Romanian Mathematical Competitions

RMC 2004

EDITORS
Mircea Becheanu
Radu Gologan

Romanian Mathematical Society & Theta Foundation

Romanian Mathematical Competitions

RMC 2004

EDITORS
Mircea Becheanu
Radu Gologan

Romanian Mathematical Society & Theta Foundation

Romanian Mathematical Competitions

RMC 2004

EDITORS
Mircea Becheanu
Radu Gologan

CONTRIBUTORS
Marian Andronache
Bogdan Enescu
Mihai Bãlunã
Cãlin Popescu
Dinu Serbãnescu
Dan Schwarz

ISBN 973–0–03572–5

ROMANIAN MATHEMATICAL COMPETITIONS

CONTRIBUTORS:

MARIAN ANDRONACHE
National College "Sf. Sava" Bucharest

BOGDAN ENESCU
National College "B.P. Haşdeu Buzău

MIHAI BĂLUNĂ
National College "Mihai Viteazul" Bucharest

MIRCEA BECHEANU – **Editor**
Department of Mathematics, University of Bucharest

RADU GOLOGAN – **Editor**
Institute of Mathematics and
University "Politehnica" Bucharest

CĂLIN POPESCU
Institute of Mathematics

DINU ŞERBĂNESCU
National College "Sf. Sava" Bucharest

DAN SCHWARZ
Board for the National Mathematical Olympiad

Societatea de Ştiinţe Matematice din România
Str. Academiei 14, 70103 Bucureşti, România
E-mail: office@rms.unibuc.ro
Home page: http://www.ssmro.com
tel/fax: +40(21)3124072

Typed and technically edited by Theta
http://www.theta.ro

Printed at NECOZ PREST
Str. Oracolului 8, Bucureşti, tel. +40(21)6106328

It became a tradition to publish the most interesting problems given at the Romanian Mathematical Olympiad and some other competitions for High School Students.

The present booklet is the eleventh in the series. The book consists of two parts. First part presents all the proposed problems for the District and Final rounds of the National Mathematical Olympiad together with the problems given for the qualification tests of the Romanian Teams participating in the International Mathematical Olympiad and the Junior and Senior Balkan Competitions, respectively. We also gathered some problems from other regional Romanian Contests and made a selection from the Shortlist that we discussed during different rounds of the Olympiad.

Most of the problems are new or, to our knowledge, do not have equivalent statements in the mathematical olympiad literature. We have to thank to the large number of teachers in mathematics, mathematicians and students who contributed during the year with more than a thousand of problems such that our selection process was not easy. Some of the problems came from different other sources: shortlisted problems from the IMO and BMO, various mathematical journals, or even from some of the large variety of Internet sources one has access now-days. The text is a joint work of the editors and the contributors.

We would like to thank SOFTWIN and Medicover,the main sponsors of the Romanian IMO Team who, beside some other important involvement in the Olympiads, supported us in part to publish the book. Thanks are due to "Gill Publisher" and to the "Sigma Foundation". Thanks are also due to Luminiţa Stafi and to Barbara Ionescu from "Theta" for helping the editors in the process of producing this booklet.

The last but not the least, we are grateful to the Board of the Institute of Mathematics "Simion Stoilow", in Bucharest, for the constant technical support in the Mathematical Olympiads and in the training seminars for students.

Bucharest,
June 22, 2004

Mircea Becheanu
Radu Gologan

CONTENTS

PART ONE

PROPOSED PROBLEMS

1.1. THE NATIONAL MATHEMATICAL OLYMPIAD

District round – March 16th 2004

7th GRADE

PROBLEM 1. Find the number of positive 6 digit integers, such that the sum of their digits is 9, and four of its digits are $1, 0, 0, 4$.

Lucian Dragomir

PROBLEM 2. Let D be a point on the side BC of a given triangle ABC. The bisector lines of the angles $\angle ADB$ and $\angle ADC$ intersect AB and AC at M and N, respectively, and the bisector lines of the angles $\angle ABD$ and $\angle ACD$ intersect DM and DN at K and L, respectively.

Prove that $AM = AN$ if and only if MN and KL are parallel.

Bogdan Enescu

PROBLEM 3. One considers the set $A = \left\{ n \in \mathbb{N}^* \mid 1 < \sqrt{1 + \sqrt{n}} < 2 \right\}$.

a) Find the set A.

b) Find the set of numbers $n \in A$ such that

$$\sqrt{n} \cdot \left| 1 - \sqrt{1 + \sqrt{n}} \right| < 1\,?$$

* * *

PROBLEM 4. In the triangle ABC we have $AB = AC$ and the points M, P on AB such that $AM = BP$. Let D be the midpoint of BC, and let R on CM and Q on BC, be such that A, R, Q are collinear and the line AQ is perpendicular on CM.

Prove that:

a) $\angle AQC \equiv \angle PQB$;

b) $\angle DRQ = 45°$.

Manuela Prajea

8th GRADE

PROBLEM 1. We say that the real numbers a şi b have property $\mathcal{P}$ if: $a^2+b \in \mathbb{Q}$ and $b^2+a \in \mathbb{Q}$.

Prove that:

a) The numbers $a = \dfrac{1+\sqrt{2}}{2}$ şi $b = \dfrac{1-\sqrt{2}}{2}$ are irrational and have property $\mathcal{P}$;

b) If a, b have property $\mathcal{P}$ and $a+b \in \mathbb{Q}\setminus\{1\}$, then a şi b are rational numbers;

c) If a, b have property $\mathcal{P}$ and $\dfrac{a}{b} \in \mathbb{Q}$, then a şi b are rational numbers.

* * *

PROBLEM 2. The real numbers a, b, c, d satisfy $a > b > c > d$ and

$$a+b+c+d = 2004 \text{ şi } a^2-b^2+c^2-d^2 = 2004.$$

Answer, with proof, to the following questions:

a) What is the smallest possible value of a?

b) What is the number of possible values of a?

Mircea Fianu

PROBLEM 3. It is said that a set of three different numbers is an *arithmetical set* if one of the three numbers is the average of the other two. Consider the set $A_n = \{1, 2, \ldots, n\}$, where n is a positive integer, $n \geqslant 3$.

a) How many arithmetical sets are in A_{10}?

b) Find the smallest n, such that the number of arithmetical sets in A_n is greater than 2004.

Lucian Dragomir

PROBLEM 4. In a given trapezoid $ABCD$, let $AB \| CD$, $\angle B = 90°$ and $AB = 2DC$. Consider points N and P on the same part of the plane (ABC), such that PA and ND are perpendicular on the trapezoid's plane, and $ND = a$, $AP = \dfrac{a}{2}$ $(a > 0)$. If M is the midpoint of BC and the triangle MNP is equilateral, find:

a) The cosine of the angle between planes (MNP) and (ABC);

b) The distance from D to the plane (MNP).

Giannina Busuioc, Niculai Solomon

9th GRADE

PROBLEM 1. Real numbers a, b, c satisfy $a^2+b^2+c^2 = 3$. Prove the inequality:

$$|a|+|b|+|c|-abc \leqslant 4.$$

Virgil Nicula

PROBLEM 2. Find the cartesian coordinates of the vertices A, B, C od a triangle ABC whose the orthocenter is $H(-3, 10)$, the circumcenter is $O(-2, -3)$, and the midpoint of BC is $D(1, 3)$.

Gabriel Popa, Mihai Bălună

PROBLEM 3. a) Prove that there are infinitely many rational positive numbers x such that:

$$\{x^2\} + \{x\} = 0,99.$$

b) Prove that there are no rational numbers $x > 0$ such that:

$$\{x^2\} + \{x\} = 1.$$

Bogdan Enescu

PROBLEM 4. A rectangle 2×4 is divided in 8 squares of side 1. Call $\mathcal{M}$ the set of the 15 vertices thus obtained.

Find the points $A \in \mathcal{M}$ satisfying: *the set* $\mathcal{M} \setminus \{A\}$ *can be arranged in pairs* $(A_1, B_1), (A_2, B_2), \ldots, (A_7, B_7)$ *such that*

$$\overrightarrow{A_1B_1} + \overrightarrow{A_2B_2} + \cdots + \overrightarrow{A_7B_7} = \overrightarrow{0}.$$

Mihai Bălună

10th GRADE

PROBLEM 1. Given a positive integer n, $n \geqslant 3$, find the number of arithmetical sets of 3 elements, contained in the set $\{1, 2, \ldots, n\}$.

* * *

PROBLEM 2. Find integers n, $n \geqslant 3$, having the property: *there are distinct integers* $a_1, a_2, \ldots, a_n$, *such that:*

$$a_1! \cdot a_2! \cdots a_{n-1}! = a_n!$$

Bogdan Enescu

PROBLEM 3. In the thetrahedron $ABCD$ consider the midpoints M, N, P, Q of sides AB, CD, AC, and BD, respectively. Prove that MN is perpendicular to both AB and CD, and PQ is perpendicular to both AC şi BD, if and only if $AB = CD$, $BC = DA$ şi $AC = BD$.

Radu Gologan

PROBLEM 4. Let $x, y \in (0, \pi/2)$. Prove that if the equality

$$(\cos x + \mathrm{i}\sin y)^n = \cos nx + \mathrm{i}\sin ny$$

is true for two consecutive integers, than it is true for all integers n.

Dinu Şerbănescu

11th GRADE

PROBLEM 1. Let $x_0 > 0$ and, for any positive integer n, consider $x_{n+1} = x_n + \dfrac{1}{\sqrt{x_n}}$.

Find a) $\lim\limits_{n\to\infty} x_n$; b) $\lim\limits_{n\to\infty} \dfrac{x_n^3}{n^2}$.

Bogdan Enescu

PROBLEM 2. Consider complex non-zero numbers $z_1, z_2, \ldots, z_{2n}$, $n \geqslant 3$, such that $|z_1| = |z_2| = \cdots = |z_{n+3}|$ and $\arg z_1 \geqslant \cdots \geqslant \arg(z_{n+3})$. Define for $i, j \in \{1, 2, \ldots, n\}$: $b_{ij} = |z_i - z_{j+n}|$, and let $B = (b_{ij}) \in \mathcal{M}_n$. Prove that $\det B = 0$.

Călin Popescu

PROBLEM 3. A function $f : \mathbb{R} \to \mathbb{R}$ is said to have property $\mathcal{P}$ if: *for every* $a, b \in \mathbb{R}$ *we have* $f\left(\dfrac{a+b}{2}\right) \in \{f(a), f(b)\}$.

a) Give an example of a non-constant function possessing $\mathcal{P}$..

b) If f has $\mathcal{P}$ and is continuous, prove that it is constant.

Dan Marinescu

PROBLEM 4. The matrix $A = (a_{ij}) \in \mathcal{M}_p(\mathbb{C})$ is defined by $a_{12} = a_{23} = \cdots = a_{p-1,p} = 1$ and $a_{ij} = 0$, for the remaining set of indices (i, j).

Prove that there are no non-zero matrices $B, C \in \mathcal{M}_n(\mathbb{C})$ such that $(I_p + A)^n = B^n + C^n$, for all non-negative integer n.

Ion Savu

12th GRADE

PROBLEM 1. Let $n \geqslant 2$ be an integer and $r \in \{1, 2, \ldots, n\}$. Consider the set

$$S_r = \{A \in \mathcal{M}_n(\mathbb{Z}_2) \mid \operatorname{rank} A = r\}.$$

a) Prove that for any $A \in S_n$ and $B \in S_r$, AB is in S_r;

b) Calculate $\sum\limits_{X \in S_r} X$.

Mihai Fulger, Valentin Vornicu

PROBLEM 2. Prove that the only continuous functions $f : [0,1] \to \mathbb{R}$, such that

$$\int_0^1 f(x)g(x)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x \cdot \int_0^1 g(x)\mathrm{d}x$$

for any non-derivable continuous function $g : [0,1] \to \mathbb{R}$, are the constant functions.

Mihai Piticari

PROBLEM 3. The ring A satisfies the following properties:

i) Its unit 1_A has order p, a prime number;

ii) There is a set $B \subset A$ containing p elements, such that: for any $x, y \in A$, there is $b \in B$ satisfying $xy = byx$.

Prove that A is commutative.

Ion Savu

PROBLEM 4. Let $a, b \in (0,1)$ and let $f : [0,1] \to \mathbb{R}$ be a continuous function, such that

$$\int_0^x f(t)\mathrm{d}t = \int_0^{ax} f(t)\mathrm{d}t + \int_0^{bx} f(t)\mathrm{d}t, \quad \text{for any } x \in [0,1].$$

a) Prove that $a + b < 1$ implies $f = 0$.

b) Prove that $a + b = 1$ implies f is constant.

Dan Marinescu

1.2. THE NATIONAL MATHEMATICAL OLYMPIAD

Final Round – Deva, April 5, 2004

7th GRADE

PROBLEM 1. On the sides AB and AD of the rhombus $ABCD$ consider the points E and F respectively, such that $AE = DF$. Lines BC and DE intersect at P, and lines CD and BF intersect at Q. Prove that:

a) $\dfrac{PE}{PD} + \dfrac{QF}{QB} = 1$;

b) Points P, A, Q are collinear.

Virginia Tică and Vasile Tică

PROBLEM 2. The side-lengths of a triangle are a, b, c.

a) Prove that there is a triangle with sides $\sqrt{a}$, $\sqrt{b}$ şi $\sqrt{c}$.

b) Prove that că $\sqrt{ab} + \sqrt{bc} + \sqrt{ac} \leqslant a + b + c < 2\sqrt{ab} + 2\sqrt{bc} + 2\sqrt{ac}$.

* * *

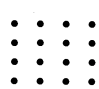
PROBLEM 3. The diagonals of the trapezoid $ABCD$ are perpendicular and intersect in O. Angle A equals $90°$, and $AB||CD$, $AB > CD$. The diagonals intersect at O. OE is the bisector line of the angle AOD, E is on the segment AD, and F on BC, such that $EF \parallel AB$. Denote by P respectively Q, the intersection points of the segment EF with the diagonals AC and BD, respectively. Prove that:

a) $EP = QF$;

b) $EF = AD$.

Claudiu-Ştefan Popa

PROBLEM 4. Sixteen points are placed in the centers of a 4×4 chess table in the following way:



a) Prove that may choose 6 points such that no isosceles triangle can be drawn with vertices at these points.

b) Prove that one cannot choose 7 points with the above property.

Radu Gologan, Dinu Şerbănescu

8th GRADE

PROBLEM 1. Find all non-negative integers n such that there are integers a and b with the property:

$$n^2 = a + b \quad \text{and} \quad n^3 = a^2 + b^2.$$

Lucian Dragomir

PROBLEM 2. Prove that the equation

$$x^2 + y^2 + z^2 + t^2 = 2^{2004},$$

where $0 \leqslant x \leqslant y \leqslant z \leqslant t$, has exactly two solutions in the set of integers.

Mihai Bălună

PROBLEM 3. Consider a frustum of a regular quadrilateral pyramid $ABCDA'B'C'D'$ in which the lines BC' are DA' orthogonal.

a) Prove that the angle between the lines AB' and DA' equals $60°$.

b) Assume that the projection of B' onto the plane (ABC) is the incenter of the triangle ABC. Prove that the distance between the lines CB' and AD' equals $\frac{1}{2}BC'$.

Mircea Fianu

PROBLEM 4. A cube of side 6 contains 1001 unit cubes with sides parallel to those of the given one. Prove that one can find two unit cubes, such that the center of one of them is inside or on the faces of the other.

Dinu Şerbănescu

9th GRADE

PROBLEM 1. Find all increasing functions $f : \{1, 2, \ldots, 10\} \to \{1, 2, \ldots, 100\}$, having the property: $x+y$ is a divisor of $xf(x)+yf(y)$, for any $x, y \in \{1, 2, \ldots, 10\}$.

Cristinel Mortici

PROBLEM 2. For any positive integer n denote by $P(n)$ the number of quadratic functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax^2 + bx + c$, having the properties:

a) $a, b, c \in \{1, 2, \ldots, n\}$;

b) The equation $f(x) = 0$ has integer roots.

Prove that $n < P(n) < n^2$, for all $n \geqslant 4$.

Laurenţiu Panaitopol

PROBLEM 3. Let H be the orthocenter of the acute triangle ABC, and let B', C' be the projection of the points B and C onto AC and AB, respectively. A variable line d, passing through H, intersects the segments BC' and CB' in M and N, respectively. The perpendiculars in M and N on d, intersect BB' and CC', in P and Q, respectively.

Find the locus of the midpoint of the segment PQ.

Gheorghe Szölösy

PROBLEM 4. Let p, q be positive integers, $p \geqslant 2$, $q \geqslant 2$. A set X has property (S) if, by definition, for any p subsets $B_i \subset X$, $i = 1, 2, \ldots, p$, not necessarily different, any of them having q elements, there is a set $Y \subset X$ having p elements, such that the intersection of Y with each B_i, $i = 1, 2, \ldots, p$, has at most one element. Prove that:

a) Any set X with $pq - q$ elements does not satisfy property (S);

b) Any set X with $pq - q + 1$ has property (S).

Dan Schwarz

10[th] GRADE

PROBLEM 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, such that $|f(x) - f(y)| \leqslant |x - y|$, for any real x, y. Prove that if the sequence $x, f(x), f(f(x)), \ldots$ is in arithmetical set, for any real x, then there is a real a such that $f(x) = x + a$, for any real number x.

Mihai Piticari

PROBLEM 2. Prove that a thetraedron in which pairs of opposite sides are equal and make equal angles, is regular.

Mircea Becheanu, Bogdan Enescu

PROBLEM 3. Let $n > 2$ be an integer and $a \in (0, \infty)$ such that

$$2^a + \log_2 a = n^2.$$

Prove that

$$2\log_2 n > a > 2\log_2 n - \frac{1}{n}.$$

Radu Gologan

PROBLEM 4. Let $(P_n)_{n\geqslant 1}$ an infinite family of planes and $(X_n)_{n\geqslant 1}$ a family of non-void sets of points, such that $X_n \subset P_n$ and the orthogonal projection of X_{n+1} onto the plane P_n is contained in X_n, for any n.

Prove that there is a sequence of points $(p_n)_{n\geqslant 1}$, such that $p_n \in P_n$ and p_n is the orthogonal projection of p_{n+1} onto the plane P_n, for any n.

Does the result remain valid if the sets X_n are infinite?

Claudiu Raicu

11[th] GRADE

PROBLEM 1. Consider an integer $n \geqslant 3$ and the parabola of equation $y^2 = 2px$, with focus F. A regular n-gone $A_1A_2\cdots A_n$ has center at F and no one of its vertices lies on the x axis. The rays $FA_1, FA_2, \ldots, FA_n$ cut the parabola at points $B_1, B_2, \ldots, B_n$.

Prove that $FB_1 + FB_2 + \cdots + FB_n > np$.

Călin Popescu

PROBLEM 2. Consider an integer n, $n \geqslant 2$.

a) Prove that there are matrices $A, B \in \mathcal{M}_n(\mathbb{C})$ such that

$$\operatorname{rang}(AB) - \operatorname{rang}(BA) = \left[\frac{n}{2}\right].$$

b) Prove that for any $X, Y \in \mathcal{M}_n(\mathbb{C})$ we have

$$\operatorname{rang}(XY) - \operatorname{rang}(YX) \leqslant \left[\frac{n}{2}\right].$$

Ion Savu

PROBLEM 3. Let $f : (a, b) \to \mathbb{R}$ be a function with the property that for any $x \in (a, b)$, there is a non-trivial interval $[a_x, b_x]$, $a < a_x \leqslant x \leqslant b_x < b$, such that f is constant on $[a_x, b_x]$.

a) Prove that the image of f is a set that is finite or countable.

b) Find all continuous functions that satisfy the given property.

* * *

PROBLEM 4. a) Construct a function $f : \mathbb{R} \to \mathbb{R}_+$ with the following property, called $\mathcal{P}$:

"Any $x \in \mathbb{Q}$ is a local strictly minimum point for f".

b) Construct $f : \mathbb{Q} \to \mathbb{R}_+$ with the property that any point is a local minimum point and f is unbounded on any set of the form $I \cap \mathbb{Q}$, where I a non-trivial interval.

c) Let $f : \mathbb{R} \to \mathbb{R}_+$ be function that is not bounded on any set of the form $I \cap \mathbb{Q}$, for any non-degenerate interval I. Prove that f has not the property $\mathcal{P}$.

Claudiu Raicu

12[th] GRADE

PROBLEM 1. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$n^2 \int_x^{x+\frac{1}{n}} f(t)\,\mathrm{d}t = nf(x) + \frac{1}{2},$$

for any $x \in \mathbb{R}$ and any positive integer n, $n \geqslant 2$.

Mihai Piticari

PROBLEM 2. Let $f \in \mathbb{Z}[X]$. For $n \in \mathbb{N}$, $n \geqslant 2$, define $f_n : \mathbb{Z}_n \to \mathbb{Z}_n$, by $f_n(\widehat{x}) = \widehat{f(x)}$, for any $x \in \mathbb{Z}$.

Find all polynomials $f \in \mathbb{Z}[X]$, such that for any $n \in \mathbb{N}$, $n \geqslant 2$, the function f_n is onto.

Bogdan Enescu

PROBLEM 3. Let $f : [0, 1] \to \mathbb{R}$ be an integrable function such that

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 xf(x)\,\mathrm{d}x = 1.$$

Prove that

$$\int_0^1 f^2(x)\,\mathrm{d}x \geqslant 4.$$

Ion Raşa

PROBLEM 4. Let K be a field of characteristics p, with $p \equiv 1 \pmod 4$.

Prove that any non-zero element in K can be written as the sum of three squares of non-zero elements from K.

Marian Andronache

1.3. THE NATIONAL MATHEMATICAL OLYMPIAD
Selection Tests for the BMO and IMO 2004

First Selection Test

1. Let a_1, a_2, a_3, a_4 be the lengths of the sides of a quadrilateral and s its semi-perimeter. Prove that

$$\sum_{i=1}^{4} \frac{1}{s + a_i} \leqslant \frac{2}{9} \sum_{1 \leqslant i < j \leqslant 4} \frac{1}{\sqrt{(s - a_i)(s - a_j)}}$$

When does equality holds?

Călin Popescu

2. Let $\mathcal{R}_i$, $i = 1, 2, \ldots, n$, be a finite family of pairwise disjoint closed rectangular regions whose sides are parallel to the coordinate axes. It is also known that the area of $\mathcal{R} = \bigcup_{i=1}^{n} \mathcal{R}_i$ is at least 4, and the projection onto Ox of their union is an interval.

Prove that $\mathcal{R}$ contains three points which are the vertices of a triangle of area 1.

Dan Ismailescu

3. Find all injective functions $f : \mathbb{N} \to \mathbb{N}$ such that for each n,

$$f(f(n)) \leqslant \frac{n + f(n)}{2}.$$

Formulated by Cristinel Mortici

4. Consider an integer $n \geqslant 2$ and a disc $\mathcal{D}$ in the complex plane. Prove that for any $z_1, z_2, \ldots, z_n \in \mathcal{D}$, there exists $z \in \mathcal{D}$ such that $z^n = z_1 z_2 \cdots z_n$.

Barbu Berceanu, Dan Schwartz, Dan Marinescu

Second Selection Test

5. A disc $\mathcal{D}$ is divided into $2n$ equal sectors; n of them are colored in red and the other n are colored in blue. Starting from an arbitrarily chosen sector, we count from 1 to n, in a clockwise order, the red sectors. We proceed in the same way with the blue sectors, but in an counterclockwise order. Prove that there exists a half-disc of $\mathcal{D}$ which contains all the numbers from 1 to n.

Kvant, M1684

6. Find nonnegative integers which can be reached by the expression

$$\frac{a^2+ab+b^2}{ab-1,}$$

when a, b are nonnegative integers and $ab \neq 1$.

Mircea Becheanu

7. Let a, b, c be integers, b odd, and consider the sequence $x_0 = 4, x_1 = 0$, $x_2 = 2c,$, $x_3 = 3b$,

$$x_n = ax_{n-4} + bx_{n-3} + cx_{n-2}, \text{ for } n \geqslant 4.$$

Prove that if p is a prime and m a positive integer, then x_{p^m} is divisible by p.

Călin Popescu

8. A square $ABCD$ is taken inside a circle γ. Inside the angle opposite to $\angle BAD$ one considers the circle tangent to the extended segments AB and AD and internally tangent to γ at A_1. The points B_1, C_1, D_1 are defined in the same way.

Prove that the straight lines AA_1, BB_1, CC_1, DD_1 are concurrent.

Radu Gologan

Third Selection Test

9. Let $n > 1$ be a positive integer and X be a set containing n elements. $A_1, A_2, \ldots, A_{101}$ are subsets of X such that the union of any 50 of them has more than $\frac{50}{51}n$ elements.

Prove that among the given subsets it is possible to choose three, such that every two of them have a non-empty intersection.

Gabriel Dospinescu

10. Prove that if n and m are integers, and m is odd, then

$$\frac{1}{3^m n}\sum_{k=0}^{m}\binom{3m}{3k}(3n-1)^k$$

is an integer.

Călin Popescu

11. The incircle of the non-isosceles triangle ABC has center I and it touches the sides BC, CA, AB, in A', B', C', respectively. The straight lines AA' and BB' intersect in P, AC and $A'C'$ in M, and $B'C'$ and BC in N.

Prove that the straight lines IP and MN are perpendicular.

Classical result

12. Let $n \geqslant 2$ be an integer and $a_1, a_2, \ldots, a_n$ real numbers. Prove that for any non-empty subset $S \subset \{1, 2, \ldots, n\}$ the following inequality holds:

$$\left(\sum_{i\in S} a_i\right)^2 \leqslant \sum_{1\leqslant i\leqslant j\leqslant n} (a_i + \cdots + a_j)^2.$$

Gabriel Dospinescu

Fourth Selection Test

13. Let m, $m \geqslant 2$, be an integer. A positive integer n is called m-good if for every positive integer a, relatively prime to n, one has $n | a^m - 1$.

Show that any m-good number is at most $4m(2^m - 1)$.

Gabriel Dospinescu

14. A point O is situated in the triangle's ABC plane. A circle $\mathcal{C}$ passing through O is cut the second time by OA, OB, OC, in P, Q, R, respectively, and $\mathcal{C}$ cuts the second time the circles $(B, O, C), (A, O, C), (A, O, B)$ in K, L, M respectively.

Prove that PK, QL, RM are concurrent.

* * *

15. Some of the n faces of a polyhedron are colored in black in such a way that any two black faces have no vertex in common. All other faces are colored in white.

Prove that the number of edges that are common borders of two white faces, is at least $n - 2$.

Călin Popescu

Fifth Selection Test

16. Consider a triangle ABC and O be an interior point of it. The straight lines OA, OB, OC meet the sides of the triangle in A_1, B_1, C_1, respectively. Let R_1, R_2, R_3 be the radii of the circles $(O, B, C), (O, C, A), (O, A, B)$ respectively and R the radius of the circumcircle of the triangle ABC. Prove that

$$\frac{OA_1}{AA_1}R_1 + \frac{OB_1}{BB_1}R_1 + \frac{OC_1}{CC_1}R_1 \geqslant R.$$

Dinu Şerbănescu

17. A *move* on a $m \times n$ board consists of:

(i) Choosing some empty squares such that no two of them are in the same row or in the same column and placing a white stone on each of the selected squares;

(ii) Placing then a black stone on each empty square that corresponds to a white stone on his row and on his column.

What is the maximum number of white stones which can appear on the board, after some moves have been made?

2004 BMO Short-list, Serbia and Montenegro

18. Let p be an odd prime number, a_i, $i = 1, 2, \ldots, p-1$ be Legendre's symbol of i relative to p (i.e. $a_i = 1$ if $i^{(p-1)/2} \equiv 1$ and $a_i = -1$ otherwise). Consider the polynomial

$$f = a_1 + a_2X + \cdots + a_{p-1}X^{p-2}.$$

a) Prove that 1 as a simple root of f if and only if $p \equiv 3 \pmod 4$.

b) Prove that if $p \equiv 5 \pmod 8$, then 1 is a root of f of order exactly two.

Călin Popescu

Supplementary Test

19. Consider a sequence of positive different integers $(a_n)_n$ such that there is $a > 0$ satisfying

$$a_n \leqslant an, \text{ for all positive integers } n.$$

Prove that:

i) If $a < 5$ the sequence contains infinitely many of numbers for which the sum of their digits (in decimal representation) is not a multiple of 5.

ii) The above result for $a = 5$.

Gabriel Dospinecu

20. Given an integer number $n \geqslant 1$, consider n distinct unit vectors in the plane, which have a common origin at point O. Suppose further that for some non-negative integer $m < n/2$, on either side of any straight line passing through O, there are at least m of these vectors. Prove that the length of the sum of all n vectors cannot exceed $n - 2m$.

Kvant

1.4. THE NATIONAL MATHEMATICAL OLYMPIAD

Selection tests for the Junior BMO

First selection test

PROBLEM 1. Find all positive real numbers a, b, c which satisfy the inequalities

$$4(ab + bc + ca) - 1 \geqslant a^2 + b^2 + c^2 \geqslant 3(a^3 + b^3 + c^3).$$

Laurenţiu Panaitopol

PROBLEM 2. Consider the numbers defined by $a_n = 3n + \sqrt{n^2 - 1}$ and $b_n = 2(\sqrt{n^2 + n} + \sqrt{n^2 - n})$, for all $n = 1, 2, \dots, 49$. Prove that there are integers A, B so that

$$\sqrt{a_1 - b_1} + \sqrt{a_2 - b_2} + \cdots + \sqrt{a_n - b_n} = A + B\sqrt{2}.$$

Titu Andreescu

PROBLEM 3. Consider a circle of center O, and let V be a point externally to the circle. The tangents from V touch the circle at points T_1, T_2. Let T be a point on the small arc T_1T_2 of the circle. The tangent at T intersects the line VT_1 in the point A and the lines TT_1 and VT_2 intersect in the point B. Let M be the intersection point of the lines OM and AB.

Prove that the lines OM and AB are perpendicular.

