Spring 2002

Ph.D. Qualifying Examination

Algebra

Part I

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. Let p and q be primes, $q > 2$. Let $G = SL(2, \mathbb{F}_p)$. (Here $\mathbb{F}_p$ is the finite field with p elements.) Suppose that q divides $|G| = p(p^2 - 1)$. Show that a q-Sylow subgroup of G is cyclic. (**Hint:** first show that G has cyclic subgroups of orders p, $p - 1$ and $p + 1$.)

2. Let R be a commutative ring with unit.

(i) Let S be a *saturated* multiplicative set of R. This means that $1 \in S$, $0 \notin S$, and $xy \in S$ if and only if $x \in S$ and $y \in S$. Show that $R - S$ is a union of prime ideals. [Hint: If $a \in R - S$ consider ideals J with $a \in J \subset R - S$.]

(ii) An element $a \in R$ is a *zero divisor* if $ab = 0$ for some $b \neq 0$. Apply (i) to show that the set of zero divisors is a union of prime ideals of R.

3. Let p be prime. Show that there exists $\alpha \in \mathbb{C}$ such that $K = \mathbb{Q}(\alpha)$ is a Galois extension of $\mathbb{Q}$ and that $\mathrm{Gal}(K/\mathbb{Q})$ is cyclic of order p. Exhibit such an α when $p = 5$.

4. Let G be a finite group and let $H \subset G$ be an abelian subgroup of prime index p. Let χ be an irreducible character of G such that $\chi(1) = p$. Prove that there exists a character ψ of H such that χ is the character of G induced from ψ.

5. (i) Let R be a principal ideal domain, and let $f, g \in R$ be coprime elements. Show that

$$R/(fg) \cong R/(f) \oplus R/(g)$$

as R-modules.

(ii) Let F be a field, and let $f(X) = X^2 + aX + b$, $g(X) = X^2 + cX + d$ be distinct irreducible polynomials over F. Let $fg = X^4 + tX^3 + uX^2 + vX + w$. Show that the matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -b & -a & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -d & -c \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -w & -v & -u & -t \end{pmatrix}$$

are conjugate in $GL(4, F)$, the group of 4×4 invertible matrices with coefficients in F.

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Part II

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

Notation: Here $\mathbb{F}_p$ denotes the finite field with p elements, and S_n denotes the symmetric group of degree n.

1. Let H be the subgroup of S_6 generated by (16425) and $(16)(25)(34)$. Let H act on S_6 by conjugation. Show that the set

$$\Sigma = \{(12)(35)(46),\ (13)(24)(56),\ (14)(25)(36),\ (15)(26)(34),\ (16)(23)(45)\}$$

is invariant under H. Hence obtain a homomorphism $\phi : H \to S_5$. Show that ϕ is an isomorphism.

2. Let Q be the group of order 8 having generators x and y such that $x^4 = y^4 = 1$, $x^2 = y^2$ and $xyx^{-1} = y^3$. Find the conjugacy classes of Q and compute its character table.

3. Let p and q be distinct primes. Show that the polynomial

$$\Phi(X) = X^{p-1} + X^{p-2} + \ldots + 1$$

has a root in $\mathbb{F}_{q^2}$ if and only if $q \equiv \pm 1$ modulo p.

4. Let V be a finite dimensional complex vector space endowed with an inner product, that is, a positive definite Hermitian form $\langle\, , \rangle$. Let $T : V \to V$ be a linear transformation which commutes with its adjoint T^*, defined by

$$\langle Tx, y \rangle = \langle x, T^* y \rangle .$$

Prove that V has a basis consisting of eigenvectors of T.

5. Let A be a commutative Noetherian ring with unit. An ideal $J \subset A$ is called a *radical ideal* if $x^n \in J$ implies that $x \in J$. Show that every proper radical ideal is a finite intersection of prime ideals. [Hint: Among counterexamples, a maximal one couldn't be prime.]