Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
January 15, 2007

- 4930: *Proposed by Kenneth Korbin, New York, NY.*

  Find an acute angle y such that $\cos(y) + \cos(3y) - \cos(5y) = \dfrac{\sqrt{7}}{2}$.

- 4931: *Proposed by Kenneth Korbin, New York, NY.*

  A Pythagorean triangle and an isosceles triangle with integer length sides both have the same length perimeter $P = 864$. Find the dimensions of these triangles if they both have the same area too.

- 4932: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let ABC be a triangle with semi-perimeter s, in-radius r and circum-radius R. Prove that
  $$\sqrt[3]{r^2} + \sqrt[3]{s^2} \le 2\sqrt[3]{2R^2}$$
  and determine when equality holds.

- 4933: *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Barcelona, Spain.*

  Let n be a positive integer. Prove that
  $$\frac{1}{n}\sum_{k=1}^{n} k\binom{n}{k}^{1/2} \le \frac{1}{2}\sqrt{(n+1)2^n}.$$

- 4934: *Proposed by Michael Brozinsky, Central Islip, NY.*

  Mrs. Moriaty had two sets of twins who were always getting lost. She insisted that one set must chose an arbitrary non-horizontal chord of the circle $x^2 + y^2 = 4$ as long as the chord went through $(1,0)$ and they were to remain at the opposite endpoints. The other set of twins was similarly instructed to choose an arbitrary non-vertical chord of the same circle as long as the chord went through $(0,1)$ and they too were to remain at the opposite endpoints. The four kids escaped and went off on a tangent (to the circle, of course). All that is known is that the first set of twins met at some point and the second set met at another point. Mrs. Moriaty did not know where to look for them but Sherlock Holmes deduced that she should confine her search to two lines. What are their equations?

- 4935: *Proposed by Xuan Liang, Queens, NY and Michael Brozinsky, Central Islip, NY.*

  Without using the converse of the Pythagorean Theorem nor the concepts of slope, similar triangles or trigonometry, show that the triangle with vertices $A(-1,0), B(m^2,0)$ and $C(0,m)$ is a right triangle.

Solutions

- 4894: *Proposed by Kenneth Korbin, New York, NY.*

  Find the dimensions of a triangle with integer length sides, and with integer area, and with perimeter 2006.

  Solution by R. P. Sealy, Sackville, New Brunswick, Canada.

  There are three such triangles. Let (a,b,c) be the sides of the triangle; then

$$\{(a,b,c,)\} = \{(493,885,628),(442,649,915),(697,531,778)\}.$$

Let s be the semi-perimeter. Then $s = 1003 = (17)(59)$. By Heron's formula

$$area = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(17)(59)(1003-a)(1003-b)(1003-c)}.$$

As a,b,c are interchangeable and the area is an integer, we may write

$$s-a = 1003-a = 17k, 1 \le k \le 59, \Rightarrow a = 1003-17k$$

$$s-b = 1003-b = 59n, 1 \le n \le 17, \Rightarrow b = 1003-59n$$

$$s-c = 1003-c = 1003-(2006-a-b) = a+b-1003 = 1003-17k-59n \Rightarrow c = 17k+59n.$$

Then $area = \sqrt{(17)(59)(17k)(59n)(1003-17k-59n)} = (17)(59)\sqrt{kn(1003-17k-59n)}$, must be an integer. A spreadsheet search gives $(k,n) = \{(30,2),(33,6)(18,8)\}$ which gives the above values for $a,b,$ and c.

Also solved by Dionne Bailey, Elsie Campbell,& Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Harry Sedinger, St. Bonaventure, NY; David Stone & John Hawkins, Statesboro, GA, and the proposer.

- 4895: *Proposed by Kenneth Korbin, New York, NY.*

  The roots of $x^3+10x^2+17x+8=0$ are the cubes of the roots of $x^3+4x^2+5x+2=0$. Find a cubic equation with roots that are the cubes of the roots of $x^3+5x^2+4x+2=0$.

  Solution by David C. Wilson, Winston-Salem, NC.

  Consider the cubic equation $x^3+ax^2+bx+c=0$ with roots $r, s,$ and t. Then $r+s+t = -a, rs+rt+st = b$, and $rst = -c$. Let $x^3+Ax^2+Bx+C=0$ be the cubic equation whose roots are $r^3, s^3,$ and t^3. Then

$$-a^3 \quad = \quad (r+s+t)^3 = r^3+s^3+t^3+3(r^2s+r^2t+rs^2+s^2t+rt^2+st^2)+6rst, \text{ and}$$

$-ab \;=\; (r+s+t)(rs+rt+st) \Rightarrow r^2s+r^2t+rs^2+s^2t+rt^2+st^2 = 3c-ab$. Thus

$$r^3+s^3+t^3 = -a^3-3c+3ab,\;\; r^3s^3+r^3t^3+s^3t^3 = b^3+3c^2-3abc, \;\; \text{and } r^3s^3t^3 = -c^3.$$

Therefore the cubic equation $x^3+Ax^2+Bx+C=0$ with roots r^3, s^3, and t^3 has $A=-r^3-s^3-t^3 = a^3+3c-3ab,\; B = r^3s^3+r^3t^3+s^3t^3 = b^3+3c^2-3abc$, and $C=-r^3s^3t^3 = c^3$. Thus, if $a=5, b=4$, and $c=2$, then $A = 5^3+3(2)-3(5)(4) = 71, B=-44$, and $C=8$. Therefore, the roots of the cubic equation $x^3+71x^2-44x+8=0$ are the cubes of the roots of the cubic equation $x^3+5x^2+4x+2=0$.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, & Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Tom Leong, Scotrun, PA; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; David Stone & John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4896: *Proposed by José Luis Díaz-Barrero and Miquel Grau, Barcelona, Spain.*

  Let n be a positive integer. Prove that

$$\prod_{k=1}^{n} \sqrt[k]{k\,!} \le \left(\frac{n+3}{4}\right)^n.$$

Solution by Ovidiu Furdui, Kalamazoo, MI.

The above inequality is equivalent to:

$$\sqrt[n]{\prod_{k=1}^{n} \sqrt[k]{k!}} \le \frac{n+1}{4}$$

We notice that in view of A-G-M inequality we get that:

$$\sqrt[k]{k!} \le \frac{1+2+\cdots+k}{k} = \frac{k+1}{2}$$

Therefore we have in view of the A-G-M and of the above inequality that:

$$\sqrt[n]{\prod_{k=1}^{n} \sqrt[k]{k!}} \le \frac{\sum_{k=1}^{n} \sqrt[k]{k!}}{n} \le \frac{1}{n}\sum_{k=1}^{n}\frac{k+1}{2} = \frac{1}{2n}\left(\frac{(n+1)(n+2)}{2}-1\right) = \frac{n+3}{4}$$

Also solved by Elsie M. Campbell, Dionne T. Bailey, & Charles Diminnie (jointly), San Angelo, TX; Jahangeer Kholdi & Boris Rays (jointly), Portsmouth, VA & Landover, MD; Tom Leong, Scotrun, PA; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH: David Stone & John Hawkins (jointly), Statesboro, GA, and the proposers.

- 4897: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let α, β and γ be the angles of triangle ABC. Prove that

$$(\csc^2\alpha + \csc^2\beta + csc^2\gamma)(1+\cos\alpha\cos\beta\cos\gamma) \ge \frac{9}{2}.$$

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX.

Since $\alpha+\beta+\gamma=\pi$, we have

$$\begin{aligned}
\cos\alpha\cos\beta\cos\gamma &= (1/2)\cos\alpha\left[\cos(\beta+\gamma)+\cos(\beta-\gamma)\right]\\
&= (1/2)\cos\alpha\left[\cos(\pi-\alpha)+\cos(\beta-\gamma)\right]\\
&= (1/2)\left[\cos\alpha\cos(\beta-\gamma)-\cos^2\alpha\right]\\
&= (1/4)\left[\cos(\alpha+\beta-\gamma)+\cos(\alpha-\beta+\gamma)-2\cos^2\alpha\right]\\
&= (1/4)\left[\cos(\pi-2\gamma)+\cos(\pi-2\beta)-2\cos^2\alpha\right]\\
&= -(1/4)\left[\cos(2\gamma)+\cos(2\beta)+2\cos^2\alpha\right]\\
&= -(1/4)\left[1-2\sin^2\gamma+1-2\sin^2\beta+2-2\sin^2\alpha\right]\\
&= (1/2)\left(\sin^2\alpha+\sin^2\beta+\sin^2\gamma\right)-1, \text{ and hence,}\\
1+\cos\alpha\cos\beta\cos\gamma &= \frac{1}{2}\left(\sin^2\alpha+\sin^2\beta+\sin^2\gamma\right).
\end{aligned}$$

Therefore, by the Arithmetic-Geometric Mean Inequality,

$$\begin{aligned}
& (\csc^2\alpha+\csc^2\beta+\csc^2\gamma)(1+\cos\alpha\cos\beta\cos\gamma)\\
= \;& (1/2)(\csc^2\alpha+\csc^2\beta+\csc^2\gamma)(\sin^2\alpha+\sin^2\beta+\sin^2\gamma)\\
\geq \;& (9/2)\sqrt[3]{\csc^2\alpha\csc^2\beta\csc^2\gamma}\sqrt[3]{\sin^2\alpha\sin^2\beta\sin^2\gamma}=\frac{9}{2}.
\end{aligned}$$

Further, equality is achieved if and only if $\sin^2\alpha=\sin^2\beta=\sin^2\gamma$. Since $0<\alpha,\beta,\gamma<\pi$ and $\alpha+\beta+\gamma=\pi$, equality occurs if and only if $\alpha=\beta=\gamma$, i.e., if and only if $\triangle ABC$ is equilateral.

Also solved by Scott H. Brown, Montgomery, AL; Ovidiu Furdui, Kalamazoo, MI; Tom Leong, Scotrun, PA; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY, and the proposer.

- 4898: *Proposed by Michael Brozinsky, Central Islip, NY.*

  a) Suppose we have $2n$ people seated around a table. In how many ways can they shake hands so that each person shakes hands with exactly one other person?

  b) Find the probability that no two of these handshakes in part a) "cross" each other.

  Solution by Paul M. Harms, North Newton, KS.

  Consider a circle with integers 1 through $2n$ placed in an increasing clockwise fashion around the circle with a person at each integer. The person at the $2n$ position has $2n-1$ possibilities for a handshake. After this person's pick take one of the other $2n-2$ people. There are $2n-3$ possibilities for a handshake. Continuing in this fashion there should be $(2n-1)(2n-3)(2n-5)\cdots(3)(1)$ possibilities for part a)

  For part b) consider a handshake on the circle mentioned above as a chord of the circle. We do not want chords to intersect. Consider $(1,2)$ as a handshake with person at 1 and 2. Let L_{2n} be the number of ways non-intersecting chords are involved with a circle of $2n$ points. Clearly $L_2=1$ and $L_4=2$, which are the sets of pairs $\{(1,4)(3,2)\}$ and $\{(3,4)(1,2)\}$. To find L_6 note that with chords $(6,1)$ or $(6,5)$ there are 4 points which have L_4 non-intersecting chords. Also $(6,2)$ cannot be used since any chord with 1 intersects chord $(6,2)$. In general, no odd number of points should be between the numbers of a chord. With $(6,3)$ there are 2 integers on each side of the chord. Thus there are L_2L_2 non-intersecting chords. Then $L_6=2L_4+1(L_2)^2=5$. We have $L_8=2L_6+2L_2L_4=10+4=14$. The 2 with L_6 can be considered for chords $(8,1)$ and $(8,5)$. The 2 with L_2L_4

can be considered for chords $(8,3)$ and $(8,5)$. Using this pattern we can find $L_{2n}$from L with smaller subscripts. We have

$$L_2 = 1, L_4 = 2, L_6 = 5, L_8 = 14,$$
$$L_{10} = 2L_8 + 2L_2L_6 + 1(L_4)^2 = 42$$
$$L_{12} = 2L_{10} + 2L_2L_8 + 2L_4L_6 = 132, \ \ etc.$$

Let the probability of part b) be denoted by P_{2n}. Then

$$P_2 = \frac{L_2}{1} = 1,\ P_4 = \frac{L_4}{3} = \frac{2}{3},\ P_6 = \frac{L_6}{5(3)} = \frac{1}{3},\ P_8 = \frac{L_8}{7(5)(3)} = \frac{2}{15}, \cdots, P_{2n} = \frac{2^n}{(n+1)!}.$$

Also solved by N. J. Kuenzi, Oshkosh, WI; Tom Leong, Scotrun, PA; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone & John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4899: *Proposed by Laszlo Szuecs, Durango, CO.*

Construct two externally tangent circles C_1, C_3 having radius $R > 0$. Let $\dfrac{R}{4} < r < R$ and construct two circles C_2, C_4 having radius r, such that each of C_2, C_4 is externally tangent to both C_1 and C_2. Construct a "framing rectangle" $ABCD$ such that AB is tangent to C_1 (only), BC is tangent to C_1 and C_2, CD is tangent to C_3 (only), and DA is tangent to C_3 and C_4. Express the dimensions of the framing rectangle in terms or R and r.

Solution by Tom Leong, Scotrun, PA.

Let O_k denote the center of $C_k, k = 1,2,3,4$; P denote the point of tangency between C_1 and C_3; and Q and S denote the projections of O_4 and O_1 respectively onto the line through O_3 and parallel to AB. If we find $\vartheta = \angle O_3O_1S$, we can find the dimensions

$$\begin{array}{rcl} AB &=& CD = 2R + O_3S = 2R + O_1O_3 \sin\vartheta = 2R + 2R\sin\vartheta \qquad \text{and} \\ BC &=& DA = 2R + O_1S = 2R + O_1O_3\cos\vartheta = 2R + 2R\cos\vartheta. \quad (*) \end{array}$$

Since $\angle QO_3P$ is exterior to triangle O_3O_1S, we have $\angle QO_3P = \vartheta + 90°$. Hence, looking at triangles QO_3O_4 and PO_3O_4, we find

$$\begin{array}{rcl} \vartheta &=& \angle QO_3P - 90° = \angle QO_3O_4 + \angle PO_3O_4 - 90° \\ &=& \cos^{-1}\dfrac{R-r}{R+r} + \cos^{-1}\dfrac{R}{R+r} - 90°. \end{array}$$

Substitution into $(*)$ and some straightforward trigonometry give

$$\begin{array}{rcl} AB &=& CD = 2R + 2R\,\dfrac{2r\sqrt{2R^2+Rr} - R(R-r)}{(R+r)^2} \qquad \text{and} \\ BC &=& DA = 2R + 2R\,\dfrac{2R\sqrt{Rr} + (R-r)\sqrt{r^2+2Rr}}{(R+r)^2} \end{array}$$

Comment. A calculation shows that, in fact, we require $R/4 < r < (3 + \sqrt{2} - 2\sqrt{2+\sqrt{2}})R$, (note $(3+\sqrt{2}-2\sqrt{2+\sqrt{2}}) \approx 0.7187$) otherwise rectangle $ABCD$ no longer "frames" the four circles with AB tangent to C_1 only (and CD tangent to C_3 only). When $r = (3+\sqrt{2}-2\sqrt{2+\sqrt{2}})R$ the framing rectangle is a square, each of whose sides is tangent to two of the circles.