Mircea Fianu

PROBLEM 4. Consider a cube and let M, N be two of its vertices. Assign the number 1 to these vertices and 0 to the other six vertices. We are allowed to select a vertex and to increase with a unit the numbers assigned to the 3 adjacent vertices and call this a *movement*. Prove that there is a sequence of movements after that all the numbers assigned to all the vertices of the cube are equal, if and only if MN is not a diagonal of a face of the cube.

Marius Ghergu, Dinu Şerbănescu

Second selection test

PROBLEM 5. Let ABC be an acute triangle and let D be a point on the side BC. The points E and F are the projections of the point D on the sides AB and AC, respectively. Lines BF and CE meet at point P. Prove that AD is the bisector line of the angle BAC if and only if the lines AP and BC are perpendicular.

Severius Moldoveanu

PROBLEM 6. An 8×8 array consists of 64 unit squares. Inside each square are written the numbers 1 or -1 so that in any 2×2 sub-array the sum of the four numbers equals 2 or -2. Prove that there exist two rows in the array which the same numbers are inscribed in the same order.

Marius Ghergu

PROBLEM 7. Consider a triangle ABC with the side lengths a, b, c so that a is the greatest. Prove that ABC is a right triangle if and only if

$$(\sqrt{a+b} + \sqrt{a-b})(\sqrt{a+c} + \sqrt{a-c}) = (a+b+c)\sqrt{2}.$$

Virgil Nicula

PROBLEM 8. Find all positive integers n for which there are distinct integer numbers $a_1, a_2, \dots, a_n$ such that

$$\frac{1}{a_1} + \frac{2}{a_2} + \ldots + \frac{n}{a_n} = \frac{a_1 + a_2 + \cdots + a_n}{2}.$$

Dinu Şerbănescu

Third selection test

PROBLEM 9. In a chess tournament each of the players plays with all the others two games, one time with the white pieces and then with the black pieces. In each game the winner gets one point, and both players receive 0.5 points if the game ends with a tie. At the end of the tournament, all the players end with the same number of points.

a) Prove that there are two players which have the same number of ties.

b) Prove that there are two players which have the same number of defeates when playing the white.

Marius Ghergu

PROBLEM 10. Consider the triangle ABC, $AB = AC$ and a variable point M on the line BC, so that B is between M and C. Prove that the sum of the in-radius of AMB and the ex-radius of AMC, corresponding to the angle M, is constant.

Virgil Nicula

PROBLEM 11. Let p, q, r be primes and let n be a positive integer such that $p^n + q^n = r^2$. Prove that $n = 1$.

Laurenţiu Panaitopol

PROBLEM 12. Let $a < b \leqslant c < d$ be positive integers so that $ad = bc$ and $\sqrt{d} - \sqrt{a} \leqslant 1$. Prove that a is a perfect square.

Dinu Şerbănescu

Fourth selection test

PROBLEM 13. Let ABC be a triangle inscribed in the circle K, and consider a point M on the arc BC which does not contain A. The tangents from M to the in-circle of ABC intersect the circle K at the points N and P. Prove that if $\angle BAC = \angle NMP$, then the triangles ABC and MNP are congruent.

Valentin Vornicu

PROBLEM 14. The real numbers $a_1, a_2, \ldots, a_n$ satisfy the equality

$$a_1^2 + a_2^2 + \cdots a_{100}^2 + (a_1 + a_2 + \cdots + a_{100})^2 = 101.$$

Prove that $|a_k| \leqslant 10$, for all $k = 1, 2, \ldots 100$.

Dinu Şerbănescu

PROBLEM 15. A finite set of positive integers is called *isolated*, if the sum of the elements in any proper subset is a number relatively prime with the sum of the elements of the isolated set. Find all non-prime integers n for which there exist positive integers a, b, so that the set $A = \{(a+b)^2, (a+2b)^2, \ldots, (a+nb)^2\}$ is isolated.

Gabriel Dospinescu

PROBLEM 16. A regular polygon with 1000 sides has its vertices colored in red, yellow or blue. A *move* consists in choosing to adjacent vertices colored differently and coloring them in the third color. Prove that there is a sequence of *moves* after which all the vertices of the polygon will have the same color.

Marius Ghergu

Fifth selection test

PROBLEM 17. Consider the triangular array

0	1	1	2	3	5	8	...
	0	1	1	2	3	5	...
		2	3	5	8	13	...
			4	7	11	18	...
				12	19	31	...

defined by the conditions:

i) on the first two rows, each element, starting with the third, is the sum of the two preceding elements;

ii) on the other rows, each element is the sum of the two elements placed above of the same column.

a) Prove that all the rows are defined according to condition i).

b) Consider 4 consecutive rows and let a, b, c, d be the first element in each of these rows, respectively. Find d in terms of a, b and c.

Dinu Şerbănescu

PROBLEM 18 . Let M, N, P be the midpoints of the sides BC, CA, AB of the triangle ABC, respectively, and let G be the centroid of the triangle. Prove that if $BMGP$ is cyclic and $2BN = \sqrt{3}AB$, then triangle ABC is equilateral.

BMO shortlist 2004

PROBLEM 19. Let A be a set of positive integers with the properties:

i) if $a \in A$, then all positive divisors of a are elements of A;

ii) if $a, b \in A$ and $1 < a < b$, then $1 + ab \in A$. Prove that if the set A has at least 3 elements, then $A = \mathbb{N}^*$.

Valentin Vornicu

PROBLEM 20. Consider a convex polygon with $n \geqslant 5$ sides. Prove that there are at most $\dfrac{n(2n-5)}{3}$ triangles of area 1 whose vertices are choosen from the vertices of the polygon.

Andrei Neguţ

1.5. THE NATIONAL MATHEMATICAL OLYMPIAD

Shortlisted Problems for the Final Round

7^{th} and 8^{th} GRADE

1. Paint in red n of the vertices of a regular octagon, $n \leqslant 8$. Find, with proof, the smallest value of n which insures the existence of an isosceles triangle having only red vertices.

Radu Gologan

2. Let a, b, c be non-zero integers such that $a > 0$, $bc > a$ and $ac + b > 3a$. Prove that $ab + c > 2a$.

Lucian Dragomir

3. For a positive integer not ending with 0, we define its reverse to be the number obtained by writing the given number in reverse order (for example, the reverse of 1234 is 4321). Find all positive integers n such that

a) n^2 can be written as the difference between a three digit number and its reverse;

b) n^3 can be written as the difference between a four-digit number and its reverse.

Valentin Vornicu

4. Let a, b be positive integers such that $\sqrt{a} + \dfrac{1}{\sqrt{b}}$ is an integer. Prove that a and b are perfect squares.

Mircea Becheanu

5. Consider an acute triangle ABC of orthocenter H and altitudes AM, BN, CP. Denote by Q and R the midpoints of BH and CH, respectively. Consider the following intersections $U = MQ \cap AB$, $V = MR \cap AC$, $T = AH \cap PN$.

Prove that T is the orthocenter of the triangle UAV.

Manuela Prajea

6. Let P be a variable point on the border of a rectangle $ABCD$, where $AB = CD = a$, $BC = AD = b$ and $a > b$.

Find the position of P for which the sum

$$PA + PB + PC + PD$$

has minimal value.

Mircea Becheanu

7. Consider a parallelogram $ABCD$ such that $m(\angle BCD) \geqslant 90^\circ$ and $BC \geqslant CD$. It is known that for any points M on DA and N on CB such that $CN \perp DN$, one has $CM \equiv DN$. Show that $ABCD$ is a square.

Alexandru Blaga

8. Consider a point M on the diagonal BD of a given rectangle $ABCD$, such that $\angle AMC = \angle CMD$. The point N is the intersection point between AM and the parallel line to CM that contains B.

Prove that the triangle BMN is equilateral if and only if $ABCD$ is a square.

Valentin Vornicu

9. Given a set X containing n elements ($n \geqslant 2$), find the number of pairs (A, B) such that $A \subseteq B \subseteq X$, $B \neq X$ and A has more that one element.?

Valentin Vornicu

10. For n, p positive integers, consider $f(n,p) = \left[\frac{n^2}{p}\right]$.

a) Prove that $f(n,3) + f(n+1,3) + f(n+2,3)$ is a perfect square.

b) Prove that if $f(n) = f(n,p) + f(n+1,p) + f(n+2,p)$ is a perfect square for any positive integer n, then $p = 3$.

Marius Burtea

11. Find all functions $f : \mathbb{R} \to \mathbb{R}$, having the property that

$$f(2x - 5) \leqslant 2x - 3 \leqslant f(2x) - 5,$$

for any real x.

Liliana Antonescu

9^{th} GRADE

12. The positive numbers a, b are given such that $a + b = 1$. Find the minimal value of

$$\frac{1}{1 - \sqrt{a}} + \frac{1}{1 - \sqrt{b}}.$$

Ioan V. Maftei

13. The positive numbers a, b, c satisfy $abc = 1$. Prove that

$$\sum \frac{a+b}{c} \geqslant 2\Big(\sum a + \sum \frac{1}{a} - 3\Big).$$

Gabriel Dospinescu

14. Let $a, b \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$, be an increasing function. Find the functions $g : [a, b] \to [f(a), f(b)]$ satisfying $g(x) - g(y) \geqslant |f(x) - f(y)|$, for any $x, y \in [a, b]$.

Călin Burdușel

15. Find the quadratic functions $f(x) = ax^2 + bx + c$, with $f(0) \in \mathbb{Z}$ having the property that

$$f\left(n + \frac{1}{n}\right) > n^2 - n + 1 \quad \text{și} \quad f\left(n + \frac{n-1}{n}\right) < n^2 + n - 1,$$

hold for infinitely many integer values of n.

Cristinel Mortici

16. For any integer n, $n \geqslant 2$, put $a_n = 2 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}}$.
Prove that $1 + \dfrac{a_2}{2} + \dfrac{a_3}{3} + \cdots + \dfrac{a_n}{n} < a_{n-1}$.

Cristinel Mortici

17. Given positive numbers a, b, c, prove the inequality

$$\sum \frac{a}{bc(c+a)} \geqslant \frac{27}{2(a+b+c)^2}.$$

Petre Bătrânețu

18. Find all quadratic functions $f(x) = ax^2 + bx + c$, where a, b, c are real numbers and $a \neq 0$, having the property that the images of the intervals $[0, 1]$ and $[4, 5]$ are two intervals having exactly one common point, their union being the interval $[1, 9]$.

Cristinel Mortici

19. For positive numbers a, b, c, prove

$$\sum \sqrt{a}(b^3 + c^3) \geqslant 2\sqrt{abc}(a^2 + b^2 + c^2).$$

Marin Chirciu

20. Find all pairs (x, y) of real numbers, that satisfy

$$x^4 + 2x^3 - y = \sqrt{3} - \frac{1}{4}, \quad y^4 + 2y^3 - x = -\sqrt{3} - \frac{1}{4}$$

Titu Andreescu

21. Prove that for real numbers a, b, c such that $a^2+b^2+c^2 = 9$, the following inequality holds

$$3 \min\{a, b, c\} \leqslant 1 + abc.$$

Virgil Nicula

22. Let P be a positive number, $q \in \{p^2 + 1, p^2 + 2, \ldots, p^2 + 2p\}$ and $x_n = [(p + \sqrt{q})^n]$, for any non-negative integer n.
Prove that there is a one-to-one correspondence between the sets

$$\{n \in \mathbb{N} \mid x_n \text{ is even}\} \quad \text{and} \quad \{n \in \mathbb{N} \mid x_n \text{is odd}\}.$$

Radu Miculescu

23. Find all positive integers n, such that the following statement holds: for any non-zero real numbers a, b, there are real numbers $x_1, x_2, \ldots, x_n$ for which

$$\begin{cases} x_1 + x_2 + \cdots + x_n = a \\ \dfrac{1}{x_1} + \dfrac{1}{x_2} + \dfrac{1}{x_n} = b. \end{cases}$$

Mihai Băluna

24. Consider a triangle ABC and a real number k, $k \geqslant 1$. Let A', B', C' be points on the sides $(BC), (AC), (AB)$ respectively, such that

$$\frac{1}{k} \leqslant \frac{AC'}{BC'} \leqslant k, \quad \frac{1}{k} \leqslant \frac{BA'}{CA'} \leqslant k, \quad \frac{1}{k} \leqslant \frac{CB'}{AB'} \leqslant k.$$

Prove that

$$\frac{\max(A'B', A'C', B'C')}{\max(AB, AC, BC)} \leqslant \frac{k}{k+1}.$$

Dan Ismailescu

25. Prove that a triangle ABC is equilateral if and only if there are points M, N, P on sides $(AB), (BC), (CA)$ respectively, such that the triangles ABC and MNP have the same centroid and the same orthocenter.

Marian Ionescu

26. Prove

$$\frac{1}{\cot 9^\circ - 3\tan 9^\circ} + \frac{3}{\cot 27^\circ - 3\tan 27^\circ} + \frac{9}{\cot 81^\circ - 3\tan 81^\circ} + \frac{27}{\cot 243^\circ - 3\tan 243^\circ} = 10\tan 9^\circ.$$

Titu Andreescu

27. In the convex quadrilateral $ABCD$ suppose that $AD = DB$, $AC = CD$ and $\angle BAC = 2\angle BDC$.
Prove that $\angle CBD = 30^\circ$ and that there is $x \in (0, 30^\circ)$ such that $\angle A = 60 + x$, $\angle B = 90 + x$, $\angle C = 150 - x$.

Virgil Nicula

28. Prove that a quadrilateral $ABCD$ is a rhombus if and only if for any point M, in its plane, the following inequality holds

$$4(MA\cdot MC + MB\cdot MD) \geqslant AC^2 + BD^2.$$

Laurenţiu Panaitopol

29. Let $a \in (0,\infty)\setminus\{1\}$. Find the real numbers x, y, z that satisfy

$$a^x + \log_a y = a$$
$$a^y + \log_a z = a$$
$$a^z + \log_a x = a.$$

* * *

30. Find all pairs (z_1, z_2) of complex numbers, satisfying the following two conditions:
(a) $|1 + z_1 + z_2| = |1 + z_1| = 1$;
(b) $|z_1 z_2(z_1 + z_2)| = 2(|z_1| + |z_2|)$.

Valentin Vornicu

31. Let x, y, z be positive numbers, such that $x + y + z = \frac{\pi}{2}$. Prove that

$$(1 - \sin x)(1 - \sin y)(1 - \sin z) \geqslant \sin x \sin y \sin z.$$

Find the cases when the equality holds.

Gheorghe Szölösy

32. Prove that a tetrahedron whose edges are in arithmetic progression and the 3 pairs of opposite edges are perpendicular, is regular.

Mircea Becheanu

33. Let M be an interior point in the tetraedron $ABCD$ and let A', B', C', D be the intersections of the lines AM, BM, CM, DM with the opposite planes, respectively. Prove that

$$\sum AM^2 \sum \frac{1}{AA'^2} \geqslant 9.$$

Andrei Chiteş

11$^{\text{th}}$ and 12$^{\text{th}}$ GRADES

34. Let $(a_n)_{n\geqslant 1}$ be a sequence of real numbers greater than 1, and suppose that $a_{n+1} \geqslant 2a_n - 1$ for any positive integer n.
a) Prove that the sequence $(x_n)_{n\geqslant 1}$, defined by

$$x_n = \frac{1}{a_1} + \frac{1}{a_1 a_2} + \cdots + \frac{1}{a_1 a_2 \cdots a_n}, \quad \forall n \geqslant 1,$$

has a finite limit x.
b) The sequence $(\alpha_n)_{n\geqslant 1}$ is defined by

$$x = x_n + \frac{\alpha_n}{a_1 a_2 \cdots a_n (a_{n+1} - 1)},$$

for all positive n. Prove that $(\alpha_n)_n$ is monotonic and find its limit.

Marian Tetiva

35. Find the set of all real numbers x, such that there is an increasing sequence consisting of positive integers $(a_n)_{n\geqslant 1}$ such that the sequence $(b_n)_{n\geqslant 1}$ defined by

$$b_n = \frac{\ln(2^{a_1} + 2^{a_2} + \cdots + 2^{a_n})}{n}$$

satisfies $\lim\limits_{n\to\infty} b_n = x$.

Radu Gologan

36. The sequence $(x_n)_{n\geqslant 1}$ is recurrently defined by $x_{n+1} = n + x_n^2$, where $x_1 = 1$. Prove that:
a) The sequence defined by $y_n = \frac{1}{2^n}\ln x_n$, $n \geqslant 1$, is bounded and monotone;
b) There is a positive real number λ such that $\lim\limits_{n\to\infty} \frac{x_n}{\lambda^{2^n}} = 1$.

Cristinel Mortici

37. Let $n \geqslant 1$ be an integer. Prove that for any $x \leqslant \frac{1}{(n+1)(2n+1)}$ there is a sequence $(a_k(x))_{n\geqslant 1}$ such that $a_k(x) \in \{0,1\}$, $k \in \mathbb{N}$, and

$$\lim_{k\to\infty}\left(\frac{a_1}{(n+1)^3}+\frac{a_2}{(n+2)^3}+\cdots+\frac{a_k}{(n+k)^3}\right)=x.$$

Marian Tetiva, Gabriel Dospinescu

38. Prove that for any $x \in [0,\infty)$ there are sequences of positive integers $(a_n)_{n\geqslant 0}$ and $(b_n)_{n\geqslant 0}$ such that

$$\lim_{n\to\infty}\left(\frac{1}{a_n+1}+\frac{1}{a_n+2}+\cdots+\frac{1}{a_n+b_n}\right)=x.$$

39. Prove that the sequence defined by $x_n = \log_n n!$, for any positive integer n, does not contain infinite arithmetic sets.

Adrian Iuga

40. Given the non-negative integers, when is the sequence defined by

$$x_{n+2} = x_{n+1}x_n, \quad x_0 = a, \quad x_1 = b, \quad a, b \geqslant 0$$

convergent?

Dan Schwarz

41. Let $(x_n)_n$ be a sequence of real numbers such that the sequences $y_n = x_n x_{n+1} x_{n+2}$ and $z_n = x_{n+1} - x_n$ are convergent.
Prove that $(x_n)_n$ is convergent.

Mihai Piticari

42. Let k be a positive integer. Prove that if a sequence $(a_n)_n$ satisfies

$$[a_{n+1}] = [a_n]^k + (k+1)[a_n] + 1.$$

for any positive n has a finite limit, then $k = 2$.

Călin Popescu, Sergiu Romaşcu

43 Let $f_n : [0,1] \to \mathbb{R}$, be a sequence of functions such that $|f_n(x) - f_n(y)| \leqslant |x - y|$ for any $x, y \in [0,1]$, and any positive integer n. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that for any rational number, $x \in [0,1]$, we have

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Prove that $\lim_{n\to\infty} f_n(x) = f(x)$ for any $x \in [0,1]$.

Radu Gologan

44. Let $f, g : R \to \mathbb{R}$ be such that g is increasing and $\lim_{t\to x} f(t) = g(f(x))$ for any x.
Prove that f is continuous.

Gabriel Dospinescu

45. Find all functions $f : (0,\infty) \to \mathbb{R}$ having the following properties:
a) $f(x) + 2x = f(3x)$ for any $x > 0$;
b) $\lim_{x\to\infty}(f(x) - x) = 0$.

Gabriel Dospinescu

46. Let $f : I \to I$, where $I \subset \mathbb{R}$ is an interval, be a continuous function such that for any $x \in I$ there is $n_x \in \mathbb{N}^*$ such that $f^{n_x}(x) = x$.
a) If $I = [a,\infty)$, prove that $f(x) = x$, for any $x \in I$.
b) Prove that if f is not the identity function in the case $I = (a,\infty)$, then $\lim_{x\to\infty} f(x) = a$.

Dan Schwarz

47. Let A, B be n by n matrices such there are non-zero real numbers k, p, q, r with the properties $A^2 = kB^2$ and $pAB + qBA = rI_n$. Prove that if $AB \neq BA$ then $p = q$.

Valentin Vornicu

48. For a given positive n, one considers the matrix

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} \end{pmatrix}$$

Prove that A is non-singular and the sum of the elements in A^{-1} is n^2.

Ioan Raşa, Mircea Dan Rus

49. Find all two by two matrices A, having integer elements, such that

$$\det(A^3 + I) = 1.$$

Mircea Becheanu

50. Let $A \in \mathcal{M}_2(\mathbb{Z})$ be a non-zero matrix. Prove that the following properties are equivalent:

a) $4\det A = (\mathrm{tr}A)^2$ and A is not of the form $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, with k an integer;

b) For any matrix $B \in \mathcal{M}_2(\mathbb{Z})$ commuting with A, the number $\det(A^2 + B^2)$ is the square of an integer.

Gabriel Dospinescu

51. Let $a > 0$ and $\alpha \geqslant 0$. Find all integrable functions $f : [0, a] \to [0, \infty)$, such that for any $x \in [0, a]$

$$f(x) \leqslant \int_0^x t^\alpha f(t)\,dt.$$

Călin Popescu, Valentin Vornicu

52. Let $(a_n)_{n\geqslant 0}$ be a sequence with the property $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} a_n = a \in \mathbb{R}$. Prove

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_k \cos\frac{k\pi}{n} = a.$$

Dan Ştefan Marinescu, Viorel Cornea

53. Consider the continuous functions $f_1, f_2, \ldots, f_n : [0, 1] \to \mathbb{R}$, that satisfy the following conditions:

a) $\int\limits_0^1 (f_k(x))^2\,dx = 1$ for $k = 1, 2, \ldots, n$;

b) $f_1(x) + f_2(x) + \cdots + f_n(x) = 0$, for all $x \in [0, 1]$.

Prove the inequality

$$\sum_{k=1}^{n} \Big(\int_0^1 (f(x) - f_k(x))^2\,dx \Big)^{1/2} \geqslant n.$$

Călin Popescu

54. On side BC, of the triangle ABC, one considers a variable point P, such that $\angle APB = t$ and $PB = x$. Prove that

$$\int_0^{|BC|} \cos t(x)\,dx = AB - AC.$$

55. Let $I \subset (0, \infty)$ be an interval and $g : I \to \mathbb{R}$ an integrable function. Prove that there are real numbers x, y such that

$$\frac{1}{\sqrt{y}} \int_x^{x+y} g(t)\,dt < 1.$$

Cristinel Mortici

56. Let : $[0, 1] \to \mathbb{R}$ be a function which has continuous derivative such that $|f'(x)| < 1$, for all $x \in [0, 1]$. Denote by $f_n = f \circ f \circ \cdots \circ f$ the n-th composition of f with itself.

a) Prove that the sequence $I_n = \int_0^1 f_n(x)\,dx$, has a finite limit.

b) Let l be the limit defined above and $(J_n)_n$ the sequence defined by

$$J_n = \sqrt[n]{\int_0^1 |f_n(x) - l|\,dx}.$$

Prove that $(J_n)_n$ has a finite limit.

Nelu Chichirim

1.6. REGIONAL MATHEMATICAL COMPETITIONS

Selected Problems

7th GRADE

1. Let m, n be positive integers. Show that $25^n - 7^m$ is divisible by 3 and find the least positive integer of the form $|25^n - 7^m - 3^m|$, when m, n run over the set of non-negative integers.

2004, Iași, Marius Ghergu

2. Let a, b be real numbers such that $|a| \geqslant 2$, $|b| \geqslant 2$. Show that

$$(a^2+1)(b^2+1) \geqslant (a+b)(ab+1)+5,$$

and find when equality holds.

2004, Iasi

3. Let ABC be a triangle, M be the foot of the altitude from C and N be the reflection of M across the line BC. The parallel line to CM through the point N intersects BC in P and AC in Q.

a) Show that $MQ \perp AP$ if and only if $AB = AC$.

b) Show that it is possible to obtain the points A, B, C when the points M, N, P are given.

2004, Iași

9th GRADE

1. Let a, b, c be real numbers. Show that

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = \sqrt[3]{a+b+c}$$

if and only if

$$a^3 + b^3 + c^3 = (a+b+c)^3.$$

2004, Focşani Contest, Bogdan Enescu

2. Let x, y, z be real numbers such that

$$x^2 + yz \leqslant 2,\ y^2 + xz \leqslant 2,\ z^2 + xy \leqslant 2.$$

Find the minimal and the maximal value of the sum $x + y + z$.

2004, Focşani Contest

3. We are given the set $A = \{1, 3, 6, 10, 15, 21, \ldots\}$. Show that there exist numbers $a_1, a_2, \ldots, a_{2004} \in A$ such that

$$a_1 + a_2 + a_3 + \cdots + a_{2003} = a_{2004}.$$

2004, Focşani, Bogdan Enescu

4. The circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect in distinct points A, B. An arbitrary line through A intersects again $\mathcal{C}_1$ in C and $\mathcal{C}_2$ in D and let M be an arbitrary point on the segment CD. The parallel line to BC through M intersects the segment BD in K and the parallel to BD through M intersects the segment BC in N. The perpendicular in N to BC intersects the arc BC of $\mathcal{C}_1$ which does not contain A in the point E. The perpendicular to BD in K intersects the arc BD of $\mathcal{C}_2$ which does not contain A in F.

Show that $\angle EMF = 90°$.

2004, Focşani Contest

5. Let n be a positive integer and a, b, c be real numbers such that $a^n = a+b$, $b^n = b + c$ and $c^n = c + a$.

Show that $a = b = c$.

2004, Iassy Contest

6. Let $ABCD$ be a convex quadrilateral and M, N, P, Q be points on the sides AB, BC, CD, DA respectively, such that

$$\frac{MA}{MB} = \frac{NB}{NC} = \frac{PD}{PC} = \frac{QA}{QD} = k,$$

where $k \neq 1$. Show that $S(ABCD) = 2S(MNPQ)$ if and only if $S(ABD) = S(BCD)$.

2004, Iassy, Petre Astafei

7. Let ABC be a right triangle such that, $\angle A = 90°$, $\angle B > \angle C$, and let D be an arbitrary point on the segment BC. The angle bisectors of $\angle ADB$ and $\angle ADC$ intersect the sides AB and AC in the points M and N, respectively. Show that the angle between the lines BC and MN is $\frac{1}{2}(B - C)$ if and only if D is the foot of the altitude from A.

2004, Iassy, Bogdan Enescu

8. Find all real numbers x, $x > 1$, such that $\sqrt[n]{[x^n]}$ is an integer number for all positive integers n, $n \geqslant 2$.

2004, Iassy, Mihai Piticari

10th GRADE

1. Find all arithmetic sequences n_1, n_2, n_3, n_4, n_5, for which $5|n_1$, $2|n_2$, $11|n_3$, $7|n_4$ and $17|n_5$.

2004, Focşani Contest

2. Let $ABCD$ be a convex quadrilateral and M, N, P, Q be the midpoints of the sides AB, BC, CD, DA respectively.

Show that if ANP and CMQ are equilateral triangles then $ABCD$ is a rhombus. Find the angles of $ABCD$.

2004, Focşani

3. Let $A = \{1, 2, 3, 4, 5\}$. Find the number of functions $f : A \to A$, with the following property: there is no triple of distinct elements $a, b, c \in A$ such that $f(a) = f(b) = f(c)$.

2004, Iassy, Adrian Zanoschi

4. Let $a \geqslant 2$ be a an integer. Consider the set

$$A = \{\sqrt{a}, \sqrt[3]{a}, \sqrt[4]{a}, \sqrt[5]{a}, \ldots\}.$$

a) Show that A does not contain an infinite geometric ratio.

b) Show that for any $n \geqslant 3$, A contains n numbers which are in a geometric ratio.

2004, Iassy, Bogdan Enescu

5. Let $ABCD$ be a tetrahedron such that the medians starting from vertex A in the triangles ABC, ABD, ACD are mutually perpendicular. Show that all edges that contain A are equal.

2004, Iassy, Dinu Şerbănescu

6. Let x, y, z be real numbers such that

$$\cos x + \cos y + \cos z = 0$$
$$\cos 3x + \cos 3y + \cos 3z = 0.$$

Prove that $cos2x \cos 2y \cos 2z \leqslant 0$.

2004, Iassy, Bogdan Enescu

11th GRADE

1. We are given a rectangle $ABCD$ and let P be an arbitrary point on the diagonal BD, $P \neq B, P \neq D$, and Q be an arbitrary point inside the triangle ABD. The perpendicular projections of P on the sides AB, AD are P_1, P_2 respectively, and the perpendicular projections of Q on the sides AB, AD are Q_1, Q_2, respectively.

Show that if $AQ_1 = \frac{1}{4}AB$ and $AQ_2 = \frac{1}{4}AD$, then the point Q does not lie inside the triangle AP_1P_2.

2004, Focşani, Mircea Becheanu

2. Let A, N be 2×2 real matrices such that $AN = NA$ and $N^m = 0$ for some positive integer m. Show that

a) $\det(A + N) = \det A$;

b) For $\det A \neq 0$, $A + N$ is invertible and $(A + N)^{-1} = (A - N)A^{-2}$.

2004, Focşani, Ion D. Ion

3. a) Prove that for all positive integers n, the following inequality holds

$$\left(1 + \frac{1}{1^2}\right)\left(1 + \frac{1}{2^2}\right)\cdots\left(1 + \frac{1}{n^2}\right) < e^{2-\frac{1}{n}}.$$

b) Show that the sequence of real numbers $(a_n)_{n \geqslant 1}$ defined by $a_1 = 1$ and

$$a_{n+1} = \frac{2}{n^2}\sum_{k=1}^{n} ka_k, \text{ for all } n \geqslant 1,$$

is monotonic increasing. Find with proof if it is a convergent sequence.

2004, Focşani

4. Let $a \in (0, 1)$ be a real number and $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies the conditions

(i) $\lim_{n \to \infty} f(x) = 0$;

(ii) $\lim_{n \to \infty} \dfrac{f(x) - f(ax)}{x} = 0$.

Show that $\lim_{n \to \infty} \dfrac{f(x)}{x} = 0$.

2004, Focşani, Mircea Becheanu

5. Let $ABCD$ be a parallelogram of unequal sides. The point E is the foot of the perpendicular from B to AC. The line through E which is perpendicular to BD intersects BC in F and AD in G.

