answer yes or no to each of 7 questions from B. He must answer at least 6 of the questions truthfully. Devise a set of 7 questions to allow B to determine the chosen integer.

35. R is a bounded region of the plane. Prove that we can find a point P and three lines through P which divide R into 6 pieces of equal area.

36. R is a bounded region of the plane. Prove that we can find a point P such that there is no line through P dividing R into R' and R'' with area R' equal to twice area R''.

37. $x_1, x_2, x_3, \ldots$ are positive reals. Prove that $\sqrt{(\sum_{i>0} x_i)} + \sqrt{(\sum_{i>1} x_i)} + \sqrt{(\sum_{i>2} x_i)} + \ldots \geq \sqrt{(\sum_{i>0} i^2 x_i)}$.

38. Prove that for any prime p we can find an integer n such that $n^8 = 16 \pmod p$.

39. All points of the plane are colored red, blue or green. Prove that we can find two points a distance 1 apart with the same color.

40. All points of the plane are colored red or blue. Prove that either we can find two red points a distance x apart for every x > 0, or we can find two blue points a distance x apart for every x > 0.

41. C is a simple plane arc of length > 1. Prove that for some n > 1 we can find at least n + 1 points on C such that the distance between each pair is at least 1/n.

42. You are given 4 coins, each of which may weigh 10gm or 9gm. Given an accurate scale (which gives the weight of a group of coins), show how to determine the weight of each coin using only 3 weighings in total.

43. α and β are reals such that $[\alpha, \beta]$ contains no integers. Prove that we can find a positive integer n such that $[n\alpha, n\beta]$ contains no integers and has $|n\alpha - n\beta| \geq 1/6$.

44. Prove that the $[(\sqrt{2} + 1)]$, $[(\sqrt{2} + 1)^2]$, $[(\sqrt{2} + 1)^3]$, ... are alternately even and odd integers.

45. Let $a_k = k + [n/k]$. Show that the smallest element of $\{a_1, a_2, \ldots, a_n\}$ equals $[\sqrt{(4n + 1)}]$.

46. α and β are positive and irrational. Show that the sets $\{[\alpha], [2\alpha], [3\alpha], \ldots\}$ and $\{[\beta], [2\beta], [3\beta], \ldots\}$ have null intersection and union $\{1, 2, 3, \ldots\}$ iff $1/\alpha + 1/\beta = 1$.

47. Select a subset $A = \{a_1, a_2, a_3, \ldots\}$ of the positive integers as follows. Take $a_1 = 1$. Then reject $a_1 + 1 = 2$. Then take a_2 to be the smallest remaining integer, 3, and reject $a_2 + 2 = 5$. In general, after selecting a_n, reject $a_n + n$, and select a_{n+1} to be the smallest remaining integer. The first few members of A are 1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, ... Find a formula for a_n.

48. A *doublet* is a positive integer all of whose prime factors occur to the second power or higher. Prove that there are infinitely many pairs of consecutive doublets.

49. Let N be the set of positive integers. If A and B are subsets of N, then we say they are *almost disjoint* iff $A \cap B$ is finite. How many subsets of N can we find every pair of which are almost disjoint?

50. n people each wish to have an equal share of a cake. A knife is available, but no measuring equipment. What procedure will satisfy each person that he has been treated fairly? For example, for n = 2, one person divides the cake into two pieces, and the other person chooses which piece to take. [Assume there are no complications with icing, shape etc. All that counts is the volume of the piece.]

51. Define the sequence a_n as follows: $a_0 = 0$, $a_1 = 1$, $a_n = a_{n-1} + a_{n-2}/2$ for n > 1. Prove that a_n is not an integer, except for n = 0, 1, 2, 4, 8.

52. Let $I_n = \{1, 2, 3, \ldots, n\}$. We define the *density* of a set T of positive integers as inf $(I_n \cap T)/n$. Let $a_1, a_2, \ldots, a_k$ be any positive integers. Let S be the set of all positive integers not divisible by any a_i. Prove that the density of S is at least $\prod (1 - 1/a_i)$.

53. Given n distinct real numbers $x_1, x_2, \ldots, x_n$, and n arbitary real numbers $y_1, y_2, \ldots, y_n$. Prove that we can find a polynomial p(x) all of whose zeros are real such that $p(x_i) = y_i$ for $1 \le i \le n$.

54. Is there a non-trivial continuous real-valued function defined on the real numbers such that f(x) + f(2x) + f(3x) = 0 for all x?

55. Exhibit: (A) an infinite group with no infinite proper subgroups; (B) a field isomorphic to a proper subfield; (C) a ring with no maximal ideals.

