# Algebra Qualifying Exam, Spring 2003: Part I

**Directions:** Work each problem in a separate bluebook. Give reasons for your answers, and make clear which facts you are assuming.

**Notation:**
$\mathbb{Z}$: Integers
$\mathbb{Q}$: Rational Field
$\mathbb{R}$: Real Field
$\mathbb{C}$: Complex Field
$\mathrm{GL}_n(R)$: Group of invertible $n \times n$ matrices with entries in the ring $R$
$\mathbb{F}_q$: Finite field with $q$ elements
$\mathbb{Z}/n$: Ring of integers mod $n$ (can also be regarded as the cyclic group of order $n$)

1. Classify all finite groups of order 140 up to isomorphism.

2. Find the degree $|E : \mathbb{Q}|$ if $E$ is the splitting field of the polynomial $X^{10} - 5 \in \mathbb{Q}[X]$. How many distinct intermediate fields $K$ exist with $\mathbb{Q} \subsetneq K \subsetneq E$?

3(a). Find all positive integers that can occur as the order of some element of $\mathrm{GL}(2, \mathbb{R})$. Exhibit an element of order 5.

(b). Find all positive integers that can occur as the order of some element of $\mathrm{GL}(3, \mathbb{F}_7)$.

(c). Find all positive integers that can occur as the order of some element of $\mathrm{GL}(4, \mathbb{Q})$. Exhibit an element of order $\neq 1$ or 2.

4(a). Find the integral closure $B$ of the integers $\mathbb{Z}$ in the field $\mathbb{Q}\left[\sqrt{-39}\right]$.

(b). Show that there are two distinct prime ideals of $B$ that contain the ideal $5B \subset B$. Give generators for these two prime ideals and show that neither is principal.

(c). How does the ideal $3B \subset B$ factor as a product of prime ideals?

5. Let $\rho : G \to \mathrm{GL}(3, \mathbb{C})$ be a 3-dimensional complex representation of a finite group $G$. Let $V$ be the vector space of all $3 \times 3$ matrices over $\mathbb{C}$. Define the adjoint representation $\rho^\frown : G \to \mathrm{GL}(V)$ by

$$\rho^\frown(g)A = \rho(g)\, A\, \rho(g^{-1})$$

for $g \in G$ and $A \in V$. Which integers can occur as the multiplicity of the trivial one dimensional representation in $\rho^\frown$?

# Algebra Qualifying Exam, Spring 2003: Part II

**Directions:** Work each problem in a separate bluebook. Give reasons for your answers, and make clear which facts you are assuming.

**Notation:**
$\mathbb{Z}$: Integers
$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$: Fields of rational, real, and complex numbers, respectively
$\mathrm{GL}_n(R)$: Group of $n \times n$ invertible matrices with entries in the ring $R$
$\mathbb{F}_q$: Finite field with $q$ elements
$\mathbb{Z}/n$: Ring of integers mod $n$ (can also be regarded as the cyclic group of order $n$)

1(a). Find all abelian groups $G$ that contain a subgroup $H$ isomorphic to $\mathbb{Z}/72\mathbb{Z}$ for which the quotient group $G/H$ is also isomorphic to $\mathbb{Z}/72\mathbb{Z}$.

(b). Find the invariant factors of the abelian group $(\mathbb{Z}/44,000\mathbb{Z})^*$, i.e., the multiplicative group of invertible elements in the ring $\mathbb{Z}/44,000\mathbb{Z}$.

2. Suppose $I \subset \mathbb{Q}[x_1, x_2, \ldots, x_n]$ is an ideal such that the set of zeroes

$$V(I) = \{\mathbf{x} \in \mathbb{C}^n : f(\mathbf{x}) = 0 \text{ for all } f \in I\}$$

is a finite set. Show that the ring $\mathbb{Q}[x_1, x_2, \ldots, x_n]/I$ is a finite dimensional vector space over $\mathbb{Q}$.

[Hint: first show that $\mathbb{Q}[x_1, x_2, \ldots, x_n]/J$ is finite dimensional, where $J = \sqrt{I}$ is the radical of $I$. Then consider powers $J \supset J^2 \supset \ldots$.]

3(a). Factor $X^5 + 7X^3 + 6X^2 + X + 5$ over the fields $\mathbb{F}_2$, $\mathbb{F}_3$, and $\mathbb{F}_5$.

[You may assume the (true) result that this polynomial has no irreducible quadratic factors over $\mathbb{F}_3$.]

(b). What are the Galois groups of $X^5 + 7X^3 + 6X^2 + X + 5$ over $\mathbb{F}_2$, $\mathbb{F}_3$, and $\mathbb{F}_5$?

(c). What are the Galois groups of $X^5 + 7X^3 + 6X^2 + X + 5$ over $\mathbb{Q}$?

4. Suppose $V$ is a finite dimensional vector space over a field $k$ and suppose $A : V \to V$ is a $k$-linear endomorphism whose minimal polynomial is *not* equal to its characteristic polynomial. Show that there exist $k$-linear endomorphisms $B, C : V \to V$ such that $AB = BA$, $AC = CA$, but $BC \neq CB$.

5(a). Produce a complex character table for the symmetric group $S_4$.

(b). The rotation group of the cube is isomorphic to $S_4$ as a permutation group of the four diagonals of the cube. Let $\rho : S_4 \to \mathrm{GL}(8, \mathbb{C})$ be the permutation representation of $S_4$ defined by the action of the rotation group on the eight vertices of the cube. Find the character of $\rho$.

(c). Decompose $\rho$ as a direct sum of irreducible representations of $S_4$.