Show that $EF = EG$ if and only if $ABCD$ is a rectangle.

2004 Iassy, Mircea Becheanu

6. Let A, B be 2-by-2 matrices with integer entries, such that $AB = BA$ and $\det B = 1$.

Prove that if $\det(A^3 + B^3) = 1$ then $A^2 = I$.

2004, Iassy, Mircea Becheanu

12^{th} GRADE

1. Let G be a group such that every element x, $x \neq 1$, has order p.

a) Show that p is a prime number.

b) Show that if any $p^2 - 1$ element subset of G contains p elements which commute one to another, then G is an Abelian group.

2004, Iassy, Claudiu Raicu

2.1. THE NATIONAL MATHEMATICAL OLYMPIAD

District round – Solutions

7^{th} GRADE

PROBLEM 1. Find the number of positive 6 digit integers such that the sum of their digits is 9, and four of its digits are $1, 0, 0, 4$.

Solution. The pair of missing digits must be $1, 2$ or $0, 3$.

In the first case the first digit can be $1, 2$ or 4. When 1 is the first digit, the remaining digits, $(1, 2, 0, 0, 4)$, can be arranged in 60 ways. When 4 or 2 is the first digit, the remaining ones can be arranged in 30 ways.

In the same way, when completing with the pair $(0, 3)$, the first digit can be $1, 3$ or 4. In each case, the remaining ones (three zeros and two distinct non-zero digits) can be arranged in 20 ways.

In conclusion, we have $60 + 2 \cdot 30 + 3 \cdot 20 = 180$ numbers which satisfy the given property.

PROBLEM 2. Let D be a point on side BC of a given triangle ABC. The bisector lines of the angles $\angle ADB$ and $\angle ADC$ intersect AB and AC at M and N, respectively, and the bisector lines of the angles $\angle ABD$ and $\angle ACD$ intersect DM and DN at K and L, respectively.

Prove that $AM = AN$ if and only if MN and KL are parallel.

Solution. In the triangle ABD, K is the incenter, so AK is the bisector line of $\angle BAD$. In the same way in the triangle ADC, AL is the bisector line of $\angle DAC$.

Using the bisector theorem in triangles AMD and ANC (for the lines AK and AL), we get:

$$\frac{AM}{AD} = \frac{MK}{KD} \text{ and } \frac{AN}{AD} = \frac{NL}{LD}.$$

But $AM = AN$ if and only if $\dfrac{MK}{KD} = \dfrac{NL}{LD}$, if and only if $KL||MN$ (by Thales theorem).

PROBLEM 3. One considers the set $A = \left\{n \in \mathbb{N}^* \mid 1 < \sqrt{1+\sqrt{n}} < 2\right\}$.

a) Describe the set A.
b) Find the set of numbers $n \in A$ such that

$$\sqrt{n} \cdot \left|1 - \sqrt{1+\sqrt{n}}\right| < 1\,?$$

Solution. a) We have $1+\sqrt{n} \geqslant 2$, that is $\sqrt{1+\sqrt{n}} > 1$.

The inequality $\sqrt{1+\sqrt{n}} < 2$ is equivalent to $1+\sqrt{n} < 4$, or $n < 9$. We get $A = \{1, 2, \ldots, 8\}$.

b) We have $\left|1 - \sqrt{1+\sqrt{n}}\right| = \sqrt{1+\sqrt{n}} - 1$. The inequality $\sqrt{1+\sqrt{n}} - 1 < \dfrac{1}{\sqrt{n}}$ is equivalent to $\sqrt{1+\sqrt{n}} < \dfrac{1+\sqrt{n}}{\sqrt{n}}$, that is $1 < \dfrac{\sqrt{1+\sqrt{n}}}{\sqrt{n}}$, or $1 < \dfrac{1}{n} + \dfrac{1}{\sqrt{n}}$. For $n \geqslant 3$ we have

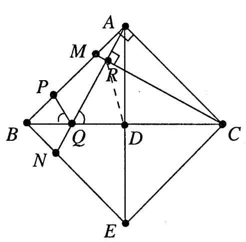
$$\frac{1}{n} + \frac{1}{\sqrt{n}} \leqslant \frac{1}{3} + \frac{1}{\sqrt{3}} < 1.$$

It follows that the given inequality is true if and only if $n \in \{1, 2\}$.

PROBLEM 4. In the triangle ABC we have $AB = AC$. Consider the points M, P on AB such that $AM = BP$. Let D be the midpoint of BC and let R on CM and Q on BC, be such that A, R, Q are collinear and the line AQ is perpendicular on CM.

Prove that:
a) $\angle AQC \equiv \angle PQB$;
b) $\angle DRQ = 45°$.

Solution. We suppose without loss of generality that the order of points is A, M, P, B.



Let E be such that $ABEC$ is a square having center D and N is the intersection point of AQ and BE.

a) As the triangles AMC and NBA are equal, we get $AM = BN$. As $AM = PB$, we obtain $PB = BN$. Because $\angle ABC = \angle CBE$ and the line QB is a common edge, we get $\Delta PQB = \Delta NQB$. It follows $\angle PQB = \angle NQB$ and, as $\angle CQA = \angle NQB$, we obtain $\angle AQC = \angle PQB$.

b) Because of the similarity of the triangles ADQ and RQC, we have $\dfrac{DQ}{RQ} = \dfrac{AQ}{CQ}$. It follows that the triangles DRQ and ACQ are similar, too. This gives that the corresponding angles are equal, thus $\angle DRQ = 45°$.

8th GRADE

PROBLEM 1. We say that the real numbers a şi b have property $\mathcal{P}$ if $a^2+b \in \mathbb{Q}$ and $b^2 + a \in \mathbb{Q}$. Prove that:

a) The numbers $a = \dfrac{1+\sqrt{2}}{2}$ şi $b = \dfrac{1-\sqrt{2}}{2}$ are irrational and satisfy $\mathcal{P}$;
b) If a, b have property $\mathcal{P}$ and $a+b \in \mathbb{Q}\setminus\{1\}$, then a şi b are rational numbers;
c) If a, b have property $\mathcal{P}$ and $\dfrac{a}{b} \in \mathbb{Q}$, then a şi b are rational numbers.

Solution. a) We have $a^2 = \dfrac{3+2\sqrt{2}}{4}$ şi $b^2 = \dfrac{3-2\sqrt{2}}{4}$, that is $a^2 + b = a + b^2 = \dfrac{5}{4} \in \mathbb{Q}$.

b) From $a^2+b-(b^2+a) \in \mathbb{Q}$ we get $(a-b)(a+b-1) \in \mathbb{Q}$. Because $a+b-1$ is a non-zero rational number, we have $a-b \in \mathbb{Q}$. As $a+b \in \mathbb{Q}$, we get $2a, 2b \in \mathbb{Q}$, that is $a, b \in \mathbb{Q}$.

c) Let $k \in \mathbb{Q} \setminus \{0,1\}$ such that $a = bk$. We infer $b(1+k^2b) \in \mathbb{Q}$ and $b(b+k) \in \mathbb{Q}$, implying $\dfrac{1+k^2b}{b+k} = r \in \mathbb{Q}$. If $r = k^2$, then $k^3 = 1$. This implies $k = 1$ and $a = b$, a contradiction. Thus $r \neq k^2$, from where $b = \dfrac{1-rk}{r-k^2} \in \mathbb{Q}$. As a result $a = \dfrac{a}{b} \cdot b \in \mathbb{Q}$.

PROBLEM 2. The real numbers a, b, c, d satisfy $a > b > c > d$ and

$$a+b+c+d = 2004 \text{ şi } a^2 - b^2 + c^2 - d^2 = 2004.$$

Answer, with proof, to the following questions:
a) What is the smallest possible value of a?
b) What is the number of possible values of a?

Solution. a) One can write $a^2 - b^2 + c^2 - d^2 = (a-b)(a+b) + (c-d)(c+d) \geqslant a+b+c+d = 2004$. If $a - b > 1$ or $c - d > 1$, we easily deduce $a^2 - b^2 + c^2 - d^2 > 2004$. Thus $a - b = 1, c - d = 1$, that is $b = a - 1, d = c - 1$, case in which $a+b+c+d = 2a + 2c - 2 = 2004$. This implies $a + c = 1003$. As $a > c$, we get $a \geqslant 502$. For $a = 503$, we obtain $b = 501, c = 501, d = 500$, thus $b = c$, which does not satisfy the given condition.

For $a = 503$ one obtains $b = 502, c = 500, d = 499$.

b) The maximal value of a is obtained for the smallest value of d. When $d = 1$ we get $c = 2$, $a = 1001$. Thus $a \in \{503, 504, \dots, 1001\}$, taking $1001 - 503 + 1 = 499$ possible values. To check that any a from the set $\{503, 504, \dots, 1001\}$ gives a solution, take $b = a - 1$, $c = 1003 - a, d = 102 - a$. It is easy to see that $a > b > c > d$, and the equalities from the hypothesis are fulfilled.

PROBLEM 3. Say that a set of three different numbers is an *arithmetical set* if one of the three numbers is the average of the other two. Consider the set $A_n = \{1, 2, \dots, n\}$, where n is a positive integer, $n \geqslant 3$.

a) How many arithmetical sets are in A_{10}?

b) Find the smallest n, such that the number of arithmetical sets in A_n is greater than 2004.

Solution. a) It easy to see, by inspection, that there are 20 arithmetical subsets of A_{10}

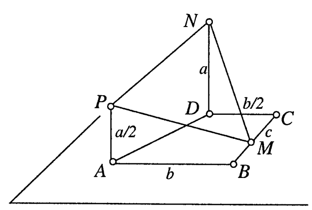
b) Consider an arithmetical set $B = \{a, a + r, a + 2r\}$. For each $r \leqslant 45$ and $1 \leqslant a \leqslant 91 - 2r$, we have $B \subset A_{91}$. As a consequence, for any $r \in \{1, 2, \dots, 45\}$ there are at least $91 - 2r$ such subsets of A_{91}. The total number of these subsets is $1 + 3 + 5 + \dots + 89 = 2025$. In the same way we can see that the number of arithmetical subsets in A_{90} is smaller than 2000.

PROBLEM 4. In a given trapezoid $ABCD$, let $AB \parallel CD$, $\angle B = 90°$ and $AB = 2DC$. Consider points N and P on the same part of the plane (ABC), such that PA and ND are perpendicular on the trapezoid plane, and $ND = a$, $AP = \dfrac{a}{2}$ $(a > 0)$. If M is the midpoint of BC and the triangle MNP is equilateral, find:

a) The cosines of the angle between the planes (MNP) and (ABC);

b) The distance from D to the plane (MNP).

Solution.



a) The orthogonal projection of the triangle PMN onto the plane (ABC) is the triangle AMD. By the projection theorem

$$\cos\alpha = \frac{S_{AMD}}{S_{PMN}} = \frac{\sqrt{3}}{3},$$

where α is the angle between planes.

b) We shall use the notations $AB = b$ and $BC = c$. Because the triangle MNP is equilateral we have

$$b^2 + \frac{c^2}{4} + \frac{a^2}{4} = \frac{b^2}{4} + \frac{c^2}{4} + a^2 = c^2 + \frac{b^2}{4} + \frac{a^2}{4},$$

and consequently $a = b = c$.

If O is the midpoint of the segment MN, then $PO \perp (DMN)$. It follows that the planes (PNM) and (DMN) are perpendicular. If $DF \perp NM$, $F \in NM$, then $DF \perp (MNP)$. Thus

$$DF = \frac{DN \cdot DM}{NM} = \frac{a\sqrt{3}}{3}.$$

9th GRADE

PROBLEM 1. Real numbers a, b, c satisfy $a^2 + b^2 + c^2 = 3$. Prove the inequality:

$$|a| + |b| + |c| - abc \leqslant 4.$$

Solution. The Cauchy-Schwarz inequality yields

$$(|a| + |b| + |c|)^2 \leqslant 3\left(a^2 + b^2 + c^2\right) = 9,$$

hence

$$|a| + |b| + |c| \leqslant 3.$$

From the AM-GM inequality it follows that

$$a^2 + b^2 + c^2 \geqslant 3\sqrt[3]{(abc)^2},$$

or $|abc| \leqslant 1$, which implies $-abc \leqslant 1$. The requested inequality is then obtained by summation.

PROBLEM 2. Find the cartesian coordinates of the vertices A, B, C of a triangle ABC, whose the orthocenter is $H(-3, 10)$, the circumcenter is $O(-2, -3)$, and the midpoint of BC is $D(1, 3)$.

Solution. We have $\overrightarrow{AH} = 2\overrightarrow{OD}$, implying that the coordinates of the point A are $(-9, -2)$ and the radius of the circumcircle equals $\sqrt{50}$. The line BC is perpendicular on OD, and its equation is $x + 2y = 7$. Since $BO = CO = \sqrt{50}$, we deduce that the coordinates of the points B, C are $(3, 2)$ and $(-1, 4)$.

PROBLEM 3. a) Prove that there are infinitely many rational positive numbers x such that:

$$\{x^2\} + \{x\} = 0,99.$$

b) Prove that there are no rational numbers $x > 0$ such that:

$$\{x^2\} + \{x\} = 1.$$

Solution. a) Since $0.99 = \frac{99}{100}$, it is natural to look for a rational x of the form $\frac{n}{10}$, for some positive integer n. It is not difficult to see that $x = \frac{13}{10}$ satisfies the given equality and then that $x = 10k + \frac{13}{10}$ also satisfies the equality for any positive integer k.

b) Suppose, by way of contradiction, that such a rational number x exists. Let $n = [x]$; then $n < x < n+1$, hence $n^2 < x^2 < n^2 + 2n + 1$. Let $[x^2] = n^2 + k$, where $0 \leqslant k \leqslant 2n$. Substituting $\{x\} = x - [x] = x - n$ and $\{x^2\} = x^2 - [x^2] = x^2 - n^2 - k$ in the given equality, yields

$$x^2 + x - n^2 - n - k - 1 = 0.$$

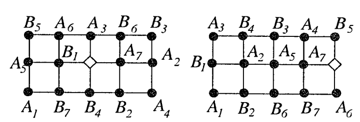
Consider the above equality as a quadratic equation for the unknown x. Since x must be rational, its discriminant $\Delta = 4n^2 + 4n + 4k + 5$ has to be an odd square, say $(2m+1)^2$. We deduce that $n^2 + n + k + 1 = m^2 + m$. Clearly, $m \geqslant n+1$, and hence $m^2 + m \geqslant n^2 + 3n + 2$. Finally, we obtain $k \geqslant 2n + 1$, a contradiction.

PROBLEM 4. A rectangle 2×4 is divided in 8 squares of side 1. Call $\mathcal{M}$ the set of the 15 vertices thus obtained.

Find the points $A \in \mathcal{M}$ satisfying the following condition: *the set $\mathcal{M} \setminus \{A\}$ can be arranged in pairs $(A_1, B_1), (A_2, B_2), \dots, (A_7, B_7)$ such that*

$$\overrightarrow{A_1B_1} + \overrightarrow{A_2B_2} + \dots + \overrightarrow{A_7B_7} = \overrightarrow{0}.$$

Solution. We will prove that the point A is either the center of the rectangle or the midpoint of one of its smaller sides. In these cases, the requested pairs can be chosen as in the figure.



For other positions of the point A, consider $\mathcal{M} = \{0,1,2,3,4\} \times \{0,1,2\}$. Suppose the partition into pairs possible, and denote $A_p(x_p, y_p)$, $B_p(z_p, t_p)$, for $p = 1, 2, \dots, 7$. Then

$$\sum_{p=1}^{7} \overrightarrow{A_pB_p} = \sum_{p=1}^{7} (z_p - x_p)\vec{i} + \sum_{p=1}^{7} (t_p - y_p)\vec{j}.$$

For the sum to equal $\vec{0}$, the numbers $\sum_{p=1}^{7}(x_p - z_p) = \sum_{p=1}^{7}(x_p + z_p) - 2\sum_{p=1}^{7} z_p$ and $\sum_{p=1}^{7}(y_p - t_p) = \sum_{p=1}^{7}(y_p + t_p) - 2\sum_{p=1}^{7} y_p$ have to be even. This happens only if the set $\mathcal{M} \setminus \{A\}$ contains an even number of points having an odd abscissa and an even number of points having an odd ordinate, which is not possible.

10th GRADE

PROBLEM 1. Given a positive integer n, $n \geqslant 3$, find the number of arithmetical progressions with 3 elements contained in the set $\{1, 2, \dots, n\}$.

Solution. Let us count the subsets according to the value of the common difference of the progressions. There are $n-2$ progressions with the common difference equal to 1: $\{1,2,3\}, \{2,3,4\}, \dots, \{n-2, n-1, n\}$. There are $n-4$ progressions with the common difference equal to 2: $\{1,3,5\}, \{2,4,6\}, \dots, \{n-4, n-2, n\}$. If n is even, say $n = 2k$, the maximal value of the common difference is $k-1$; there are two such progressions: $\{1, k, 2k-1\}$ and $\{2, k+1, 2k\}$. Thus, in this case, the total number of subsets is $2 + 4 + \dots + 2k - 2 = k(k-1)$.

If n is odd, say $n = 2k+1$, the maximal value of the common difference is k and there is only one such progression: $\{1, k+1, 2k+1\}$. The total number of subsets is $1 + 3 + \ldots + (2k-1) = k^2$.

In both cases, the number of subsets equals $\left[\frac{n}{2}\right] \cdot \left[\frac{n-1}{2}\right]$.

PROBLEM 2. Find integers n, $n \geqslant 3$, having the property: there are distinct integers $a_1, a_2, \dots, a_n$, such that:

$$a_1!a_2! \cdots a_{n-1}! = a_n!$$

Solution. We prove inductively that for all $n \geqslant 3$ such numbers exist. Indeed, for $n = 3$ we have $3! \cdot 5! = 6!$. Assume that for some n there exist $a_1 < a_2 < \dots < a_n$ such that

$$a_1! \cdot a_2! \cdot \ldots \cdot a_{n-1}! = a_n!.$$

Let $b = a_n! - 1$. Then $a_n < b$ and

$$a_1! \cdot a_2! \cdots a_{n-1}! \cdot b! = a_n! \cdot (a_n! - 1)! = (a_n!)!$$

Thus, denoting $b = a_n$, $a_n! = a_{n+1}$, we have

$$a_1! \cdot a_2! \cdots a_{n-1}! \cdot a_n! = a_{n+1}!,$$

as desired.

PROBLEM 3. Let M, N, P, and Q, be the midpoints of the edges AB, CD, AC, and BD, respectively, of the thetrahedron $ABCD$. It is known that MN is perpendicular to both AB and CD and PQ is perpendicular to both AC and BD. Prove that $AB = CD, BC = DA$, and $AC = BD$.

Solution. Denote $\overrightarrow{AB} = x$, $\overrightarrow{AC} = y$, and $\overrightarrow{AD} = z$. Then $\overrightarrow{MN} = \dfrac{-x+y+z}{2}$ and since $\overrightarrow{MN} \cdot \overrightarrow{AB} = \overrightarrow{MN} \cdot \overrightarrow{CD} = 0$, we obtain

$$\begin{aligned} xz + xy - x^2 &= 0 \\ y^2 - z^2 - xy + xz &= 0. \end{aligned}$$

Substracting these equalities yields $z^2 = (x-y)^2$, hence $AD = BC$. Adding them up yields $y^2 = (x-y)^2$, thus $AC = BD$.

In the same way, the second condition of perpendicularity leads to $AB = CD$.

Alternative solution. It easy to see that MN is a median and an altitude in the triangle ABN; thus the triangle is isosceles. This implies $BN = AN$. In the same way one can prove that $AQ = CQ$, $MC = MD$ and $BQ = QD$. Using the formula for the median line, we easily get the equalities of pairs of edges.

PROBLEM 4. Let $x, y \in (0, \frac{\pi}{2})$. Prove that if the equality

$$(\cos x + \mathrm{i} \sin y)^n = \cos nx + \mathrm{i} \sin ny$$

is true for two consecutive integers, then it is true for all integers n.

Solution. Suppose

$$\begin{aligned} (\cos x + \mathrm{i} \sin y)^n &= \cos nx + \mathrm{i} \sin ny \\ (\cos x + \mathrm{i} \sin y)^{n+1} &= \cos (n+1) x + \mathrm{i} \sin (n+1) y. \end{aligned}$$

Multiplying the first relation by $\cos x + \mathrm{i} \sin x$ and substracting the second, we get

$$\sin x \sin nx = \sin y \sin ny.$$

Suppose $x < y$. Then $\sin x < \sin y$, and hence $|\sin nx| \geqslant |\sin ny|$.

Considering the absolute values in the equality $(\cos x + \mathrm{i} \sin y)^n = \cos nx + \mathrm{i} \sin ny$, we obtain

$$\left(\cos^2 x + \sin^2 y\right)^n = \cos^2 nx + \sin^2 ny.$$

Thus, we have

$$\begin{aligned} 1 = (\cos^2 x + \sin^2 x)^n &< \left(\cos^2 x + \sin^2 y\right)^n \\ &= \left(\cos^2 nx + \sin^2 ny\right)^n \leqslant \cos^2 nx + \sin^2 nx = 1, \end{aligned}$$

a contradiction. A similar contradiction is reached by assuming $x > y$. It follows that $x = y$, which implies the conclusion.

11th GRADE

PROBLEM 1. Let $x_0 > 0$ and for any positive integer n, consider $x_{n+1} = x_n + \dfrac{1}{\sqrt{x_n}}$.

Find: a) $\lim\limits_{n\to\infty} x_n$; b) $\lim\limits_{n\to\infty} \dfrac{x_n^3}{n^2}$.

Solution. a) It is not difficult to see that $(x_n)_{n\geqslant 0}$ is strictly increasing, hence $\lim x_n = L$ exists. If $L \in \mathbb{R}$, then the recursive relation gives $L = L + \dfrac{1}{\sqrt{L}}$, which is impossible. It follows that $L = \infty$.

b) First, observe that

$$\frac{x_{n+1}}{x_n} = 1 + \frac{1}{x_n\sqrt{x_n}},$$

from which we deduce that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1.$$

We calculate $\lim\limits_{n\to\infty} \dfrac{\sqrt{x_n^3}}{n}$, using the Stolz lemma:

$$\begin{aligned} \lim_{n\to\infty} \frac{\sqrt{x_n^3}}{n} &= \lim_{n\to\infty} \left(\sqrt{x_{n+1}^3} - \sqrt{x_n^3}\right) \\ &= \lim_{y\to\infty} \left(\sqrt{\left(y + y^{\frac{-1}{2}}\right)^3} - \sqrt{y^3}\right) = \frac{3}{2}, \end{aligned}$$

that is $\lim\limits_{n\to\infty} \dfrac{x_n^3}{n^2} = \dfrac{9}{4}$.

PROBLEM 2. Consider complex non-zero numbers $z_1, z_2, \ldots, z_{2n}$, $n \geqslant 3$, such that $|z_1| = |z_2| = \cdots = |z_{n+3}|$ and $\arg z_1 \geqslant \cdots \geqslant \arg(z_{n+3})$. Define for $i, j \in \{1, 2, \ldots, n\}$: $b_{ij} = |z_i - z_{j+n}|$ and let $B = (b_{ij}) \in \mathcal{M}_n$. Prove that $\det B = 0$.

Solution. Consider $z = r(\cos 2x + \mathrm{i} \sin 2x)$ and $w = r(\cos 2y + \mathrm{i} \sin 2y)$, then $|z - w| = 2r\,|\sin(x-y)|$. Using the hypothesis we can reduce $\det B$ to the form

$$\begin{vmatrix} \sin(x_1 - x_{n+1}) & \sin(x_1 - x_{n+2}) & \sin(x_1 - x_{n+3}) & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \sin(x_n - x_{n+1}) & \sin(x_n - x_{n+2}) & \sin(x_n - x_{n+3}) & \ldots \end{vmatrix}.$$

Writing it as a sum of determinants, we reach the desired conclusion.

PROBLEM 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with the property $\mathcal{P}$: *for every* $a, b \in \mathbb{R}$ *we have* $f\left(\frac{a+b}{2}\right) \in \{f(a), f(b)\}$.

a) Give an example of a non-constant function possesing $\mathcal{P}$.
b) If f satisfies $\mathcal{P}$ and is continuous, prove that it is constant.

Solution. a) An example is the function

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geqslant 1. \end{cases}$$

b) Observe that if $c < d$ and $f(c) = f(d)$, then f is constant on the interval $[c, d]$. Indeed, let m_1 be the midpoint of $[c, d]$, let m_2 and m_3 be the midpoints of $[c, m_1]$ and $[m_1, d]$ and so forth. The set of these points is dense in $[c, d]$, and since f is continuous, the conclusion follows.

Now, suppose there exist $a < b$ such that $f(a) \neq f(b)$, and let $a_1 = \sup\{x \in [a, b] \mid f(a) = f(x)\}$ and $b_1 = \inf\{x \in [a, b] \mid f(b) = f(x)\}$. Then $f(a) = f(a_1)$, $f(b) = f(b_1)$, hence f is constant on both intervals $[a, a_1]$ and $[b_1, b]$. But $f\left(\frac{a_1+b_1}{2}\right) \notin \{f(a_1), f(b_1)\}$, a contradiction.

PROBLEM 4. The matrix $A = (a_{ij}) \in \mathcal{M}_p(\mathbb{C})$ is defined by $a_{12} = a_{23} = \cdots = a_{p-1,p} = 1$ and $a_{ij} = 0$ for the remaining set of indices (i, j).

Prove that there are no non-zero matrices $B, C \in \mathcal{M}_n(\mathbb{C})$ such that $(I_p + A)^n = B^n + C^n$ for all non-negative integer n.

Solution. From $(I_p + A)^2 = B^2 + C^2$ and $(I_p + A)^3 = B^3 + C^3$ we obtain $BC = CB = 0_p$. The equality $I_p + A = B + C$ implies $B + AB = B^2 + BC = B^2$ and also $B + BA = B^2 + CB = B^2$. Hence $AB = BA$ and, similarily, $AC = CA$. It is easy to see that it follows that there exist b, c such that

$$B = \begin{pmatrix} b & & * \\ 0 & b & \\ 0 & & b \end{pmatrix}, \quad C = \begin{pmatrix} c & & * \\ 0 & c & \\ 0 & & c \end{pmatrix}$$

with $b + c = 1$. The equality $BC = 0_p$ implies $bc = 0$, so either b or c equals 0. If, for instance, $b = 0$, then it follows that $c = 1$, hence C is invertible. In this case, the equality $BC = 0_p$ implies $B = 0_p$, as desired.

12th GRADE

PROBLEM 1. Let $n \geqslant 2$ be an integer and $r \in \{1, 2, \ldots, n\}$. Consider the set

$$S_r = \{A \in \mathcal{M}_n(\mathbb{Z}_2) \mid \operatorname{rank} A = r\}.$$

a) Prove that for any $A \in S_n$ and $B \in S_r$, AB is in S_r;
b) Calculate $\sum_{X \in S_r} X$.

Solution. a) $\operatorname{rank} A = n$ implies A invertible. As $\operatorname{rank} AB \leqslant \operatorname{rank}(B) = r$ and $\operatorname{rank} AB \geqslant \operatorname{rank}(A^{-1}AB) = \operatorname{rank} B = r$, we get $\operatorname{rank}(AB) = r$.

b) Consider $A \in S_n$ and the function $f : S_r \to S_r$, $f(X) = AX$. It easy to see that A is one-to-one (and thus onto) if and only if f is one-to-one and onto (the set S_r isfinite).

We thus get $S = \sum_{X \in S_r} X = \sum_{X \in S_r} AX = AS$, which can be written $(A - I)S = 0$.

In order to prove $S = 0$, it suffices to show that there is an $A_n \in S_n$, such that $A_n - I$ is invertible. Take for example, for $n = 2$ and $n = 3$: $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. In general consider matrices of the form

$$A_{2n} = \begin{pmatrix} A_2 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_2 \end{pmatrix}, \quad A_{2n+1} = \begin{pmatrix} A_3 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_2 \end{pmatrix}.$$

PROBLEM 2. Prove that constant functions are the only continuous functions $f : [0, 1] \to \mathbb{R}$, such that

$$\int_0^1 f(x)g(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x \cdot \int_0^1 g(x)\,\mathrm{d}x,$$

for any non-derivable continuous function $g : [0, 1] \to \mathbb{R}$.

Solution. If f verifies the given property, then, for every real c, the relation is also verified by $f - c$. Thus, the function

$$h(x) = f(x) - \int_0^1 f(t)\mathrm{d}t$$

verifies the equality

$$(1) \qquad \int_0^1 h(x)g(x)\mathrm{d}x = 0, \quad \text{for all } g \text{ continuous and non-derivable.}$$

If, by contradiction, h is not the zero function, take $x_0 \in (0, 1)$, such that $h(x_0) \neq 0$. Consider an interval of the form $V = [x_0 - \varepsilon, x_0 + \varepsilon] \subset (0, 1)$ where $\varepsilon > 0$, on which h has constant sign. Such an interval exists by the intermediate value property. The function defined on $[0, 1]$ by $g(x) = (x - x_0)^2 - \varepsilon^2$ for $x \in V$ and $g(x) = 0$ otherwise, is not derivable. The integral in (1) is not null, a contradiction.

PROBLEM 3. A ring A satisfyies the following properties:
i) Its unit 1_A has order p, a prime number;

ii) There is a set $B \subset A$ with p elements such that: *for any* $x, y \in A$, *there is* $b \in B$ *satisfying* $xy = byx$.

Prove that A is commutative.

Solution. As k and p are mutually prime, by Euclid theorem there are $a, b \in \mathbb{Z}$ such that $ka + pb = 1$, that is $(k1_A)(a1_A) = 1_A$. For $x = y = 1_A$ we get $1_A = b \in B$.