Also solved by Jahangeer Kholdi & Boris Rays (jointly), Portsmouth, VA and Landover, MD, and by the proposer.

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- 4936: *Proposed by Kenneth Korbin, New York, NY.*

  Find all prime numbers P and all positive integers a such that $P - 4 = a^4$.

- 4937: *Proposed by Kenneth Korbin, New York, NY.*

  Find the smallest and the largest possible perimeter of all the triangles with integer-length sides which can be inscribed in a circle with diameter 1105.

- 4938: *Proposed by Luis Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b and c be the sides of an acute triangle ABC. Prove that

$$\csc^2 \frac{A}{2} + \csc^2 \frac{B}{2} + \csc^2 \frac{C}{2} \geq 6 \left[\prod_{cyclic} \left(1 + \frac{b^2}{a^2} \right) \right]^{1/3}$$

- 4939: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  For any positive integer n, prove that

$$\left\{ 4^n + \left[\sum_{k=0}^{n-1} 3^{k+1/2} \binom{2n}{2k+1} \right]^2 \right\}^{1/2}$$

  is a whole number.

- 4940: *Proposed by Michael Brozinsky, Central Islip, NY and Leo Levine, Queens, NY .*

  Let $S = \{n \in N | n \geq 5\}$. Let $G(x)$ be the fractional part of x, i.e., $G(x) = x - [x]$ where $[x]$ is the greatest integer function. Characterize those elements T of S for which the function

$$f(n) = n^2\Big(G\Big(\frac{(n-2)!}{n}\Big)\Big) = n.$$

- 4941: *Proposed by Tom Leong, Brooklyn, NY.*

The numbers $1, 2, \cdots, 2006$ are randomly arranged around a circle.
(a) Show that we can select 1000 adjacent numbers consisting of 500 even and 500 odd numbers.
(b) Show that part (a) need not hold if the numbers were randomly arranged in a line.

Solutions

- 4900: *Proposed by Kenneth Korbin, New York, NY.*

Find three pairs of positive integers (a, b) with $a < b$ such that triangles with sides $(a, b, 25)$ can be inscribed in a circle with diameter 65.

Solution by David E. Manes, Oneonta, NY.

Five such pairs of positive integers are $(16, 39), (33, 52), (39, 56), (52, 63)$, and $(60, 65)$. Assume the triangle has vertices A, B, and C with opposite sides a, b, and c respectively. Then one can argue geometrically that $\sin(\angle ACB) = \dfrac{c}{2R}$, where R is the radius of the circumscribed circle. Thus, $\sin(\angle ACB) = \dfrac{25}{65} = \dfrac{5}{13}$ so that $\cos(\angle ACB) = \pm\dfrac{12}{13}$. If $\cos(\angle ACB) = \dfrac{12}{13}$, then by the law of cosines,

$$615 = a^2 + b^2 - \frac{24ab}{13} \quad \text{or} \quad 13\mathrm{a}^2 - 24\mathrm{ab} + 13\mathrm{b}^2 - 625 \cdot 13 = 0.$$

Note that the quadratic equation is symmetric in a and b. Solving for a, one obtains

$$a = \frac{24b \pm 10\sqrt{4225 - b^2}}{26}. \quad \mathbf{(1)}$$

Since a is an integer, it follows that $4225 - b^2 = x^2$ for some integer x. This equation has a finite number of solutions for b; namely $b = 16, 25, 33, 39, 52, 56, 60, 63$, or 65. Substituting the values $39, 52, 56, 63$, and 65 for b in $\mathbf{(1)}$ and using the negative sign for the square root yields the five stated solutions. Finally, if $\cos(\angle ACB) = \dfrac{-12}{13}$, then no solutions for triangles are obtained.

Also solved by Dionne Bailey, Elsie Campbell,& Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Tom Leong, Brooklyn, NY; Peter E. Liley, Lafayette, IN, and the proposer.

- 4901: *Proposed by Kenneth Korbin, New York, NY.*

Given pentagon $ABCDE$ with sides $\overline{AB} = 468$, $\overline{BC} = 580$, $\overline{CD} = 1183$, and $\overline{DE} = 3640$. Find the length of side $\overline{AE}$ so that the area of the pentagon is maximum.

Solution by Tom Leong, Brooklyn, NY.

Pentagon $ABCDE$ along with its reflection about line AE yield an octagon all of whose sides are given. Pentagon $ABCDE$ has maximum area if and only if the octagon has maximum area. It is well-known that the maximum area of a polygon with prescribed sides occurs when the polygon is inscribed in a circle (see, for example, G. Polya, *Mathematics* *and Plausible Reasoning*, Princeton University Press, 1990). Hence pentagon $ABCDE$ has maximum area when it is inscribed in a (semi)circle with AE as diameter.

Let O denote the center of this circle and put $a = \overline{AB}, b = \overline{BC}, c = \overline{CD}, d = \overline{DE}, x = \overline{AE}$ and $\vartheta_a = \angle AEB, \vartheta_b = \angle BEC, \vartheta_c = \angle CAD, \vartheta_d = \angle DAE$. The (extended) Law of Sines in triangle AEB gives $\sin\vartheta_a = a/x$ and consequently $\cos\vartheta_a = \sqrt{x^2-a^2}/x$. We obtain similar formulas for $\vartheta_b, \vartheta_c, \vartheta_d$ by looking at triangles BEC, CAD, DAE. Since $(\vartheta_a+\vartheta_b)+(\vartheta_c+\vartheta_d) = \frac{1}{2}\angle AOC + \frac{1}{2}\angle EOC = 90^\circ$, we have $\sin(\vartheta_a+\vartheta_b) = \cos(\vartheta_c+\vartheta_d)$. Thus $\sin\vartheta_a\cos\vartheta_b + \cos\vartheta_a\sin\vartheta_b = \cos\vartheta_c\cos\vartheta_d - \sin\vartheta_c\sin\vartheta_d$
$\frac{a}{x}\cdot\frac{\sqrt{x^2-b^2}}{x} + \frac{\sqrt{x^2-a^2}}{x}\cdot\frac{b}{x} = \frac{\sqrt{x^2-c^2}}{x}\cdot\frac{\sqrt{x^2-d^2}}{x} - \frac{c}{x}\cdot\frac{d}{x}$
$a\sqrt{x^2-b^2} + b\sqrt{x^2-a^2} = \sqrt{(x^2-c^2)(x^2-d^2)} - cd$. Clearing radicals, we would obtain a quartic equation in x^2 which in theory is solvable. However, using a computer algebra system is quicker and easier. Using the obvious bounds $3640 = d < x < a+b+c+d = 5871$, we obtain $x = 4225$ which can be verified as the exact answer.

Also solved by the proposer.

- 4902: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Prove that

$$F_nF_{n+1} \le \frac{2}{n+1}\sum_{k=1}^{n} kF_k^2$$

where F_k is the k^{th} Fibonacci number defined by $F_0 = 0, F_1 = 1$ and for $k \ge 2, F_k = F_{k-1} + F_{k-2}$.

Solution by Brian D. Beasely, Clinton, SC.

It is straightforward to show that the given inequality holds for $n \in \{1,2,3,4\}$. For $n \ge 5$, we prove the stronger inequality

$$F_nF_{n+1} \le \frac{2}{n+1}(nF_n^2), \quad \text{or equivalently} \quad F_{n+1} \le \frac{2n}{n+1}F_n.$$

Since $5/3 \le 2n/(n+1)$ for $n \ge 5$, it suffices to show that $F_{n+1} \le (5/3)F_n$ for $n \ge 5$. We use the Binet formula $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ for $n \ge 0$, where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Then

$$\begin{aligned} 3F_{n+1} \le 5F_n \quad &\iff \quad \frac{3(\alpha^{n+1}-\beta^{n+1})}{\sqrt{5}} \le \frac{5(\alpha^n-\beta^n)}{\sqrt{5}} \\ &\iff \quad (5-3\beta)\beta^n \le (5-3\alpha)\alpha^n, \end{aligned}$$

where we note that both $5-3\beta$ and $5-3\alpha$ are positive. Since $\beta < 0 < \alpha$, this last inequality holds for odd n. For even n, it holds when

$$n \ge \frac{\log((5-3\beta)/(5-3\alpha))}{\log(\alpha/|\beta|)} = 4,$$

so we are done.

Addendum. Using the Binet formula again and noting that $\alpha > 1$ while $|\beta| < 1$, we have the corresponding asymptotic result $F_{n+1} \sim \dfrac{\alpha n}{n+1}F_n$.

Also solved by the proposer Dionne Bailey, Elsie Campbell & Charles Diminnie, San Angelo, TX; N. J. Kuenzi, Oshkosh, WI; Tom Leong, Brooklyn, NY; Carl Libis, Kingston, RI; Charles McCracken, Dayton, OH, and the proposer.

- 4903: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let n be a nonnegative integer. Prove that

$$(n!)^4(n^2+6n+11)^n \geq 2^{n+2}3^{2n+1}(n+1)^{n-3}(n+2)^{n-2}(n+3)^{n-1}$$

Solution by Paul M. Harms, North Newton, KS.

When n is a positive integer an inequality from a Stirling Formula is $n! > \sqrt{2n\pi}(n/e)^n$. Replacing the factorial in the problem by this (Stirling) inequality, it is shown below that for large enough n,

$$2^2n^2\pi^2n^{4n}e^{-4n}(n^2+6n+11)^n \geq \frac{2^n2^23^{2n}3(n+1)^n[(n+2)(n+3)]^n}{(n+1)^3(n+2)^2(n+3)}.$$

Multiplying by positive numbers, simplifying and using $n^2+6n+11=(n+3)^2+2$, the last inequality is equivalent to the following inequality:

$$(n+1)^3(n+2)^2(n+3)n^2\pi \geq [2e^43^2/n^3]^n[3/\pi][(n+1)/n]^n[(n+2)(n+3)/\{(n+3)^2+2\}]^n.$$

Note that $2e^43^2 < 1000, 3/\pi < 1, \quad (n+2)(n+3)/\{n+3)^2+2\} < 1$ and $[(\mathrm{n}+1)/\mathrm{n}]^{\mathrm{n}}$ approaches e from below as n increases.

When n is a positive integer greater than 9,

$$(n+1)^3(n+2)^2(n+3)n^2\pi > [1000/n^3]^n(1)(3)1^n$$

$$> [2e^43^2/n^3]^n[3/\pi][(n+1)/n]^n[(n+2)(n+3)/\{(n+3)^2+2\}]^n.$$

This means that the original problem inequality holds when n is an integer greater than 9, To complete the problem show that the original problem inequality holds for $n = 0, 1, 2, \cdot\cdot, 9$.

Also solved by Tom Leong, Brooklyn, NY, and the proposer.

- 4904: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

  Let S be a set of positive integers such that for any element p in S which is sufficiently large, either $p-1$ or $p+1$ is composite. Such a set is called an UP-DOWN set. The set of primes is obviously in this category. Show that the set of perfect numbers, whether even or odd, is an UP-DOWN set.

Solution by Charles McCracken, Dayton, OH.

If n is odd, then $n-1$ and $n+1$ are even and hence composite.

If n is even, $n = 2^{p-1}(2^p-1)$ where p is prime. Now

$$\begin{aligned} n = 2^{p-1}(2^p-1) = 2^{2p-1}-2^{p-1} &= 2^{\text{odd}}-2^{\text{even}} \\ &\equiv 2-1 \equiv 1 \equiv 1(\text{mod}3). \end{aligned}$$

Therefore $n-1 \equiv 0(\text{mod}3)$ and hence composite.

Note we exclude the case where $p = 2$ and $n = 6$ which is the one exception to the general statement.

Also solved by Charles Ashbacher, Cedar Rapids, IA; Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, & Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Jahangeer Kholdi, Portsmouth, VA; Kenneth Korbin, New York, NY; N. J. Kuenzi, Oshkosh, WI; Tom Leong, Brooklyn, NY; David E. Manes, Oneonta, NY; Boris Rays, Landover, MD; R. P. Sealy, Sackville, New Brunswick, Canada, and the proposer.

- 4905: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

  Consider a set S of positive integers in which the elements range over all possible numbers of digits (such as the set of repunit numbers). Such a set S is called *digitally complete.* Which of the following are digitally complete?

  1. The set of factorials? 2. The set of primes?

Solution by N. J. Kuenzi, Oshkosh, WI.

1. Consider the set of factorials. For any positive integer n, let $L(n)$ be the length of the digital representation of $n!$
Examples: $3! = 6$ so $L(3) = 1$, $5! = 120$ so $L(5) = 3$, and $10! = 3,628,800$ so $L(10) = 7$.

For $n > 10$, if $m > n$ then $L(m) > L(n)$. Now $100! = 100(99!)$ and so $L(100) = L(99) + 2$. It follows that there isn't any positive integer n for which the length of the digital representation of $n!$ is $L(99) + 1$.

If you are willing to do some multiplications you can numerically verify that $L(14) = 11$ and $L(15) = 13$. So there isn't any positive integer n for which the length of the digital representation of $n!$ is 12. The set of factorials is not digitally complete.

2. Consider the set of primes. Primes less than 10 have a single digit representation. Primes between 10 and 100 have a two digit representation. In general, any prime number p between 10^{n-1} and 10^n will have a digital representation of length n.

It is known that for $x > 3$ there is at least one prime number between x and $2x - 2$. (See Beiler, Albert H. *Recreations in the Theory of Numbers: The Queen of Mathematics Entertains,* Dover Publications, Inc. 1964, p.227).

It follows from this result that there is at least one prime number between 10^{n-1} and 10^n and so there is a prime number which has digital representation of length n. The set of primes is digitally complete.

Also solved by Brian D. Beasley, Clinton, SC; Russell Euler & Jawad Sadek (jointly), Maryville, MO; Kenneth Korbin, New York, NY; Tom Leong, Brooklyn, NY; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; R. P. Sealy, Sackville, New Brunswick, Canada, and the proposer.

Late Solutions

Late solutions were received from **R.P. Sealy of Sackville, New Brunswick, Canada** to problem 4889, and from **David C. Wilson of Winston-Salem, NC** to problem 4891.

Problems

Ted Eisenberg, Section Editor

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2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
March 15, 2007

- 4942: *Proposed by Kenneth Korbin, New York, NY.*

  Given positive integers a and b. Find the minimum and the maximum possible values of the sum $(a+b)$ if $\dfrac{ab-1}{a+b} = 2007$.

- 4943: *Proposed by Kenneth Korbin, New York, NY.*

  Given quadrilateral $ABCD$ with $\overline{AB} = 19$, $\overline{BC} = 8$, $\overline{CD} = 6$, and $\overline{AD} = 17$. Find the area of the quadrilateral if both $\overline{AC}$ and $\overline{BD}$ also have integer lengths.

