Stanford University Mathematics Department
Algebra Qualifying Exam, Fall 2003
Part I

1. Classify finite groups of order $2p^2$ up to isomorphism, where p is an odd prime.

2. Let $A \in M(n, K)$ be an $n \times n$ matrix over a field K such that the minimal polynomial of A has degree n. Show that every matrix in $M(n, K)$ that commutes with A is a K-linear combination of the identity matrix and powers of A.

3(a). Suppose $\rho : G \to \mathrm{GL}(V)$ is an irreducible complex representation of a finite group G. Show that if $\mathrm{Z}(G) \subset G$ is the center of G, then there is a homomorphism $\chi : \mathrm{Z}(G) \to \mathbf{C}^*$ such that $\rho(g)v = \chi(g)v$ for all $g \in \mathrm{Z}(G)$ and $v \in V$.

(b). Conversely, show that for any homomorphism $\chi : \mathrm{Z}(G) \to \mathbf{C}^*$, there exists an irreducible $\rho : G \to \mathrm{GL}(V)$ such that $\rho(g)v = \chi(g)v$ for all $g \in \mathrm{Z}(G)$ and $v \in V$.

4. Suppose $g(T) \in \mathbf{Z}[T]$ is a monic polynomial with roots $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbf{C}$. If $|\alpha_j| = 1$ for all j, show that each α_j is a root of unity. Show by example that this conclusion may fail to hold if $g(T)$ is not assumed monic.

5. Let R be a ring with identity, and let M be a left R-module. Prove there exist submodules $M = M_0 \supset M_2 \supset \cdots \supset M_n = (0)$ so that M_j/M_{j+1} is a simple R-module (for all j) if and only if M satisfies both the ascending chain condition and the descending chain condition for submodules. [Hints: For the "if" direction, begin by using one chain condition to get a maximal or a minimal proper submodule. For the "only if" direction, use an induction on n.]

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Part II

1. If $p < q < r$ are primes and G is a finite group with $|G| = pqr$, prove that the Sylow-r subgroup of G is normal. [Hint: First get some normal Sylow subgroup.]

2. Suppose V is a vector space of dimension 20 over the field $\mathbf{Q}$, and $A : V \to V$ is a linear transformation with minimal polynomial $(T^2 + 1)^2(T^3 + 2)^2$.

(a). How many distinct similarity classes of such A exist?

(b). If V is generated by two elements as a $\mathbf{Q}[T]$ module, where T acts on V as the linear transformation A, and if $p(T) \in \mathbf{Q}[T]$, what integers can occur as the dimension of the kernel of $p(T) : V \to V$ as a rational vector space?

3. Find the Galois groups of the polynomials $X^6 + 3$ and $X^6 + X^3 + 1$ over the fields $\mathbf{Q}$ and $\mathbf{F}_7$.

4. Suppose k is a field of characteristic $\neq 2$, and let R denote the polynomial ring $k[x_1, x_2, \ldots, x_n]$. Suppose $f(x_1, x_2, \ldots, x_n) \in R$ is a non-constant polynomial that is not divisible by the square of any non-constant polynomial in R. Show that the ring $S = R[T]/(T^2 f)$ is the integral closure of R in the field of fractions of S.

5. Suppose G is the group of order 12 with presentation

$$G = \langle\, x, y \mid x^4 = y^3 = 1,\ xyx^{-1} = y^2 \,\rangle .$$

Find the complex character table of G.