Suppose A is non-commutative, that is, there are $x, y \in A$, $b \in B \setminus \{1_A\}$ such that $xy = byx$. Thus $x + k1_A$ and y do not commute for $k = 0, 1, \ldots, p-1$. As $B \setminus \{1_A\}$ consists of $p-1$ elements, there are $s < r$ in $\{1, 2, \ldots, p-1\}$ and $a \in B \setminus \{1_A\}$, such that $(x + r1_A)y = ay(x + r1_A)$ and $(x + s1_A)y = ay(x + s1_A)$. Denoting by $z = x + s1_A$ and $t = r - s \in \{1, 2, \ldots, p-1\}$ we get $zy = ayz$ and $(z + t1_A)y = ay(z + t1_A)$. This gives $y = ay$, or $zy = ayz = yz$. In conclusion $(x + s1_A)y = y(x + s1_A)$ implying $xy = yx$, a contradiction.

PROBLEM 4. Let $a, b \in (0,1)$ and let $f : [0,1] \to \mathbb{R}$ be a continuous function, such that

$$\int_0^x f(t)\mathrm{d}t = \int_0^{ax} f(t)\mathrm{d}t + \int_0^{bx} f(t)\mathrm{d}t, \quad \text{for any } x \in [0,1].$$

a) Prove that $a + b < 1$ implies $f = 0$.

b) Prove that $a + b = 1$ implies that f is constant.

Solution. a) As f is continuous, the functions in x defined by the given equality, have finite derivatives. Taking derivatives on both sides, we get

$$f(x) = af(ax) + bf(bx),$$

for all $x \in [0,1]$. Consider $M = \sup_{t \in [0,1]} |f(t)|$. By the mean theorem, we have $|f(x)| \leqslant (a+b)M$, that is $M \leqslant (a+b)M$. Thus $M = 0$ implying $f = 0$.

b) Let x_0 be a maximum point of f. Then

$$f(x_0) = af(ax_0) + bf(bx_0) \leqslant (a+b)f(x_0).$$

It follows that ax_0 is a maximum point too. By iteration, $a^n x_0$ is a maximum point, for any n. As f is continuous, and $\lim\limits_{n\to\infty} a^n x_0 = 0$, we infer that 0 is a maximum point. In thesame way, if x_1 is a minimum point for f, we get by induction that $a^n x_1$ is a minimum point, for any n. Considering limits, we conclude that 0 is a minimum point. As 0 is in the same time maximum and minimum point for f, we get $f(x) = f(0)$, for all $x \in [0,1]$, thus f is constant.

2.2. THE NATIONAL MATHEMATICAL OLYMPIAD

Final Round – Solutions

7th GRADE

PROBLEM 1. On the sides AB and AD of the rhombus $ABCD$, consider the points E and F respectively, such that $AE = DF$. Lines BC and DE intersect at P, and lines CD and BF intersect at Q. Prove that:

a) $\dfrac{PE}{PD} + \dfrac{QF}{QB} = 1$;

b) Points P, A, Q are collinear.

Solution. a) From $BE \parallel CD$ we have $\dfrac{PE}{PD} = \dfrac{BE}{CD}$, and from $DF \parallel BC$ we get $\dfrac{QF}{QB} = \dfrac{FD}{BC}$. These imply

$$\frac{PE}{PD} + \frac{QF}{QB} = \frac{BE}{CD} + \frac{FD}{BC} = 1.$$

b) From $FD \parallel BC$, we deduce $\dfrac{FQ}{QB} = \dfrac{FD}{BC} = \dfrac{EA}{AB}$. Thales theorem gives $EF \parallel AQ$.

From $BE \parallel CD$, we get $\dfrac{PE}{PD} = \dfrac{BE}{CD} = \dfrac{AF}{AD}$, and by Thales theorem we deduce $EF \parallel AP$. It follows that the points P, A, Q are collinear.

PROBLEM 2. The side-lengths of a triangle are a, b, c.

a) Prove that there is a triangle with sides $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$.

b) Prove that $\sqrt{ab} + \sqrt{bc} + \sqrt{ac} \leqslant a + b + c < 2\sqrt{ab} + 2\sqrt{bc} + 2\sqrt{ac}$.

Solution. a) From $a + b > c$ one obtains $a + b + 2\sqrt{ab} > c$, or $(\sqrt{a} + \sqrt{b})^2 > c$, that is $\sqrt{a} + \sqrt{b} > \sqrt{c}$. By symmetry, this proves that $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are the sides of a triangle.

b) The inequality $\sqrt{ab} + \sqrt{bc} + \sqrt{ac} \leqslant a + b + c$ is equivalent to $2\sqrt{ab} + 2\sqrt{bc} + 2\sqrt{ac} \leqslant 2a + 2b + 2c$, or $(a - 2\sqrt{ab} + b) + (a - 2\sqrt{ac} + c) + (b - 2\sqrt{bc} + c) \geqslant 0$, that is $(\sqrt{a} - \sqrt{b})^2 + (\sqrt{a} - \sqrt{c})^2 + (\sqrt{b} - \sqrt{c})^2 \geqslant 0$, obviously true.

Because $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are the sides of a triangle, we have:

$$\sqrt{a} < \sqrt{b} + \sqrt{c}, \quad \sqrt{b} < \sqrt{a} + \sqrt{c}, \quad \sqrt{c} < \sqrt{a} + \sqrt{b}.$$

Multiply the above relations by $\sqrt{a}, \sqrt{b}, \sqrt{c}$, respectively. Summing up we get $a + b + c < 2\sqrt{ab} + 2\sqrt{bc} + 2\sqrt{ac}$.

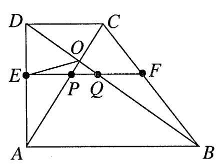
PROBLEM 3. The diagonals of the trapezoid $ABCD$ are perpendicular and intersect in O. Angle A equals $90°$, $AB \parallel CD$ and, $AB > CD$. The diagonals intersect at O. OE is the bisector line of the angle AOD, E is on the segment AD, and F is on BC, such that $EF \parallel AB$. Denote by P, Q, the intersection points of the segment EF with the diagonals AC and BD, respectively. Prove that:
a) $EP = QF$;
b) $EF = AD$.

Solution. a) The triangles EPA and DCA are similar. We get

$$\frac{EP}{DC} = \frac{AE}{AD}. \tag{1}$$

From the similarity of the triangles FQB and CDB we also obtain

$$\frac{QF}{DC} = \frac{BF}{BC}. \tag{2}$$



By Thales theorem $\frac{AE}{AD} = \frac{BF}{BC}$ and from (1) and (2) we obtain $\frac{EP}{DC} = \frac{QF}{DC}$, that is $EP = QF$.

b) Because $\angle AOQ = \angle AEQ = 90°$, the quadrilateral $AEOQ$ is cyclic, thus $\angle EQA = \angle EOA) = 45°$. It follows that the triangle EAQ is isosceles and $AE = EQ$.

The quadrilateral $DEPO$ is cyclic because $m(\widehat{DEP}) = m(\widehat{DOP}) = 90°$. This implies $\angle DPE = \angle DOE = 45°$, that is DEP is isosceles and $DE = EP$.

But $EP = QF$. We get $EF = EP + PF = ED + EA = AD$.

PROBLEM 4. Sixteen points are placed in the centers of a 4×4 chess table in the following way:

•	•	•	•
•	•	•	•
•	•	•	•
•	•	•	•

a) Prove that one may choose 6 points such that no isosceles triangle can be drawn with vertices at these points.
b) Prove that one cannot choose 7 points with the above property.

Solution. a) Any of the following configurations has the asked property:

•	○	○	○		•	•	○	○
○	•	•	•		•	•	•	•
○	•	•	•		○	•	•	○
○	•	•	•		○	•	○	•

b) Suppose that we select 7 points such that there are no isosceles triangles having the vertices at these points. There are exactly two possibilities:

1) All chosen points are on the border of the big square. Partition the set of 12 points in three sets that are the vertices of three squares as seen in the figure:

○	◇	+	○
+	•	•	◇
◇	•	•	+
○	+	◇	○

Observe that we cannot choose three points with the same symbol that are not the vertices of an isosceles triangle.

2) There is at least an interior chosen point. With the other 15 points form five sets as seen in the figure:

*	*	*	○
*	•	◇	⊗
*	◇	◇	○
○	⊗	○	⊗

From the points marked with $\circ, \otimes, \diamond$, we can choose only one point and from those marked with $*$, we can choose two points. In total we can choose at most 6 points.

8th GRADE

PROBLEM 1. Find all non-negative integers n such that there are integers a and b with the property:

$$n^2 = a + b \quad \text{and} \quad n^3 = a^2 + b^2.$$

Solution. From the inequality $2(a^2 + b^2) \geqslant (a + b)^2$ we get $2n^3 \geqslant n^4$, that is $n \leqslant 2$. Thus:
– for $n = 0$, we choose $a = b = 0$,
– for $n = 1$, we take $a = 1$, $b = 0$, and,
– for $n = 2$, we may take $a = b = 2$.

PROBLEM 2. Prove that the equation

$$x^2 + y^2 + z^2 + t^2 = 2^{2004},$$

where $0 \leqslant x \leqslant y \leqslant z \leqslant t$, has exactly two solutions in the set of integers.

Solution. The solutions are $(0,0,0,2^{1002})$ and $(2^{1001},2^{1001},2^{1001},2^{1001})$.

In order to prove the statement, let (x,y,z,t) be a solution. Observe that for odd a we have $a = 4n \pm 1$, and a^2 gives the remainder 1 when divided by 8. Thus the equation has no solution with an odd component.

We thus must have $x = 2x_1$, $y = 2y_1$, $z = 2z_1$, $t = 2t_1$, where $0 \leqslant x_1 \leqslant y_1 \leqslant z_1 \leqslant t_1$ are integers and $x_1^2 + y_1^2 + z_1^2 + t_1^2 = 2^{2002}$. By the same argument $x_1 = 2x_2$, $y_1 = 2y_2$, $z_1 = 2z_2$, $t_1 = 2t_2$, where $0 \leqslant x_2 \leqslant y_2 \leqslant z_2 \leqslant t_2$ are integers and $x_2^2 + y_2^2 + z_2^2 + t_2^2 = 2^{2002}$.

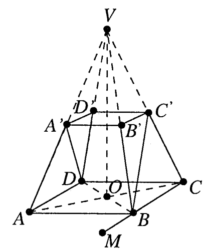
Recursively, $x = 2^{2001}a$, $y = 2^{2001}b$, $z = 2^{2001}c$, $t = 2^{2001}d$, where $0 \leqslant a \leqslant b \leqslant c \leqslant d$ are integers and $a^2 + b^2 + c^2 + d^2 = 4$. This relation simply implies the conclusion.

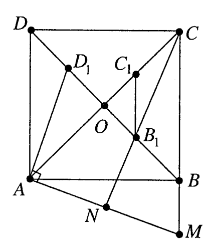
PROBLEM 3. Consider a frustum of a regular quadrilateral pyramid $ABCDA'B'C'D'$ in which the lines BC' are DA' orthogonal.

a) Prove that the angle between the lines AB' and DA' equals $60°$.

b) If the projection of B' onto the plane (ABC) is the incenter of the triangle ABC, prove that the distance between the lines CB' and AD' equals $\frac{1}{2}BC'$.

Solution. a) The diagonals BC' and DA' are contained in perpendicular planes to (ABC). They are parallel and contain the points B and D, respectively. The diagonals CD' and AB' belong to planes which are perpendicular to (ABC) and contain C and A respectively.





Consider a point in the space and draw through it parallel lines to the diagonals BC', CD', DA' and AB' respectively. One obtains a regular quadrilateral pyramid; its diagonal section is an isosceles right triangle (thus equal to "half" of the base square).

We infer that the lateral edges of this pyramid are equal to the base edges, meaning that the lateral faces are equilateral triangles. We thus get that the angle made by DA' and AB' equals $60°$.

b) Let B_1C_1 be the projection of the segment $B'C'$ onto the plane (ABC) and D_1 the projection of point D' onto the plane (ABC). Then

$$CB_1 \parallel AD_1. \tag{1}$$

The triangles B_1C_1C and ABC_1 are isosceles and $B_1C_1 = CC_1$, $AB = AC_1$. Thus $B_1C_1 + BC = AC$.

Let $M \in CB$ $(B \in CM)$ be such that $BM = B_1C_1 = B'C'$ and N is the midpoint of AM. We find out that $MB'C'B$ is a parallelogram, thus $MB' = BC'$, and $MB' = AM$.

In the triangle ACM we get $CN \perp AM$, and $D_1A \perp AM$ from (1). We see that AM is perpendicular on the planes $(CB'B_1)$ and $(AD'D_1)$, thus the segment AN equals the common perpendicular of the diagonals CB' and AD'. It follows $AN = \frac{1}{2}AM = \frac{1}{2}BC'$.

PROBLEM 4. A cube of side 6 units contains 1001 unit cubes with sides parallel to those of the given one. Prove that one can find two unit cubes such that the center of one of them is inside or on the faces of the other.

Solution. Partition the cube into 12^3 cubes of sides $\frac{1}{2}$ each. The centers of the 1001 unit cubes are at least $\frac{1}{2}$ apart from the initial cube faces. As a consequence, they cannot be inside the cubes of side $\frac{1}{2}$ which have a face on one of the faces of the given cube. It results that the 1001 centers are inside the 1000 cubes of side $\frac{1}{2}$ which are inside the given cube. By the pigeon-hole principle, two of centers of the unit cubes are inside or on the faces of the same cube of side $\frac{1}{2}$. This proves the result.

9^{th} GRADE

PROBLEM 1. Find all increasing functions $f : \{1,2,\ldots,10\} \to \{1,2,\ldots,100\}$, having the property: $x+y$ is a divisor of $xf(x)+yf(y)$, for any $x, y \in \{1,2,\ldots,10\}$.

Solution. As $x+y$ is a divisor of both $xf(x)+yf(y)$ and $xf(y)+yf(y)$, we get by subtraction $x+y|x(f(y)+f(x))$.

For $y = x+1$ we obtain that $2x+1$ is a divisor of $x(f(x+1)-f(x))$ and, because $(2x+1,x)=1$, we deduce $2x+1|f(x+1)-f(x)$.

In particular $f(x+1)-f(x) \geqslant 2x+1$, for all $x \in 1,2,\ldots,9$, thus

$$\sum_{x=1}^{9}(f(x+1)-f(x)) \geqslant \sum_{x=1}^{9}(2x+1).$$

This implies $f(10) \geqslant f(1)+99 \geqslant 100$, thus $f(10)=100$ and $f(1)=1$.

From $f(x+1)-f(x) = 2x+1$, for any $x = 1,2,\ldots,9$, we conclude $f(x) = x^2$.

PROBLEM 2. For any positive integer n denote by $P(n)$ the number of quadratic functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax^2+bx+c$, having the properties:

a) $a,b,c \in \{1,2,\ldots,n\}$;

b) The equation $f(x)=0$ has integer roots.

Prove that $n < P(n) < n^2$, for all $n \geqslant 4$.

Solution. As the quadratic equations $x^2+kx+k-1=0$, $k=2,3,\ldots,n$ and $2x^2+4x+2=0$, $x^2+4x+4=0$ have integer roots, we infer $P(n) > n$.

For f satisfying the given conditions, we write $f(x) = a(x+\alpha_1)(x+\alpha_2)$ with $\alpha_1, \alpha_2 \in \mathbb{Z}$, a, $a(\alpha_1+\alpha_2)$, $a\alpha_1\alpha_2 \in \{1, 2, \ldots, n\}$.

In particular, $\alpha_2 \leqslant \dfrac{n}{a\alpha_1}$. We find

$$P(n) \leqslant \sum_{\substack{1\leqslant\alpha_1\leqslant n\\ 1\leqslant a\leqslant n}} \frac{n}{a\alpha_1} = n\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}\right)^2.$$

As $1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n} < \sqrt{n}$ for $n \geqslant 5$ (induction on n), and $P(4) = 5$ we conclude $P(n) < n^2$.

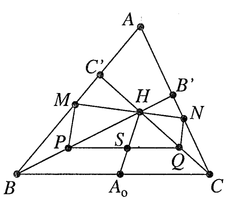
Remark. A more careful analysis of the above solution implies a better bound, namely $P(n) < \dfrac{n^2}{4}$, for $n \geqslant 4$.

PROBLEM 3. Let H be the orthocenter of the acute triangle ABC, and let B', C' be the projection of the points B and C onto AC and AB, respectively. A variable line d, passing through H, intersects the segments BC' and CB' in M and N, respectively. The perpendiculars in M and N on d, intersect BB' and CC', in P and Q, respectively.

Find the locus of the midpoint of the segment PQ.

Solution. By the following similarities:

$$\Delta HMP \sim \Delta HB'N, \quad \Delta HNQ \sim \Delta HC'M, \quad \Delta HBC' \sim \Delta HCB',$$



we obtain $\dfrac{HP}{HN} = \dfrac{HM}{HB'}$, $\dfrac{HQ}{HM} = \dfrac{HN}{HC'}$, $\dfrac{HB'}{HC'} = \dfrac{HC}{HB}$. Therefore $\dfrac{HP}{HB} = \dfrac{HQ}{HC}$.

As a consequence, the segments PQ and BC are parallel. Thus, the midpoint S of the segment PQ is situated on the fixed line HA_0, where A_0 is the midpoint of BC.

Conversely, let S be a point of the locus and PQ the corresponding segment. The corresponding point M is one of the points M_1, M_2, where the circle of diameter HP intersects the segment BC. The line M_kH will intersect AC at N_k, $k = 1, 2$. The perpendicular line N_k on M_kN_k will cut CC' at Q_k, $k = 1, 2$. By the first part of the proof, $PQ_k \parallel BC$, $k = 1, 2$. But $PQ \parallel BC$, so $Q_k = Q$, $k = 1, 2$. Therefore, we find out that one of the limiting configurations corresponds to the case when the circle of diameter HP is tangent to BC'.

Let M_0 be the tangent point, and ω_0 its center. The lines ω_0M_0 and HC' are parallel (they are both perpendicular on BC'). As a consequence $\angle\omega_0M_0H \equiv \angle M_0HC'$. But $\angle\omega_0M_0H \equiv \angle\omega_0HM_0$, implying that HM_0 is the interior bisector line of $\angle BHC'$.

Let S_0 be the point corresponding to this configuration. The locus will be the segmet A_0S_0 situated on the line HA_0, and the limiting position A_0 will be obtained when d coincides with one of the perpendicular lines BB' or CC'.

Remark: The preceeding discussion implies that when d rotates around H, remaining inside the angles $\angle BHC'$ and $\angle CHB'$, the point S moves along the segment A_0S_0 in two ways: to both lines $d_1 = M_1HN_1$ and $d_2 = M_2HN_2$, $M_1, M_2 \in BC'$ and $N_1, N_2 \in CB'$, that make the same angles with the bisector line HM_0 of $\angle BHC'$, there correspond the same points P and Q, that is the same point S.

PROBLEM 4. Let p, q be positive integers, $p \geqslant 2$, $q \geqslant 2$. A set X has property (S) if, by definition, for any p subsets $B_i \subset X$, $i = 1, 2, \ldots, p$, not necessarily different, any of them having q elements, there is a set $Y \subset X$ having p elements, such that the intersection of Y with each B_i, $i = 1, 2, \ldots, p$, has at most one element. Prove that:

a) Any set X with $pq - q$ elements does not satisfy property (S);

b) Any set X with $pq - q + 1$ has property (S).

Solution. a) If X has $pq - q = (p-1)q$ elements, chose $p-1$ subsets B_i, $i = 1, \ldots, p-1$ with q elements each. They form a partition of X, and let B_p arbitrary, containing q elements. By the pigeon-hole principle, any subset Y with p elements, will intersect at least one of the sets B_i, $i = 1, \ldots, p-1$ in at least two elements,

b) For given i, observe that $\bigcup_{j\neq i} B_j$ contains at most $(p-1)q$ elements. As X has $pq - q + 1 = (p-1)q + 1$ elements, we will find at least one element at our disposal, say x_i, which does not belong to any B_j, $j \neq i$.

Apply the above remark for $i = 1$: if $x_1 \in B_1$ we will continue; if not, we replace an element, say $y_1 \in B_1$, by x_1. Continuing with $i = 2$, 3, etc., we observe that at each step the chosen x_i will be different from all the preceding ones.

Consider $Y = \{x_i \mid i = 1, \ldots, p\}$. The set Y will intersect each new B_i in exactly one element, that is x_i. Replace now x_i with y_i, to obtain the original B_i. If $y_i \notin Y$, the intersection of Y with B is empty; if $y_i \in Y$, the intersection would contain an element, that has to be y_i.

10th GRADE

PROBLEM 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, such that $|f(x) - f(y)| \leqslant |x - y|$, for any real x, y. Prove that if the sequence $x, f(x), f(f(x)), \ldots$ arithmetic, for any real x, then there is a real a such that $f(x) = x + a$, for any real number x.

Solution. Suppose the existence of $x, y \in \mathbb{R}$ such that $f(x) - x > f(y) - y$. Denoting by $p = f(x) - x$, and $q = f(y) - y$, we get $f(x + np) = x + (n+1)p$,

$f(y+nq) = y+(n+1)q$, for all $n \in \mathbb{N}$. As we must have $|f(x+np) - f(y+nq)| \leqslant |x+np-y-nq|$, we deduce $|x-y+(n+1)(p-q)| \leqslant |x-y+n(p-q)|$.

For suficiently large n, we will have $x-y+(n+1)(p-q) \leqslant x-y+n(p-q)$ implying $p \leqslant q$, a contradiction. Thus $f(x)-x = f(y)-y$, for all $x, y \in \mathbb{R}$, that is $f(x) - x = f(0) - 0$, a constant.

PROBLEM 2. Prove that any thetrahedron in which pairs of opposite sides are equal and make equal angles, is regular.

Solution. Denote by $ABCD$ the given thetrahedron. We shall use the following obvious vectorial relation

$$\overrightarrow{AB} \cdot \overrightarrow{CD} + \overrightarrow{AC} \cdot \overrightarrow{DB} + \overrightarrow{AD} \cdot \overrightarrow{BC} = 0.$$

Let a the common angle between opposite edges. We get

$$\cos a \cdot (\pm AB \cdot CD \pm AC \cdot BD \pm AD \cdot BC) = 0,$$

for some choice of the sequence of $+, -$ signes.

If $\cos a \neq 0$, Then $\pm AB \cdot CD \pm AC \cdot BD \pm AD \cdot BC = 0$, and we find that one of the numbers $AB \cdot CD$, $AC \cdot BD$, $AD \cdot BC$ equals the sum of the other two, contradicting the Ptolemei relation.

Thus, $\cos a = 0$, that is, opposite edges are orthogonal. It is an easy argument to show that $AB^2 + CD^2 = AC^2 + BD^2 = AD^2 + BC^2$ (take for example AD' parallel and equal to BD and observe that triangles ACD' and DCD' are right triangles). As by the hypothesis $AB = CD$, $AC = BD$ and $AD = BC$, we conclude that all edges are equal, so $ABCD$ is equilateral.

Alternative solution. One can easily prove that $2|\cos a| \cdot AB \cdot CD = |AD^2 + BC^2 - AC^2 - BD^2|$. Denote by x, y, z the edges, we get

$$\frac{|x^2 - y^2|}{z^2} = \frac{|y^2 - z^2|}{x^2} = \frac{|z^2 - x^2|}{y^2},$$

which implies $x = y = z$.

PROBLEM 3. Let $n > 2$ be an integer and $a \in (0, \infty)$ such that

$$2^a + \log_2 a = n^2.$$

Prove that

$$2\log_2 n > a > 2\log_2 n - \frac{1}{n}.$$

Solution. It is clear that $a > 1$, so $n^2 = 2^a + \log_2 a > 2^a$, which gives $2\log_2 n > a$.

In order to prove the second inequality observe that

$$n^2 = 2^a + \log_2 a < 2^a + \log_2(2\log_2 n),$$

which implies

$$a > \log_2(n^2 - \log_2(2\log_2 n)) = 2\log_2 n + \log_2\left(1 - \frac{\log_2(2\log_2 n)}{n^2}\right).$$

We have to prove that

$$\log_2\left(1 - \frac{\log_2(2\log_2 n)}{n^2}\right) > -\frac{1}{n}$$

or

$$\left(1 - \frac{\log_2(2\log_2 n)}{n^2}\right)^n > \frac{1}{2}.$$

Using the Bernoulli inequality, it is sufficient to prove $1 - \frac{\log_2(2\log_2 n)}{n} > \frac{1}{2}$, that is $\frac{\log_2(2\log_2 n)}{n} < \frac{1}{2}$.

The last inequality is equivalent to $2\log_2 n < 2^{\frac{n}{2}}$, or $\log_2 n < 2^{\frac{n-2}{2}}$. It is a routine induction argument to finish the proof.

PROBLEM 4. Let $(P_n)_{n \geqslant 1}$ an infinite family of planes and $(X_n)_{n \geqslant 1}$ a family of non-void sets of points such that $X_n \subset P_n$ and the orthogonal projection of X_{n+1} onto the plane P_n is contained in X_n, for any n.

Prove that there is a sequence of points $(p_n)_{n \geqslant 1}$ such that $p_n \in X_n$ and p_n is the orthogonal projection of p_{n+1} onto the plane P_n, for any n.

Does the result remain valid if the sets X_n are infinite?

Solution. Consider positive integers n, p. Denote by $A_{n,p}$ the image in P_n of the set X_{n+p}, after succesive projections onto the planes $P_{n+p-1}, \ldots, P_n$. $A_{n,p}$ is a non-empty subset of X_n. As X_{n+p+1} projects onto the plane P_{n+p} as a subset of X_{n+p}, we deduce that $A_{n,p+1} \subseteq A_{n,p}$. For any n we get a decreasing sequence of non-empty subsets $X_n \supseteq A_{n,1} \supseteq A_{n,2} \supseteq \cdots \supseteq A_{n,p} \supseteq \cdots$.

As X_n is finite, one can find $p \geqslant 1$ with the property that $A_{n,k} = A_{n,p}$, for any $k \geqslant p$. Denote by T_n this subset of X_n. For large p, $A_{n,p} = T_n$, $A_{n+1,p-1} = T_{n+1}$, thus T_n is the projection of T_{n+1} onto the plane P_n. Chose an arbitrary p_1 in T_1. Then, there is p_2 în T_2 such that p_1 is the projection of p_2 onto P_1. Inductively, construct the sequence $(p_n)_{n \geqslant 1}$ such that $p_n \in T_n$, $p_{n+1} \in T_{n+1}$ and such that p_n is the projection of p_{n+1} on P_n,. This concludes the proof.

If the sets are infinite, consider the planes P_n with equation $z = n$ and the sets $X_n = \{(x, 0, n) \mid x \geqslant n\}$. It is easy to check that the conclusion fails.

11^{th} GRADE

PROBLEM 1. Consider an integer $n \geqslant 3$ and the parabola of equation $y^2 = 2px$, with focus F. A regular n-gone $A_1A_2\cdots A_n$ has center at F and no one of its vertices lies the x axis. The rays $FA_1, FA_2, \ldots, FA_n$ cut the parabola at points $B_1, B_2, \ldots, B_n$.

Prove that $FB_1 + FB_2 + \cdots + FB_n > np$.

Solution. Let $FB_k = t_k$, α the angle made by FB_1 with Ox and let $\alpha_k = \alpha + \frac{2(k-1)\pi}{n}$. It follows that the coordinates of B_k are $\left(\frac{p}{2} + t_k\cos\alpha_k, t_k\sin\alpha_k\right)$.

We have $t_k^2\sin^2\alpha_k = p^2 + 2pt_k\cos\alpha_k$. The positive root of the last equation is $t_k = \dfrac{p}{1-\cos\alpha_k}$. We get

$$\sum_{k=1}^{n}\frac{1}{t_k} = \frac{n}{p} - \frac{1}{p}\sum_{k=1}^{n}\cos\left(\alpha + \frac{2(k-1)\pi}{n}\right) = \frac{n}{p}.$$

Using the Cauchy inequality we obtain

$$\Big(\sum_{k=1}^{n}\frac{1}{t_k}\Big)\Big(\sum_{k=1}^{n}t_k\Big) \geqslant n^2,$$

that gives $\sum\limits_{k=1}^{n} t_k \geqslant np$.

The equality implies $t_1 = t_2 = \cdots = t_n = r$, which is impossible because the circle of center F and radius r intersects the parabola in at most two points.

Remark. It can be shown that $\displaystyle\sum_{k=1}^{n} t_k = \frac{n^2p}{1-\cos n\alpha}$.

PROBLEM 2. Consider an integer n, $n \geqslant 2$.

a) Prove that there are matrices $A, B \in \mathcal{M}_n(\mathbb{C})$ such that

$$\operatorname{rank}(AB) - \operatorname{rank}(BA) = \left[\frac{n}{2}\right].$$

b) Prove that for any $X, Y \in \mathcal{M}_n(\mathbb{C})$ we have

$$\operatorname{rank}(XY) - \operatorname{rank}(YX) \leqslant \left[\frac{n}{2}\right].$$

Solution. a) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. One can easily check that $\operatorname{rang}(AB) - \operatorname{rang}(BA) = 1$.