56. $a_1, a_2, a_3, \ldots$ is a sequence of positive integers. It satisfies $a_{n+1} > a_{a_n}$ for n > 0. Prove that $a_n = n$.

57. Let ABC be a triangle with angle BAC = 90. $P_1, P_2, \ldots, P_n$ is a finite set of points inside ABC. Prove that we can relabel the points $Q_1, \ldots, Q_n$ so that $\sum_{0<i<n} Q_iQ_{i+1}^2 \le BC^2$.

58. In some game a player's *batting average* for a given period is defined as his total *score* for the period divided by his total *at-bats*. [Scores must be non-negative integers, and at-bats must be positive integers.] Player A has a higher batting average than player B for both the first half of the season and the second half of the season. Does he necessarily have a higher batting average for the season as a whole?

59. Let $e^{e^x} = \sum a_n x^n$. Estimate a_n. [In other words, find upper and lower bounds for a_n.]

60. Define $x_0, x_1, x_2, \ldots$ by $x_0 = 1$, $x_n = x_{n-1} + 1/x_{n-1}$ (for n > 0). Prove $x_n \to \infty$. How fast? [In other words, find upper and lower bounds for x_n.]

61. Prove $1 + x + x^4 + x^9 + \ldots + x^{n^2} + \ldots \to \infty$ as $x \to 1$. How fast? [In other words, find upper and lower bounds for x_n.]

62. Show that the polynomial $x^n + x = 1$ has a unique positive root k_n. Show that $k_n \to 1$ as $n \to \infty$. How fast?

63. Define $y_1 = 1$, $y_n = \sin y_{n-1}$ for n > 1. Prove $y_n \to 0$ as $n \to \infty$. How fast?

64. Define $f(x) = 1/(1 + x) + 1/(2 + x) + 1/(4 + x) + \ldots + 1/(2^n + x) + \ldots$. Prove $f(x) \to 0$ as $x \to \infty$. How fast?

65. Let U be a set with n elements. Let K be a set of subsets of U, each with three elements, such that the intersection of any two distinct elements of K is either empty or contains just one element. Let f(n) be the largest possible number of elements that K can have. Estimate f(n). [In other words, find upper and lower bounds for f(n).]

66. Let S be the sequence of positive integers divisible only by 2 or 3 arranged in ascending order. [The sequence starts: 1, 2, 3, 4, 6, 8, ...] Prove that the ratio of successive terms tends to 1.

67. We say that the lattice point P is *visible* from the origin O if the segment OP does not contain any other lattice points. Show that the proportion of lattice points in the square $0 < x, y \le n$ which are visible from the origin tends to a limit and find it. [*Lattice points* are points in the Euclidean plane with integral coordinates.]

68. Let f(k) be the number of lattice points in the disk radius k centered on the origin. Prove that $f(k) = \pi k^2 + O(k)$ as $k \to \infty$.

69. A triangle has lattice points as vertices and contains no other lattice points. Prove that its area is 1/2.

70. Let S be the set of all positive multiples of k. Find an asymptotic expression for $f(x) = \sum_{n \in S} x^n/n!$ [In other words, find g(x) in closed form so that $f(x)/g(x) \to 1$ as $n \to \infty$. What is the error term?]

71. S is a subset of {1, 2, ... , n} such that if $k \in S$, then $2k \notin S$. Let f(n) be the maximum possible number of elements in such an S. Estimate f(n). [In other words, find upper and lower bounds for f(n), or an asymptotic expression for f(n) as $n \to \infty$, as appropriate. g(n) is an *asymptotic expression* for f(n), or $f(n) \sim g(n)$, if $f(n)/g(n) \to 1$ as $n \to \infty$.]

72. Let $a_1, a_2, a_3, \ldots$ be the powers [positive integers of the form n^m with m and n positive integers and m > 2] arranged in increasing order. The first few terms are: 1, 4, 8, 9, 16, 25, ... Find an asymptotic expression for a_n.

73. Let $S = \{P_i\}$ be a set of n points in the unit square. Relabel the points $Q_1, Q_2, \ldots, Q_n$ so that the path length $Q_1Q_2 + Q_2Q_3 + \ldots + Q_{n-1}Q_n$ is as short as possible. Let this length be f(S). Let g(n) = sup f(S), where the supremum is taken over all possible sets S with n points (in the unit square). Estimate g(n).

74. The Euclidean algorithm for finding the greatest common divisor d of two positive integers m and n involves replacing the pair (m, n) by the pair (k, m), where k is the remainder when n is divided by m. The algorithm terminates when k = 0. The number of steps is the number of replacements. For example, (35, 20) requires 4 steps: (35, 20), (20, 35), (15, 20), (5, 15), (0, 5). Estimate the maximum number of steps for (m, n) with m, n < N.

