Real Analysis Qualifying Exam: Part I (Autumn 2001)

1. Let $K(x,y) \in \mathcal{C}^0([0,1] \times [0,1])$. Prove that the mapping

$$g(x) \mapsto f(x) = (Kg)(x) = \int_0^1 K(x,y)g(y)\,dy$$

is a compact mapping from $C([0,1])$ to $C([0,1])$.

2. Prove the Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(x+2\pi n) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}$$

for all f in the Schwartz space

$$\mathcal{S} \equiv \{f : (1+x^2)^m |f^{(n)}(x)| \le C_{m,n} \quad \text{for all} \quad m,n \ge 0\}.$$

Here $\hat{f}(\xi) = \int_{\mathbf{R}} f(x)e^{-ix\xi}\,dx$.

3. Let p be a number with $1 \le p < \infty$. Assume that f and f_n, $n = 1, 2, \ldots$ are functions in $L^p(\mathbf{R}^n)$ (with respect to standard Lebesgue measure), and that $f_n \to f$ almost everywhere. Prove that $\|f_n - f\|_{L^p} \to 0$ if and only if $\|f_n\|_{L^p} \to \|f\|_{L^p}$.

4. Suppose that the real-valued function $f(x)$ is nondecreasing on the interval $[0,1]$. Prove that there exists a sequence of continuous functions $f_n(x)$ such that $f_n \to f$ pointwise on this interval.

5. Suppose $f \in L^1([0,1])$ but $f \notin L^2([0,1])$. Prove that there exists a complete orthonormal basis ϕ_n for $L^2([0,1])$ such that for each n, ϕ_n is continuous and moreover

$$\int_0^1 f(x)\phi_n(x)\,dx = 0.$$

6. Let λ_n be an arbitrary discrete sequence in $\mathbf{R}$. Define

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{i\lambda_n x}}{n^2}.$$

Prove that $f \in \mathcal{C}^0(\mathbf{R})$, and that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\,dx$$

exists.

Real Analysis Qualifying Exam: Part II (Autumn 2001)

1. Does there exist a function $f \in \mathcal{C}^0([0,1])$ such that

$$\int_0^1 xf(x)\,dx = 1, \text{ and}$$
$$\int_0^1 x^n f(x)\,dx = 0 \quad \text{for } n = 0, 2, 3, \ldots ?$$

2. Let $\{U_n\}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Let $\{V_n\} \subset \mathcal{H}$ be such that $\sum \|V_n - U_n\|^2 = S < \infty$. Show that the linear span of $\{V_n\}$ is a subspace of finite codimension. Prove in fact that when $S < 1$, then $\{V_n\}$ is a basis for $\mathcal{H}$.

3. Let $f_n \in L^p([0,1])$, $\|f_n\|_p \leq 1$ and assume that $f_n(x) \to 0$ almost everywhere. Prove that $f_n \to 0$ weakly.

4. Let (X_j, d_j) be metric spaces, $j = 1, 2$. Let $f : (X_1, d_1) \to (X_2, d_2)$ be distance nondecreasing, i.e. $d_2(f(y), f(z)) \geq d_1(y, z)$ for all $y, z \in X_1$. Are either of the following two implications true? Prove or give a counterexample.

(a) If (X_1, d_1) is complete, then (X_2, d_2) is complete.
(b) If (X_2, d_2) is complete, then (X_1, d_1) is complete.

5. Assume f real-valued and measurable on a probability measure space (e.g., on $[0,1]$ with Lebesgue measure) and write

$$\Phi(\lambda) = \mu(\{x : f(x) < \lambda\})\,.$$

Prove that for any continuous function g on $\mathbf{R}$,

(a) $g \circ f$ is measurable.
(b) $g \circ f$ is integrable if and only if g is integrable with respect to the measure $d\Phi$, and

$$\int g \circ f\,d\mu = \int_{\mathbf{R}} g(\lambda)\,d\Phi(\lambda)\,.$$

6. Prove that for almost all $x \in [0,1]$, there are at most finitely many rational numbers with reduced form p/q such that $q \geq 2$ and $|x - p/q| < 1/(q \log q)^2$.

Hint: Consider intervals of length $2/(q \log q)^2$ centered at rational points p/q.