In the case when $n = 2p$, choose

$$A_{2p} = \begin{pmatrix} A & & & 0 \\ & A & & \\ & & \ddots & \\ 0 & & & A \end{pmatrix}, \quad B_{2p} = \begin{pmatrix} B & & & 0 \\ & B & & \\ & & \ddots & \\ 0 & & & B \end{pmatrix}.$$

When $n = 2p+1$, choose $A_{2p+1} = \begin{pmatrix} A_{2p} & 0 \\ 0 & 0 \end{pmatrix}$, $B_{2p+1} = \begin{pmatrix} B_{2p} & 0 \\ 0 & 0 \end{pmatrix}$.

b) Suppose $\operatorname{rang}(XY) \geqslant \operatorname{rang}(YX)$. We have

$$(1) \qquad \operatorname{rang}(XY) - \operatorname{rang}(YX) \leqslant \operatorname{rang}(XY) \leqslant \operatorname{rang}(X).$$

From Sylvester inequality, $\operatorname{rang}(YX) \geqslant \operatorname{rang}(X) + \operatorname{rang}(Y) - n$, which implies $\operatorname{rang}(XY) - \operatorname{rang}(YX) \leqslant \operatorname{rang}(Y) + n - \operatorname{rang}(X) - \operatorname{rang}(Y)$, that is

$$(2) \qquad \operatorname{rang}(XY) - \operatorname{rang}(YX) \leqslant n - \operatorname{rang}(X).$$

From (1) and (2) we get $\operatorname{rang}(XY) - \operatorname{rang}(YX) \leqslant \left[\frac{n}{2}\right]$

PROBLEM 3. Let $f : (a,b) \to \mathbb{R}$ be a function with the property that for any $x \in (a,b)$, there is a non-trivial interval $[a_x, b_x]$, $a < a_x \leqslant x \leqslant b_x < b$, such that f is constant on $[a_x, b_x]$.

a) Prove that the image of f is a set which is finite or countable.

b) Find all continuous functions satisfying the given property.

Solution. a) Let $f(x) \in \operatorname{Im} f$, with $x \in (a,b)$. Choose $r_x \in \mathbb{Q} \cap [a_x, b_x]$. By the given condition, $f(x) = f(r_x)$. Thus $\operatorname{Im} f = \{f(r_x) \mid x \in (a,b)\}$.

If we denote $\{r_x \mid x \in (a,b)\}$ by $\{q_1, q_2, \ldots, q_n, \ldots\}$, then the set $\operatorname{Im} f = \{f(q_n)\}_{n\in\mathbb{N}^*}$ is at most countable.

b) If f is continuous, we shall prove that it is constant. Suppose that f takes at least two different values, say λ and μ, with $\lambda < \mu$. The function f has the intermediate value property, that is $[\lambda, \mu] \subset \operatorname{Im} f$. This is in contradiction with the fact that $\operatorname{Im} f$ is countable.

PROBLEM 4. a) Construct a function $f : \mathbb{R} \to \mathbb{R}_+$ with the following property, called $\mathcal{P}$: Any $x \in \mathbb{Q}$ is a local strictly minimum point for f.

b) Construct $f : \mathbb{Q} \to \mathbb{R}_+$, with the property that any point is a local strictly minimum point and f is unbounded on any set of the form $I \cap \mathbb{Q}$, where I is a non-trivial interval.

c) Let $f : \mathbb{R} \to \mathbb{R}_+$ be function which is not bounded on any set of the form $I \cap \mathbb{Q}$, where I is a non-degenerate interval. Prove that f has not the property $\mathcal{P}$.

Solution. a) Define

$$f(x) = \begin{cases} 1 - \frac{1}{p}, & x = \frac{n}{p},\ (n,p) = 1,\ p > 0 \\ 1, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

One can see that $x = \dfrac{n}{p}$ is a local strictly minimum point, because there is an open interval around x that doesn't contain fractions with denominators in the set $1, 2, \ldots, p$.

b) The function $f : \mathbb{Q}^+ \to \mathbb{R}$ defined by $f\left(\dfrac{n}{p}\right) = p$ has the given properties.

c) Suppose that f has property $\mathcal{P}$. Consider a non-degenerate interval $I_0 = [\alpha_0, \beta_0]$. By induction, we shall define a decreasing sequence of closed intervals $I_k = [\alpha_k, \beta_k]$, $k = 0, 1, \dots$, such that $f(I_k) \subset [k, \infty)$. Suppose we have constructed $I_1, I_2, \dots, I_{n-1}$ with the given properties. We can find $x_n \in \mathbb{Q} \cap I_{n-1}$ with $f(x_n) \geqslant n$. Take a of x_n, $I_n = [\alpha_n, \beta_n] \subset I_{n-1}$, such that for $x \in I_n$, $f(x) \geqslant f(x_n) \geqslant n$. This proves the induction step.

From the fact that α_k is increasing, β_k is decreasing and $\alpha_k < \beta_k$, we deduce the existence of an element $\gamma \in \bigcap_{n \geqslant 0} I_n$, that is $f(\gamma) \geqslant n$ for all integers n, a contradiction.

12th GRADE

PROBLEM 1. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$n^2 \int_x^{x+\frac{1}{n}} f(t)\,\mathrm{d}t = nf(x) + \frac{1}{2},$$

for any $x \in \mathbb{R}$ and any positive integer n, $n \geqslant 2$.

Solution. As f is continuous on $\mathbb{R}$, consider an anti-derivative F. The given relation is equivalent to

$$(1) \qquad n^2 \left(F\left(x + \frac{1}{n}\right) - F(x) \right) = nf(x) + \frac{1}{2},$$

for any $x \in \mathbb{R}$ and all $n \in \mathbb{N}^*$.

As a consequence, f has finite derivative at any point. Taking derivatives in (1), we get

$$(2) \qquad n \left(f\left(x + \frac{1}{n}\right) - f(x) \right) = f'(x)$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}^*$.

From (2) it follows that f is twice derivable and the following relation holds

$$n \left(f'\left(x + \frac{1}{n}\right) - f'(x) \right) = f''(x),$$

implying that f'' is continuous.

Consider $x \in \mathbb{R}$. By Lagrange theorem, relation (2) implies the existence of a point $c_n = c_n(x) \in \left(x, x + \frac{1}{n}\right)$ with $f'(c_n) = f'(x)$. By Rolle theorem, for f' on the interval with endpoints x and c_n, we get a point ζ_n belonging to the interval with endpoints x and c_n and having the property that

$$(3) \qquad f''(\zeta_n) = 0.$$

As $\lim_{n\to\infty} \zeta_n = x$, the continuity of f'', and (3), give together

$$f''(x) = \lim_{n\to\infty} f''(\zeta_n) = 0.$$

As x was arbitrary, we deduce that f has the form $f(x) = ax + b$, where $a, b \in \mathbb{R}$. Substituting in the given relation, we find $a = 1$.

We conclude $f(x) = x + b$, with b arbitrary.

PROBLEM 2. Let $f \in \mathbb{Z}[X]$. For $n \in \mathbb{N}$, $n \geqslant 2$, define $f_n : \mathbb{Z}_n \to \mathbb{Z}_n$, by $f_n(\widehat{x}) = \widehat{f(x)}$, for any $x \in \mathbb{Z}$.

Find all polynomials $f \in \mathbb{Z}[X]$ such that for any $n \in \mathbb{N}$, $n \geqslant 2$, the function f_n is onto.

Solution. a) If $\widehat{x} = \widehat{y}$ in $\mathbb{Z}_n$, we get $k \in \mathbb{Z}$ such that $x - y = kn$. We can write

$$f(x) - f(y) = (x - y)g(x, y),$$

where $g(x, y)$ is an integer. Thus $f(x) - f(y)$ is a multiple of n, and as a consequence, $\widehat{f(x)} = \widehat{f(y)}$, i.e. $f_n(\widehat{x}) = f_n(\widehat{y})$.

b) First, we show that $\deg f \leqslant 1$. Suppose, by way of contradiction, that $\deg f = k \geqslant 2$. The polynomial $g(X) = f(X + 1) - f(X)$ has degree $k - 1$. As $k - 1 \geqslant 1$, one can find $x \in \mathbb{Z}$, such that $|g(x)| \geqslant 2$. Consider $n = |g(x)| = |f(x + 1) - f(x)|$.

In the ring $\mathbb{Z}_n$ we have $\widehat{f(x+1)} = \widehat{f(x)}$, or $f_n(\widehat{x+1}) = f_n(\widehat{x})$. The last equality shows that f_n is not injective and as a consequence nor onto. As f can not be constant, let $m \in \mathbb{Z}^*$, $a \in \mathbb{Z}$, such that for any $x \in \mathbb{Z}$, $f(x) = mx + a$.

We shall now show that $|m| = 1$; for if $|m| \geqslant 2$, the map $f_{|m|} : \mathbb{Z}_{|m|} \to \mathbb{Z}_{|m|}$ is constant, thus not onto, a contradiction.

It is easy to see that the polynomials $X + a$ and $-X + a$ satisfy the given conditions.

PROBLEM 3. Let $f : [0, 1] \to \mathbb{R}$ be an integrable function such that

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 xf(x)\,\mathrm{d}x = 1.$$

Prove that

$$\int_0^1 f^2(x)\,\mathrm{d}x \geqslant 4.$$

Solution. The degree 1 polynomial p satisfying the relations $\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 xf(x)\,\mathrm{d}x = 1$, is $p(x) = 6x - 2$. Thus

$$\int_0^1 (f(x) - p(x))\,\mathrm{d}x = \int_0^1 x(f(x) - p(x))\,\mathrm{d}x = 0.$$

We get

$$\int_0^1 p(x)(f(x)-p(x))\,\mathrm{d}x = 0.$$

This implies

$$\begin{aligned}0 \leqslant \int_0^1 (f(x)-p(x))^2\,\mathrm{d}x &= \int_0^1 f(x)(f(x)-p(x))^2\,\mathrm{d}x\\ &= \int_0^1 f^2(x)\,\mathrm{d}x - 6\int_0^1 xf(x)\,\mathrm{d}x + 2\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 f^2(x)\,\mathrm{d}x - 4,\end{aligned}$$

whence the conclusion.

PROBLEM 4. Let K be a field of characteristics p, with $p \equiv 1 \pmod 4$. Prove that any non-zero element in K can be written as the sum of three squares of non-zero elements from K.

Solution. Let $m \in \mathbb{N}^*$ such that $p = 4m+1$.

a) By Wilson theorem $p|(p-1)!+1$, thus $(p-1)!+1 = 0$ in K. This gives $(4m)! = -1$ in K. On the other hand $1 = -4m, 2 = -(4m-1), \dots, 2m = -(2m+1)$ in K, giving

$$-1 = (-1)^{2m}(4m)^2(4m-1)^2\cdots(2m+1)^2.$$

We get

$$-1 = [(4m)(4m-1)\cdots(2m+1)]^2,$$

in K. We conclude that -1 is a perfect square in K.

b) Consider $a, b \in K^*$ two commuting elements. We have

(1) $$(a+b)^2 = a^2+b^2+2ab$$

and

(2) $$(a-b)^2 = a^2+b^2-2ab.$$

Consider $b = 2^{-1}$. If $a \in K$ then a and 2^{-1} commute.

If $a \neq -2^{-1}$, from (1) we get $a = (a+2^{-1})^2 - a^2 - b^2 = (a+2^{-1})^2 + a_1^2 + b_1^2$, because -1 is a perfect square in K.

If $a = -2^{-1}$, then $a \neq b$. By (2) we have $a = a^2+b^2-(a-b)^2 = a^2+b^2+c^2$, because -1 is a perfect square in K.

Thus any $a \in K^*$ can be written as a sum of three non-zero squares of some elements in K.

2.3. THE NATIONAL MATHEMATICAL OLYMPIAD
Selection Tests for the BMO and IMO 2004

First Selection Test

PROBLEM 1. Let a_1, a_2, a_3, a_4 be the lengths of the sides of a quadrilateral and s be its semi-perimeter. Prove that

$$\sum_{i=1}^{4} \frac{1}{s+a_i} \leqslant \frac{2}{9} \sum_{1\leqslant i<j\leqslant 4} \frac{1}{\sqrt{(s-a_i)(s-a_j)}}$$

When does equality hold?

Solution. Numbers $s - a_i$ are positive, and the AM-GM inequality gives

(1) $$\frac{1}{3}\sum_{j\neq i}(s-a_j) \geqslant \Big(\prod_{j\neq i}(s-a_j)\Big)^{1/3}.$$

for every $i = 1,2,3,4$. Since $\sum_{j\neq i}(s-a_j) = s+a_i$, relation (1) leads to

(2) $$\frac{3}{s+a_i} \leqslant \Big(\prod_{j\neq i}\frac{1}{\sqrt{s-a_j}}\Big)^{2/3}.$$

On the other hand, the AM-GM inequality gives

(3) $$\begin{aligned}\frac{1}{3}\sum_{j<k,\, j,k\neq i}\frac{1}{\sqrt{(s-a_j)(s-a_k)}} &\geqslant \Big(\prod_{j<k,\, j,k\neq i}\frac{1}{\sqrt{(s-a_j)(s-a_k)}}\Big)^{1/3}\\ &= \Big(\prod_{j\neq i}\frac{1}{\sqrt{s-a_j}}\Big)^{2/3}.\end{aligned}$$

Combining (2) and (3) and adding we get the required inequality. The equality takes place if and only if the quadrilateral is a rhombus.

Alternative solution. We can write, using AM-GM-HM inequalities

$$\frac{2}{9}\sum_{i<j}\frac{1}{\sqrt{(s-a_i)(s-a_j)}} \geqslant \frac{4}{9}\sum_{i<j}\frac{1}{a_i+a_j} = \frac{2}{9}\sum_{i=1}^{4}\Big(\sum_{i\neq j}\frac{1}{a_i+a_j}\Big)$$
$$\geqslant \frac{2}{9}\sum_{i=1}^{4}\frac{9}{2a_i+2s} = \sum_{i=1}^{4}\frac{1}{s+a_i}.$$

PROBLEM 2. Let $\mathcal{R}_i$, $i=1,2,\ldots,n$, be pairwise disjoint closed rectangular regions whose sides are parallel to the coordinate axes. It is also known that the area of $\mathcal{R} = \bigcup_{i=1}^{n} \mathcal{R}_i$ is at least 4 and the projection onto Ox of their union is an interval.

Prove that $\mathcal{R}$ contains three points which are the vertices of a triangle of area 1.

Solution. Let $[a,b]$ the projection of $\mathcal{R}$ onto Ox. For $x \in [a,b]$, define $m(x) = \min\{y \mid (x,y) \in \mathcal{R}\}$ and $M(x) = \max\{y \mid (x,y) \in \mathcal{R}\}$. Since $\mathcal{R}$ is included in $\{(x,y) \mid x \in [a,b],\ m(x) \leqslant y \leqslant M(x)\}$ and has area at least 4, it follows that there exists $x_0 \in [a,b]$ such that

$$(b-a)\,(M(x_0)-m(x_0)) \geqslant 4.$$

Consider the points A and B having coordinates $(x_0, m(x_0))$ and $(x_0, M(x_0))$ respectively, and suppose, for instance, $x_0 \leqslant \dfrac{a+b}{2}$. Then, for

$$x_1 = x_0 + \frac{2}{M(x_0)-m(x_0)}$$

we have $x_0 < x_1 \leqslant x_0 + \frac{b-a}{2} \leqslant b$, therefore one can find a point $C \in \mathcal{R}$ having coordinates (x_1, y_1) such that area(ABC)= 1.

Remark. The above solution is an elementary version of the obvious one given in terms of the Fubini theorem for a double integral. It is thus clear that the fact that the regions are rectangles is not important, the main point being the interval-projection property.

As the author pointed out to us, the problem comes from a conjecture of Erdős.

PROBLEM 3. Find all injective functions $f : \mathbb{N} \to \mathbb{N}$ such that, for each n,

$$f(f(n)) \leqslant \frac{n+f(n)}{2}.$$

Solution. It is clear that if we prove that $f(n) \leqslant n$ for any non-negative n we shall get by easy induction that $f(n) = n$ for any n. Indeed, from $f(0) \leqslant 0$ we get $f(0) = 0$ and if we suppose $f(k) = k$ for any $k \leqslant n$, from $f(n+1) \leqslant n+1$ and the injectivity condition, we get $f(n+1) = n+1$.

Suppose, by way of contradiction, that $f(n) > n$ for some non-negative n and denote $f^k = f \circ f \circ \cdots \circ f$, the k-composition of f with itself, $k = 2,3,\ldots$. We easily obtain by induction that $f^k(n) < f(n)$ for any $k = 2,3,\ldots$. Indeed if $f^p(n) < n$ for $p = 1,2,\ldots,k$ we have

$$f^{p+1}(n) \leqslant \frac{f^p(n)+f^{p-1}(n)}{2} < f(n).$$

Thus the sequence $(f^k(n))_k$ is a bounded sequence of non-negative integers and so $f^p(n) = f^q(n)$ for some $p < q$. Since f is injective, it follows that $f^{q-p}(n) = n$, therefore $f^{q-p+1}(n) = f(n)$, a contradiction.

It is obvious that the identity function verifies the given relation and is injective.

Remark. There are many other ways for proving the result and maybe the simplest is to make connections with a well-known fact on limits of sequences: if a sequence of non-negative numbers, say $(a_n)_n$ is sub-additive, that is $a_{n+1} \leqslant \dfrac{a_n + a_{n-1}}{2}$ for any n, then the sequence has a finite limit.

In our case, for any n the sequence $\left(f^k(n)\right)_k$ is sub-additive, and being a convergent sequence of integers is constant beginning from some term. This contradicts the injectivity, as above, if f is not the identity function.

PROBLEM 4. Consider an integer $n \geqslant 2$ and a disc $\mathcal{D}$ in the complex plane. Prove that for any $z_1, z_2, \ldots, z_n \in \mathcal{D}$, there exists $z \in \mathcal{D}$ such that $z^n = z_1 z_2 \cdots z_n$.

Solution. First, consider the case $n = 2$. In this case, if $u^2 = z_1 z_2$, then the two complex roots of second order of z_1, z_2 are $\pm u$ and satisfy

$$|z_1 - u|\cdot|z_2+u| + |z_1+u|\cdot|z_2-u| = 2|u|\cdot|z_1-z_2|,$$

which shows that $z_1, u, z_2, -u$ are (in this order) the vertices of a cyclic quadrilateral. This shows that u and $-u$ are on the two arcs determined by z_1 and z_2 on a circle $\mathcal{C}$; since at least one of these arcs is included in $\mathcal{D}$, it follows that u or $-u$ id is $\mathcal{D}$.

In the case $n \geqslant 3$ we may suppose that $r_k = |z_k| \neq 0$ for $k = 1,2,\ldots,n$, and we will denote $r = \sqrt[n]{r_1 r_2 \cdots r_n}$. An argument based on continuity (there exists w_1 on the segment $[z_1, z_2]$ such that $|w_1|^2 = |z_1|\cdot|z_2|$, there exists w_2 on the segment $[w_1, z_3]$ such that $|w_2|^3 = |w_1|^2|z_3|$, and so on shows that there exists at least one complex number of modulus r inside $\mathcal{D}$, therefore the intersection of the circle $\mathcal{C}(O,r)$ and the interior of $\mathcal{D}$ is not empty.

If $\mathcal{C}(O,r)$ is included in $\mathcal{D}$, then every root of order n of $z_1 z_2 \cdots z_n$ belongs to $\mathcal{D}$.

If $\mathcal{C}(O,r)$ is not included in $\mathcal{D}$ then, for some sufficiently small interval (a,b) which contains r, the intersections of $\mathcal{D}$ with the circles $\mathcal{C}(O,a)$ and $\mathcal{C}(O,b)$ are arcs $\mathcal{A}_1, \mathcal{A}_2$ of these circles. Let $\mathcal{A}$ be the region of $\mathcal{D}$ situated between $\mathcal{A}_1$ and $\mathcal{A}_2$, that is $\mathcal{A} = \{z \in \mathcal{D} \mid a < |z| < b\}$. Then there exist $\alpha, \beta \in \mathbb{R}$ such that every

$z \in \mathcal{A}$ has an unique argument between α and β. The case $n = 2$ and the lemma below allow replacing in a finite number of steps $(z_1, z_2, \dots, z_n)$ with $w_j \in \mathcal{A}$ for $j = 1, 2, \dots, n$ and $z_1 z_2 \cdots z_n = w_1 w_2 \cdots w_n$. Then we can take

$$z = r(\cos\varphi + \mathrm{i}\sin\varphi), \quad \varphi = \frac{\sum_{j=1}^{n} \arg(w_j)}{n}.$$

Lemma. A transformation of a family $(r_1, r_2, \dots, r_n)$ of positive numbers consists of replacing $r_i = \max\{r_p | p = 1, 2, \dots, n\}$ and $r_j = \min\{r_p | p = 1, 2, \dots, n\}$ with $\sqrt{r_i r_j}$ and $\sqrt{r_i r_j}$.

If $a < r = \sqrt[n]{r_1 r_2 \cdots r_n} < b$, then, after a finite number of such transformations, we get a family $(s_1, s_2, \dots, s_n)$ with $s_p \in (a, b)$ for $p = 1, 2, \dots, n$.

Proof. Let $\alpha = \ln a, \beta = \ln b, x = \ln r$ and $x_p = \ln r - p$, $p = 1, 2, \dots, n$. Denote d_s the value of the sum $\sum |x_p - x|$ after s transformations.

We suppose, for sake of simplicity, that $x_1 \leqslant \cdots \leqslant x_k \leqslant x \leqslant x_{k+1} \leqslant \cdots \leqslant x_n$. Then $(x - x_1) + \cdots + (x - x_k) = (x_{k+1} - x) + \cdots + (x_n - x)$ and

$$d_s = 2\sum_{p \leqslant k}(x - x_p) = 2\sum_{p \geqslant k+1}(x_p - x).$$

When replacing (x_1, x_n) with $\left(\frac{x_1 + x_n}{2}, \frac{x_1 + x_n}{2}\right)$, d_s is modified by

$$2\left|\frac{x_1 + x_n}{2} - x\right| - (x - x_1) - (x_n - x) = -2\min\{x - x_1, x_n - x\}.$$

But $d_s \leqslant 2n(x - x_2)$ and $d_s \leqslant 2n(x_n - x)$, implies $2\min\{x - x_1, x_n - x\} \geqslant \frac{1}{n} d_s$, whence $d_{s+1} \leqslant \frac{n-1}{n} d_s$. This shows that $(d_s)_s$ tends to 0, and the conclusion follos immediately.

Second Selection Test

PROBLEM 5. A disc $\mathcal{D}$ is divided into $2n$ equal sectors, n of them are colored in red and the other n are colored in blue. Starting from an arbitrarily chosen sector we count from 1 to n, in a clockwise order, the red sectors. We proceed in the same way with the blue sectors, but in an counterclockwise order.

Prove that there exists a half-disc of $\mathcal{D}$ which contains all the numbers from 1 to n.

Solution. Let r_i and respectively b_i the red, respectively the blue, sector which carries the number i, $i = 1, 2, \dots, n$.

Consider for each i the arc a_i, taken in a clockwise order, which starts "just after r_i ", and ends "just before b_i" and assume that the smallest of the a_i-s is a_k.

Then, all the sectors corresponding to a_k are of the same color (otherwise a_{k+1} is less then a_k), for instance red.

It is now easy to see that the half-disc starting "just after r_k" and taken clockwise fulfils the requirement. Indeed, if this half-disc contains p red sectors and $n-p$ blue sectors, then, with indices taken modulo n, it contains $r_{k+1}, r_{k+2}, \dots, r_{k+p}$ and $b_k, b_{k-1}, \dots, b_{k-n+p+1}$, hence it contains all numbers from 1 to n.

PROBLEM 6. Find all non-negative integers which can be reached by the expression

$$\frac{a^2 + ab + b^2}{ab - 1},$$

when a, b are non-negative integers and $ab \neq 1$.

Solution. Let $\frac{a^2 + ab + b^2}{ab - 1} = k$, $k \in \mathbb{N}$.

In the case $a = b$, the number $k = 3 + \frac{3}{a^2 - 1}$ is a positive integer if $a = 0$ or $a = 2$, whence $k = 0$ or $k = 4$.

The case $b = 0$ leads to $k = -a^2$, hence $a = 0$ and $k = 0$.

We will look now for the solutions (a, b) such that $a > b > 0$. Since the given relation is equivalent to $a^2 - (k-1)ab + b^2 + k = 0$, we see that if (a, b) is a solution and $b > (k-1)b - a > 0$, then $(b, (k-1)b - a)$ is also a solution. The inequality $(k-1)b - a > 0$ is true in all cases because it becomes successively

$$k > \frac{a+b}{b}, \quad \frac{a^2 + ab + b^2}{ab - 1} > \frac{a+b}{b}, \quad b^3 > -a - b.$$

In the same way, the inequality $b > (k-1)b - a$ is successively equivalent to

$$k < \frac{2b+a}{b}, \quad \frac{a^2 + ab + b^2}{ab - 1} < \frac{2b+a}{b}, \quad a > b + \frac{3b}{b^2 - 1} \quad \text{for } b > 1.$$

If $3b < b^2 - 1$, that is $b \geqslant 4$, the previous inequality is true, therefore for each solution (a, b) with $a > b \geqslant 4$ we find a solution (b, c) with $b > c > 0$. Thus, the descent method shows that each solution corresponds to a solution with $b \leqslant 3$.

For $b = 1$ we get $k = a + 2 - \frac{3}{a-1}$, $a = 4$ or $a = 2$, $k = 7$.

For $b = 2$ we get $4k = 2a + 5 + \frac{21}{2a-1}$, $a = 4$ or $a = 11$, $k = 4$ or $k = 7$.

For $b = 3$ we get $9k = 3a + 10 + \frac{91}{3a-1}$ which does not yield any solution.

Thus, the answer is $k \in \{0, 4, 7\}$.

PROBLEM 7. Let a, b, c be integers, b odd and consider the sequence $x_0 = 4, x_1 = 0$, $x_2 = 2c, x_3 = 3b$,

$$x_n = ax_{n-4} + bx_{n-3} + cx_{n-2}, \quad \text{for } n \geqslant 4.$$

Prove that if p is a prime and m a positive integer, then x_{p^m} is divisible by p.

Solution. Consider the characteristic polynomial

$$f = X^4 - cX^2 - bX - a$$

associated with the given recursion formula. We can check that all the roots of f are simple: if α is a root for both f and $f' = 4X^3 - 2cX - b$, then α is a root of $g = 4f - Xf' = -2cX^2 - 3bX - 4a$, a root of $h = cf' + 2Xg = -6bX^2 - (2c^2 + 8a)X - bc$ and a root of $l = 3bg - ch = (2c^3 + 8ac - 9b^2)X + bc^2 - 12ab$. Since $l(\alpha) = 0$ and $2c^3 + 8ac - 9b^2$ is odd, it follows that α is rational and, because f is monic and $f(\alpha) = 0$, we deduce that α is an integer, whence $f'(\alpha) = 4\alpha^3 - 2c\alpha - b$ is odd, a contradiction.

The above result shows that there exist coefficients a_1, a_2, a_3, a_4, uniquely determined by x_0, x_1, x_2, x_3, such that

$$x_n = a_1\alpha_1^n + a_2\alpha_2^n + a_3\alpha_3^n + a_4\alpha_4^n,$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the roots of f. We notice that

$$\begin{aligned} x_0 = 4 &= \alpha_1^0 + \alpha_2^0 + \alpha_3^0 + \alpha_4^0 \\ x_1 = 0 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ x_2 = 2c &= \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 = -2\sum \alpha_1\alpha_2 \\ x_3 = 3b &= \alpha_1^3 + \alpha_2^3 + \alpha_3^3 + \alpha_4^3 = 3\sum \alpha_1\alpha_2\alpha_3, \end{aligned}$$

which proves that $a_1 = a_2 = a_3 = a_4 = 1$.

Remark now that if p is a prime, then $p \mid \binom{p}{k}$ for each $k = 1, 2, \ldots, p-1$, hence

$$(X+Y)^p = X^p + Y^p + pQ(X, Y),$$

where Q is a symmetric integer polynomial in two variables. This easily extends to

$$(X+Y+Z+T)^p = X^p + Y^p + Z^p + T^p + pR(X, Y, Z, T),$$

where R is a symmetric polynomial in four variables.

In order to prove the required conclusion, we shall use induction on m. For $m = 1$, we notice that

$$x_p = \alpha_1^p + \alpha_2^p + \alpha_3^p + \alpha_4^p = -pR(\alpha_1, \alpha_2, \alpha_3, \alpha_4),$$

and since R can be represented as an integer polynomial in symmetric fundamental polynomials, $R(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is an integer.

Assume now that x_{p^m} is divisible by p. Then

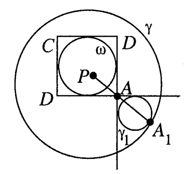
$$\begin{aligned} x_{p^{m+1}} &= \alpha_1^{p^{m+1}} + \alpha_2^{p^{m+1}} + \alpha_3^{p^{m+1}} + \alpha_4^{p^{m+1}} \\ &= pR(\alpha_1^{p^m}, \alpha_2^{p^m}, \alpha_3^{p^m}, \alpha_4^{p^m}) - (\alpha_1^{p^m} + \alpha_2^{p^m} + \alpha_3^{p^m} + \alpha_4^{p^m})^p \\ &= pS(\alpha_1, \alpha_2, \alpha_3, \alpha_4) - x_{p^m}^p \end{aligned}$$

is also divisible by p, because S is also a symmetric integer polynomial (hence $S(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is an integer).

PROBLEM 8. A square $ABCD$ is taken inside a circle γ. Inside the angle opposite to $\angle BAD$ is taken the circle tangent to the extended segments AB and AD and internally tangent to γ at A_1. The points B_1, C_1, D_1 are defined in the same way.

Prove that the straight lines AA_1, BB_1, CC_1, DD_1 are concurrent.

Solution. The common point of AA_1, BB_1, CC_1, DD_1 is the center P of the homotethy $\mathcal{H}$ of negative ratio which sends the incircle ω of $ABCD$ into γ.



This follows from the fact that the circle γ_1 tangent to γ at A_1 can be obtained from ω using an homotethy $\mathcal{H}_1$ of center A and negative ratio, and γ can be obtained from γ_1 through an homotethy $\mathcal{H}_1'$ of center A_1 and positive ratio. Therefore $\mathcal{H} = \mathcal{H}_1 \circ \mathcal{H}_1'$ has its center P on AA_1. In the same way P belongs to BB_1, CC_1, DD_1.

Third Selection Test

PROBLEM 9. Let $n > 1$ be a positive integer and X be a set containing n elements. $A_1, A_2, \ldots, A_{101}$ are subsets of X such that the union of any 50 of them has more than $\frac{50}{51}n$ elements.

Prove that among the given subsets it is possible to choose three, such that every two of them have a non-empty intersection.