75. How many edges can a triangle-free graph of n points have?

76. Let $f(n) = n - [n/2] + [n/3] - \ldots$. Prove that $f(n)/n \to \ln 2$ as $n \to \infty$.

77. f(x, y) is a polynomial with real coefficients having degree m in x and degree n in y. Prove that $f(x, e^x) = 0$ has at most mn + m + n real roots.

78. Find a real valued function f on the real line such that $f(x)/x^2 \to 1$ as $x \to \infty$, but $f'(x)/(2x)$ does not tend to 1 as $x \to \infty$. Prove that such an f cannot be convex.

79. f and g are real-valued functions on [0, 1] and differentiable on (0, 1]. g'(x) > 0 on (0 ,1] and $\lim_{x\to 0^+} f'(x)/g'(x)$ exists (but may be $\pm \infty$). Prove that even if f(0)/g(0) is indeterminate, $\lim_{x\to 0^+} f(x)/g(x)$ exists (but may be $\pm \infty$).

80. f is a real valued function on the reals and $f(x) + f'(x) \to 0$ as $x \to \infty$. Prove that $f(x) \to 0$ and $f'(x) \to 0$ as $x \to \infty$.

81. F: $[0, \infty) \to (0, \infty)$ is monotonic increasing. y = f(x) is a solution of y'' + F(x) y = 0. Prove f(x) is bounded as $x \to \infty$.

82. f is a real-valued twice differentiable function on the reals. f and f '' are bounded. Prove that f ' is also bounded.

83. For the positive real number a, define $x_0 = a$, $x_n = a^{x_{n-1}}$ for n > 0, and define $f(a) = \lim_{n\to\infty} x_n$. Show that $f(\sqrt{2}) = 2$, but that f(x) = 4 has no solution. What is the supremum of the values k for which f(x) = k has a solution?

84. f is a continuous real-valued function on the reals, but is not necessarily differentiable. It satisfies $\lim_{h\to 0^+}$ (f(x + 2h) - f(x + h))/h = 0 for all x. Prove that f is constant.

85. Show that f:[a, b] → R (the reals) has a continuous derivative iff $\lim_{h\to 0}$ (f(x+h) - f(x))/h exists uniformly on [a, b].

86. Let f(x) = |sin x sin 2x sin 4x ... sin 2^nx|. Prove that f(x) ≤ 2/√3 f(π/3).

87. x_1, x_2, x_3, ... is a sequence of positive reals such that $x_n < x_{n+1} + x_n^2$. Prove that $\Sigma\ x_n$ diverges.

88. Find all possible ways of labeling the faces of two dice with positive integers so that the probability of throwing a total score of N when the two dice are thrown together is the same as with normal dice (so we require p(2) = p(12) = 1/36, p(3) = p(11) = 2/36, p(4) = p(10) = 3/36, p(5) = p(9) = 4/36, p(6) = p(8) = 5/36, p(7) = 6/36, where p(N) is the probability of a total score of N). Note that the two dice do not have to be labeled in the same way.

89. Find two disjoint sets A and B whose union is the non-negative integers such that every positive integer can be expressed as the sum of two distinct elements of A in the same number of ways as it can be expressed as the sum of two distinct elements of B.

90. Let N be the set of positive integers. Prove that we cannot find an integer n > 1 and subsets $N_1, N_2, \ldots, N_n$ of N, such that (1) the subsets are disjoint, (2) they have union N, (3) each N_i is an arithmetic progression $\{a_i, a_i + d_i, a_i + 2d_i, \ldots\}$, and (4) each d_i is different.

91. Can we find a set of positive integers S, such that all sufficiently large integers can be expressed in the same number of ways as a sum a + b, with a ≤ b and a, b ∈ S.

92. We are interested in sequences of integers $a_1 < a_2 < \ldots < a_n$, such that the set of n(n - 1)/2 positive differences $\{a_i - a_j: i > j\}$ is just {1, 2, 3, ... , n(n - 1)/2}. Such a sequence is possible for n = 4: 1, 3, 6, 7. Is it possible for any n > 4?

93. Show that the number of partitions of N into odd (positive) integers equals the number of partitions of N into distinct (positive) integers.

94. Let $f(n) = \int_0^\infty (1 + x/n)^n e^{-x}\,dx$. Estimate f(n) as n → ∞.

95. Let $f(n) = \int_0^\infty t^n e^{-t}\,dt$. Show that $f(n) \sim n^n e^{-n}\sqrt{(2\pi n)}$. [Given functions f(n) and g(n), f(n) ~ g(n) means that f(n)/g(n) → 1 as n → ∞.]