- 4944: *Proposed by James Bush, Waynesburg, PA.*

  Independent random numbers a and b are generated from the interval $[-1,1]$ to fill the matrix $A = \begin{pmatrix} a^2 & a^2+b \\ a^2-b & a^2 \end{pmatrix}$. Find the probability that the matrix A has two real eigenvalues.

- 4945: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Prove that

  $$17 + \sqrt{2}\sum_{k=1}^{n}\left(L_k^4 + L_{k+1}^4 + L_{k+2}^4\right)^{1/2} = L_n^2 + 3L_{n+1}^2 + 5L_nL_{n+1}$$

  where L_n is the n^{th} Lucas number defined by $L_0 = 2, L_1 = 1$ and for all $n \geq 2, L_n = L_{n-1} + L_{n-2}$.

- 4946: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Let z_1, z_2 be nonzero complex numbers. Prove that

$$\left(\frac{1}{|z_1|}+\frac{1}{|z_2|}\right)\left(\left|\frac{z_1+z_2}{2}+\sqrt{z_1z_2}\right|+\left|\frac{z_1+z_2}{2}-\sqrt{z_1z_2}\right|\right)\geq 4.$$

- 4947: *Proposed by Tom Leong, Brooklyn, NY.*

  Define a set S of positive integers to be *among composites* if for any positive integer n, there exists an $x \in S$ such that all of the $2n$ integers $x \pm 1, x \pm 2, \ldots, x \pm n$ are composite. Which of the following sets are among composites? (a) The set $\{a + dk | k \in N\}$ of terms of any given arithmetic progression with $a, d \in N, d > 0$. (b) The set of squares. (c) The set of primes. (d)* The set of factorials.

Solutions

- 4906: *Proposed by Kenneth Korbin, New York, NY.*

  Given hexagon $ABCDEF$ with sides $\overline{AB} = \overline{BC} = \overline{CD} = \overline{DE} = \overline{EF} = 1$.
  Find the length of side $\overline{AF}$ so that the area is maximum.

  Solution by Harry Sedinger, St. Bonaventure, NY.

  It is well known that a polygon with sides of equal length has maximum area when it is also regular. In this case, the hexagon is half of a ten sided polygon P with sides of length one. Thus the area of the hexagon is maximum when P is regular. In this case, the side from A to F is the diameter of P and is well known to be equal to $1/\sin 18^o$.

  Comment by David E. Manes, Oneonta, NY. The solution follows from a more general result; namely, given an $n-gon$ with $n-1$ sides of unit length, then the length of the remaining side that maximizes the area of the $n-gon$ is $\csc(\pi/(2n-2))$. A beautiful proof of this result attributed to Murray Klamkin is given in *In Polya's Footsteps,* (R. Honsberger, MAA, 1997, p. 30). Note that n=6 yields the stated result.

  Also solved by Tom Leong, Brooklyn, NY; Boris Rays & Jahangeer, Kholdi (jointly), Landover, MD & Portsmouth, MD (respectively), and the proposer.

- 4907: *Proposed by Kenneth Korbin, New York, NY.*

  (a) Find the dimensions of all integer sided triangles with perimeter $P > 200$ and with area $K = 2P$.

  (b) Find the dimensions of all integer sided triangles with perimeter $P > 2000$ and with area $K = 3P$.

  Solution by David Stone and John Hawkins (jointly), Statesboro,GA.

  Surprisingly, there don't seem to be many solutions. Triangles with integer sides and integer area are known as integer Heronian triangles (see Math Wold at wolfram.com). There are infinitely many such triangles, but the given conditions place severe restrictions on their size.

  If a, b, c are the sides of such a triangle, with perimeter $a + b + c$ and semiperimeter $s = (a+b+c)/2$, then the area is given by Heron's Formula as $K = \sqrt{s(s-a)(s-b)(s-c)}$. For (a) we believe there are three solutions:

$$\begin{array}{ccc} (18, 289, 305) & P = 612 & K = 1224 \\ (19, 153, 170) & P = 342 & K = 684 \\ (21, 85, 104) & P = 210 & K = 420. \end{array}$$

And for (b) there is exactly one solution:

$$(38, 1369, 1405) \quad P = 2812 \quad K = 8436.$$

We have much supporting evidence, but no conclusive proof, that these are all of the solutions. We also have a general conjecture.

Conjecture: For any fixed scaling fctor r, there are only finitely many integer Heronian triangles with $K = rP$. The largest such triangle is:

$$(4r^2 + 2, (4r^2 + 1)^2, 16r^4 + 12r^2 + 1) \quad P = 4(2r^2 + 1)(4r^2 + 1) \quad K = 4r(2r^2 + 1)(4r^2 + 1).$$

Here is the method: Noting that each side of the triangle must be less than the semiperimeter s, we introduce a new parameter: Let $s = c + e$. where $e \geq 1$. That is, $(a + b + c)/2 = c + e$, so $c = a + b - 2e, P = 2(a + b - e)$ and $s = a + b - e$. Moreover, $K^2 = s(s - a)(s - b)(s - c) = (a + b - e)(b - e)(a - e)e$.

Now, imposing the condition $K = rP$ or $K^2 = r^2 P^2$ we have $(a+b-e)(b-e)(a-e)e = r^2 2^2 (a + b - e)^2$ and solving for b, we find that $b = e + \dfrac{4r^2 a}{ae - e^2 - 4r^2}$. Because b and e are integers, this implies that $ae - e^2 - 4r^2$ must be a divisor of $4r^2 a$. There are many such divisors, but the simplest is 1: set $ae - e^2 - 4r^2 = 1$, so $e(a - e) = 4r^2 + 1$.

Again there may be many ways for e and $a - e$ to achieve a factorization of $4r^2 + 1$, but we select a simple factorization: set $e = 4r^2 + 1$ and $a - e = 1$. This forces $a = e + 1 = 4r^2 + 2$. We can then compute $b = 16r^4 + 12r^2 + 1$, and $c = (4r^2 + 1)^2$, and finally $P = 4(2r^2 + 1)(4r^2 + 1)$ and $K = 4r(2r^2 + 1)(4r^2 + 1)$ which indeed equals rP.

By choosing other divisors and factorizations, we find other solutions (with much duplication!). However they all seem to be smaller than the one just demonstrated–thus our conjecture.

For instance,with $r = 2$, we can make another easy choice for the divisor: let $ae - e^2 - 4r^2 = 2$, so we have $e(a - e) = 18$. Letting $e = 18$ and $a - e = 1$ produces the triangle $(19, 153, 170)$ noted earlier. Or by selecting $ae - e^2 - 16 = 4$ and $e = 20$ we find the triangle $(21, 85, 104)$. All other choices we have made lead to triangles with perimeter < 200. Likewise with $r = 3$, all other choices lead to triangles with perimeter< 2000,(except for the one noted above (38,1369,1405)). If we want to put our trust in a machine, a computer search found no other $K = 3P$ triangles with $P > 2000$.

A connection to the Golden Ratio: Assuming the truth of our conjecture, the longest side of the largest $K = rP$ triangle is $16r^4 + 12r^2 + 1$, which factors, not nicely using integers, but nicely using the Golden Ratio: $16r^4 + 12r^2 + 1 = (4r^2 + \alpha^2)(4r^2 + \beta^2)$, where α and β are the roots of $x^2 - x - 1$.

Comment by editor: **Charles Diminnie of San Angelo, TX** calls our attention to the article "Pythagorean Triples and the Problem A=mP for Triangles" in the April 06 issue of the Mathematics Magazine by Lubomir P. Markov. This article shows the degree of complexity which is inherent in this problem. The problem though, was built by Ken Korbin from formulas in an article by K. R. S. Sastry entitled "Heron Problems" (Journal of Mathematics and Computer Education, 29(2), Spring, 1995).

Also solved by Dionne Bailey, Elsie Campbell, & and Charles Diminnie (jointly), San Angelo, TX; David E. Manes, Oneonta, NY, and the proposer.

- 4908: *Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Barcelona, Spain.*

Evaluate $\displaystyle\int_0^1 e^{x^2}\,dx + \int_1^e \sqrt{\ln x}\,dx.$

Solution I by N. J. Kuenzi, Oshkosh, WI.

The rectangle with corners at $(0,0),(1,0),(1,e)$, and $(0,e)$ has area of e square units. The curve $y=e^{x^2}$ (or $x=\sqrt{\ln y}$) splits the rectangle into two regions. The area of the lower region is given by $\int_0^1 e^{x^2}dx$ and the area of the upper region is given by $\int_1^e \sqrt{\ln(y)}dy$. The sum of the areas of the two regions is the area of the given rectangle which is e.

$$\int_0^1 e^{x^2}dx + \int_1^e \sqrt{\ln(y)}dy = \int_0^1 e^{x^2}dx + \int_1^e \sqrt{\ln(x)}dx = e.$$

Solution II by R. P. Sealy, Sackville, New Brunswick, Canada.

The answer is e. Making the substitution $x=e^{t^2}$ in the second integral and then integrating by parts $\int_1^e \sqrt{\ln x}dx = \int_0^1 2t^2e^{t^2}dt = e - \int_0^1 e^{t^2}dt = e - \int_0^1 e^{x^2}dt.$

Also solved by Brain D. Beasley, Clinton, SC; John Boncek, Montgomery, AL; Michael Bronzinsky, Central Islip, NY; Elsie M. Campbell, Dionne T. Bailey, & Charles Diminnie (jointly), San Angelo, TX; Ben Diener & Neil Long (jointly), Upland, IN; Paul M. Harms, North Newton, KS; Karl Havlak, San Angelo, TX; Daryl Henry, David Kasper, & Rebekah Bergens (jointly), Upland, IN; Julia Hess, Cassandra Johnston, & Peter Schweitzer (jointly), Upland, IN; Jahangeer Kholdi, Portsmouth, VA; William R. Klinger, Upland, IN; Tom Leong (two solutions), Brooklyn, NY; Kevin Little, Jonas Herum, & Aaron Hoesli (jointly), Upland, IN; Brian Lucarelli, Waynesburg, PA; David E. Manes, Oneonta, NY; Boris Rays, Landover, MD; Harry Sedinger, St. Bonaventure, NY; David Stone & John Hawkins (jointly), Statesboro, GA; Aziz Zahraoui, Portsmouth, VA, and the proposers.

- 4909: *Proposed by José Luis Díaz-Barrero Barcelona, Spain.*

Prove that
$$(a^{2b}+b^{2a})(a^{2c}+c^{2a})(b^{2c}+c^{2b}) > \frac{1}{8}$$
for any $a,b,c\in(0,1)$.

Solution by Michael Brozinsky, Central Islip, NY.

Let (x,y) be interior to the square S bounded by $x=0, x=1, y=0$, and $y=1$. Let C denote an arbitrary constant in $[0,1)$. The function $F(x,y)=x^{2y}+y^{2x}$ is symmetric about the line $y=x$ and $F(x,x+C)>F(x,x)$ if $C>0$ since $x^{2(x+C)}>x^{2x}$ and $(x+C)^{2x}>x^{2x}$. Hence, in determining the greatest lower bound to $F(x,y)$ on S, it suffices to consider C=0; i.e., the function $g(x)=F(x,x)=2x^{2x}$.

Now $g'(x)=2x^{2x}(2\ln(x)+2)$ and so (by the first derivative test), $g(x)$ has an absolute minimum on $(0,1)$ when $\ln(x)=-1$; i.e., when $x=1/e$, g(x)= $2(1/2)^{2/e}=A$. The expression in the problem is merely $F(a,b)\cdot F(a,c)\cdot F(b,c)$ and its absolute minimum is thus A^3 which is approximately 0.88, which is greater than 7/8.

Also solved by Tom Leong (two solutions), Brooklyn, NY; David E. Manes, Oneonta, NY; David Stone & John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4910: *Proposed by Karl Havlak, San Angelo, TX.*

  A man began an evening with \$10. He visited 10 casinos and doubled his money at each casino. Upon exiting one of the casinos, he found a couple of paper bills on the ground of U.S. currency (\$1, \$2, \$5, \$10, \$20, \$50, \$100). If he left the last casino with \$10,656, can we determine exactly how much money he found on the ground and when he found it?

  Solution by Carl Libis, Kingston, RI.

  \$10 doubled 10 times is $10(2^{10}) = \$10,240$. The difference between \$10,240 and \$10,656 is \$416. Keep dividing \$416 by 2 until the value is not an integer. These values are 416, 208, 104, 52, 26, 13, and 6.5. The only value that is the sum of U.S. currency is \$52=\$50+\$2. Therefore the man found \$52 on the ground after leaving the seventh casino.

  Also solved by Brian D. Beasley, Clinton, SC; John Boncek, Montgomery, AL; Elsie M. Campbell, Dionne T. Bailey, & Charles Diminnie (jointly) , San Angelo, TX; Ben Diener & Neil Long (jointly), Upland, IN; Paul M. Harms, North Newton, KS; Tom Leong, Brooklyn, NY; Julia Hess, Cassandra Johnston, & Peter Schweitzer (jointly), Upland, IN; David Kasper, Rebekah Bergens, & Daryl Henry (jointly), Upland, IN; Kevin Little, Aaron Hoesli, & Jonas Herum (jointly), Upland, IN; Jahangeer Kholdi, Portsmouth, VA; William R. Klinger, Upland, IN; Kenneth Korbin, NY, NY; N. J. Kuenzi, Oshkosh, WI; Peter E. Liley, Lafayette, IN; Susan Malkowski, Richmond, KY; David E. Manes, Oneonta, NY; Melfried Olson, Honolulu, HI; Jennifer Pevley, Richmond, KY; Boris Rays, Landover, MD; R. P. Sealy, Sackville, New Brunswick, Canada; Harry Sedinger, St. Bonaventure, NY; Tonya Simmons, Montgomery, AL, and the proposer.

- 4911: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

  It is easy to show, if zero factors are ignored, that the product of the squares of the six trigonometric functions is 1. Is it possible for the sum of these squares also to equal 1?

  Solution by Brian D. Beasley, Clinton, SC.

  Yes, provided that we define the trigonometric functions for complex variables. We seek z such that $\sin^2 z+\cos^2 z+\tan^2 z+\cot^2 z+\sec^2 z+\csc^2 z = 1$, or $\dfrac{\sin^2 z}{\cos^2 z}+\dfrac{\cos^2 z}{\sin^2 z}+\dfrac{1}{\cos^2 z}+\dfrac{1}{\sin^2 z} = 0$. This is equivalent to $\sin^4 z + (1-\sin^2 z)^2 + 1 = 0$, so we need a solution of the equation $\sin z = \pm\sqrt{\dfrac{1}{2}(1\pm i\sqrt{3})} = \pm\dfrac{1}{2}(\sqrt{3}\pm i)$. Since $\sin z = (e^{iz}-e^{-iz})/(2i)$, we have $e^{iz} = \pm\dfrac{1}{2}\pm\dfrac{i\sqrt{3}}{2}\pm\sqrt{\dfrac{1}{2}\pm\dfrac{i\sqrt{3}}{2}}$, where the second and fourth plus/minus signs are the same. Focusing on only one solution for z, we find $z = \dfrac{\pi}{4} - i\ln\left(\dfrac{\sqrt{2}(1+\sqrt{3})}{2}\right)$.
  For this value of z, it is straightforward to verify that $\sin z = (\sqrt{3}-i)/2$, and hence the sum of the squares of the six trigonometric functions at z will equal 1.