Solution. Consider a graph Γ with vertices $A_1, A_2, \ldots, A_{101}$ and edges between the vertices whose intersection is non-empty.

If this graph has no triangles, the it has 51 vertices with degree at most 50. Indeed, if there are at most 50 vertices with degree at most 50, then there are 51 vertices with degree at least 51, so two of them, say A and B, must have a common edge. Since A and B are connected each with 50 vertices among the remaining 99, there exists a vertex C connected with both of them. We therefore obtain a triangle ABC, which is a contradiction.

Denote now by $A_{i_1}, A_{i_2}, \ldots, A_{i_{51}}$ the 51 vertices of degree at most 50. Each of the A_{i_k} has common elements with at most 50 subsets, so there exist 50 subsets such that A_{i_k} is contained in the complementary of their union. Since each union of 50 subsets has more than $\frac{50}{51}n$ elements, it follows that each A_{i_k} has less than

$\frac{1}{51}n$ elements. But in this case $A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_{50}}$ has less than $\frac{50}{51}n$ elements, false.

PROBLEM 10. Prove that if n and m are integers, and m is odd, then

$$\frac{1}{3^m n}\sum_{k=0}^{m}\binom{3m}{3k}(3n-1)^k$$

is an integer.

Solution. Let $\omega = \mathrm{e}^{\frac{2\pi i}{3}}$. Then

$$(1)\qquad \begin{aligned} &3\sum_{k=0}^{m}\binom{3m}{3k}(3n-1)^k \\ &= \left(1+\sqrt[3]{3n-1}\right)^{3m} + \left(1+\omega\sqrt[3]{3n-1}\right)^{3m} + \left(1+\omega^2\sqrt[3]{3n-1}\right)^{3m}. \end{aligned}$$

The right side of the above equality is the sum of the $3m$-th power of the roots x_1, x_2, x_3 of the polynomial

$$(X-1)^3 - (3n-1) = X^3 - 3X^2 + 3X - 3n.$$

Let $s_k = x_1^k + x_2^k + x_3^k$. Then $s_0 = s_2 = s_2 = 3$ and

$$(2)\qquad s_{k+3} = 3s_{k+2} - 3s_{k+1} + 3ns_k.$$

It follows by induction that each s_k is an integer divisible by $3^{[\frac{k}{3}]+1}$. A repeated application of (2) yields

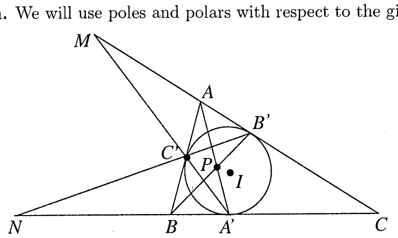
$$s_{k+7} = 63ns_{k+2} - 9(n^2-3n-3)s_{k+1} + 27n(2n+1)s_k.$$

Since $s_3 = 9n$, it follows inductively that s_{6k+3} is divisible by $3^{2k+2}n$ for all non-negative integers k, and the conclusion follows by (1).

PROBLEM 11. The incircle of the non-isosceles triangle ABC has center I and it touches in A', B', C' the sides BC, CA, AB, respectively. The straight lines AA' and BB' intersect in P, AC and $A'C'$ in M and $B'C'$ and BC in N.

Prove that the straight lines IP and MN are perpendicular.

Solution. We will use poles and polars with respect to the given circle.



The polar of B is $A'C'$ and M belongs to $A'C'$, therefore the polar m of M passes through B. It follows that m is BB'. In the same way the polar of N is AA'. These show that the pole of MN is P, the intersection of the lines AA' and BB', therefore $MN \perp IP$.

PROBLEM 12. Let $n \geqslant 2$ be an integer and $a_1, a_2, \ldots, a_n$ be real numbers. Prove that for any non-empty subset $S \subset \{1, 2, \ldots, n\}$ the following inequality holds:

$$\Big(\sum_{i\in S} a_i\Big)^2 \leqslant \sum_{1\leqslant i\leqslant j\leqslant n}(a_i+\cdots+a_j)^2.$$

Solution. Let $S = \{i_1, i_1+1, \ldots, j_1, i_2, i_2+1, \ldots, j_2, \ldots, i_p, \ldots, j_p\}$ the ordering of S where $j_k < i_{k+1} - 1$ for $k = 1, 2, \ldots, p-1$. Put $s_p = a_1 + a_2 + \cdots + a_p$, $s_0 = 0$. Then

$$\sum_{i\in S} a_i = s_{j_p} - s_{i_p-1} + s_{j_{p-1}} - s_{i_{p-1}-1} + \cdots + s_{j_1} - s_{i_1-1}$$

and

$$\sum_{1\leqslant i\leqslant j\leqslant n}(a_i+\cdots+a_j)^2 = \sum_{0\leqslant i\leqslant j\leqslant n}(s_j - s_i)^2.$$

It suffices to prove an inequality of the form

$$(1)\qquad (x_1 - x_2 + \cdots + (-1)^{l+1}x_l)^2 \leqslant \sum_{1\leqslant i<j<l}(x_j-x_i)^2 + \sum_{i=1}^{l} x_i^2,$$

because this means neglecting some non-negative terms in the right-hand member of the given inequality.

Inequality (1) reduces to

$$4\sum_{\substack{1\leqslant i<j\leqslant l\\ j-i=\text{even}}} x_i x_j \leqslant (l-1)\sum_{i=1}^{l} x_i^2.$$

This can be obtained by adding up inequalities of the form $4x_ix_j \leqslant 2(x_i^2 + x_j^2)$, $i<j$, $j-i =$ even (for an odd i, x_i takes part in $\left[\frac{l-1}{2}\right]$ such inequalities, and for an even i, x_i takes part in $\left[\frac{l}{2}\right] - 1$ of them).

Fourth Selection Test

PROBLEM 13. Let m, $m \geqslant 2$, be an integer. A positive integer n is called m-good if for every positive integer a, relatively prime to n, one has $n | a^m - 1$.
Show that any m-good number is at most $4m(2^m - 1)$.

Solution. If m is odd then $n|(n-1)^m - 1$ implies $n|2$, hence $n \leqslant 2$.
Take now $m = 2^t q$, $t \geqslant 1$, q odd. If $n = 2^u(2v+1)$ is m-good, then $(2v+1)|(2v-1)^m - 1$, whence $(2v+1)|2^m - 1$. Also, if $a = 8v+5$ then $(a, n) = 1$, so

$$2^u | (a^q)^{2^t} - 1 = (a^q - 1)(a^q + 1)(a^{2q} + 1) \cdots (a^{2^{t-1}q} + 1).$$

But $a^q \equiv 5 \pmod 8$ implies that the exponent of the factor 2 in the last product is $t+2$, therefore $u \leqslant t+2$, whence $n \leqslant 4 \cdot 2^t(2v+1) \leqslant 4m(2^m - 1)$.

PROBLEM 14. A point O is situated in the triangle's ABC plane. A circle $\mathcal{C}$ passing through O meets the second time OA, OB, OC in P, Q, R, respectively, and $\mathcal{C}$ intersects the second time the circles $(B,O,C), (A,O,C), (A,O,B)$ in K, L, M respectively.
Prove that PK, QL, RM are concurrent.

Solution. An inversion of center O transforms the circle $\mathcal{C}$ into a straight line d. Points P, Q, R become the intersections of d with the straight lines OA, OB, OC and K, L, M become the common points of d with BC, CA, AB respectively.
Thus the conclusion reduces to proving that the circles $(O,P,K), (O,Q,L)$, (O,R,M) have a second common point. In order to do this, it is enough to show that they have a common radical axis.
Let X be the intersection of the radical axis of the circles $(O,Q,L), (O,R,M)$ with d. Then, considering oriented segments, we get $\overline{XQ} \cdot \overline{XL} = \overline{XR} \cdot \overline{XM}$ and it is enough to prove that $\overline{XP} \cdot \overline{XK} = \overline{XQ} \cdot \overline{XL}$.
By Menelaus theorem for the triangles OPQ, OQR, ORP and straight lines AB, BC, CA respectively, we get

$$\frac{\overline{OA}}{\overline{PA}} \cdot \frac{\overline{PM}}{\overline{QM}} \cdot \frac{\overline{QB}}{\overline{OB}} = 1, \quad \frac{\overline{OB}}{\overline{QB}} \cdot \frac{\overline{QK}}{\overline{RK}} \cdot \frac{\overline{RC}}{\overline{OC}} = 1, \quad \frac{\overline{OC}}{\overline{RC}} \cdot \frac{\overline{RL}}{\overline{PL}} \cdot \frac{\overline{PA}}{\overline{OA}} = 1$$

whence $\overline{PM} \cdot \overline{QK} \cdot \overline{RL} = \overline{QM} \cdot \overline{RK} \cdot \overline{PL}$.
Take now coordinates on d, with X as origin (denoted with same but small letters). Then $(m-p)(k-q)(l-r) = (m-q)(k-r)(l-p)$ and $ql = mr$, which leads to $(pk - ql)(r + m - q - l) = 0$.
If $r + m = q + l$ we get $\{M, R\} = \{Q, L\}$, in which case the conclusion is obvious; otherwise $pk = ql$, whence $\overline{XP} \cdot \overline{XK} = \overline{XQ} \cdot \overline{XL}$.

PROBLEM 15. Some of the n faces of a polyhedron are colored in black, in such a way that any two black faces have no vertex in common. All other faces are colored in white.
Prove that the number of edges that are common borders of two white faces is at least $n - 2$.

Solution. By Euler's formula, $E = V + F - 2$, where E, V and F respectively, denote the number of edges, vertices and faces of the polyhedron. Let further V_0 and E_0 denote the number of vertices and edges belonging to black faces. Since every vertex from V_0 belongs to only one black face, it follows that $V_0 = E_0$. Therefore

$$E - E_0 = V - E_0 + F - 2 = V - V_0 + F - 2 \geqslant F - 2 = n - 2,$$

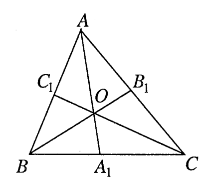
so there are at least $n - 2$ edges which do not belong to a black face.

Fifth Selection Test

PROBLEM 16. Consider a triangle ABC and O be a point in its interior. The straight lines OA, OB, OC meet the sides of the triangle in A_1, B_1, C_1, respectively. Let R_1, R_2, R_3 be the radii of the circles $(O,B,C), (O,C,A), (O,A,B)$ respectively and R the radius of the circumcircle of the triangle ABC. Prove that

$$\frac{OA_1}{AA_1}R_1 + \frac{OB_1}{BB_1}R_1 + \frac{OC_1}{CC_1}R_1 \geqslant R.$$

Solution.



Notice that

$$\frac{OA_1}{AA_1} = \frac{\text{area}(OBC)}{\text{area}(ABC)} = \frac{OB \cdot OC \cdot BC}{4R_1} \cdot \frac{4R}{AB \cdot AC \cdot BC},$$

so we have to prove that

$$\sum OB \cdot OC \cdot BC \geqslant AB \cdot AC \cdot BC.$$

Consider the complex coordinates $O(0), A(a), B(b), C(c)$. The conclusion reads

$$\sum |b| \cdot |c| \cdot |b - c| \geqslant |a - b| \cdot |b - c| \cdot |c - a|,$$

that is

$$\sum |b^2 c - c^2 b| \geqslant |ab^2 + bc^2 + ca^2 - a^2 b - b^2 c - c^2 a|,$$

which is obvious.

PROBLEM 17. A *move* on a $m \times n$ board consists of:

(i) Choosing some empty squares such that no two of them are in the same row or in the same column and placing a white stone on each of the selected squares;

(ii) Placing then a black stone on each empty square which has a white stone on his row and on his column.

What is the maximum number of white stones which can appear on the board after some moves have been made?

Solution. The answer is $m+n-1$.

We can obtain $m+n-1$ white stones by making succesively the moves $(1,1),(1,2),\ldots,(1,n),(2,n),\ldots,(m,n)$.

Take now a board on which stones have been placed according to the rules, and remove all the lines and colums which contain less than two white stones; in this way we remove at most $m+n-1$ white stones. If some white stone is remaining, then on every remaining line and column there are at least two white stones. Among the remaining white stones, consider one which has been placed with the last move. Then, with the same move, at least another white stone must have been placed on the same line or on the same column, which contradicts the established rules.

PROBLEM 18. Let p be an odd prime number and a_i, $i = 1,2,\ldots,p-1$, be the Legendre symbol of i relative to p (i.e. $a_i = 1$ if $i^{\frac{(p-1)}{2}} \equiv 1$ and $a_i = -1$ otherwise). Consider the polynomial

$$f = a_1 + a_2X + \cdots + a_{p-1}X^{p-2}.$$

a) Prove that 1 is a simple root of f if and only if $p \equiv 3 \pmod 4$.

b) Prove that if $p \equiv 5 \pmod 8$, then 1 is a root of order exactly two for f.

Solution. We shall call "residue" and "non-residue", the elements of the set $R = \{1,2,\ldots,p-1\}$ which are quadratic residues mod p and, respectively, non-quadratic residues mod p. Let Q be the set of residues and N be the set of non-residues. By definition, $a_i = 1$ if i is a residue and $a_i = -1$ if i is a non-residue. Since for $x, y \in \mathbb{R}$ the congruence $x^2 \equiv y^2 \pmod p$ is equivalent to $x = y$ or $x+y = p$, it follows that $Q = \{x^2 \,|\, x \in R\}$ has $\frac{p-1}{2}$ elements, that is $|Q| = |N|$, hence $f(1) = 0$ and f is divisible by $X-1$.

a) We have

$$f'(1) = \sum_{i=1}^{p-1}(i-1)a_i = \sum_{i=1}^{p} ia_i - \sum_{i=1}^{p} a_i$$
$$= \sum_{i=1}^{p} ia_i = \sum_{i\in Q} i - \sum_{i\in N} i.$$

It follows that $f'(1) = 0$ if and only if the sum of the residues equals the sum of non-residues. Since $\sum_{i\in Q} i + \sum_{i\in N} i = \frac{p(p-1)}{2}$, the latter is possible only if $p \equiv 1 \pmod 4$. In this case $-1 \in Q$ and $x \in Q$ implies $p - x \in Q$, therefore the $\frac{p-1}{2}$ elements of Q can be grouped in $\frac{p-1}{4}$ pairs, each pair having sum p, hence

$$\sum_{i\in Q} i = \frac{p(p-1)}{4} = \frac{1}{2}\sum_{i\in R} i.$$

b) We have already proved that in this case $(X-1)^2$ is a divisor of f. Notice that

$$f''(1) = \sum_{i=1}^{p-1}(i-1)(i-2)a_i$$
$$= \sum_{i=1}^{p-1} i^2a_i - 3\sum_{i=1}^{p-1} ia_i + 2\sum_{i=1}^{p-1} a_i = \sum_{i=1}^{p-1} i^2a_i.$$

We will use the fact that for a prime p of the form $8k+5$, we have $2 \in N$. Indeed

$$2\cdot 4\cdot 6\cdots(p-1) \equiv (2\cdot 4\cdots(4k+2))\cdot((4k+4)\cdots(8k+4))$$
$$\equiv (2\cdot 4\cdots(4k+2))\cdot((-4k-1)\cdots(-1)) \pmod p,$$

hence $2^{4k+2}\cdot(4k+2)! \equiv (-1)^{2k+1}\cdot(4k+2)! \pmod p$, so $2^{4k+2} \equiv -1 \pmod p$.

Consider the sets $Q_< = \{r \in Q \mid r < \frac{p}{2}\}$, $Q_> = \{r \in Q \mid \frac{p}{2}\}$, $N_< = \{r \in N \mid r < \frac{p}{2}\}$, $N_> = \{r \in N \mid r > \frac{p}{2}\}$.

Since $-1 \in Q$, sending $r \in R$ to $p - r \in R$ defines a one-to-one function from Q to Q, from N to N, form $Q_<$ to $Q_>$, and from $N_<$ to $N_>$. Therefore both $Q_<$ and $Q_>$ have $\frac{p-1}{4}$ elements and

$$\sum_{r\in Q_>} r^2 = \sum_{r\in Q_<}(p-r)^2 = \frac{p^2(p-1)}{4} - 2p\sum_{r\in Q_<} r + \sum_{r\in Q_<} r^2,$$

so

(1) $$\sum_{r\in Q} r^2 = \frac{p^2(p-1)}{4} - 2p\sum_{r\in Q_<} r + 2\sum_{r\in Q_<} r^2.$$

In the same way

(2) $$\sum_{r\in N} r^2 = \frac{p^2(p-1)}{4} - 2p\sum_{r\in N_<} r + 2\sum_{r\in N_<} r^2.$$

By the same arguments, sending r to $2r$ (respectively to $p-2r$) defines one-to-one maps from $Q_<$ to the sets N_{even} (respectively, N_{odd}) of even (respectively, odd) non-residues. Therefore,

$$\sum_{r\in N} r^2 = \sum_{r\in R_<} (p-2r)^2 + 4\sum_{r\in Q_<} r^2 = \frac{p^2(p-1)}{4} - 4p\sum_{r\in Q_<} r + 8\sum_{r\in Q_<} r^2. \tag{3}$$

A similar argument yields

$$\sum_{r\in Q} r^2 = \frac{p^2(p-1)}{4} - 4p\sum_{r\in N_<} r + 8\sum_{r\in N_<} r^2. \tag{4}$$

Combining (1) and (3), then (2) and (4), we get

$$\sum_{r\in Q} r^2 - \sum_{r\in N} r^2 = 2\Big(p\sum_{r\in Q_<} r - 3\sum_{r\in Q_<} r^2\Big),$$
$$\sum_{r\in Q} r^2 - \sum_{r\in N} r^2 = -2\Big(p\sum_{r\in N_<} r - 3\sum_{r\in N_<} r^2\Big).$$

Suppose now that $f''(1) = 0$, that is

$$\sum_{r\in Q} r^2 = \sum_{r\in N} r^2. \tag{5}$$

Using (1) and (4) we get

$$\sum_{r\in Q} r^2 = \frac{p^2(p-1)}{4} - \frac{4}{3}p\sum_{r\in Q_<} r$$
$$\sum_{r\in Q} r^2 = \frac{p^2(p-1)}{4} - \frac{4}{3}p\sum_{r\in N_<} r,$$

which leads to $\sum_{r\in Q_<} r = \sum_{r\in N_<} r$. On the other hand

$$\sum_{r\in R_<} r + \sum_{r\in N_<} r = \sum_{r<p/2} r = \frac{p^2-1}{8} = \text{odd},$$

a contradiction. This shows that the assumption in (5) is false.

Supplementary Test

PROBLEM 19 Consider a sequence of positive different integers $(a_n)_n$ such that there is $a > 0$ satisfying

$$a_n \leqslant an, \text{ for all positive integers } n.$$

Prove that:

i) If $a < 5$ the sequence contains infinitely many of numbers for which the sum of their digits (in decimal representation) is not a multiple of 5.

ii) The above result for $a = 5$.

Solution. Consider $B = (b_n)_n$ the strictly increasing sequence of all the nonnegative integers having the sum of their digits a multiple of 5, i.e.:

$$b_0 = 0, b_1 = 5, b_2 = 14, b_3 = 19, b_4 = 23, \ldots$$

We shall use the following easy result.

Lemma. For all positive n we have $5n \leqslant b_n \leqslant 5n+4$.

Proof. Consider the last digit of the numbers $5n, 5n+1, 5n+2, 5n+3, 5n+4$ and notice that the union of the numerical intervals $[5n, 5n+4]$ is the set of non-negative integers.

Let us now assume that a_n takes only a finite number of values with sum of digits not a multiple of 5, i.e. there exists an integer $M > 0$ such that $n \geqslant M$ implies $a_n \in B$. But then, for any integer $N > 0$, consider the numbers $M, M+1, \ldots, M+N$. Since $a_{M+k} \in B$ for $k = 0, 1, 2, \ldots, N$, and because a_n takes each value at most once, we have

$$5N \leqslant b_N \leqslant \max_{0\leqslant k\leqslant N} a_{M+k} \leqslant \max_{0\leqslant k\leqslant N} a(M+k) = a(M+N),$$

hence $N \leqslant \dfrac{aM}{5-a}$, absurd for $a < 5$, as N is arbitrarily large.

For $a = 5$, take $a_0 = 1$ and $a_n = b_{n-1}$, for $n > 0$; we have $b_{n-1} \leqslant 5(n-1) + 4 = 5n - 1 < 5n$, so $a_n < 5n$ for $n > 0$ but $a_n \in B$.

PROBLEM 20. Given an integer number $n \geqslant 1$, consider n distinct unit vectors in the plane, which have a common origin at some point O. Suppose further that for some non-negative integer $m < \frac{n}{2}$, on either side of any straight line passing through O lie at least m of these vectors. Prove that the length of the sum of all n vectors cannot exceed $n - 2m$.

Solution. Throughout the proof vectors are written in bold-face characters. Let $\mathbf{v}$ denote the sum of all n vectors. Leaving aside the trivial case $\mathbf{v} = \mathbf{0}$, assume henceforth $\mathbf{v} \neq \mathbf{0}$ and consider a standard orthogonal frame xOy whose positive x-axis is directed along $\mathbf{v}$. The plane punctured at O comes out with a natural partition into four quadrants :

$$Q_1 = \{(x,y) \mid x > 0 \text{ and } y \geqslant 0\},$$
$$Q_2 = \{(x,y) \mid x \leqslant 0 \text{ and } y > 0\},$$
$$Q_3 = \{(x,y) \mid x < 0 \text{ and } y \leqslant 0\},$$
$$Q_4 = \{(x,y) \mid x \geqslant 0 \text{ and } y < 0\}.$$

Let further $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be an anti-clockwise sequential order of the given vectors. Thus, for some integers p, q and r, $0 \leqslant p \leqslant q \leqslant r \leqslant n$,

a) the $\mathbf{v}_i$ lie in Q_1 for $1 \leqslant i \leqslant p$, and the order is $\mathbf{v}_1, \ldots, \mathbf{v}_p$;
b) the $\mathbf{v}_i$ lie in Q_2 for $p < i \leqslant q$, and the order is $\mathbf{v}_{p+1}, \ldots, \mathbf{v}_q$;
c) the $\mathbf{v}_i$ lie in Q_3 for $q < i \leqslant r$, and the order is $\mathbf{v}_{q+1}, \ldots, \mathbf{v}_r$;
d) the $\mathbf{v}_i$ lie in Q_4 for $r < i \leqslant n$, and the order is $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$.

The cases $p = 0$, or $p = q$, or $q = r$, or $r = n$ are not necessarily excluded; they simply mean that no $\mathbf{v}_i$ lies in Q_1, or in Q_2, or in Q_3, or in Q_4, respectively. Applied to the lines $x = 0$ and $y = 0$, the condition that on either side lie at least m of the $\mathbf{v}_i$ yields

$$(1) \qquad m + p \leqslant r\,, \quad \text{and} \quad m + r \leqslant n + p\,,$$

on the one hand, and

$$(2) \qquad m \leqslant q\,, \quad \text{and} \quad m + q \leqslant n\,,$$

on the other. By (2) above, the vectors

$$\mathbf{v}_{q-m+i} + \mathbf{v}_{q+i}\,, \quad 1 \leqslant i \leqslant m\,,$$

are well defined. We claim that they always point at the left half-plane $x \leqslant 0$; that is, they have a non-positive x-component :

$$(3) \qquad \mathbf{v} \cdot (\mathbf{v}_{q-m+i} + \mathbf{v}_{q+i}) \leqslant 0\,, \quad 1 \leqslant i \leqslant m\,.$$

To prove (3), fix an index i, $1 \leqslant i \leqslant m$, and note that $\mathbf{v}_{q-m+i}$ always lies in $Q_1 \cup Q_2$, and $\mathbf{v}_{q+i}$ always lies in $Q_3 \cup Q_4$. Clearly, only the cases where $\mathbf{v}_{q-m+i}$ lies in Q_1 or $\mathbf{v}_{q+i}$ lies in Q_4 need to be dealt with.

In the first case, $q - m + i \leqslant p$, so $q + i \leqslant m + p \leqslant r$ by (1); that is, $\mathbf{v}_{q+i}$ must lie in Q_3. Recalling the way the vectors have been ordered and that at least m of them lie on either side of the line along $\mathbf{v}_{q-m+i}$, we deduce that $\mathbf{v}_{q+i} = \mathbf{v}_{(q-m+i)+m}$ cannot lie outside the angle $\angle(-\mathbf{v}, -\mathbf{v}_{q-m+i})$ situated in Q_3 and (3) follows.

In the second case, $r + 1 \leqslant q + i$, so $p + 1 \leqslant r - m + 1 \leqslant q - m + i$ by (1); that is, $\mathbf{v}_{q-m+i}$ must lie in Q_2. As in the first case, we then deduce that $\mathbf{v}_{q-m+i} = \mathbf{v}_{(q+i)-m}$ cannot lie outside the angle $\angle(-\mathbf{v}, -\mathbf{v}_{q+i})$ situated in Q_2 and (3) follows again.

Finally, since the length of the sum of the remaining $n - 2m$ unit vectors cannot exceed $n - 2m$, we conclude by (3) that neither can the length of $\mathbf{v}$.

2.4. THE NATIONAL MATHEMATICAL OLYMPIAD

Selection Tests for the JBMO 2004

PROBLEM 1. Find all positive real numbers a, b, c which satisfy the inequalities

$$4(ab + bc + ca) - 1 \geqslant a^2 + b^2 + c^2 \geqslant 3(a^3 + b^3 + c^3).$$

Solution. Using the Chebyshev inequality we derive

$$(a + b + c)(a^2 + b^2 + c^2) \leqslant 3(a^3 + b^3 + c^3),$$

hence $a + b + c \leqslant 1$. On the other hand,

$$4(ab + bc + ca) - 1 \geqslant a^2 + b^2 + c^2 \geqslant ab + bc + ca,$$

therefore $ab + bc + ca \geqslant 1$. As

$$3(ab + bc + ca) \leqslant (a + b + c)^2 \leqslant 1,$$

we obtain $a + b + c \geqslant 1$, thus $a + b + c = 1$. Consequently, $a + b + c = 1$ and $3(ab + bc + ca) = (a + b + c)^2$, which imply

$$a = b = c = \frac{1}{3}.$$

PROBLEM 2. Consider the numbers defined by $a_n = 3n + \sqrt{n^2 - 1}$ and $b_n = 2(\sqrt{n^2 + n} + \sqrt{n^2 - n})$, for all $n = 1, 2, \ldots, 49$. Prove that there are integers A, B so that

$$\sqrt{a_1 - b_1} + \sqrt{a_2 - b_2} + \cdots + \sqrt{a_n - b_n} = A + B\sqrt{2}.$$

Solution. The key idea is to observe that

$$a_n - b_n = \frac{1}{2}\left(\sqrt{n - 1} - 2\sqrt{n} + \sqrt{n + 1}\right)^2.$$

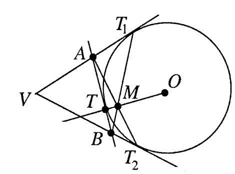
As $2\sqrt{n} \geqslant \sqrt{n-1} + \sqrt{n-1}$, it follows that the sum is $4\sqrt{2} - 5$.

PROBLEM 3. Consider a circle of center O and V a point externally to the circle. The tangents from V touch the circle at points T_1, T_2. Let T be a point on the small arc T_1, T_2 of the circle. The tangent at T intersects the line VT_1 in the point A and the lines TT_1 and VT_2 intersect in the point B. Let M be the intersection point of the lines BT_1 and AT_2.

Prove that lines OM and AB are perpendicular.

Solution. The approach of the problem is to see no circles in the figure. Instead, recall that a quadrilateral $ABCD$ is orthogonal if and only if

$$AB^2 + CD^2 = AD^2 + BC^2.$$



Using succesively the Pytagoras theorem we have

$$\begin{aligned} BA^2 - BT_2^2 &= BA^2 - (BO^2 - OT_2^2) = BA^2 - (BO^2 - OT_1^2) \\ &= BA^2 - (BA^2 - AT_1^2) = AT_1^2 = AO^2 - OT_1^2 = OA^2 - OT_2^2, \end{aligned}$$

so the conclusion follows from the relation.

PROBLEM 4. Consider a cube and let M, N be two of its vertices. Assign the number 1 to these vertices and 0 to the other six vertices. We are allowed to select a vertex and to increase with a unit the numbers written in the 3 adjiacent vertices and call this a *movement*. Prove that there is a sequence of movements after that all the numbers assigned to all the vertices of the cube are equal, if and only if MN is not a diagonal of a face of the cube.

Solution. Color the 8 vertices of the cube in black or white so that the 4 vertices of the 2 regular thetrahedrons have the same color; notice that the 3 vertices adjiacent to a vertex have its opposite color. Therefore, each movement increase the sum of the numbers assingned to the vertices sharing the same color by 3. Consider the cases:

1) MN is a diagonal of a face of the cube. Then M and N have the same color, say black. Assume by contradiction that there is a sequence of movements after which the same number n is assigned to all the vertices. Let k_1 and k_2 be the number of white, respectively black vertices that were selected to perform the movements. Then

$$4n = 3k_1 + 2 = 3k_2,$$

a contradiction.

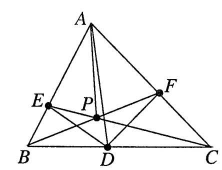
2) MN is a diagonal of the cube. Selecting the vertices M, then N, and performing these 2 movements, to all the vertices the number 1 will be assigned, as needed.

3) MN is a side of the cube. The same outcome as in the previous case will occur after 2 movements when selecting the diagonally opposite vertices of M and N.

This provides us with a full solution.

PROBLEM 5. Let ABC be an acute triangle and let D be a point on the side BC. Points E and F are the projections of the point D on the sides AB and AC, respectively. Lines BF and CE meet at point P. Prove that AD is the bisector line of the angle BAC if and only if lines AP and BC are perpendicular.

Solution. Let $a, b, c, x.y$ be the lengths of the sides BC, CA, AB, BD, DC, respectively and let A' be the foot of the altitude from A in the triangle ABC. Notice that $x + y = a$.