96. Let $f(n) = \sum_{i=0}^n n^i / i!$ Show that $f(n) \sim e^n/2$.

97. Estimate $\int_0^1 \cos^n(x^2)\,dx$ as n → ∞. [In other words, find suitable upper and lower bounds or an asymptotic expression.]

98. A and B repeatedly play a fair game of chance with unit stake. In other words, at each turn, either A wins 1 unit from B or A loses 1 unit to B, and each outcome has probability 1/2, independent of all other turns. If A starts with m units and B starts with n units, what is the expected number of turns before one player is wiped out?

99. A and B repeatedly play a game of chance with unit stake. At each turn, either A wins 1 unit from B with probability 0.51, or A loses 1 unit to B with probability 0.49. Both A and B have unlimited capital. What is A's expected peak cumulative loss?

100. A plays a game of chance. At each turn he wins an amount k which is uniformly distributed on the interval [0, 1]. The game stops when his cumulative win is at least 1. What is the expected number of turns?

101. An experiment has n possible outcomes, all equally likely. Each trial is independent. We repeat until we have two trials with the same outcome. For example, we would stop after getting 1, 4, 3, 4. What is the expected number of trials f(n)? Find an asymptotic expression for f(n).

102. I have n fair coins. I toss them. I leave those that came up heads and toss those that came up tails. And so on. Each time I toss just those that came up tails the previous time, and I stop if there were none. What is the expected number of tosses f(n)? [For example, with 3 coins I might get: HTT, HT, T, H, 4 tosses.] Estimate f(n). [In other words find upper and lower bounds, or an asymptotic expression].

103. Toss a fair coin repeatedly. Suppose the outcome of the nth toss is E_n. We have some decision rule whereby depending on E_1, E_2, ... , E_n we either (1) toss again, (2) declare outcome A and stop, or (3) declare outcome not A and stop. The decision rule is arranged so that the probability of A is 1/3. Show that the expected number of tosses is at least 2.

104. We are given n distinct items a_1, a_2, ... , a_n. They are arranged in a random order. What is the expected number of items for which we have a_i in the *i*th place?

105. Define the n x n matrix A by $A_{ij} = i\,j$. The elements of A are integers in the range 1 to n^2. But some numbers appear more than once and some do not appear at all. Let B be the set of numbers which appear at least once and let $f(n) = |B|/n^2$. Show that $f(n) \to 0$ as $n \to \infty$. [You may find it helpful to assume the following result. Define g(n) as the number of prime factors of n counting multiplicity, so $g(p_1^{r_1} p_2^{r_2} \dots p_m^{r_m}) = \sum r_i$. Then $g(n) \sim \ln \ln n$.]

106. f is a continuous function on the non-negative reals. f has the property that $f(x + a) - f(x) \to 0$ as $x \to \infty$ for each a. Prove that we can find functions g and h such that: (1) f(x) = g(x) + h(x) for all x; (2) $g(x) \to 0$ as $x \to \infty$; (3) h is differentiable; and (4) $h'(x) \to 0$ as $x \to \infty$.

107. f is continuous on the non-negative reals. For each positive k the sequence f(k), f(2k), f(3k), ... tends to zero. Is it necessarily true that $f(x) \to 0$ as $x \to \infty$.

108. Let C be the set of all functions f on the non-negative reals with derivatives of all orders such that f(x) >= 0, f '(x) <= 0, f ''(x) >= 0, f '''(x) <= 0, ... for all x. Show that C is *convex* [In other words, if $f_i \in C$, $\lambda_i \geq 0$ and $\sum_{i=0}^n \lambda_i = 1$, then $\sum_{i=0}^n \lambda_i f_i \in C$.]. The *extreme* points of C are the functions f such that if $f = \sum_{i=0}^n \lambda_i f_i$ with $f_i \in C$, $\lambda_i \geq 0$ and $\sum_{i=0}^n \lambda_i = 1$, then all but one λ_i are zero. Show that the extreme points are the functions $a\,e^{-bx}$ with $a, b \geq 0$.

109. Place a positive real $a_{i\,j}$ at each lattice point (i, j) of the plane such that $a_{i\,j} = (a_{i+1\,j} + a_{i\,j+1} + a_{i-1\,j} + a_{i\,j-1})/4$ for all i, j. Show that all $a_{i\,j}$ are equal.

REFERENCES

1. John Scholes, www.kalva.demon.co.uk/

2. Kiran S. Kedlaya, www.unl.edu/amc/