  Comment by editor: **R. P. Sealy of Sackville, New Brunswick, Canada** is the only other individual of the 26 who submitted a solution to this problem that considered complex values for the argument. When restricted to the real domain, it is easily shown that the sum of the squares cannot be equal to one.

Problems

Ted Eisenberg, Section Editor

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- 4948: *Proposed by Kenneth Korbin, New York, NY.*

  The sides of a triangle have lengths x_1, x_2, and x_3 respectively. Find the area of the triangle if
  $$(x-x_1)(x-x_2)(x-x_3) = x^3 - 12x^2 + 47x - 60.$$

- 4949: *Proposed by Kenneth Korbin, New York, NY.*

  A convex pentagon is inscribed in a circle with diameter d. Find positive integers a, b, and d if the sides of the pentagon have lengths a, a, a, b, and b respectively and if $a > b$. Express the area of the pentagon in terms of a, b, and d.

- 4950: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers such that $abc = 1$. Prove that
  $$\frac{a+b}{\sqrt[4]{a^3}+\sqrt[4]{b^3}} + \frac{b+c}{\sqrt[4]{b^3}+\sqrt[4]{c^3}} + \frac{c+a}{\sqrt[4]{c^3}+\sqrt[4]{a^3}} \ge 3.$$

- 4951: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let α, β, and γ be the angles of an acute triangle ABC. Prove that
  $$\pi \sin\sqrt{\frac{\alpha^2+\beta^2+\gamma^2}{\pi}} \ge \alpha\sin\sqrt{\alpha} + \beta\sin\sqrt{\beta} + \gamma\sin\sqrt{\gamma}.$$

- 4952: *Proposed by Michael Brozinsky, Central Islip, NY & Robert Holt, Scotch Plains, NJ.*

  An archeological expedition discovered all dwellings in an ancient civilization had $1, 2$, or 3 of each of k independent features. Each plot of land contained three of these houses such that the k sums of the number of each of these features were all divisible by 3. Furthermore, no plot contained two houses with identical configurations of features and no two plots had the same configurations of three houses. Find **a)** the maximum number of plots that a house with a given configuration might be located on, and **b)** the maximum number of distinct possible plots.

- 4953: *Proposed by Tom Leong, Brooklyn, NY.*

  Let$\pi(x)$ denote the number of primes not exceeding x. Fix a positive integer n and define sequences by $a_1 = b_1 = n$ and

$$a_{k+1} = a_k - \pi(a_k) + n, \qquad b_{k+1} = \pi(b_k) + n + 1 \quad \text{for k} \geq 1.$$

  a) Show that $\lim\limits_{k\to\infty} a_k$ is the n^{th} prime.

  b) Show that $\lim\limits_{k\to\infty} b_k$ is the n^{th} composite.

Solutions

- 4912: *Proposed by Kenneth Korbin, New York, NY.*

  Find an explicit formula for the N^{th} term for the sequence

$$2, 15, 88, 475, 2472, \cdots, t_N, \cdots$$

  where $t_N = 10t_{N-1} - 31t_{N-2} + 30t_{N-3}$.

Solution by Bryce Duncan (student), Auburn University-Montgomery, AL.

The characteristic equation for the linear recurrence is

$$\lambda^3 - 10\lambda^2 + 31\lambda - 30 = (\lambda - 2)(\lambda - 3)(\lambda - 5) = 0.$$

The general formula for t_N will be given by $t_N = \alpha_1(2^N) + \alpha_2(3^N) + \alpha_3(5^N)$ for some α_i. Using the initial conditions with $2, 15$, and 88 corresponding to $N = 0, 1, 2$ respectively, we arrive at the linear system:

$$\begin{aligned} 2 &= \alpha_1 + \alpha_2 + \alpha_3 \\ 15 &= 2\alpha_1 + 3\alpha_2 + 5\alpha_3 \\ 88 &= 4\alpha_1 + 9\alpha_2 + 25\alpha_3. \end{aligned}$$

Solving this yields $\alpha_1 = -2/3,\ \alpha_2 = -3/2,\ +\alpha_3 = 25/6$. And so

$$t_N = -\frac{2}{3}(2^N) - \frac{3}{2}(3^N) + \frac{25}{6}(5^N). \qquad t_N = \frac{-2^{N+2} - 3^{N+2} + 5^{N+2}}{6}.$$

Also solved by Brian D. Beasley, Clinton, SC; Dionne Bailey, Elsie Campbell, & Charles Diminnie (jointly), San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; Rebecca Duff (student), Waynesburg College, PA; Paul M. Harms, North Newton, KS; N. J. Kuenzi, Oshkosh, WI; David E. Manes,

Oneonta, NY; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone & John Hawkins (jointly), Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 4913: *Proposed by Kenneth Korbin, New York, NY.*

The number $256 = 2^8$ has 9 positive integer divisors which are $1, 2, 4, 8, 16, 32, 64, 128, 256$. Find the smallest number with 256 positive integer divisors.

Solution by Paul M. Harms, North Newton, KS.

Each prime number has two positive integer divisors. We also see that any prime factor raised to the power n has $n+1$ positive integer divisors. If we multiply prime factors, the quantity of positive integer divisors is the product of the number of positive integer divisors of each prime factor. For example, $2^3(3)(5)$ has 2^4positive integer divisors. For this problem we need prime factors raised to powers which have 2^n positive integer divisors where n is a positive integer. A number like 2^3 has $4 = 2^2$ positive integer divisors and 2^7 has $8 = 2^3$ positive integer divisors.

To find the number satisfying the problem, I will start with all prime numbers raised to the first power and then decrease the number by changing powers of the smaller primes and excluding the large primes. The numbers $2(3)(5)(7)(11)(13)(17)(19)$ has 2^8 positive integer divisors. The following number have 2^8 positive integers divisors: $2^3(3)(5)(7)(11)(13)(17)$ and $N = 2^3(3)(5)(7)(11)(13)$. The next highest power of 2 or 3 that we need is the power of 7. Clearly, numbers like $2^7 3^3(5)(7)(11), 2^3 3^7(5)(11)$ or $2^3 3^3 5^5(7)(11)$ are all greater than N. The smallest number which satisfies the problem is $N = 2^3 3^3(5)(7)(11)(13) = 1,081,080$.

Also solved by Brian D. Beasley, Clinton, SC; Pat Costello, Richmond, KY; Bryce Duncan, Montgomery, AL; Jeff Herrin, Richmond, KY; Bryan Howard, Wetumpka, AL; N. J. Kuenzi, Oshkosh, WI; Carl Libis, Kingston, RI; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Yair Mulian, Beer-Sheva, Israel; R. P. Sealy, Sackville, New Brunswick, Canada; April Spears, Richmond, KY; David Stone & John Hawkins (jointly), Statesboro, GA;David C. Wilson, Winston-Salem, NC, and the proposer.

- 4914: *Proposed by Kenneth Korbin, New York, NY.*

Find three primitive Pythagorean triangles which all have the same length perimeter 14280.

Solution by David C. Wilson, Winston-Salem, NC.

The three sides of a PPT are $m^2 - n^2, 2mn, m^2 + n^2$ where$(m, n) = 1$ and m and n have different parity. Since the perimeter must be 14280, $(\mathrm{m}^2 - n^2) + 2mn + (m^2 + n^2) = 14280 \Longrightarrow m^2 + mn = 7140 \Longrightarrow n = \dfrac{7140}{m} - m$. Thus, m must be a divisor of $7140 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 17$. There are 48 divisors of 7140, but we need to check only the 24 divisors between 1 and 85.The only three that work are $m = 60, 68, 84$. If $m = 60$, then $n = 59$ and the sides are $119, 7080, 7081$; if $m = 68$, then $n = 37$ and the sides are $3255, 5032, 5993$; and if $m = 84$, then $n = 1$ and the sides are $7055, 168, 7057$.

Also solved by Charles Ashbacher, Cedar Rapids, IA; Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, & Charles Diminnie (jointly), San Angelo, TX; Pat Costello, Richmond, KY; William R. Klinger, Up-

land, IN; N. J. Kuenzi, Oshkosh, WI; Peter E. Liley, Lafayette, IN; David E. Manes,Oneonta, NY; Charles McCracken, Dayton, OH; Amihai Menuhin, Beer-Sheva, Israel; John Nord, Spokane, WA; R. P. Sealy, Sackville, New Brunswick, Canada; Harry Sedinger, St. Bonaventure, NY; Boris Rays & Jahangeer Kholdi (jointly), Landover, MD & Portsmouth, VA (respectively); David Stone & John Hawkins (jointly), Statesboro, GA; and the proposer.

This problem was also solved by the following students at Taylor University: Ben Diener & Neil Long (jointly), Kevin Little, Aaron Hoesli, & Jonas Herum (jointly), David Kasper, Rebekah Bergens & Daryl Henry (jointly); and by the following students at Eastern Kentucky University: Charles Groce, Ceyhun Ferik &Yongbok Lee (jointly); April Spears, and Martina Bray.

- 4915: *Proposed by Isabel Díaz Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Find the following sums:

$$(a) \quad \sum_{n=1}^{\infty} \frac{1003n2^{n+1}}{(n+2)!}, \qquad (b) \quad \sum_{n=1}^{\infty} \frac{n}{n^4+4n^2+16}.$$

Solution by Brian D. Beasley, Clinton, SC.

(a) We have

$$\sum_{n=1}^{\infty} \frac{1003n2^{n+1}}{(n+2)!} = 1003\sum_{n=1}^{\infty}\left(1-\frac{2}{n+2}\right)\frac{2^{n+1}}{(n+1)!} = 1003\sum_{n=1}^{\infty}\left(\frac{2^{n+1}}{(n+1)!}-\frac{2^{n+2}}{(n+2)!}\right),$$

so the series telescopes to the sum $1003(2) = 2006$.

(b) We have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n}{n^4+4n^2+16} &= \sum_{n=1}^{\infty}\left(\frac{1/4}{n^2-2n+4}-\frac{1/4}{n^2+2n+4}\right)\\ &= \sum_{n=1}^{\infty}\left(\frac{1/4}{(n-2)^2+2(n-2)+4}-\frac{1/4}{n^2+2n+4}\right),\end{aligned}$$

so the series telescopes to the sum $\frac{1}{4}\left(\frac{1}{3}+\frac{1}{4}\right) = \frac{7}{48}$.

Also solved by Chris Boucher, Salem, MA; Elsie M. Campbell, Dionne T. Bailey, & Charles Diminnie, San Angelo, TX; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; David Stone & John Hawkins, Statesboro, GA; Jon Welch, Pensacola, FL; David C. Wilson, Winston-Salem, NC, and the proposers.

- 4916: *Proposed by Isabel Díaz Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Let n be a positive integer. Prove that

$$\ln(1+F_n^2)\ln(1+L_n^2) \le \ln^2(1+F_{2n})$$

where F_n is the n^{th} Fibonacci number and L_n is the n^{th} Lucas number.

Solution by Charles R. Diminnie, San Angelo, TX.

To begin, let $f(x) = \ln(1+e^x) - e^x$. Since, $\lim_{x\to-\infty} f(x) = 0$ and

$$f'(x) = -\frac{e^{2x}}{1+e^x} < 0$$

for all x, we have $f(x) < 0$ for all x. Next, if $g(x) = \ln[\ln(1+e^x)]$, then for all x,

$$g'(x) = \frac{e^x}{(1+e^x)\ln(1+e^x)} \text{ and } \mathrm{g}''(\mathrm{x}) = \frac{\mathrm{e^x}[\ln(1+\mathrm{e^x}) - \mathrm{e^x}]}{(1+\mathrm{e^x})^2\ln^2(1+\mathrm{e^x})} = \frac{\mathrm{e^x f(x)}}{(1+\mathrm{e^x})^2\ln^2(1+\mathrm{e^x})} < 0.$$

Hence, $g(x)$ is concave down for all x and we get

$$\begin{aligned}
g\left(\frac{x+y}{2}\right) &\geq \frac{g(x)+g(y)}{2} \\
\Longrightarrow \ln\left[\ln\left(1+e^{\frac{x+y}{2}}\right)\right] &\geq \frac{\ln[\ln(1+e^x)] + \ln[\ln(1+e^y)]}{2} \qquad (1) \\
\Longrightarrow \ln\left[\ln^2\left(1+e^{\frac{x}{2}}e^{\frac{y}{2}}\right)\right] &\geq \ln[\ln(1+e^x)\ln(1+e^y)] \\
\Longrightarrow \ln^2\left(1+e^{\frac{x}{2}}e^{\frac{y}{2}}\right) &\geq \ln(1+e^x)\ln(1+e^y)
\end{aligned}$$

for all x, y. If $\alpha, \beta > 0$, substituting $x = 2\ln\alpha$ and $y = 2\ln\beta$ in (1) yields

$$\ln^2(1+\alpha\beta) \geq \ln\left(1+\alpha^2\right)\ln\left(1+\beta^2\right). \qquad (2)$$

Since $F_{2n} = F_n L_n$ and $F_n, L_n > 0$ for all $n \geq 1$, it follows by (2) that

$$\ln\left(1+F_n^2\right)\ln\left(1+L_n^2\right) \leq \ln^2(1+F_n L_n) = \ln^2(1+F_{2n}).$$

Remark. Statement (2) is a special case of Problem 3099 in the December, 2005 issue of **Crux Mathematicorum**.

Also solved by David Stone & John Hawkins (jointly), Statesboro, GA, and the proposers.

- 4917: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let n be a positive integer. Prove that $\displaystyle\sum_{k=0}^{n}\left\{\frac{1}{n}\sum_{\substack{j=0\\j\neq k}}^{n}\binom{n}{j}^2\right\}^{1/2} \geq 2^n.$

Solution by Dionne Bailey, Elsie Campbell & Charles Diminnie, San Angelo, TX.

We will use the well-known result $\displaystyle\sum_{i=0}^{n}\binom{n}{i} = 2^n$. By the Arithmetic Mean - Root Mean Square Inequality, we have $\displaystyle\left\{\frac{1}{n}\sum_{\substack{j=0\\j\neq k}}^{n}\binom{n}{j}^2\right\}^{\frac{1}{2}} \geq \frac{1}{n}\sum_{\substack{j=0\\j\neq k}}^{n}\binom{n}{j} = \frac{2^n - \binom{n}{k}}{n}$ for $\mathrm{k} = 0, 1, \ldots, \mathrm{n}$.