Due to the Ceva theorem, the claim is equivalent to

$$\frac{BD}{DC} \cdot \frac{CF}{FA} \cdot \frac{AE}{EB} = 1, \text{ which is equivalent to } \frac{AB}{AC} = \frac{BD}{DC}.$$

As $CF = y\cos C, FA = b - y\cos C, BE = x\cos B, AE = c - x\cos B, BA' = c\cos B$ and $A'C = b\cos C$, the equivalence rewrites

$$cy(c - x\cos B) = bx(b - y\cos C),$$

which is the same as $xb = cy$. Indeed, we have

$$c^2y - cxy\frac{a^2 + c^2 - b^2}{2ac} = b^2x - bxy\frac{a^2 + b^2 - c^2}{2ab}$$

equivalent to $a(c^2y - b^2x) = xy(c^2 - b^2)$, or $c^2y(a - x) = b^2x(a - y)$. This gives $c^2y^2 = b^2x^2$, equivalent to $cy = bx$, as claimed.

PROBLEM 6. An array 8×8 consists of 64 unit squares. Inside each square are written the numbers 1 or -1 so that in any 2×2 subarray the sum of the four numbers equals 2 or -2. Prove that there exist two rows in the array which the same numbers are inscribed in the same order.

Solution. The main idea is to observe that two consecutive rows have exactly 4 equal elements, namely those lying on the columns $1,3,5,7$ or $2,4,6,8$. Moreover, on the other 4 columns the elements are different. Without loss of generality, assume that rows 1 and 2 are equal with respect to the columns $1,3,5,7$ and different on the column $2,4,6,8$; we call these rows *odd equal*. If rows 2 and 3 are also odd equal, then rows 1 and 3 are equal, as needed. If not, then rows 2 and 3 are *even equal*. Now consider the rows 3 and 4; we are done if the rows are even equal, so assume that they are odd equal. Finnaly, if rows 4 and 5 are odd equal,then rows 3 and 5 are equal, and if rows 4 and 5 are even equal, then rows 1 and 5 are equal. This concludes the proof.

PROBLEM 7. Consider a triangle ABC with the side lenghts a,b,c so that a is the greatest. Prove that the triangle is rightangled if and only if

$$\left(\sqrt{a+b}+\sqrt{a-b}\right)\left(\sqrt{a+c}+\sqrt{a-c}\right)=(a+b+c)\sqrt{2}.$$

Solution. Squaring both sides of the equality yields

$$2\left(a+\sqrt{a^2-b^2}\right)\left(a+\sqrt{a^2-c^2}\right)=(a+b+c)^2.$$

It is easy to observe that the equality holds if $a^2=b^2+c^2$. To prove the converse statement, assume that $a^2\geqslant b^2+c^2$. Then $\sqrt{a^2-b^2}>c$ and $\sqrt{a^2-c^2}>b$, hence

$$\begin{aligned}2\left(a+\sqrt{a^2-b^2}\right)\left(a+\sqrt{a^2-c^2}\right)&>2(a+b)(a+c)=2a^2+2(ab+bc+ca)\\ >a^2+b^2+c^2+2(ab+bc+ca)&=(a+b+c)^2,\end{aligned}$$

false. The case $a^2\leqslant b^2+c^2$ leads similarily to a contradiction, and we are done.

PROBLEM 8. Find all positive integers n for which there are distinct integer numbers $a_1,a_2,\cdots,a_n$ such that

$$\frac{1}{a_1}+\frac{2}{a_2}+\ldots+\frac{n}{a_n}=\frac{a_1+a_2+\cdots+a_n}{2}.$$

Solution. Rearrange the numbers $a_1,a_2,\cdots,a_n$ in ascending order: $b_1<b_2<\ldots<b_n$. Obviously, $k\leqslant b_k$, and substituting b_k with k, the left-hand side term increases. Futhermore, by the rearrangements inequality we infer that the maximum value of the left-hand side term is

$$\frac{1}{n}+\frac{2}{n-1}+\cdots+\frac{n}{1}.$$

On the other side, the right-hand side term is greater than or equal to

$$\frac{1+2+\cdots+n}{2}=\frac{n(n+1)}{4}.$$

We have

$$\begin{aligned}\frac{1}{n}+\frac{2}{n-1}+\cdots+\frac{n}{1}&=\sum_{k=1}^{n}\frac{n-k+1}{k}=(n+1)\sum_{k=1}^{n}\frac{1}{k}-n\\ &=1+(n+1)\sum_{k=2}^{n}\frac{1}{k}=(n+1)\sum_{k=2}^{n+1}\frac{1}{k}.\end{aligned}$$

For $n>6$ we prove by induction on n that

$$\frac{n}{4}\geqslant\sum_{k=2}^{n+1}\frac{1}{k},$$

which implies that the given equality cannot hold. Indeed, for $n=7$ we have $\frac{7}{4}=1.75\geqslant\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{8}=1.51\ldots$. If the inequality holds for $n>7$ then it is true for $n+1$, as $\frac{1}{4}\geqslant\frac{1}{n+1}$.

We are left with the cases when $n=2,3,4,5,6$. Clearly, the case case $n=2$ is impossible.

For $n=3$ we have the numbers $a_1=1, a_2=2$ and $a_3=3.$, so $n=3$ is a solution.

If $n=4$, then

$$a_1+a_2+a_3+a_4=2\left(\frac{1}{a_1}+\frac{2}{a_2}+\frac{3}{a_3}+\frac{4}{a_4}\right)\leqslant2\left(\frac{1}{4}+\frac{2}{3}+\frac{3}{2}+\frac{4}{1}\right)<13,$$

so $a_1+a_2+a_3+a_4\leqslant12$. By inspection, all the cases: $\{a_1,a_2,a_3,a_4\}=\{1,2,3,4\}$, $\{1,2,3,5\}$, $\{1,2,4,5\}$ and $\{1,2,3,6\}$ fail to satisfy the required relation.

If $n=5$, then

$$\begin{aligned}a_1+a_2+a_3+a_4+a_5&=2\left(\frac{1}{a_1}+\frac{2}{a_2}+\frac{3}{a_3}+\frac{4}{a_4}+\frac{5}{a_5}\right)\\ &\leqslant2\left(\frac{1}{5}+\frac{2}{4}+\frac{3}{3}+\frac{4}{2}+\frac{5}{1}\right)<17.4,\end{aligned}$$

so $a_1+a_2+a_3+a_4+a_5\leqslant17$. We study the cases $\{a_1,a_2,a_3,a_4,a_5\}=\{1,2,3,4,5\}$, $\{1,2,3,4,6\}$ and $\{1,2,3,5,6\}$ with no succes (for an easy argument, observe that 5 must be a_5 and so on).

Finally, for $n=6$ we have $a_1+a_2+\cdots+a_6\leqslant22$, thus $\{a_1,a_2,a_3,a_4,a_5,a_6\}$ can be $\{1,2,3,4,5,6\}$ or $\{1,2,3,4,5,7\}$. The last case fails immediately because of 7, and the same outcame is for the first one.

Therefore $n=3$.

PROBLEM 9. In a chess tournament each of the players have played with all the others two games, one time with the white and then with the black. In each game the winners gets one point and both players receive 0.5 points if the game ends with a tie. At the end of the tournament, all the players end with the same number of points.

a) Prove that there are two players which have the same number of ties.

b) Prove that there are two players which have the same number of defeates when playing the white.

Solution. Let n be the number of players in the tournament. The total numbers of matches is $n(n-1)$, hence each player end up with $n-1$ points.

a) Assume by contradiction that each player has a different number of ties. As a tie gives 0.5 points, it follows that each player has an odd number of ties. Since the possible cases are: $0, 2, 4, \ldots, 2(n-1)$, we infer that each of these numbers is assigned to each of the players. Consider A the player with 0 ties and B the player with $2(n-1)$ ties. Each player has played $2(n-1)$ matches, hence B obtained a tie in each match played. The match A-B thus ended with a tie, a contradiction, since A has no ties.

b) Suppose the contrary. Then each of the n players has $0, 1, \ldots, n-1$ losses when playing the white. Let X and Y be the players with 0 and $n-1$ losses, respectively. The player Y has no points when playing the white and $n-1$ points, so he won all the matches with the black pieces. This implies that the match X-Y is won by Y, so is lost by X, a contradiction, since X has 0 losses with the white pieces.

PROBLEM 10. Consider the triangle ABC with $AB = AC$ and a variable point M on the line BC so that B is between M and C. Prove that the sum of the inradius of AMB and the exradius of AMC corresponding to the angle M is constant.

Solution. The Stewart relation gives $AM^2 \cdot BC + AC^2 \cdot MB = AB^2 \cdot MC + MB \cdot MC \cdot BC$, so $AM^2 \cdot BC = AB^2(MC - MB) + MB \cdot MC \cdot BC$, hence $AM^2 = AB^2 + MB \cdot MC$. Let r be the inradius of triangle AMB and R the exradius of triangle AMC corresponding to the angle $\angle M$.

Since $r = \dfrac{2\text{area}\, AMB}{AM + MB + AB}$ and $R = \dfrac{2\text{area}\, AMC}{MA + MB - AB}$, then $\dfrac{r}{h} = \dfrac{MB}{AM + MB + AB}$ and $\dfrac{R}{h} = \dfrac{MC}{AM + MC - AB}$. Thus $r + R = h$ is equivalent to

$$MB(MA + MB - AB) + MC(MA + MB + AB)$$
$$= (MA + MB + AB)(MA + MB - AB),$$

which turns to be equivalent to $MB(MA+MB-AB) = (MA+MB+AB)(MA-AB)$, or to $MB \cdot MC + MB(MA - AB) = MA^2 - AB^2 + MB(MA - AB)$. The last equality reduces to

$$MA^2 = AB^2 + MB \cdot MC,$$

so the claim holds.

PROBLEM 11. Let p, q, r be primes and let n be a positive integer such that

$$p^n + q^n = r^2.$$

Prove that $n = 1$.

Solution. Clearly one of the primes p, q or r is equal to 2. If $r = 2$ then $p^n + q^n = 4$, false, so assume that $p > q = 2$.

Consider the case when $n > 1$ is odd; we have

$$(p+2)(p^{n-1} - 2p^{n-2} + 2^2p^{n-3} - \cdots + 2^{n-1}) = r^2.$$

Notice that $p^{n-1} - 2p^{n-2} + 2^2p^{n-3} - \cdots + 2^{n-1} = 2^{n-1} + (p-2)(p^{n-2} + p^{n-4} + \cdots) > 1$ and $p+2 > 1$ hence both factors are equal to r. This rewrites as $p^n + 2^n = (p+2)^2 = p^2 + 4p + 4$, which is false for $n \geqslant 3$.

Consider the case when $n > 1$ is even and let $n = 2m$. It follows that $p^m = a^2 - b^2, 2^m = 2ab$ and $r = a^2 + b^2$, for some integers a, b with $(a, b) = 1$. Therefore, a and b are powers of 2, so $b = 1$ and $a = 2^{m-1}$. This implies $p^m = 4^{m-1} - 1 < 4^m$, so p must be equal to 3. The equality $3^m = 4^{m-1} - 1$ fails for $m = 1$ and also for $m \geqslant 2$, as $4^{m-1} > 3^m + 1$, by induction.

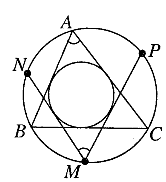
Consequently $n = 1$; take for example $p = 23$, $q = 2$ and $r = 5$.

PROBLEM 12. Let $a < b \leqslant c < d$ be positive integers so that $ad = bc$ and $\sqrt{d} - \sqrt{a} < 1$. Prove that a is a perfect square.

Solution. Consider the integers $0 < m \leqslant n < p$ so that $b = a + n, c = a + m$ and $d = a + p$. Then $a(a+p) = (a+m)(a+n)$ and $a + p \leqslant a + 1 + 2\sqrt{a}$. As $p = m + n + \frac{mn}{a}$ is an integer, then $a \leqslant mn$ and $p \geqslant m + n + 1$. On the other hand, $1 + 2\sqrt{a} \geqslant n + m + 1 \Rightarrow \sqrt{a} \geqslant \frac{m+n}{2} \geqslant \sqrt{mn}$, hence $a \geqslant mn$. Consequently, $a = mn, m = n$, so a is a square.

PROBLEM 13. Let ABC be a triangle inscribed in the circle K and consider a point M on the arc BC which does not contain A. The tangents from M to the incircle of ABC intersect the circle K at the points N and P. Prove that if $\angle BAC = \angle NMP$, then triangles ABC and MNP are congruent.

Solution.



Let Q be the intersection point of the line segments AB and MP. The tangents from A and M to the incirle are equal (as they are $r \cdot \cot \frac{A}{2}$). Moreover, the tangents from Q to the incircle are equal, so $AQ = MQ$. This implies $\angle QMA = \angle QAM$, so the arcs AP and BM are equal. In the trapezoid $APBM$, the diagonals AB and MP are equal, and likewise $AC = MN$. This concludes the proof.

PROBLEM 14. The real numbers $a_1, a_2, \ldots, a_n$ satisfies the relation

$$a_1^2 + a_2^2 + \cdots + a_{100}^2 + (a_1 + a_2 + \cdots + a_{100})^2 = 101.$$

Prove that $|a_k| \leqslant 10$, for all $k = 1, 2, \ldots, 100$.

Solution. Assume by contradiction that $|a_k| > 10$, for some k. Wlog, let $k = 1$. Then $a_1^2 > 100$ and $a_2^2 + a_3^2 + \ldots + a_{100} + s^2 < 1$, where $s = a_1 + a_2 + \ldots + a_{100}$. On the other hand, the Cauchy-Schwarz inequality yields

$$a_1^2 = (s - a_2 - a_3 + \ldots - a_{100})^2 \leqslant 100(a_2^2 + a_3^2 + \ldots + a_{100}^2 + s^2) < 100,$$

a contradiction.

PROBLEM 15. A finite set of positive integers is called *isolated* if the sum of the elements in any proper subset is a number relatively prime with the sum of the elements of the isolated set. Find all non-prime integers n for which there exist positive integers a, b so that the set $A = \{(a+b)^2, (a+2b)^2, \ldots, (a+nb)^2\}$ is isolated.

Solution.
The sum of the elements of the set A is $S = na^2 + n(n+1)b + \frac{n(n+1)(2n+1)}{6}b^2$. Assume that n has a prime divisor $p > 3$. Then $p|S$ and $p|(a+b)^2 + (a+2b)^2 + \cdots + (a+pb)^2 = pa^2 + p(p+1)b + \frac{p(p+1)(2p+1)}{6}b^2$, a contradiction. It remains $n = 2^k 3^l$, for some integers k, l. Suppose that $k > 1$. Then $2|S$ and so all elements of A must be odd. Taking the subset given by any pair we reach the contradiction. Finally, suppose that $l > 1$, so $3|S$. If one of the numbers $a+b, a+2b, a+3b$ is divisible by 3, then we have a contradiction; if not, then $3|b$. Then $3|(a+b)^2 + (a+2b)^2 + (a+3b)^2 = 3a^2 + 12ab + 14b^2$, again a contradiction.
We are left with $n = 6$. It satisfies the claim: the set $A = \{4, 9, 16, 25, 36, 49\}$ is isolated, because the sum of its elements is a prime number (139).

PROBLEM 16. A regular polygon with 1000 sides has its vertices colored in red, yellow or blue. A *move* consists in choosing to adjiacent vertices colored differently and coloring them in the third color. Prove that there is a sequence of *moves* after which all the vertices of the polygon will have the same color.

Solution. Let $A_1, A_2, \ldots A_{1000}$ be the vertices of the polygon. We start with two lemmas.

Lemma 1. Three of four consecutive vertices have the same color. Then after a sequence of moves all vertices will have the color of the fourth vertex.

Proof. Let the colors be 0,1 and 2. We have two cases:
a) $1110 \longrightarrow 1122 \longrightarrow 1002 \longrightarrow 2202 \longrightarrow 2112 \longrightarrow 0000$.
b) $1011 \longrightarrow 1221 \longrightarrow 0000$.

Lemma 2. Any 4 consecutive vertices will turn after several moves in the same color.

Proof. Form two pairs of consecutive vertices and change them in the same color - if they do not already have it. Then follow the sequence $1122 \longrightarrow 1002 \longrightarrow 2202 \longrightarrow 2112 \longrightarrow 0000$.

By the second lemma, after several moves the vertices A_1, A_2, A_3, A_4 will have the same color, say red. Likewise, A_5, A_6, A_7, A_8 will have the same color. Consider now the vertices A_4, A_5, A_6, A_7; the first is red and the other three have the same color. By the first lemma they all will turn red – of course, we do nothing if they were already red. We move on with this procedure until $A_1, A_2, \ldots A_{997}$ turn red (note that $997 = 4 + 3 \cdot 332$, so this requires 332 steps). Now consider the vertices $A_{998}, A_{999}, A_{1000}, A_1$; by the second lemma they all will share the same color. If this is red, we are done. If not, say that they are blue, and taking the vertices $A_{997}, A_{998}, A_{999}, A_{1000}$ we obtain - using the first lemma – all vertices to be red, except for A_1, which is blue. Now A_1, A_2, A_3, A_4 turn blue, then A_5, A_6, A_7, A_8 and so on. This time, after 333 steps, all the 1000 vertices ($1000 = 1 + 3 \cdot 333$) will be colored in blue.

Comment. Substituting colors with digits, notice that all moves: $01 \longrightarrow 22$, $02 \longrightarrow 11$ and $12 \longrightarrow 00$ preserve the sum (mod 3). This means that the final color is unique and, of course, is given by the sum of the digits assigned to the vertices of the initial configuration.

PROBLEM 17. Consider the triangular array

0	1	1	2	3	5	8	...
	0	1	1	2	3	5	...
		2	3	5	8	13	...
			4	7	11	18	...
				12	19	31	...

defined as follows:

i) on the first two rows, each element starting with the third is the sum of the two preceding elements;

ii) on the other rows each element is the sum of the two elements placed above of the same column.

a) Prove that all the rows are defined according to condition i).

b) Consider 4 consecutive rows and let a, b, c, d be the first element in each of these rows, respectively. Find d in terms of a, b and c.

Solution. a) In the array below

a	b	c
d	e	f
g	h	i

where i is an element of the third row, observe that $i = c + f = (a+b) + (d+e) = (a+d) + (b+c) = g + h$. The same argument holds for all the other rows, by induction.

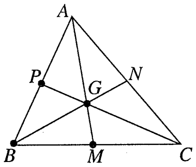
b) We prove that $d = 2b + 2c - a$. Indeed, from the array

$$\begin{array}{cccc} a & x & y & z \\ & b & t & u \\ & & c & v \\ & & & d \end{array},$$

we derive $d = u + v = b + t + (u + z) = 2(b + t) + (x + y) = 2b + 2(t + y) + x - y = 2b + 2c + x - (x + a) = 2b + 2c - a$.

PROBLEM 18. Let M, N, P be the midpoints of the sides BC, CA, AB of the triangle ABC, respectively, and let G be the centroid of the triangle. Prove that if $BMGP$ is cyclic and $2BN = \sqrt{3}AB$, then triangle ABC is equilateral.

Solution. By the power of a point theorem we have $AG \cdot AM = AP \cdot AB$, so $4MA^2 = 3AB^2$ and thus $AM = \frac{\sqrt{3}}{2}AB = BN$.



Then $AG = GB$, so the median GP is also an altitude in the triangle AGB. This implies $\angle BPG = 90°$, and since $BMGP$ is cyclic, $\angle GMA = 90°$. It follows that $BC = CA$ and $AB = AC$, so the triangle is equilateral.

PROBLEM 19. Let A be a set of positive integers with the properties:
i) if $a \in A$, then all positive divisors of a are elements of A;
ii) if $a, b \in A$ and $1 < a < b$, then $1 + ab \in A$.
Prove that if the set A has at least 3 elements, then $A = \mathbb{N}^*$.

Solution. It is obvious that $1 \in A$, since 1 is a divisor of any integer. Consider a, b two elements of A with $1 < a < b$. Since at least one of a, b or $1 + ab$ is even, then 2 is an element of A.

We induct on $n \geqslant 6$ to prove that $n \in A$. Assume that $k \in A$ for all $k = 1, 2, \ldots, n - 1$. If n is odd, then $n = 2p + 1$ with $1 < 2 < p \in A$, hence $n \in A$. If n is even, then $n = 2p$. As above, $2p - 1$ and $2p + 1$ are elements of A and consequently $1 + (2p - 1)(2p + 1) = 4p^2 \in A$. The first property implies $n = 2p \in A$, as needed.

To complete the proof, we need to show that $3, 4, 5 \in A$. For this, consider $a > 2$ an element of A. Then $1 + 2 \cdot a \in A$, $1 + 2(1 + 2a) = 3 + 4a \in A$ and $1 + (1 + 2a)(3 + 4a) = 4 + 10a + 8a^2 \in A$. If a is even, then $4 | 4 + 10a + 8a^2$ and so $4 \in A$. If a is odd, then choose a to be $4 + 10a + 8a^2$ and again $4 \in A$. Next, as $1 < 2 < 4 \in A$ we have $1 + 2 \cdot 4 = 9 \in A$ and so $3 \in A$. Finally, $7 = 1 + 2 \cdot 3 \in A, 15 = 1 + 2 \cdot 7 \in A$, hence $5 \in A$ and we are done.

PROBLEM 20. Consider a convex polygon with $n \geqslant 5$ sides. Prove that there are at most $\dfrac{n(2n-5)}{3}$ triangles of area 1 with whose vertices are choosen from the vertices of the polygon.

Solution. Let $A_1, A_2, \ldots, A_n$ be the vertices of the polygon. We start with the following

Lemma: Each segment A_iA_j belongs to at most 2 triangle of area 1 located on the same side of the line A_iA_j.

Proof of the lemma. Indeed, suppose that on the same side of the line A_iA_j exist the vertices A_m, A_n, A_p so that the triangles $A_iA_jA_m$, $A_iA_jA_n$ and $A_iA_jA_p$ have the area 1. Then the points A_m, A_n, A_p will be at the same distance to the line A_iA_j, hence colinear. This is a contradiction, since the polygon is convex.

Consider first the n sides of the polygon. Each of them can form at most 2 triangles of area 1, as all the vertices lie on the same side, hence we have by now at most $2n$ such triangles.

Consider now the n diagonals A_iA_{i+2} – with the cyclic notations: $A_{n+j} = A_j$. Each of them can form at most 3 triangles of area 1, one with A_{i+1} and two with the vertices lying on the other side. Thus we have at most $5n = 2n + 3n$ triangles.

Finally, consider the other diagonals of the polygon. They are $\frac{n(n-5)}{2}$, and each of them can form at most 4 triangles. The final counting is $5n + 4\frac{n(n-5)}{2} = n(2n - 5)$, except that we have counted each triangle three times, one time for each side. Therefore, there are at most $\frac{n(2n-5)}{2}$ triangles, as claimed.

2.5. REGIONAL MATHEMATICAL COMPETITIONS

Problems and Solutions

7th GRADE

1. Let m, n be positive integers. Show that $25^n - 7^m$ is divisible by 3 and find the least positive integer of the form $|25^n - 7^m - 3^m|$, when m, n run over the set of non-negative integers.

Solution. Since $25 \equiv 1 \pmod 3$ and $7 \equiv 1 \pmod 3$, it follows that $25^n - 7^m \equiv o \pmod 3$.

For the second part of the problem, we first remark that if m is odd, then any number $a = 25^n - 7^m - 3^m$ is divisible by 15. This follows from the first part toghether with

$$7^m + 3^m \equiv 2^m + (-2)^m \equiv 0 \pmod 5.$$

Moreover, for $m = n = 1$ one obtains $25 - 7 - 3 = 15$.

Assume now that m is even, say $m = 2k$. Then

$$\begin{aligned} 7^m + 3^m = 7^{2k} + 3^{2k} &\equiv \left((-3)^{2k} + 3^{2k}\right) \pmod{10} \\ &\equiv 2 \cdot 9^k \pmod{10} \equiv \pm 2 \pmod{10} \equiv 2 \text{ or } 8 \pmod{10}. \end{aligned}$$

So, the last digit of the number $25^n - 7^m - 3^m$ is either 3 or 7. Because the number $25^n - 7^m - 3^m$ is divisible by 3, the required number cannot be 7. The situation $|25^n - 7^m - 3^m| = 3$ can also not occur, because $25^n - 7^m - 3^m \equiv 1 \pmod 8$.

2. Let a, b be real numbers such that $|a| \geqslant 2$, $|b| \geqslant 2$. Show that

$$(a^2+1)(b^2+1) \geqslant (a+b)(ab+1) + 5,$$

and find when equality holds.

Solution. We have

$$(a^2+1)(b^2+1) - (ab+1)(a-b) - 5 = (a^2b^2 - a^2b - ab^2 + ab) + (a^2 + b^2 - a - b - ab) - 4.$$

Observe that

$$a^2b^2 - a^2b - ab^2 + ab = ab(a-1)(b-1) \geqslant 4$$

and

$$a^2 + b^2 - a - b - ab = \frac{1}{2}\left((a-b)^2 + a(a-2) + b(b-2)\right) \geqslant 0.$$

The inequality becomes an equality for $a = b = 2$.

3. Let ABC be a triangle, M be the foot of the altitude from C and N be the reflection of M across the line BC. The parallel line to CM through the point N intersects BC in P and AC in Q.

a) Show that $MQ \perp AP$ if and only if $AB = AC$.

b) Show that it is possible to obtain the points A, B, C when the points M, N, P are given.

Solution. a) We have $AB \perp PQ$. Therefore $MQ \perp AP$ if and only if M is the orthocenter of the triangle APQ, that is if and only if $MP \perp AC$. Because $\angle MPC = \angle NPC = \angle MCP$, it follows that $PM \perp AC$ if and only if $\angle B = \angle C$.

b) In the given conditions, B is the orthocenter of the triangle MNP, C is the intersection of the altitude from P with the parallel line through M to PN and A is the intersection of MB with the perpendicular bisector of the segment BC.

9th GRADE

1. Let a, b, c be real numbers. Show that

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = \sqrt[3]{a+b+c}$$

if and only if

$$a^3 + b^3 + c^3 = (a+b+c)^3.$$

Solution. Using the well-known identity

$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3(a+b)(b+c)(c+a),$$

we see that $(a+b+c)^3 = a^3 + b^3 + c^3$ if and only if either $a+b=0$, or $b+c=0$, or $c+a=0$. It follows that

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = \sqrt[3]{a+b+c}.$$

For the converse, one may apply the same argument for the numbers $\sqrt[3]{a}$, $\sqrt[3]{b}$, $\sqrt[3]{c}$.

2. Let x, y, z be real numbers such that

$$x^2 + yz \leqslant 2,\ y^2 + xz \leqslant 2,\ z^2 + xy \leqslant 2.$$

Find the minimal and the maximal value of the sum $x + y + z$.

Solution. Adding up the given inequalities, one obtains

$$x^2+y^2+z^2+xy+yz+zx \leqslant 6.$$

Taking into account that $xy+yz+zx \leqslant x^2+y^2+z^2$, one obtains $2(x+y+z)^2 \leqslant 18$. Therefore, $-3 \leqslant x+y+z \leqslant 3$.

The values $x=y=z=1$ and $x=y=z=-1$, respectively, show that both lower and upper bounds are attainable.

3. We are given the set $A=\{1,3,6,10,15,21,\ldots\}$. Show that there exist numbers $a_1,a_2,\ldots,a_{2004} \in A$ such that

$$a_1+a_2+a_3+\cdots+a_{2003}=a_{2004}.$$

Solution. The set A is the set of the so-called triangular numbers:

$$A=\{T_k=1+2+3+\cdots+k \mid k \geqslant 1\}.$$

We will prove by induction the following statement: for any k, $k \geqslant 2$, there exist triangular numbers $T_{i_1}, T_{i_2}, \ldots, T_{i_{k+1}}$, such that

$$T_{i_1}+T_{i_2}+\cdots+T_{i_k}=T_{i_{k+1}}.$$

For $k=2$, we have $T_3+T_5=T_6$. Assume that

$$T_{i_1}+T_{i_2}+\cdots+T_{i_k}=T_p.$$

Since $T_p=\frac{p(p+1)}{2}$, let

$$T_{\frac{p(p+1)}{2}-1}=1+2+\cdots+\left(\frac{p(p+1)}{2}-1\right).$$

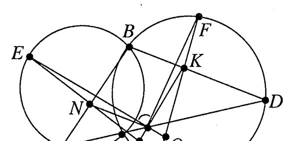
Then

$$T_{\frac{p(p+1)}{2}-1}+T_p=T_{\frac{p(p+1)}{2}}=T_{i_1}+\cdots+T_{i_k}+T_{\frac{p(p+1)}{2}-1}.$$

4. The circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect in distinct points A, B. An arbitrary line through A intersects again $\mathcal{C}_1$ in C and $\mathcal{C}_2$ in D and let M be an arbitrary point on the segment CD. The parallel line to BC through M intersects the segment BD in K and the parallel to BD through M intersects the segment BC in N. The perpendicular in N to BC intersects the arc BC of $\mathcal{C}_1$ which does not contain A in the point E. The perpendicular to BD in K intersects the arc BD of $\mathcal{C}_2$ which does not contain A in F.

Show that $\angle EMF=90°$.

Solution.



Let P,Q be the intersection point of EN, KM and FK, NM respectively. The triangles NPM and KQM are right angled, satisfy $\angle NMP=\angle KMQ$, therefore they are similar. It follows that $\angle PNM=\angle MKQ$ and also $\angle ENM=\angle FKM$.

5. Let n be a positive integer and a,b,c be real numbers such that $a^n=a+b$, $b^n=b+c$ and $c^n=c+a$.

Show that $a=b=c$.

Solution. If $n=1$ the conclusion is obvious. let $n \geqslant 2$. We have $a^n-b^n=a-c, b^n-c^n=b-a, c^n-a^n=c-b$. Distinguish two cases:

Case 1. n is odd. Then the function $f(x)=x^n$ is monotonic increasing on the real axis. If $a>b$, one obtains $a>c, c<b$ and then $a<b$, which is a contradiction.

