# Real Analysis Exam: Part I (Fall 2002)

1. Let $G$ be an unbounded open set in $(0, \infty)$, and let

$$D = \{x : nx \in G \text{ for infinitely many natural numbers } n\}.$$

Prove that $D$ is dense in $(0, \infty)$.

2. Suppose $S$ is a linear subspace of $\mathbf{C}[0,1]$ such that for all $f \in S$,

$$\|f\|_\infty \leq \lambda \|f\|_2$$

where $\|f\|_\infty$ and $\|f\|_2$ are the $\mathcal{L}^\infty$ (or sup) norm and the $\mathcal{L}^2$ norm, respectively. Let $v_1, v_2, \ldots, v_n$ be orthonormal functions in $S$.

(a) For every $x$, show that there are numbers $a_i(x)$ $(1 \leq i \leq n)$ such that $\sum |a_k(x)|^2 = 1$ and such that

$$\sum a_k(x) v_k(x) = \left(\sum |v_k(x)|^2\right)^{1/2}.$$

(b) Show that $\sum |v_k(x)|^2 \leq \lambda^2$.

(c) Show that $\dim S \leq \lambda^2$.

3. Let $f \in \mathbf{C}^1(\mathbf{R})$ and suppose that $f(x+1) = f(x)$ for all $x$. Prove that

$$\|f\|_\infty \leq \int_0^1 |f(t)|\,dt + \int_0^1 |f'(t)|\,dt.$$

4. Let $\mu$ be a non-negative Borel measure on $\mathbf{R}^2$ with $\mu(\mathbf{R}^2) = 1$. Suppose that every set consisting of a single point has $\mu$ measure 0. Show that for every $\lambda \in (0,1)$, there is Borel set $E$ such that $\mu(E) = \lambda$.

5(a). Let $f$ be a function in $\mathcal{L}^1(\mathbf{R})$ such that $f = f * f$, where $*$ denotes convolution. Prove that $f = 0$ almost everywhere.

5(b). Find all functions $f$ on $\mathbf{T}$ (the reals mod 1) such that $f = f * f$.

# Real Analysis Exam: Part II (Fall 2002)

1. Let $f : \mathbf{R} \to \mathbf{R}$ be a function that is continuous except at a countable set of points. Prove that there is a sequence $g_n$ of continuous functions such that $g_n(x) \to f(x)$ for all $x$.

Hint: Use piecewise linear functions.

2. Let $f : \mathbf{R} \to \mathbf{R}$ be a $C^\infty$ function with compact support. Suppose there is an infinite set $S$ of positive integers such that

$$|f^{(n)}(x)| \leq n!$$

for all $n \in S$ and for all $x \in \mathbf{R}$. Prove that $f \equiv 0$.

3. Let $g : \mathbf{R} \to \mathbf{R}$ be a $C^1$ function such that $g(x+1) = g(x)$ for all $x$. Let

$$f(x) = \sum_{k=1}^{\infty} 2^{-k} g(2^k x).$$

Show that there is a number $A < \infty$ such that

$$|f(x) - f(y)| \leq A\,|x-y| \cdot |\log|x-y||$$

for all $x, y$ with $|x-y| \leq \frac{1}{2}$.

Hint: Assume $2^{-n-1} \leq |x-y| \leq 2^{-n}$ and divide the series for $f$ into two parts.

4(a). For every $\epsilon > 0$, show that there is function $f \in \mathcal{L}^1[0,1]$ such that: $f(x) \geq 0$, $f(x) = 0$ on a set of measure $\geq 1-\epsilon$, and

$$\int_a^b f(x)\,dx > 0$$

for every interval $0 < a < b < 1$.
4(b). Show that there is an absolutely continuous, strictly increasing function $h$ on $[0,1]$ such that $h'(x) = 0$ on a set of measure $\geq 1 - \epsilon$.

5. Let $f \in C_0^\infty(\mathbf{R})$ be a function and let $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\,dx$ be its Fourier transform. Assume that $\hat{f}(\xi) = 0$ if $|\xi| \geq 1/2$. Show that

(i)
$$\hat{f}(\xi) = \sum_n f(n) e^{-2\pi i n \xi}$$

for $|\xi| \leq \frac{1}{2}$ and that

(ii)
$$f(x) = \sum_n f(n) \frac{\sin \pi(x-n)}{\pi(x-n)}.$$