Therefore,

$$\sum_{k=0}^{n}\left\{\frac{1}{n}\sum_{\substack{j=0\\j\neq k}}^{n}\binom{n}{j}^2\right\}^{\frac{1}{2}} \geq \frac{1}{n}\sum_{k=0}^{n}\left[2^n - \binom{n}{k}\right] = \frac{1}{n}\left[(n+1)2^n - 2^n\right] = 2^n.$$

Also solved by David E. Manes, Oneonta, NY; David Stone & John Hawkins, two solutions, (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before May 1, 2007

- 4954: *Proposed by Kenneth Korbin, New York, NY.*

  Find four pairs of positive integers (a, b) that satisfy

  $$\frac{a+i}{a-i} \cdot \frac{b+i}{b-i} = \frac{111+i}{111-i}$$

  with $a < b$.

- 4955: *Proposed by Kenneth Korbin, New York, NY.*

  Between 100 and 200 pairs of red sox are mixed together with between 100 and 200 pairs of blue sox. If three sox are selected at random, then the probability that all three are the same color is 0.25. How many pairs of sox were there altogether?

- 4956: *Proposed by Kenneth Korbin, New York, NY.*

  A circle with radius $3\sqrt{2}$ is inscribed in a trapezoid having legs with lengths of 10 and 11. Find the lengths of the bases.

- 4957: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $\{a_n\}_{n\geq 0}$ be the sequence defined by $a_0 = 1, a_1 = 2, a_2 = 1$ and for all $n \geq 3$, $a_n^3 = a_{n-1}a_{n-2}a_{n-3}$. Find $\lim\limits_{n\to\infty} a_n$.

- 4958: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $f : [a, b] \to R$ ($0 < a < b$) be a continuous function on $[a, b]$ and derivable in (a, b).

Prove that there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{1}{c\sqrt{ab}} \cdot \frac{\ln(ab/c^2)}{\ln(c/a) \cdot \ln(c/b)}.$$

- 4959: *Proposed by Juan-Bosco Romero Márquez, Valladolid, Spain.*

Find all numbers $N = ab$, were $a, b = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, such that

$$[S(N)]^2 = S(N^2),$$

where S(N)=a+b is the sum of the digits. For example:

$$\begin{array}{rcll} N & = & 12 & N^2 = 144 \\ S(N) & = & 3 & S(N^2) = 9 \quad \text{and } [\text{S(N)}]^2 = \text{S(N}^2\text{)}. \end{array}$$

Solutions

- 4918: *Proposed by Kenneth Korbin, New York, NY.*

Find the dimensions of an isosceles triangle that has integer length inradius and sides and which can be inscribed in a circle with diameter 50.

Solution by Paul M. Harms, North Newton, KS.

Put the circle on a coordinate system with center at $(0, 0)$ and the vertex associated with the two equal sides at $(0, 25)$. Also make the side opposite the $(0, 25)$ vertex parallel to the x-axis. Using (x, y) as the vertex on the right side of the circle, we have $x^2 + y^2 = 25^2 = 625$. Let d be the length of the equal sides. Using the right triangle with vertices at $(0, 25), (0, y)$, and (x, y) we have $(25 - y)^2 + x^2 = d^2$.

Then $d^2 = (25 - y)^2 + (25^2 - y^2) = 1250 - 50y$; the semi-perimeter $s = x + d$ and the inradius $r = \sqrt{\dfrac{x^2(d - x)}{d + x}}$. Using $x^2 + y^2 = 25^2$, we will check to see if $x = 24$ and $y = 7$ satisfies the problem. The number $d^2 = 900$, so $d = 30$. The inradius $r = \sqrt{\dfrac{24^2(6)}{54}} = 8$. Thus the isosceles triangle with side lengths $30, 30, 48$ and $r = 8$ satisfies the problem. If $x = 24$ and $y = -7$, then $d = 40$ and $r = 12$. The isosceles triangle with side lengths $40, 40, 48$ and $r = 12$ also satisfies the problem.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; David Stone and John Hawkins, Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 4919: *Proposed by Kenneth Korbin, New York, NY.*

Let x be any even positive integer. Find the value of

$$\sum_{k=0}^{x/2} \binom{x-k}{k} 2^k.$$

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX.

To simplify matters, let $x = 2n$ and

$$S(n) = \sum_{k=0}^{n} \binom{2n-k}{k} 2^k.$$

Since

$$\binom{m}{i} = \binom{m-1}{i-1} + \binom{m-1}{i}$$

for $m \geq 2$ and $1 \leq i \leq m-1$, we have

$$\begin{aligned}
\binom{2n+4-k}{k} &= \binom{2n+3-k}{k-1} + \binom{2n+3-k}{k} \\
&= \binom{2n+3-k}{k-1} + \binom{2n+2-k}{k-1} + \binom{2n+2-k}{k} \\
&= \binom{2n+3-k}{k-1} + \binom{2n+3-k}{k-1} - \binom{2n+2-k}{k-2} + \binom{2n+2-k}{k} \\
&= \binom{2n+2-k}{k} + 2\binom{2n+3-k}{k-1} - \binom{2n+2-k}{k-2}
\end{aligned}$$

for $n \geq 1$ and $2 \leq k \leq n+1$.

Therefore, for $n \geq 1$,

$$\begin{aligned}
S(n+2) &= \sum_{k=0}^{n+2} \binom{2n+4-k}{k} 2^k \\
&= 1 + (2n+3) \cdot 2 + \sum_{k=2}^{n+1} \binom{2n+4-k}{k} 2^k + 2^{n+2} \\
&= 1 + (2n+3) \cdot 2 + \sum_{k=2}^{n+1} \binom{2n+2-k}{k} 2^k + 2\sum_{k=2}^{n+1} \binom{2n+3-k}{k-1} 2^k \\
&\quad - \sum_{k=2}^{n+1} \binom{2n+2-k}{k-2} 2^k + 2^{n+2} \\
&= 4 + \sum_{k=0}^{n+1} \binom{2n+2-k}{k} 2^k + 2\sum_{k=1}^{n} \binom{2n+2-k}{k} 2^{k+1} - \sum_{k=0}^{n-1} \binom{2n-k}{k} 2^{k+2} + 2^{n+2} \\
&= S(n+1) + 4\sum_{k=0}^{n+1} \binom{2n+2-k}{k} 2^k - \sum_{k=0}^{n-1} \binom{2n-k}{k} 2^{k+2} - 2^{n+2} \\
&= 5S(n+1) - 4\sum_{k=0}^{n} \binom{2n-k}{k} 2^k \\
&= 5S(n+1) - 4S(n).
\end{aligned}$$

To solve for $S(n)$, we use the usual techniques for solving homogeneous linear difference equations with constant coefficients. If we look for a solution of the form $S(n) = t^n$, with $t \neq 0$, then

$$S(n+2) = 5S(n+1) - 4S(n)$$

becomes

$$t^2 = 5t - 4,$$

whose solutions are $t = 1, 4$. This implies that the general solution for $S(n)$ is

$$S(n) = A \cdot 4^n + B \cdot 1^n = A \cdot 4^n + B,$$

for some constants A and B. The initial conditions $S(1) = 3$ and $S(2) = 11$ yield $A = \dfrac{2}{3}$ and $B = \dfrac{1}{3}$. Hence,

$$S(n) = \frac{2}{3} \cdot 4^n + \frac{1}{3} = \frac{2^{2n+1}+1}{3}$$

for all $n \geq 1$. The final solution is

$$\sum_{k=0}^{x/2} \binom{x-k}{k} 2^k = \frac{2^{x+1}+1}{3}$$

for all even positive integers x.

Also solved by David E. Manes, Oneonta, NY, David Stone, John Hawkins, and Scott Kersey (jointly), Statesboro, GA, and the proposer.

- 4920: *Proposed by Stanley Rabinowitz, Chelmsford, MA.*

Find positive integers a, b, and c (each less than 12) such that

$$\sin\frac{a\pi}{24} + \sin\frac{b\pi}{24} = \sin\frac{c\pi}{24}.$$

Solution by John Boncek, Montgomery, AL.

Recall the standard trigonometric identity:

$$\sin(x+y) + \sin(x-y) = 2\sin x \cos y.$$

Let $x + y = \dfrac{a\pi}{24}$ and $x - y = \dfrac{b\pi}{24}$. Then

$$\sin\frac{a\pi}{24} + \sin\frac{b\pi}{24} = 2\sin\frac{(a+b)\pi}{48}\cos\frac{(a-b)\pi}{48}.$$

We can make the right hand side of this equation equal to $\sin\dfrac{c\pi}{24}$ if we let $a - b = 16$ and $a + b = 2c$, or in other words, by choosing a value for c and then taking $a = 8 + c$ and $b = c - 8$.

Since we want positive solutions, we start by taking $c = 9$. This gives us $a = 17$ and $b = 1$. Since $\sin\dfrac{17\pi}{24} = \sin\dfrac{7\pi}{24}$, replace $a = 17$ by $a = 7$ and we have a solution $a = 7, b = 1$ and $c = 9$.

By taking $c = 10$ and $c = 11$ and using the same analysis, we obtain two additional triples that solve the problem, namely: $a = 6, b = 2, c = 10$ and $a = 5, b = 3, c = 11$.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Kenneth Korbin, NY, NY; Peter, E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4921: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Evaluate $\displaystyle\int_0^{\pi/2} \frac{\cos^{2006} x + 2006 \sin^2 x}{2006 + \sin^{2006} x + \cos^{2006} x} dx.$

Solution by Michael C. Faleski, Midland, MI.

Call this integral I. Now, substitute $\sin^2 x = 1 - \cos^2 x$ and add to the numerator $\sin^{2006} x - \sin^{2006} x$ to give

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{2006 + \sin^{2006} x + \cos^{2006} x - (2006 \cos^2 x + \sin^{2006} x)}{2006 + \sin^{2006} x + \cos^{2006} x} dx \\ &= \int_0^{\pi/2} dx - \int_0^{\pi/2} \frac{2006 \cos^2 x + \sin^{2006} x}{2006 + \sin^{2006} x + \cos^{2006} x} dx. \end{aligned}$$

The second integral can be transformed with $u = \pi/2 - x$ to give

$$\int_0^{\pi/2} \frac{2006 \cos^2 x + \sin^{2006} x}{2006 + \sin^{2006} x + \cos^{2006} x} dx = -\int_{\pi/2}^{0} \frac{\cos^{2006} u + 2006 \sin^2 u}{2006 + \sin^{2006} u + \cos^{2006} u} du = I.$$

Hence, $I = \int_0^{\pi/2} dx - I \Longrightarrow 2I = \dfrac{\pi}{2} \Longrightarrow I = \dfrac{\pi}{4}.$

$$\int_0^{\pi/2} \frac{\cos^{2006} x + 2006 \sin^2 x}{2006 + \sin^{2006} x + \cos^{2006} x} dx = \frac{\pi}{4}.$$

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Ovidiu Furdui, Kalamazoo, MI; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4922: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b be real numbers such that $0 < a < b$ and let $f : [a, b] \to R$ be a continuous function in $[a, b]$ and derivable in (a, b). Prove that there exists $c \in (a, b)$ such that

$$cf(c) = \frac{1}{\ln b - \ln a} \int_a^b f(t)\, dt.$$

Solution by David E. Manes, Oneonta, NY.

For each $x \in [a, b]$, define the function $F(x)$ so that $F(x) = \int_a^x f(t)dt$. Then $F(b) = \int_a^b f(t)dt, F(a) = 0$ and, by the Fundamental Theorem of Calculus, $F'(x) = f(x)$ for each $x \in (a, b)$.

Let $g(x) = \ln(x)$ be defined on $[a, b]$. Then both functions F and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) and $g'(x) = \dfrac{1}{x} \neq 0$

for each $x \in (a,b)$. By the Extended Mean-Value Theorem, there is at least one number $c \in (a,b)$ such that

$$\frac{F'(c)}{g'(c)} = \frac{F(b)-F(a)}{g(b)-g(a)} = \frac{\displaystyle\int_a^b f(t)dt}{\ln b - \ln a}.$$

Since $\dfrac{F'(c)}{g'(c)} = cf(c)$, the result follows.

Also solved by Michael Brozinsky, Central Islip, NY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4923: *Proposed by Michael Brozinsky, Central Islip, NY.*

  Show that if $n \geq 6$ and is composite, then n divides $(n-2)!$.

  Solution by Brian D. Beasley, Clinton, SC.

  Let n be a composite integer with $n \geq 6$. We consider two cases:

  (i) Assume n is not the square of a prime. Then we may write $n = ab$ for integers a and b with $1 < a < b < n-1$. Thus a and b are distinct and are in $\{2, 3, \ldots, n-2\}$, so $n = ab$ divides $(n-2)!$.

  (ii) Assume $n = p^2$ for some odd prime p. Then $n - 2 = p^2 - 2 \geq 2p$, since $p > 2$. Hence both p and $2p$ are in $\{3, 4, \ldots, n-2\}$, so $n = p^2$ divides $(n-2)!$.

  Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Luke Drylie (student, Old Dominion U.), Chesapeake, VA; Kenneth Korbin, NY, NY; Paul M. Harms, North Newton, KS; Jahangeer Kholdi, Portsmouth, VA; N. J. Kuenzi, Oshkosh, WI; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4924: *Proposed by Kenneth Korbin, New York, NY.*

  Given $\displaystyle\sum_{N=1}^{\infty} \frac{F_N}{K^N} = 3$ where F_N is the N^{th} Fibonacci number. Find the value of the positive number K.

  Solution by R. P. Sealy, Sackville, New Brunswick, Canada.

  The ratio test along with the fact that $\displaystyle\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$ implies $\displaystyle\sum_{n=1}^{\infty} \frac{F_n}{K^n}$ converges for $K > \dfrac{1+\sqrt{5}}{2}$. Then

$$\begin{aligned} 3 = \sum_{n=1}^{\infty} \frac{F_n}{K^n} &= \frac{1}{K} + \frac{1}{K^2} + \sum_{n=3}^{\infty} \frac{F_n}{K^n} \\ &= \frac{1}{K} + \frac{1}{K^2} + \sum_{n=3}^{\infty} \frac{F_{n-1}+F_{n-2}}{K^n} \\ &= \frac{1}{K} + \frac{1}{K^2} + \frac{1}{K}\sum_{n=3}^{\infty} \frac{F_{n-1}}{K^{n-1}} + \frac{1}{K^2}\sum_{n=3}^{\infty} \frac{F_{n-2}}{K^{n-2}} \end{aligned}$$

$$= \frac{1}{K} + \frac{1}{K^2} + \frac{1}{K}\left(3 - \frac{1}{K}\right) + \frac{3}{K^2}$$
$$= \frac{4}{K} + \frac{3}{K^2} \Rightarrow K = \frac{2+\sqrt{13}}{3}.$$

Also solved by Brian D. Beasley, Clinton, SC; Sam Brotherton (student, Rockdale Magnet School For Science and Technology), Conyers, GA; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; Luke Drylie (student, Old Dominion U.), Chesapeake, VA; Paul M. Harms, North Newton, KS; Jahangeer Kholdi and Boris Rays (jointly), Portsmouth, VA & Chesapeake,VA (respectively); N. J. Kuenzi, Oshkosh, WI; Tom Leong, Scotrun, PA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4925: *Proposed by Kenneth Korbin, New York, NY.*

In the expansion of

$$\frac{x^4}{(1-x)^3(1-x^2)} = x^4 + 3x^5 + 7x^6 + 13x^7 + \cdots$$

find the coefficient of the term with x^{20} and with x^{21}.