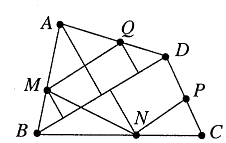
Case 2. n is even. If $a,b,c \geqslant 0$ the conclusion follows like above. Since $a+b \geqslant 0, b+c \geqslant 0, c+a \geqslant 0$, we cannot have negative numbers. Assume that $a<0$ and $b \geqslant 0, c \geqslant 0$. Let $a=-x, x>0$. Then $x^n=-x+b, b^n=b+c, c^n=c-x$. It follows $b>x, c>x$. Since $x^n-c^n=b-c$ we get $b<c$. On the other hand $b^n-c^n=b+x>0$, that is $b>c$, a contradiction.

6. Let $ABCD$ be a convex quadrilateral and M,N,P,Q be points on the sides AB,BC,CD,DA respectively, such that

$$\frac{MA}{MB}=\frac{NB}{NC}=\frac{PD}{PC}=\frac{QA}{QD}=k,$$

where $k \neq 1$. Show that $S(ABCD)=2S(MNPQ)$ if and only if $S(ABD)=S(BCD)$.

Solution.



From $\dfrac{QA}{QD} = \dfrac{MA}{MB} = k$, we get $\dfrac{AM}{AB} = \dfrac{MQ}{BD} = \dfrac{k}{k+1}$ and from $\dfrac{BN}{NC} = \dfrac{PD}{DC} = k$, we get $\dfrac{NC}{BC} = \dfrac{PN}{BD} = \dfrac{1}{k+1}$.

Hence $\dfrac{MQ+PN}{BD} = \dfrac{k+1}{k+1} = 1$, implying $BD = MQ + PN$.

Let h_1, h_2 be the length of the perpendicular from A, C respectively, to BD and u, v the length of the perpendiculars from M, N respectively, to BD. Since the triangle AMQ is similar to with ABD, and CPN is similar to CDB, we get $\dfrac{k}{k+1} = \dfrac{h_2 - u}{h_2}$ and $\dfrac{1}{k+1} = \dfrac{h_1 - v}{h_1}$. Therefore, $u = \dfrac{h_2}{k+1}$ and $v = \dfrac{h_1 k}{k+1}$.

Since $MNPQ$ is a trapezoid,

$$S(MNPQ) = \frac{1}{2}(PN + MQ)\left(\frac{h_2}{k+1} + \frac{kh_1}{k+1}\right) = \frac{BD}{2(k+1)}(h_2 + kh_1).$$

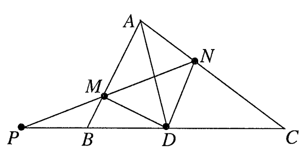
It is clear that $S(ABCD) = \frac{BD}{2}(h_1 + h_2)$.

Therefore, $S(ABCD) = 2S(MNPQ)$ which is equivalent to $(k+1)(h_1 + h_2) = 2(h_2 + kh_1)$, or to $h_1 = h_2$, that is the same with $S(ABD) = S(BCD)$.

7. Let ABC be a right triangle such that, $\angle A = 90^\circ$, $\angle B > \angle C$, and let D be an arbitrary point on the segment BC. The angle bisectors of $\angle ADB$ and $\angle ADC$ intersect the sides AB and AC in the points M and N, respectively. Show that the angle between the lines BC and MN is $\frac{1}{2}(B - C)$ if and only if D is the foot of the altitude from A.

Solution. Let P be the intersection point of BC and MN. Since $\angle MDN = 90^\circ$, $AMDN$ is a cyclic quadrilateral. It follows that

$$\angle AMN = \angle ADN = \frac{\angle ADC}{2} \quad \text{and} \quad \angle ANM = \frac{\angle ADB}{2}.$$



We have $\angle MPB = \angle ABC - \angle AMN$. Hence, $\angle MPB = \frac{1}{2}(B - C)$ is equivalent to $\angle ABC = \angle ADC - \angle BCD$, which is the same as $90^\circ - \angle BCD = \angle ADC - \angle BCD$, or $\angle ADC = 90^\circ$.

8. Find all real numbers x, $x > 1$, such that $\sqrt[n]{[x^n]}$ is an integer number for all positive integers n, $n \geqslant 2$.

Solution. Put $\sqrt[n]{[x^n]} = a_n$. Then $[x^n] = a_n^n$ and $a_n^n \leqslant x^n < a_n^n + 1$. Taking roots, one obtains $a_n \leqslant x < \sqrt[n]{a + n^n + 1}$. This shows that $[x] = a_n$.

We will show that all positive integers x, $x \geqslant 2$, satisfy the condition. Assume, by way of contradiction, that there is a solution x which is not a nonnegative integer. Put $x = a + \alpha$, $a \geqslant 1, 0 < \alpha < 1$.

It follows that $a^n < (a+\alpha)^n < a^n + 1$, and therefore,

$$1 < \left(1 + \frac{\alpha}{a}\right)^n < 1 + \frac{1}{a^n} \leqslant 2.$$

On the other hand, by Bernoulli inequality,

$$\left(1 + \frac{\alpha}{a}\right)^n \geqslant 1 + n\frac{\alpha}{a} > 2,$$

for sufficiently large n, a contradiction.

10th GRADE

1. Find all arithmetic sequences n_1, n_2, n_3, n_4, n_5, for which $5|n_1, 2|n_2, 11|n_3, 7|n_4$ and $17|n_5$.

2004, Focşani Contest

Solution. We take $n_1 = 5a, n_2 = 2b, n_3 = 11c, n_4 = 7d$ and $n_5 = 17e$. We require the conditions: $n_1 + n_3 = 2n_2, n_2 + n_4 = 2n_3, n_3 + n_5 = 2n_4$. From these, we derive the system of linear equations:

$$\begin{cases} 5a & -4b & +11c & & & = 0 \\ & 2b & -22c & +7d & & = 0 \\ & & 11c & -14d & +17e & = 0 \end{cases}$$

We solve it in rational numbers by expanding a, b, c in terms of d, e, and one obtains

(1) $$c = \frac{14d - 17e}{11}, \quad b = \frac{21}{2}d - 17e \quad a = \frac{28d - 51e}{5}.$$

From these, one has: $d = 2x$, $6x - 6e \equiv 0 \pmod{11}$ and $x - e \equiv 0 \pmod 5$. Therefore $x - e \equiv 0 \pmod{55}$, $e = x - 55y$, where $x, y \in \mathbb{Z}$. Thus

$$a = x + 561y, \quad b = 4x + 935y, \quad c = x + 85y, \quad d = 2x, \quad e = x - 55y.$$

It follows that d is even, $d = 2x$, and

$$c = \frac{28x - 17e}{11} \quad a = \frac{56x - 51}{5}.$$

The required sequence is

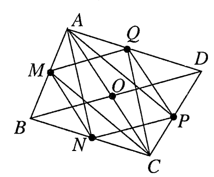
$$\begin{aligned} n_1 &= 5x + 2805y \\ n_2 &= 8x + 1870y \\ n_3 &= 11x + 935y \\ n_4 &= 14x \\ n_5 &= 17x - 935y. \end{aligned}$$

We remark that the ratio of the sequence is $3x - 935y$.

2. Let $ABCD$ be a convex quadrilateral and M, N, P, Q be the midpoints of the sides AB, BC, CD, DA respectively.

Show that if ANP and CMQ are equilateral triangles then $ABCD$ is a rhombus. Find the angles of $ABCD$.

Solution. Let O be the intersection point of the diagonals AC and BD. Since $ABCD$ is a convex quadrilateral, O is an interior point.



We shall first give a solution using complex coordinates. Let a, b, c, d be the complex coordinates of A, B, C, D respectively, such that O is the origin of the complex plane. Then M, N, P, Q have complex coordinates $m = \dfrac{a+b}{2}, n = \dfrac{b+c}{2}$, $p = \dfrac{c+d}{2}, q = \dfrac{d+a}{2}$ respectively. The condition that ANP and CMQ are equilateral triangles gives us the equalities

$$(1) \qquad \begin{aligned} 2c + (a+b)\varepsilon + (d+a)\varepsilon^2 &= 0 \\ 2a + (c+d)\varepsilon + (b+c)\varepsilon^2 &= 0, \end{aligned}$$

where $\varepsilon = \cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}$. Taking into account that $\varepsilon^2 + \varepsilon + 1 = 0$, the above equation yields

$$\varepsilon = \frac{2c - a - d}{d - b} = \frac{2a - b - c}{b - d}.$$

It follows that $c + a = b + d$. As AC and BD intersect at O, this gives $a + c = b + d = 0$. Therefore $ABCD$ is a parallelogram and $c = -a, d = -b$.

The last equalities, when introduced in (1), give

$$2a + (-a - b)\varepsilon + (b - a)\varepsilon^2 = 0, \quad \text{or} \quad 2a - a\varepsilon - b\varepsilon + (b - a)(-1 - \varepsilon) = 0.$$

We obtain $\dfrac{a}{b} = \mathrm{i}\dfrac{\sqrt{3}}{3}$. This shows that OA and OB are perpendicular and

$$\frac{OA}{OB} = \tan(\angle OBA) = \frac{\sqrt{3}}{3}.$$

Therefore $ABCD$ is a rhombus and $\angle B = \angle D = 60°$ and $\angle A = \angle C = 120°$.

Alternative solution. This is an essentially metric argument. Observe that one can easily check that in the triangle ABC the medians AN and CM are equal. This implies that the triangle is isosceles (using, for example, the median formula). In the same way the triangle DAC is isosceles. The median formula imply that the two triangles are equal, that is $ABCD$ is a rhombus. Checking the angles is now an easy task.

3. Let $A = \{1, 2, 3, 4, 5\}$. Find the number of functions $f : A \to A$, with the following property: there is no triple of distinct elements $a, b, c \in A$ such that $f(a) = f(b) = f(c)$.

Solution. There are three types of functions which do not posses the given property:

1) Functions which assign to three distinct elements a, b, c an element $a' \in A$, and to the remaining two elements, arbitrary elements from $A \setminus \{a'\}$. The number of such functions is $\dbinom{5}{3} \cdot 5 \cdot 4^2$.

2) Functions which assign to four distinct elements from A an element $a' \in A$ and to the remaining fifth element in A some element from $A \setminus \{a'\}$. The number of such functions is $\dbinom{5}{4} \cdot 5 \cdot 4$.

3) Constant functions; their number is 5.

The required number is thus, $5^5 - \left(\binom{5}{3} \cdot 5 \cdot 4^2 - \binom{5}{3} \cdot 5 \cdot 4 - 5\right) = 2220$.

4. Let $a \geqslant 2$ be a an integer. Consider the set

$$A = \left\{\sqrt{a}, \sqrt[3]{a}, \sqrt[4]{a}, \sqrt[5]{a}, \ldots\right\}.$$

a) Show that A does not contain an infinite geometric ratio.

b) Show that for any $n \geqslant 3$, A contains n numbers which are in a geometric ratio.

Solution. a) Observe that $a \subset [1, \sqrt{a}]$. Assume that $(x_n)_n$ is an infinite geometric ratio of ratio q. Then $1 \leqslant x_1 q^n \leqslant \sqrt{a}$, for all $n \geqslant 1$. In the case $q < 1$ the inequality $1 \leqslant x_1 q^n$ gives a contradiction and in the case $q > 1$, the inequality $q^n x_1 \leqslant \sqrt{a}$ gives a contradiction.

b) The numbers $a^{\frac{1}{n!}}, a^{\frac{2}{n!}}, \ldots, a^{\frac{n}{n!}}$ are in geometric ratio.

5. Let $ABCD$ be a tetrahedron such that the medians starting from vertex A in the triangles ABC, ABD, ACD are mutually perpendicular. Show that all edges that contain A are equal.

Solution. We shall use vectors. Put $\overrightarrow{AB} = b, \overrightarrow{AC} = c, \overrightarrow{AD} = d$. Hence, the medians are given by the vectors $\frac{1}{2}(b+c), \frac{1}{2}(c+d), \frac{1}{2}(d+b)$. The scalar products of each pair is zero. We thus obtain

$$\begin{aligned} b^2 + (bc + cd + db) = 0 \\ c^2 + (bc + cd + db) = 0 \\ d^2 + (bc + cd + db) = 0, \end{aligned}$$

which easily gives $b^2 = c^2 = d^2$, that is $|b| = |c| = |d| = 0$.

6. Let x, y, z be real numbers such that

$$\begin{aligned} \cos x + \cos y + \cos z = 0 \\ \cos 3x + \cos 3y + \cos 3z = 0. \end{aligned}$$

Prove that $\cos 2x \cos 2y \cos 2z \leqslant 0$.

Solution. We shall use the formula $\cos 3x = 4\cos^3 x - 3\cos x$. Summing up the given equalities yields

$$\cos^3 x + \cos^3 y + \cos^3 z = 0.$$

From the algebraic decomposition formula

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca),$$

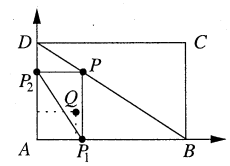
we deduce $\cos x \cos y \cos z = 0$. Assume $\cos z = 0$. Then $-\cos y = \cos x$ and $\cos 2x = \cos 2y$. Since $\cos 2z = -1$, it follows that $\cos 2x \cos 2y \cos 2z = -\cos^2 2x \leqslant 0$, and we are done.

11^{th} GRADE

1. We are given a rectangle $ABCD$ and let P be an arbitrary point on the diagonal BD, $P \neq B, P \neq D$, and Q be an arbitrary point inside the triangle ABD. The perpendicular projections of P on the sides AB, AD are P_1, P_2 respectively, and the perpendicular projections of Q on the sides AB, AD are Q_1, Q_2, respectively.

Show that if $AQ_1 = \frac{1}{4}AB$ and $AQ_2 = \frac{1}{4}AD$, then the point Q does not lie inside the triangle AP_1P_2.

Solution. We will prove the statement by using coordinates. Take a system of coordinates such that $A(0,0), B(a,0), D(0,b)$. Consider P of coordinates $P(p,q)$. One has $Q\left(\frac{a}{b}, \frac{b}{4}\right)$.



The equation of the line BD is $\frac{x}{a} + \frac{y}{b} = 1$. Therefore, we have

$$\frac{p}{a} + \frac{q}{b} = 1. \tag{1}$$

Write (1) in the form $\frac{1}{\frac{a}{p}} + \frac{1}{\frac{b}{q}} = 1$ and use Cauchy-Schwarz inequality to obtain

$$1 = \frac{1}{\frac{a}{p}} + \frac{1}{\frac{b}{q}} \geqslant \frac{(1+1)^2}{\frac{a}{p} + \frac{b}{q}} \quad \text{or} \quad \frac{a}{p} + \frac{b}{q} \geqslant 4,$$

that is

$$\frac{\frac{a}{4}}{p} + \frac{\frac{a}{4}}{q} \geqslant 1.$$

The equation of the line P_1P_2 is $\frac{x}{p} + \frac{y}{q} = 1$. The inequality from above shows that th point $Q\left(\frac{a}{4}, \frac{b}{4}\right)$ does not lie in the half plane which contains the triangle AP_1P_2.

2. Let A, N be 2×2 real matrices such that $AN = NA$ and $N^m = 0$ for some positive integer m. Show that

a) $\det(A + N) = \det A$;

b) If $\det A \neq 0$, then $A + N$ is invertible and $(A+N)^{-1} = (A - N)A^{-2}$.

Solution. It is easy to show that if $N^m = 0$ for some $m > 1$, then $N^2 = 0$. As A commutes with N, we have also $(NA^{-1})^2 = 0$. The following easy consequence of the characteristic equation will be used.

Remark. If B is a 2×2 real matrix such that $B^2 = 0$ then $\mathrm{tr}(B + I) = 2$ and $\det(B + I) = 1$.

Proof. The fact that $\mathrm{tr}\,(B + I) = 2$ is a consequence of the formula of trace. The characteristic equation for $B + I$ gives

$$(I + B)^2 - 2(I + B) + \det(I + B)I = 0,$$

from which the conclusion of the remark follows.

In the case the given matrix A is invertible, use the preceding remark for $B = NA^{-1}$. Thus

$$\det(A + N) = \det\left(A(I + NA^{-1})\right) = \det A \det(I + NA^{-1}) = \det A.$$

In the case when A is not invertible, consider the polynomials $f(x) = \det(xI + A)$ and $g(x) = \det(xI+A+N)$. They are monic and quadratic and since $f \neq 0$ and $f(x) = g(x)$ for infinitely many x, it follows that $f = g$. In particular $f(0) = g(0)$, which gives $\det(A) = \det(A+N)$.

It is a routine to verify

$$(A-N)A^{-2}(A+N) = (A^2 - N^2)A^{-2} = A^2A^{-2} = I,$$

which proves $(A-N)A^{-2} = (A+N)^{-1}$.

3. a) Prove that for all positive integers n, the following inequality holds

$$\left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\cdots\left(1+\frac{1}{n^2}\right) < e^{2-\frac{1}{n}}.$$

b) Show that the sequence of real numbers $(a_n)_{n\geqslant 1}$ defined by $a_1 = 1$ and

$$a_{n+1} = \frac{2}{n^2}\sum_{k=1}^{n} ka_k, \text{ for all } n \geqslant 1,$$

is monotonic increasing. Find with proof if it is a convergent sequence.

Solution. a) We prove the statement by induction on n. For $n = 1$ we have $2 < e$, obviously true.

Assume that the statement is true for n. To prove it for $n+1$ it suffices to prove that

$$e^{2-\frac{1}{n}}\left(1+\frac{1}{(n+1)^2}\right) < e^{2-\frac{1}{n+1}}.$$

The inequality is equivalent to

$$1+\frac{1}{(n+1)^2} < e^{\frac{1}{n}-\frac{1}{n+1}} = e^{\frac{1}{n(n+1)}},$$

or

$$\left(1+\frac{1}{(n+1)^2}\right)^{n(n+1)} < e.$$

The last is an easy consequence of $n(n+1) < (n+1)^2$ and $\left(1+\dfrac{1}{m}\right)^m < e$, for any positive m.

b) From the equalities

$$n^2a_{n+1} = \sum_{k=1}^{n} ka_k$$

$$(n-1)^2a_n = \sum_{k=1}^{n-1} ka_k,$$

one obtains by subtraction $n^2a_{n+1} - (n-1)^2a_n = 2na_n$, or $a_{n+1} = a_n\dfrac{n^2+1}{n^2} > a_n$. This proves that $(a_n)_n$ is monotonically increasing. Since $a_{n+1} = \left(1+\dfrac{1}{n^2}\right)a_n$, we get

$$a_{n+1} = \prod_{k=1}^{n}\left(1+\frac{1}{k^2}\right) < e^{2-\frac{1}{n}} < e^2,$$

thus it is bounded and convergent.

4. Let $a \in (0,1)$ be a real number and $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies the conditions:

(i) $\lim\limits_{n\to\infty} f(x) = 0$;

(ii) $\lim\limits_{n\to\infty} \dfrac{f(x) - f(ax)}{x} = 0$.

Show that $\lim\limits_{n\to\infty} \dfrac{f(x)}{x} = 0$.

Solution. From condition (ii) we have: for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(ax)| < \varepsilon|x|$, for all $x \in (-\delta, \delta)$. It follows that for all positive integers n and all $x \in (-\delta, \delta)$, one has

$$|f(x) - f(a^nx)| \leqslant |f(x) - f(ax)| + |f(ax) - f(a^2x)| + \cdots + |f(a^{n-1}x) - f(a^nx)|$$
$$< \varepsilon|x|(1 + a + a^2 + \cdots + a^{n-1}) = \varepsilon\frac{a^n-1}{a-1}|x| \leqslant \frac{\varepsilon}{a-1}|x|.$$

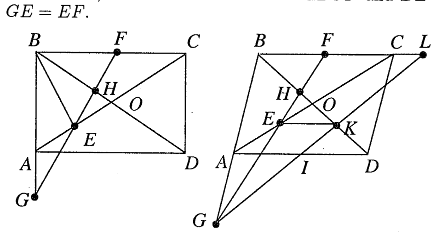
Since $\varepsilon > 0$ was arbitrary and f has limit 0 at $x_0 = 0$, it follows that

$$\lim_{x\to 0}\frac{f(x)}{x} = 0.$$

5. Let $ABCD$ be a parallelogram of unequal sides. The point E is the foot of the perpendicular from B to AC. The line through E which is perpendicular to BD intersects BC in F and AD in G.

Show that $EF = EG$ if and only if $ABCD$ is a rectangle.

Solution. First, assume that $ABCD$ is a rectangle. Let H be the intersection point of FG and BD. In the right triangles ABC and FBG, the segments BE and BH are altitudes, respectively. Then $\angle ABE = \angle ACB$ and $\angle BGF = \angle HBC$. Since $\angle HBC = \angle ACB$, it follows that $\angle GBE = \angle BGF$ and $BE = GE$. This proves that $GE = EF$.



For the second implication we shall present two solutions.

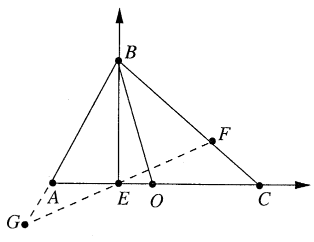
A geometric solution. Let O be the center of the parallelogram. The line FO intersects AD in I and GI intersects BD in K and BC extended in L. Thus $EO \parallel GI$. Therefore, the quadrilateral $GECL$ is a trapezoid. It is known that in a trapezoid the midpoints of parallel sides and the intersection point of nonparallel sides are three collinear points. In our case, it follows that K is the midpoint of GL.

Consider now the triangle FGL. Since E, K are midpoints of the sides GF, GL respectively, it follows that $EK \parallel FL$.

Finally, we look at the triangle GBK in which GH and BE are altitudes. It follows that E is the orthocenter and EK is perpendicular on GB. Since $EK \parallel BC$, the conclusion follows.

A solution using coordinates. We formulate the problem as: give a triangle ABC, let BE be the altitude from B and O the midpoint of the side BC. The perpendicular from E on BO is such that the line segments GE and EF are equal. Then, the angle B is a right angle.

Take the origin of the orthogonal axis to be E and the line EB to be on the Oy axis. Assume that coordinates of points are $A(-a,0), B(0,b), C(c,0)$ where $a,b,c > 0$. We have to prove that $b^2 = ac$.



By standard calculations, we get the following equations and coordinates:

– line GF: $y = \dfrac{c-a}{2b}x$;

– line BC: $\dfrac{x}{c} + \dfrac{y}{b} = 1$;

– point F: $x_F = \dfrac{2b^2c}{2b^2 - c^2 - ac}$, $y_F = \dfrac{cb(c-a)}{2b^2 + c^2 - ac}$;

– line AB: $-\dfrac{x}{a} + \dfrac{y}{b} = 1$;

– point G: $x_G = \dfrac{21b^2}{-2b^2 + ac - a^2}$, $y_G = \dfrac{ab(c-a)}{-2b^2 + ac - a^2}$.

The condition $EG = EF$ is equivalent to $x_F = -x_G$, that is

$$\frac{2b^2c}{2b^2 + c^2 - ac} = \frac{2ab^2}{2b^2 - ac + a^2},$$

which easily gives $b^2 = ac$.

6. Let A, B be 2-by-2 matrices with integer entries, such that $AB = BA$ and $\det B = 1$.

Prove that if $\det(A^3 + B^3) = 1$ then $A^2 = I$.

Solution. As $\det B = 1$, B is invertible and B^{-1} has integer entries. From $A^3 + B^3 = \left((AB^{-1})^3 + I\right) B^3$, it follows that $\det\left((AB^{-1})^3 + I\right) = 1$. We will show that $(AB^{-1})^2 = 0$. It will be, thus, sufficient to treat the case $B = I$.

From the decomposition $A^3 + I = (A+I)(A+\varepsilon I)(A+\varepsilon^2 I)$, where ε is the complex cubic root of unity, it follows that $P(-1)P(-\varepsilon)P(\varepsilon^2) = 1$, where $P(X) = X^2 - \operatorname{Tr} X + \det(A) = X^2 - mX + n$ is the characteristic polynomial of A. Since $P(-\varepsilon^2) = \overline{P(-\varepsilon)}$, it follows that $P(-\varepsilon)P(-\varepsilon^2) \in \mathbb{N}$. Therefore $P(-1) = P(-\varepsilon)P(-\varepsilon^2) = 1$.

We get $1 + m + n = 1$ and $(\varepsilon^2 + m\varepsilon + n)(\varepsilon + m\varepsilon^2 + n) = 1$, which give $m = n = 0$.

12^{th} GRADE

1. Let G be a group such that every element x, $x \neq 1$, has order p.

a) Show that p is a prime number.

b) Show that if any $p^2 - 1$ element subset of G contains p elements which commute one to another, then G is an Abelian group.

Solution. a) Assume by contradiction that p has the decomposition $p = ab$, $a > 1, b > 0$. Then, the order of x^a is b, a contradiction.

b) We start with the following

Lemma. Let $x, y \in G$ such that $x^i y^j = y^j x^i$ for all $2 \leqslant i, j \leqslant p-1$. Then $xy = yx$.

Proof. Since $(j,p) = 1$, there exists a, $1 \leqslant a \leqslant p-1$, such that $aj \equiv 1 \pmod p$. Then $x^i y^{aj} = y^{aj} x^i$. Since $aj = 1 + mp$, $y^{aj} = y^{1+mp} = y$. Therefore $x^i y = y x^i$.

In the same way, since $(i,p) = 1$, one can find b, $1 \leqslant b \leqslant p-1$, such that $bi \equiv 1 \pmod p$. Since $x^i y = y x^i$, we deduce $x^{bi} y = y x^{bi}$ and since $bi - 1 = np$, $x^{bi} = x$. Therefore $xy = yx$, that proves the lemma.

Now, we prove the second statement. Assume by way of contradiction, that there exist $x, y \in G$ such that $xy \neq yx$. Consider the set $A \subset G$, $A = \{x^i y^j \mid 0 \leqslant i, j \leqslant p-1\} \setminus \{1\}$. Formally, A has $p^2 - 1$ elements. By the lemma and the assumption it has indeed.

Let $B \subset A$ be a p-element subset, such that any two elements of B commute. If there exists i, $i \geqslant 1$, such that $x^i \in B$, B contains at least one element of the form $x^a y^b$, $0 \leqslant a \leqslant p-1, 0 \leqslant b \leqslant p-1$. From $x^i(x^a y^b) = (x^a y^b)x^i$, we get $x^i y^b = y^b x^i$ and by the lemma, $xy = yx$. The same argument shows that if some element y^j is in B, then $xy = yx$.

By the previous considerations, we are led to consider that B contains p elements from the array bellow:

$$\begin{array}{cccc} xy & xy^2 & \cdots & xy^{p-1} \\ x^2y & x^2y^2 & \cdots & x^2y^{p-1} \\ \cdots & \cdots & \cdots & \cdots \\ x^{p-1}y & x^{p-1}y^2 & \cdots & x^{p-1}y^{p-1} \end{array}.$$

By the pigeon-hole principle, there are two elements in B which lie on the same line. Let they be x^ay^b a nd x^ay^c. It follows that $(x^ay^b)(x^ay^c) = (x^ay^c)(x^ay^b)$. By assuming that $b > c$, one has $y^{b-c}x^a = x^ay^{b-c}$. The lemma implies $xy = yx$, a contradiction.

The ROMANIAN MATHEMATICAL SOCIETY (Societatea de Ştiinţe Matematice din România) is a centennial professional organization of Romanian mathematicians. It is the founder, more than hundred years ago of the Romanian Mathematical Olympiad. The monthly journal "Gazeta Matematică" is worldwide known as well as other mathematical texts published by the Society during years.

http://www.ssmro.com

The THETA FOUNDATION is a non-for-profit organization promoting mathematical knowledge.
The Theta Foundation is the editor of the "Journal of Operator Theory", a world-wide known mathematical periodical, and of some important texts in mathematical research. A series of translations into Romanian of some fundamental monographs in mathematics is also one of the Theta achievements.

http://www.theta.ro

SOFTWIN is a noteworthy example of a Romanian software and services company with extensive experience and great expertise in the education and training field, particularly in developing intelligent content solutions - such as the Intuitext(tm) interactive lessons - meant to improve quality and innovation of the teaching/learning process in the long run. SOFTWIN's enduring sponsorship of Romanian students participating in the International Mathematics Olympiads is another way of supporting and encouraging excellence in the educational environment in Romania.

SOFTWIN
Software and Services

intuitext
enjoy learning !

MEDICOVER
Your personal physician

SPONSORS OF THE ROMANIAN TEAM FOR THE IMO 2004

ISBN 973-0-03572-5

The ROMANIAN MATHEMATICAL SOCIETY (Societatea de Ştiinţe Matematice din România) is a centennial professional organization of Romanian mathematicians. It is the founder, more than hundred years ago of the Romanian Mathematical Olympiad. The monthly journal "Gazeta Matematică" is worldwide known as well as other mathematical texts published by the Society during years.

http://www.ssmro.com

The THETA FOUNDATION is a non-for-profit organization promoting mathematical knowledge.
The Theta Foundation is the editor of the "Journal of Operator Theory", a world-wide known mathematical periodical, and of some important texts in mathematical research. A series of translations into Romanian of some fundamental monographs in mathematics is also one of the Theta achievements.

http://www.theta.ro

SOFTWIN is a noteworthy example of a Romanian software and services company with extensive experience and great expertise in the education and training field, particularly in developing intelligent content solutions - such as the Intuitext(tm) interactive lessons - meant to improve quality and innovation of the teaching/learning process in the long run. SOFTWIN's enduring sponsorship of Romanian students participating in the International Mathematics Olympiads is another way of supporting and encouraging excellence in the educational environment in Romania.

SOFTWIN
Software and Services

intuitext
enjoy learning !

MEDICOVER
Your personal physician

SPONSORS OF THE ROMANIAN TEAM FOR THE IMO 2004

ISBN 973–0–03572–5