Solution 1 by Brian D. Beasley, Clinton, SC.

We have

$$\begin{aligned}\frac{1}{(1-x)^3(1-x^2)} &= \frac{1}{(1-x)^4(1+x)} \\ &= (1-x+x^2-x^3+\cdots)(1+x+x^2+x^3+\cdots)^4 \\ &= (1-x+x^2-x^3+\cdots)(1+2x+3x^2+4x^3+\cdots)^2 \\ &= (1-x+x^2-x^3+\cdots)(1+4x+10x^2+20x^3+\cdots),\end{aligned}$$

where the coefficients of the second factor in the last line are the binomial coefficients $C(k,3)$ for $k = 3,4,5,\ldots$. Hence, allowing for the x^4 in the original numerator, the desired coefficient of x^{20} is

$$\sum_{k=3}^{19} C(k,3)(-1)^{19-k} = 525.$$

Similarly, the desired coefficient of x^{21} is

$$\sum_{k=3}^{20} C(k,3)(-1)^{20-k} = 615.$$

Solution 2 by Tom Leong, Scotrun, PA.

Equivalently, we find the coefficients of x^{16} and x^{17} in

$$\frac{1}{(1-x)^3(1-x^2)}. \tag{1}$$

We use the following well-known generating functions:

$$\begin{aligned}\frac{1}{1-x^2} &= 1+x^2+x^4+x^6+\cdots\\ \frac{1}{(1-x)^{m+1}} &= \binom{m}{m}+\binom{m+1}{m}x+\binom{m+2}{m}x^2+\binom{m+3}{m}x^3+\cdots.\end{aligned}$$

A decomposition of (1) is

$$\frac{1}{(1-x)^3(1-x^2)}=\frac{1}{2}\frac{1}{(1-x)^4}+\frac{1}{4}\frac{1}{(1-x)^3}+\frac{1}{8}\frac{1}{(1-x)^2}+\frac{1}{8}\frac{1}{(1-x)}.$$

Thus the coefficient of x^n is

$$\frac{1}{2}\binom{n+3}{3}+\frac{1}{4}\binom{n+2}{2}+\frac{1}{8}\binom{n+1}{1}+\frac{1}{8}=\frac{(n+2)(n+4)(2n+3)}{24}\quad \text{if } n \text{ is even}$$

or

$$\frac{1}{2}\binom{n+3}{3}+\frac{1}{4}\binom{n+2}{2}+\frac{1}{8}\binom{n+1}{1}=\frac{(n+1)(n+3)(2n+7)}{24}\quad \text{if } n \text{ is odd.}$$

So the coefficient of x^{16} is $\dfrac{18\cdot 20\cdots 35}{24}=525$ and the coefficient of x^{17} is $\dfrac{18\cdot 20\cdots 41}{24}=615$.

Solution 3 by Paul M. Harms, North Newton, KS.

When

$$-1<x<1,\quad \frac{1}{1-x}=1+x+x^2+\cdots.$$

Taking two derivatives, we obtain for

$$-1<x<1,\quad \frac{2}{(1-x)^3}=2+3(2)x+4(3)x^2+\cdots.$$

When

$$-1<x<1,\quad \frac{x^4}{1-x^2}=x^4+x^6+x^8+\cdots.$$

The series for $\dfrac{x^4}{(1-x)^3(1-x^2)}$ can be found by multiplying

$$\frac{1}{2}\cdot\frac{2}{(1-x)^3}\cdot\frac{x^4}{(1-x^2)}=\frac{1}{2}\Big[2+3(2)x+4(3)x^2+\cdots+18(17)x^{16}+19(18)x^{17}+\cdots\Big]\Big[x^4+x^6+x^8+\cdots\Big].$$

The coefficient of x^{20} is

$$\frac{1}{2}\Big[18(17)+16(15)+14(13)+\cdots 4(3)+2\Big]=525.$$

The coefficient of x^{21} is

$$\frac{1}{2}\Big[19(18)+17(16)+15(14)+\cdots 5(4)+3(2)\Big]=615.$$

Comment: Jahangeer Kholdi and Boris Rays noticed that the coefficients in $x^4+3x^5+7x^6+13x^7+22x^8+34x^9+50x^{10}+\cdots$, are the partial sums of the alternate triangular numbers. I.e., $1, 3, 1+6, 3+10, 1+6+15, 3+10+21, \cdots$, which leads to the coefficients of x^{20} and x^{21} being 525 and 615 respectively.

Also solved by Michael Brozinsky, Central Islip, NY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; Jahangeer Kholdi and Boris Rays (jointly), Portsmouth, VA & Chesapeake,VA (respectively); Peter E. Liley, Lafayette, IN; John Nord, Spokane, WA; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4926: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Calculate
$$\sum_{n=1}^{\infty} \frac{nF_n^2}{3^n}$$
where F_n is the n^{th} Fibonacci number defined by $F_1 = F_2 = 1$ and for $n \geq 3, F_n = F_{n-1} + F_{n-2}$.

Solution by David Stone and John Hawkins, Statesboro, GA.

By Binet's Formula, $F_n = \dfrac{\alpha^n - \beta^n}{\sqrt{5}}$, where α and β are the solutions of the quadratic equation $x^2 - x - 1 = 0$; $\alpha = \dfrac{1+\sqrt{5}}{2}$, $\beta = \dfrac{1-\sqrt{5}}{2}$.

Note that $a - b = \sqrt{5}$, $\alpha \cdot \beta = -1$, $\alpha^2 + \beta^2 = 3$, and $\alpha^6 + \beta^6 = 18$. Also recall from calculus that $\displaystyle\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$. Thus we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{nF_n^2}{3^n} &= \sum_{n=1}^{\infty} \frac{n}{3^n} \frac{\alpha^{2n} - 2\alpha^n\beta^n + \beta^{2n}}{5} \\
&= \sum_{n=1}^{\infty} \frac{n}{3^n} \frac{\alpha^{2n} - 2(-1)^n + \beta^{2n}}{5} \\
&= \frac{1}{5}\left\{ \sum_{n=1}^{\infty} n\left(\frac{\alpha^2}{3}\right)^n - 2\sum_{n=1}^{\infty} n\left(\frac{-1}{3}\right)^n + \sum_{n=1}^{\infty} n\left(\frac{\beta^2}{3}\right)^n \right\} \\
&= \frac{1}{5}\left\{ \frac{\frac{\alpha^2}{3}}{\left[1 - \frac{\alpha^2}{3}\right]^2} - 2\frac{\frac{-1}{3}}{\left[1 + \frac{1}{3}\right]^2} + \frac{\frac{\beta^2}{3}}{\left[1 - \frac{\beta^2}{3}\right]^2} \right\}, \quad \text{valid because } \frac{\beta^2}{3} < \frac{\alpha^2}{3} < 1; \\
&= \frac{1}{5}\left\{ \frac{3\alpha^2}{[3-\alpha^2]^2} + \frac{3}{8} + \frac{3\beta^2}{[3-\beta^2]^2} \right\} \\
&= \frac{3}{5}\left\{ \frac{\alpha^2}{[\beta^2]^2} + \frac{1}{8} + \frac{\beta^2}{[\alpha^2]^2} \right\} \text{ because } \alpha^2 + \beta^2 = 3, \\
&= \frac{3}{5}\left\{ \frac{1}{8} + \frac{\alpha^6 + \beta^6}{\alpha^4\beta^4} \right\} \text{ by algebra,}
\end{aligned}$$

$$= \frac{3}{5}\left\{\frac{1}{8}+\frac{18}{1}\right\} = \frac{87}{8}.$$

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Tom Leong, Scotrun, PA, and the proposer.

- 4927: *Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Barcelona, Spain.*

  Let k be a positive integer and let

$$A=\sum_{n=0}^{\infty}\frac{(-1)^n}{2k(2n+1)} \qquad \text{and} \qquad B=\sum_{n=0}^{\infty}(-1)^n\left\{\sum_{m=0}^{2k}\frac{(-1)^m}{(4k+2)n+2m+1}\right\}.$$

  Prove that $\dfrac{B}{A}$ is an even integer for all $k\geq 1$.

  Solution by Tom Leong, Scotrun, PA.

  Note that inside the curly braces in the expression for B, the terms of the (alternating) sum are the reciprocals of the consecutive odd numbers from $(4k+2)n+1$ to $(4k+2)n+(4k+1)$. As $n=0,1,2,\ldots$, the reciprocal of every positive odd number appears exactly once in this sum (disregarding its sign). Thus

$$B=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{2k}\frac{(-1)^{m+n}}{(4k+2)n+2m+1}\right\}=\sum_{i=0}^{\infty}\frac{(-1)^i}{2i+1}$$

  from which we find $\dfrac{B}{A}=2k$. (In fact, it is well-known that $B=\pi/4$.)

  Comment by Editor: This problem was incorrectly stated when it was initially posted in the May, 06 issue of SSM. The authors reformulated it, and the correct statement of the problem and its solution are listed above. The corrected version was also solved by **Paul M. Harms of North Newton, KS.**

- 4928: *Proposed by Yair Mulian, Beer-Sheva, Israel.*

  Prove that for all natural numbers n

$$\int_0^1\frac{2x^{2n+1}}{x^2-1}dx=\int_0^1\frac{x^n}{x-1}+\frac{1}{x+1}dx.$$

  Comment by Editor: The integrals in their present form do not exist, and I did not see this when I accepted this problem for publication. Some of the readers rewrote the problem in what they described as "its more common form;" i.e., to show that $\displaystyle\int_0^1\frac{2x^{2n+1}}{x^2-1}-\left(\frac{x^n}{x-1}+\frac{1}{x+1}\right)dx=0$. But I believe that one cannot legitimately recast the problem in this manner, because the $\int_a^b(f(x)+g(x))dx=\int_a^b f(x)dx+\int_a^b g(x)dx$ if, and only if, $f(x)$ and $g(x)$ is each integrable over these limits. So as I see it, the problem as it was originally stated is not solvable. Mea culpa, once again.

• 4929: *Proposed by Michael Brozinsky, Central Islip, NY.*

An archaeological expedition uncovered 85 houses. The floor of each of these houses was a rectangular area covered by mn tiles where $m \le n$. Each tile was a 1 unit by 1 unit square. The tiles in each house were all white, except for a (non-empty) square configuration of blue tiles. Among the 85 houses, all possible square configurations of blue tiles appeared once and only once. Find all possible values of m and n.

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX.

Assume that each configuration of blue tiles is a $k \times k$ square. Since $m \le n$ and each such configuration was non-empty, it follows that $k = 1, 2, \ldots, m$. For each value of k, there are $(m-k+1)(n-k+1)$ possible locations for the $k \times k$ configuration of blue tiles. Since each arrangement appeared once and only once among the 85 houses, we have

$$\begin{aligned}
85 &= \sum_{k=1}^{m} (m-k+1)(n-k+1) \\
&= \sum_{k=1}^{m} (m+1)(n+1) - (m+n+2)\sum_{k=1}^{m} k + \sum_{k=1}^{m} k^2 \\
&= m(m+1)(n+1) - (m+n+2)\frac{m(m+1)}{2} + \frac{m(m+1)(2m+1)}{6} \\
&= \frac{m(m+1)}{6}\left[3n-(m-1)\right]
\end{aligned}$$

or

$$m(m+1)\left[3n-(m-1)\right] = 510. \tag{1}$$

This implies that m and $m+1$ must be consecutive factors of 510. By checking all 16 factors of 510, we see that the only possible values of m are $1, 2, 5$. If $m = 2$, (1) does not produce an integral solution for n. If $m = 1$ or 5, equation (1) yields $n = 85$ or 7 (respectively). Therefore, the only solutions are $(m, n) = (1, 85)$ or $(5, 7)$.

Also solved by Tom Leong, Scotrun, PA; Paul M. Harms, North Newton, KS; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
June 1, 2007

- 4960: *Proposed by Kenneth Korbin, New York, NY.*

  Equilateral triangle ABC has an interior point P such that

  $$\overline{AP} = \sqrt{5},\ \overline{BP} = \sqrt{12},\ \text{and } \overline{\text{CP}} = \sqrt{17}.$$

  Find the area of $\triangle APB$.

- 4961: *Proposed by Kenneth Korbin, New York, NY.*

  A convex hexagon is inscribed in a circle with diameter d. Find the area of the hexagon if its sides are $3, 3, 3, 4, 4$ and 4.

- 4962: *Proposed by Kenneth Korbin, New York, NY.*

  Find the area of quadrilateral $ABCD$ if the midpoints of the sides are the vertices of a square and if $AB = \sqrt{29}$ and $CD = \sqrt{65}$.

- 4963: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Calculate

  $$\lim_{n\to\infty} \sum_{1\le i<j\le n} \frac{1}{3^{i+j}}.$$

- 4964: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let x, y be real numbers and we define the law of composition

  $$x \perp y = x\sqrt{1+y^2} + y\sqrt{1+x^2}.$$

Prove that $(R,+)$ and $(R,\perp)$ are isomorphic and solve the equation $x \perp a = b$.

- 4965: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let h_a, h_b, h_c be the heights of triangle ABC. Let P be any point inside $\triangle ABC$. Prove that

$$(a)\quad \frac{h_a}{d_a}+\frac{h_b}{d_b}+\frac{h_c}{d_c}\ge 9, \quad (b)\quad \frac{d_a^2}{h_a^2}+\frac{d_b^2}{h_b^2}+\frac{d_c^2}{h_c^2}\ge \frac{1}{3},$$

  where d_a, d_b, d_c are the distances from P to the sides BC, CA and AB respectively.

Solutions

- 4930: *Proposed by Kenneth Korbin, New York, NY.*

  Find an acute angle y such that $\cos(y)+\cos(3y)-\cos(5y)=\dfrac{\sqrt{7}}{2}$.

  Solution by Brian D. Beasley, Clinton, SC.

  Given an acute angle y, let $c=\cos(y)>0$. We use $\cos(3y)=4c^3-3c$ and $\cos(5y)=16c^5-20c^3+5c$ to transform the given equation into

$$-16c^5+24c^3-7c=\frac{\sqrt{7}}{2}.$$

  Since this equation in turn is equivalent to

$$32c^5-48c^3+14c+\sqrt{7}=(8c^3-4\sqrt{7}c^2+\sqrt{7})(4c^2+2\sqrt{7}c+1)=0,$$

  we need only determine the positive zeros of $f(x)=8x^3-4\sqrt{7}x^2+\sqrt{7}$. Applying $\cos(7y)=64c^7-112c^5+56c^3-7c$, we note that the six zeros of

$$64x^6-112x^4+56x^2-7=f(x)(8x^3+4\sqrt{7}x^2-\sqrt{7})$$

  are $\cos(k\pi/14)$ for $k\in\{1,3,5,9,11,13\}$. We let $g(x)=8x^3+4\sqrt{7}x^2-\sqrt{7}$ and use $g'(x)=24x^2+8\sqrt{7}x$ to conclude that g is increasing on $(0,\infty)$, and hence has at most one positive zero. But $g(1/2)>0$, $\cos(\pi/14)>1/2$, and $\cos(3\pi/14)>1/2$, so $\cos(\pi/14)$ and $\cos(3\pi/14)$ must be zeros of $f(x)$ instead. Thus we may take $y=\pi/14$ or $y=3\pi/14$ in the original equation.

  Also solved by: Dionne Bailey, Elsie Campbell, and Charles Dimminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayete, IN; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Satesboro, GA, and the proposer.

- 4931: *Proposed by Kenneth Korbin, New York, NY.*

  A Pythagorean triangle and an isosceles triangle with integer length sides both have the same length perimeter $P=864$. Find the dimensions of these triangles if they both have the same area too.

  Solution by David Stone and John Hawkins (jointly), Statesboro, GA.

  Surprisingly, there exists only one such pair of triangles: the (primitive) Pythagorean tiangle $(135, 352, 377)$ and the isosceles triangle $(366, 366, 132)$. Each has a perimeter 864 and area $23,760$.

  By Heron's Formula (or geometry), an isosceles triangle with given perimeter P and sides

(a, a, b) has area

$$A = \frac{b}{4}\sqrt{4a^2 - b^2} = \frac{P-2a}{4}\sqrt{P(4a-P)}, \text{ where } \frac{P}{4} \le a \le \frac{P}{2}.$$

In our problem, $P = 864$. We can analyze possibilities to reduce the number of cases to check or we can use a calculator or computer to check all possibilities. In any case, there are only a few such triangles with integer length sides:

$$\left\{\begin{array}{ccc} a & b & A \\ 222 & 420 & 15,120 \\ 240 & 384 & 27,648 \\ 270 & 324 & 34,992 \\ 312 & 240 & 34,560 \\ 366 & 132 & 23,760 \end{array}\right\}$$

Now, if (a, b, c) is a Pythaorean triangle with given perimeter P and given area A, we can solve the equations

$$\begin{aligned} P &= a + b + c \\ c^2 &= a^2 + b^2 \\ A &= \frac{ab}{2} \end{aligned}$$

to obtain $a = \dfrac{(P^2 + 4A) \pm \sqrt{P^4 - 24P^2A + 16A^2}}{4P}, \quad b = \dfrac{2A}{a}, \ c = P - a - \dfrac{2A}{a}.$

We substitute $P = 864$ and the values for A from the above table. Only with $A = 23,760$ do we find a solutions $(135, 352, 377)$. (Note that the two large values of A each produce a negative under the radical because those values of A are too large to be hemmed up by a perimeter of 864, while the first two values of A produce right triangles with non-integer sides.)

Also solved by Brain D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; Amihai Menuhin, Beer-Sheva, Israel, Harry Sedinger, St. Bonaventure, NY, and the proposer.

- 4932: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let ABC be a triangle with semi-perimeter s, in-radius r and circum-radius R. Prove that

$$\sqrt[3]{r^2} + \sqrt[3]{s^2} \le 2\sqrt[3]{2R^2}$$

and determine when equality holds.

Solution by the proposer.

From Euler's inequality for the triangle $2r \le R$, we have $r/R \le 1/2$ and

$$\left(\frac{r}{R}\right)^{2/3} \le \left(\frac{1}{2}\right)^{2/3} \qquad (1)$$

Next, we will see that

$$\frac{s}{R} \le \frac{3\sqrt{3}}{2} \qquad (2)$$

In fact, from Sine's Law
$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R,$$
we have
$$\frac{a+b+c}{\sin A+\sin B+\sin C}=2R$$
or
$$\frac{s}{R}=\frac{a+b+c}{2R}=\sin A+\sin B+\sin C\leq\frac{3\sqrt{3}}{2}$$
as claimed. Notice that the last inequality is an immediate consequence of Jensen's inequality applied to the function $f(x)=\sin x$ that is concave in $[0,\pi]$.

Finally, from (1) and (2), we have
$$\left(\frac{r}{R}\right)^{2/3}+\left(\frac{s}{R}\right)^{2/3}\leq\left(\frac{1}{2}\right)^{2/3}+\left(\frac{3\sqrt{3}}{2}\right)^{2/3}=2\sqrt[3]{2}$$
from which the statement immediately follows as desired. Note that equality holds when $\triangle ABC$ is equilateral, as immediately follows from (1) and (2).

- 4933: *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Barcelona, Spain.*

Let n be a positive integer. Prove that
$$\frac{1}{n}\sum_{k=1}^{n}k\binom{n}{k}^{1/2}\leq\frac{1}{2}\sqrt{(n+1)2^n}.$$

Solution by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX .

By the Binomial Theorem,
$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}x^k &= (1+x)^n\\
\frac{d}{dx}\sum_{k=0}^{n}\binom{n}{k}x^k &= \frac{d}{dx}(1+x)^n\\
\sum_{k=1}^{n}k\binom{n}{k}x^{k-1} &= n(1+x)^{n-1}\\
\sum_{k=1}^{n}k\binom{n}{k}x^{k} &= nx(1+x)^{n-1}\\
\frac{d}{dx}\sum_{k=1}^{n}k\binom{n}{k}x^{k} &= \frac{d}{dx}\left[nx(1+x)^{n-1}\right]\\
\sum_{k=1}^{n}k^2\binom{n}{k}x^{k-1} &= n(1+x)^{n-2}(nx+1) \qquad (1).
\end{aligned}$$

Evaluating (1) when $x=1$,
$$\begin{aligned}
\sum_{k=1}^{n}k^2\binom{n}{k} &= n(n+1)2^{n-2}\\
\frac{1}{n}\sum_{k=1}^{n}k^2\binom{n}{k} &= \frac{(n+1)2^n}{4} \qquad (2).
\end{aligned}$$

By the Root Mean Square Inequality and (2),

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} k\binom{n}{k}^{1/2} &\le \sqrt{\frac{\sum\limits_{k=1}^{n} k^2\binom{n}{k}}{n}} \\
&= \sqrt{\frac{(n+1)2^n}{4}} \\
&= \frac{1}{2}\sqrt{(n+1)2^n}.
\end{aligned}$$

Also solved by the proposer.

- 4934: *Proposed by Michael Brozinsky, Central Islip, NY.*

  Mrs. Moriaty had two sets of twins who were always getting lost. She insisted that one set must chose an arbitrary non-horizontal chord of the circle $x^2 + y^2 = 4$ as long as the chord went through $(1, 0)$ and they were to remain at the opposite endpoints. The other set of twins was similarly instructed to choose an arbitrary non-vertical chord of the same circle as long as the chord went through $(0, 1)$ and they too were to remain at the opposite endpoints. The four kids escaped and went off on a tangent (to the circle, of course). All that is known is that the first set of twins met at some point and the second set met at another point. Mrs. Moriaty did not know where to look for them but Sherlock Holmes deduced that she should confine her search to two lines. What are their equations?

  Solution by R. P. Sealy, Sackville, New Brunswick, Canada

  The equations of the two lines are $x = 4$ for the first set of twins and $y = 4$ for the second set of twins.

  The vertical chord through the point (1,0) meets the circle at points $(1, \sqrt{3})$ and $(1, -\sqrt{3})$. The slopes of the tangent lines are $-\dfrac{1}{\sqrt{3}}$ and $\dfrac{1}{\sqrt{3}}$. So the equations of the tangent lines are

$$y = -\frac{1}{\sqrt{3}}x + \frac{4}{\sqrt{3}} \quad \text{and} \quad y = \frac{1}{\sqrt{3}}x - \frac{4}{\sqrt{3}}.$$

  These tangent lines meet at the point (4,0). Otherwise, a non-vertical (and non-horizontal) chord through the point (1,0) intersects the circle at points (a, b) and (c, d), $bd \neq 0$, $b \neq d$. The slopes of the tangent lines are $-\dfrac{a}{b}$ and $-\dfrac{c}{d}$. So the equations of the tangent lines are

$$y = -\frac{a}{b}x + \frac{4}{b} \quad \text{and} \quad y = -\frac{c}{d}x + \frac{4}{d}.$$

  The x-coordinate of the point of intersection of the tangent lines is $\dfrac{4(d-b)}{ad-bc}$. And since the points $(a, b), (c, d)$ and (1,0) are on the chord, we have

$$\frac{b-0}{a-1} = \frac{d-0}{c-1}$$

  or

$$d - b = ad - bc.$$

  Therefore, the x-coordinate of the point of intersection of the tangent lines is 4.

Similar calculations apply to position of the second set of twins.

Also solve by Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4935: *Proposed by Xuan Liang, Queens, NY and Michael Brozinsky, Central Islip, NY.*

Without using the converse of the Pythagorean Theorem nor the concepts of slope, similar triangles or trigonometry, show that the triangle with vertices $A(-1,0), B(m^2,0)$ and $C(0,m)$ is a right triangle.

Solution by Harry Sedinger, St. Bonaventure, NY.

Let $O=(0,0)$. The area of $\triangle ABC$ is

$$\begin{aligned}\frac{1}{2}\Big(|OB|\Big)\Big(|AC|\Big) &= \frac{1}{2}m(m^2+1)=\frac{1}{2}m\sqrt{m^2+1}\sqrt{m^2+1}\\ &= \frac{1}{2}\sqrt{m^4+m^2}\sqrt{m^2+1}=\frac{1}{2}\Big(|BC|\Big)\Big(|AB|\Big).\end{aligned}$$

Thus if AB is considered the base of $\triangle ABC$, its height is $|BC|$, so $AB\perp BC$ and $\triangle ABC$ is a right triangle.

Also solved by Charles Ashbacher, Cedar Rapis, IA; Brian D. Beasley, Clinton, SC; Grant Evans (student, Saint George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Jahangeer Kholdi, Portsmouth, VA; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA; William Weirich (student Virginia Commonwealth University), Richmond, VA, and the proposers.

Editor's comment: Several readers used the distance formula or the law of cosines, or the dot product of vectors in their solutions; but to the best of my knowledge, these notions are obtained with the use of the Pythagorean Theorem.

Problems

Ted Eisenberg, Section Editor

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2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
July 1, 2007

- 4966: *Proposed by Kenneth Korbin, New York, NY.*

  Solve:
  $$16x + 30\sqrt{1-x^2} = 17\sqrt{1+x} + 17\sqrt{1-x}$$
  with $0 < x < 1$.

- 4967: *Proposed by Kenneth Korbin, New York, NY.*

  Given equilateral triangle ABC with an interior point P such that $\overline{\text{AP}}^2 + \overline{\text{BP}}^2 = \overline{\text{CP}}^2$, and with an exterior point Q such that $\overline{\text{AQ}}^2 + \overline{\text{BQ}}^2 = \overline{\text{CQ}}^2$, where points C, P, and Q are in a line. Find the lengths of $\overline{\text{AQ}}$ and $\overline{\text{BQ}}$ if $\overline{\text{AP}} = \sqrt{21}$ and $\overline{\text{BP}} = \sqrt{28}$.

- 4968: *Proposed by Kenneth Korbin, New York, NY.*

  Find two quadruples of positive integers (a, b, c, d) such that
  $$\frac{a+i}{a-i} \cdot \frac{b+i}{b-i} \cdot \frac{c+i}{c-i} \cdot \frac{d+i}{d-i} = \frac{a-i}{a+i} \cdot \frac{b-i}{b+i} \cdot \frac{c-i}{c+i} \cdot \frac{d-i}{d+i}$$
  with $a < b < c < d$ and $i = \sqrt{-1}$.

- 4969: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers such that $abc = 1$. Prove that
  $$\frac{1}{a^2\left(\dfrac{1}{a} + \dfrac{1}{c}\right)} + \frac{1}{b^2\left(\dfrac{1}{b} + \dfrac{1}{a}\right)} + \frac{1}{c^2\left(\dfrac{1}{c} + \dfrac{1}{b}\right)} \geq \frac{3}{2}$$

- 4970: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $f : [0,1] \longrightarrow \mathbf{R}$ be a contintuous convex function. Prove that

$$\frac{3}{4}\int_0^{1/5} f(t)dt + \frac{1}{8}\int_0^{2/5} f(t)dt \geq \frac{4}{5}\int_0^{1/4} f(t)dt.$$

- 4971: *Proposed by Howard Sporn, Great Neck, NY and Michael Brozinsky, Central Islip, NY.*

  Let $m \geq 2$ be a positive integer and let $1 \leq x < y$. Prove:

$$x^m - (x-1)^m < y^m - (y-1)^m.$$

Solutions

- 4936: *Proposed by Kenneth Korbin, New York, NY.*

  Find all prime numbers P and all positive integers a such that $P - 4 = a^4$.

Solution 1 by Daniel Copeland (student, Saint George's School), Spokane, WA.

$$\begin{aligned} P &= a^4 + 4 \\ &= (a^2+2)^2 - 4a^2 \\ &= (a^2 - 2a + 2)(a^2 + 2a + 2). \end{aligned}$$

Since P is a prime, one of the factors of P must be 1. Since a is a positive integer, $a^2 - 2a + 2 = 1$ which yields the only positive solution $a = 1, P = 5$.

Solution 2 by Timothy Bowen (student, Waynesburg College), Waynesburg, PA.

The only solution is $P = 5$ and $a = 1$.
Case 1: Integer a is an even integer. For $a = 2n$, note $P = a^4 + 4 = (2n)^4 + 4 = 4\cdot(4n^4+1)$. Clearly, P is a composite for all natural numbers n.

Case 2: Integer a is an odd integer. For $a = 2n+1$, note that $P = a^4 + 4 = (2n+1)^4 + 4 = (4n^2 + 8n + 5)(4n^2 + 1)$. P is prime only for $n = 0$ (corresponding to $a = 1$ and $P = 5$). Otherwise, P is a composite number for all natural numbers n.

Solution 3 by Jahangeer Kholdi & Robert Anderson (jointly), Portsmouth, VA.

The only prime is $P = 5$ when $a = 1$. Consider $P = a^4 + 4$. If a is an even positive integer, then clearly P is even and hence a composite integer. Moreover, if a is a positive integer ending in digits $\{1, 3, 7 \text{ or } 9\}$, then P is a positive integer ending with the digit of 5. This also implies P is divisible by 5 and hence a composite. Lastly, assume $a = 10k+5$ where $k = 0$ or $k > 0$; that is a is a positive integer ending with a digit of 5. Then $P = (10k+5)^4 + 4$. But

$$P = (10k+5)^4 + 4 = (100k^2 + 80k + 17)(100k^2 + 120k + 37).$$

Hence, for all positive integers $a > 1$ the positive integer P is composite.

Also solved by Brian D. Beasley, Clinton, SC; Dionne Bailey, Elsie Campbell and Charles Diminnie (jointly), San Angelo, TX; Pat Costello, Richmond, KY;

Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; Vicki Schell, Pensacola, FL; R. P. Sealy, Sackville, New Brunswick, Canada; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins of Statesboro, GA jointly with Chris Caldwell of Martin, TN, and the proposer.

- **4937:** *Proposed by Kenneth Korbin, New York, NY.*

Find the smallest and the largest possible perimeter of all the triangles with integer-length sides which can be inscribed in a circle with diameter 1105.

Solution by Paul M. Harms, North Newton, KS.

Consider a radius line from the circle's center to one vertex of an inscribed triangle. Assume at this vertex one side has a length a and subtends a central angle of $2A$ and the other side making this vertex has a length b and subtends a central angle of $2B$.

Using the perpendicular bisector of chords, we have $\sin A = \dfrac{a/2}{1105/2} = \dfrac{a}{1105}$ and $\sin B = \dfrac{b}{1105}$. Also, the central angle of the third side is related to $2A+2B$ and the perpendicular bisector to the third side gives

$$\begin{aligned}\sin(A+B) = \frac{c}{1105} &= \sin A\cos B + \sin B\cos A\\ &= \frac{a}{1105}\frac{\sqrt{1105^2-b^2}}{1105} + \frac{b}{1105}\frac{\sqrt{1105^2-a^2}}{1105}\\ \text{Thus}\quad c &= \frac{1}{1105}\left(a\sqrt{1105^2-b^2} + b\sqrt{1105^2-a^2}\right).\end{aligned}$$

From this equation we find integers a and b which make integer square roots. Some numbers which do this are $\{47{,}1104\ 105,\ 1100,\ 169,\ 1092, \text{ etc. }\}$. Checking the smaller numbers for the smallest perimeter we see that a triangle with side lengths $\{105{,}169{,}272\}$ gives a perimeter of 546 which seems to be the smallest perimeter.

To find the largest perimeter we look for side lengths close to the lengths of an inscribed equilateral triangle. An inscribed equilateral triangle for this circle has side length close to 957. Integers such as $884, 943, 952, 975,$ and 1001 make integer square roots in the equation for c. The maximum perimeter appears to be 2870 with a triangle of side lengths $\{943{,}952{,}975\}$.

Comment: David Stone and John Hawkins of Statesboro, GA used a slightly different approach in solving this problem. Letting the side lengths be $a, b,$ and c and noting that the circumradius is 552.5 they obtained

$$\frac{1105}{2} = \frac{abc}{4\sqrt{(a+b+c)(a+b-c)(a-b+c)(b+c-a)}}$$

which can be rewritten as

$$\sqrt{(a+b+c)(a+b-c)(a-b+c)(b+c-a)} = \frac{abc}{(2)(5)(13)(17)}.$$

They then used that part of the law of sines that connects in any triangle ABC, side length $a, \angle A$ and the circumradius R; $\dfrac{a}{\sin A} = 2R$. This allowed them to find that $c^2 =$

$a^2+b^2 \mp \dfrac{2ab\sqrt{1105^2-c^2}}{1105}$. Noting that the factors of a,b, and c had to include the primes 2,5,13 and 17 and that 1105^2-c^2 had to be a perfect square, (and similarly for 1105^2-b^2 and 1105^2-a^2) they put EXCEL to work and proved that $\{105,\ 272,\ 169\}$ gives the smallest perimeter and that $\{952,\ 975,\ 943\}$ gives the largest. All in all they found 101 triangles with integer side lengths that can be inscribed in a circle with diameter 1105.

Also solved by the proposer.

- 4938: *Proposed by Luis Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b and c be the sides of an acute triangle ABC. Prove that

$$\csc^2\frac{A}{2}+\csc^2\frac{B}{2}+\csc^2\frac{C}{2}\ge 6\left[\prod_{cyclic}\left(1+\frac{b^2}{a^2}\right)\right]^{1/3}$$

Solution by proposers.

First, we claim that $a^2 \ge 2(b^2+c^2)\sin^2(A/2)$. In fact, the preceding inequality is equivalent to $a^2 \ge (b^2+c^2)(1-\cos A)$ and

$$\begin{aligned} a^2-(b^2+c^2)(1-\cos A) &= b^2+c^2-2bc\cos A-(b^2+c^2)+(b^2+c^2)\cos A \\ &= (b-c)^2\cos A \ge 0. \end{aligned}$$

Similar inequalities can be obtained for b and c. Multiplying them up, we have

$$a^2b^2c^2 \ge 8(a^2+b^2)(b^2+c^2)(c^2+a^2)\sin^2(A/2)\sin^2(B/2)\sin^2(C/2). \tag{1}$$

On the other hand, from GM-HM inequality we have

$$\begin{aligned} \sin^2(A/2)\sin^2(B/2)\sin^2(C/2) &\ge \left(\frac{3}{1/\sin^2(A/2)+1/\sin^2(B/2)+1/\sin^2(C/2)}\right)^3 \\ &= \left(\frac{3}{\csc^2(A/2)+\csc^2(B/2)+\csc^2(C/2)}\right)^3. \end{aligned}$$

Substituting into the statement of the problem yields

$$\begin{aligned} \left(\csc^2\frac{A}{2}+\csc^2\frac{B}{2}+\csc^2\frac{C}{2}\right)^3 &\ge 216\left(\frac{a^2+b^2}{c^2}\right)\left(\frac{b^2+c^2}{a^2}\right)\left(\frac{c^2+a^2}{b^2}\right) \\ &= 216\prod_{cyclic}\left(1+\frac{b^2}{a^2}\right). \end{aligned}$$

Notice that equality holds when $A=B=C=\pi/3$. That is, when $\triangle ABC$ is equilateral and we are done.

- 4939: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  For any positive integer n, prove that

$$\left\{4^n+\left[\sum_{k=0}^{n-1}3^{k+1/2}\binom{2n}{2k+1}\right]^2\right\}^{1/2}$$

is a whole number.

Solution by David E. Manes, Oneonta, NY.

Let $W = 4^n + \left[\sum_{k=0}^{n-1} 3^{k+1/2}\begin{pmatrix} 2n \\ 2k+1\end{pmatrix}\right]^2$ and notice that it suffices to show that $\sqrt{W}$ is a whole number. Expanding $(\sqrt{3}+1)^{2n}$ and $(\sqrt{3}-1)^{2n}$ using the Binomial Theorem and subtracting the second expansion from the first, one obtains

$$\sum_{k=0}^{n-1} 3^{k+1/2}\begin{pmatrix} 2n \\ 2k+1\end{pmatrix} = \frac{(\sqrt{3}+1)^{2n}-(\sqrt{3}-1)^{2n}}{2}.$$

Therefore,

$$\begin{aligned}
W &= 4^n + \left[\frac{(\sqrt{3}+1)^{2n}-(\sqrt{3}-1)^{2n}}{2}\right]^2 \\
&= 4^n + \frac{(\sqrt{3}+1)^{4n}-2^{2n+1}+(\sqrt{3}-1)^{4n}}{4} \\
&= \frac{2^{2n+2}+(\sqrt{3}+1)^{4n}-2^{2n+1}+(\sqrt{3}-1)^{4n}}{4} \\
&= \frac{(\sqrt{3}+1)^{4n}+2^{2n+1}+(\sqrt{3}-1)^{4n}}{4} \\
&= \left[\frac{(\sqrt{3}+1)^{2n}-(\sqrt{3}-1)^{2n}}{2}\right]^2.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sqrt{W} &= \frac{(\sqrt{3}+1)^{2n}-(\sqrt{3}-1)^{2n}}{2} = \sum_{k=0}^{n}\binom{2n}{2k}(\sqrt{3})^{2k} \\
&= \sum_{k=0}^{n}\binom{2n}{2k}3^k, \text{ a whole number.}
\end{aligned}$$

Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul H. Harms, North Newton, KS, and the proposer.

- 4940: *Proposed by Michael Brozinsky, Central Islip, NY and Leo Levine, Queens, NY .*

Let $S = \{n \in N | n \geq 5\}$. Let $G(x)$ be the fractional part of x, i.e., $G(x) = x - [x]$ where $[x]$ is the greatest integer function. Characterize those elements T of S for which the function

$$f(n) = n^2\left(G\left(\frac{(n-2)!}{n}\right)\right) = n.$$

Solution by R. P. Sealy, Sackville, New Brunswick, Canada

T is the set of primes in S. One form of Wilson's Theorem states: A necessary and sufficient condition that n be prime is that $(n-1)! \equiv -1(\text{mod n})$. But $(n-1)! = (n-1)(n-2)!$ with $n-1 \equiv -1(\text{mod n})$. Therefore $(n-2)! \equiv 1(\text{mod n})$ if, and only if, n is prime. Therefore

$$f(n) = n^2\left(G\left(\frac{(n-2)!}{n}\right)\right) = n^2\cdot\frac{1}{n} = n \ \text{ if, and only if, } n \geq 5 \text{ is prime.}$$

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- **4941:** *Proposed by Tom Leong, Brooklyn, NY.*

The numbers $1, 2, \cdots, 2006$ are randomly arranged around a circle.
(a) Show that we can select 1000 adjacent numbers consisting of 500 even and 500 odd numbers.
(b) Show that part (a) need not hold if the numbers were randomly arranged in a line.

Solution 1 by Paul Zorn, Northfield, MN.

Claim: Suppose we have 1003 0's and 1003 1's arranged in a circle, like a 2006-hour clock. Then there must be a stretch of length of 1000 containing 500 of each.

Proof: Call the clock positions $1, 2, \cdots, 2006$ as on an ordinary clock, and let $a(n)$ be 0 or 1, depending on what's at position n. Let $S(n) = a(n) + a(n+1) + \cdots + a(n+999)$, where addition in the arguments is mod 2006.

Note that $S(n)$ is just the number of 1's in the 1000-hour stretch starting at n, and we're done if $S(n) = 500$ for some n.

Now $S(n)$ has two key properties, both easy to show:
i) $S(n+1)$ differs from $S(n)$ by at most 1
ii) $S(1) + S(2) + S(3) + \cdots S(2006) = 1000\cdot$(sum of *all* the 1's around the circle) $=1000(1003)$.

From **i)** and **ii)** it follows that if $S(j) > 500$ and $S(k) < 500$ for some j and k, then $S(n) = 500$ for some n *between* j and k. So suppose, toward contradiction, that (say) $S(n) > 500$ for all n. Then

$$S(1) + S(2) + S(3) + \cdots + S(2006) > 2006 \cdot 501 = 1003(1002),$$

which contradicts ii) above.

Solution 2 by Harry Sedinger, St. Bonaventure, NY.

Denote the numbers going around the circle in a given direction as $n_1, n_2, \cdots, n_{206}$ where n_i and n_{i+1} are adjacent for each i and n_{2006} and n_1 are also adjacent. Let S_i be the set of 1,000 adjacent numbers going in the same direction and starting with n_i. Let $E(S_i)$ be the number of even numbers in S_i. It is easily seen that each number occurs in exactly 1000 such sets. Thus the sum S of occurring even numbers in all such sets is 1,003 (the number of even numbers) times 1000 which is equal to 1,003,000.

a) Suppose that $E(S_i) \neq 500$ for every i. Clearly $E(S_i)$ and $E(S_{i+1})$ differ by at most one, (as do $E(S_{2006})$ and $E(S_1)$), so either $E(S_i) \leq 499$ for every i or $E(S_i) \geq 501$ for every i. In the first case $S \leq 499 \cdot 2,006 < 1003,000$, a contradiction, and in the second case $S \geq 501 \cdot 2006 > 1,003,000$, also a contradiction. Hence $E(S_i) = 500$ for some k and the number of odd numbers in S_k is also 500.

b) It is easily seen that a) does not hold if the numbers are sequenced by 499 odd, followed by 499 even, followed by 499 odd, followed by 499 even, followed by 4 odd, and followed by 4 even.

Also solved by Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Apologies Once Again

I inadvertently forgot to mention that David Stone and John Hawkins of Statesboro, GA jointly solved problems 4910 and 4911. But worse, in my comments on 4911 (Is is possible for the sums of the squares of the six trigonometric functions to equal one), I mentioned that only two of the 26 solutions that were submitted considered the problem with respect to complex arguments. (For real arguments the answer is no; but for complex arguments it is yes.) David and John's solution considered both arguments–which makes my omission of their name all the more embarrassing. So once again, mea-culpa.

Problems

Ted Eisenberg, Section Editor

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- 4972: *Proposed by Kenneth Korbin, New York, NY.*

  Find the length of the side of equilateral triangle ABC if it has a cevian $\overline{\text{CD}}$ such that

  $$\overline{\text{AD}} = x, \quad \overline{\text{BD}} = x + 1 \quad \overline{\text{CD}} = \sqrt{y}$$

  where x and y are positive integers with $20 < x < 120$.

- 4973: *Proposed by Kenneth Korbin, New York, NY.*

  Find the area of trapezoid ABCD if it is inscribed in a circle with radius R=2, and if it has base $\overline{\text{AB}} = 1$ and $\angle\text{ACD} = 60^o$.

- 4974: *Proposed by Kenneth Korbin, New York, NY.*

  A convex cyclic hexagon has sides a, a, a, b, b, and b. Express the values of the circumradius and the area of the hexagon in terms of a and b.

- 4975: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Solve in R the following system of equations

  $$\left.\begin{aligned} 2x_1 &= 3x_2\,\sqrt{1+x_3^2} \\ 2x_2 &= 3x_3\,\sqrt{1+x_4^2} \\ &\cdots\cdots \\ 2x_n &= 3x_1\,\sqrt{1+x_2^2} \end{aligned}\right\}$$

- 4976: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers. Prove that
  $$\frac{a^2+3b^2+9c^2}{bc}+\frac{b^2+3c^2+9a^2}{ca}+\frac{c^2+3a^2+9b^2}{ab} \geq 27.$$

- 4977: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $1 < a < b$ be real numbers. Prove that for any $x_1, x_2, x_3 \in [a, b]$ there exist $c \in (a, b)$ such that
  $$\frac{1}{\log x_1}+\frac{1}{\log x_2}+\frac{1}{\log x_3}+\frac{3}{\log x_1x_2x_3}=\frac{4}{\log c}.$$

Solutions

- 4942: *Proposed by Kenneth Korbin, New York, NY.*

  Given positive integers a and b. Find the minimum and the maximum possible values of the sum $(a+b)$ if $\dfrac{ab-1}{a+b}=2007$.

Solution by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX.

If $\dfrac{ab-1}{a+b}=2007$, then
$$\begin{aligned} ab-1 &= 2007(a+b) \\ ab-2007a-2007b &= 1 \\ ab-2007a-2007b+2007^2 &= 1+2007^2 \\ (a-2007)(b-2007) &= 2\cdot 5^2\cdot 13\cdot 6197 \qquad (\mathbf{1}). \end{aligned}$$

Since **(1)** and the sum $(a+b)$ are symmetric in a and b, then we will assume that $a < b$. By the prime factorization in **(1),** there are exactly 12 distinct values for $(a-2007)$ and $(b-2007)$ which are summarized below.

$a-2007$	$b-2007$	a	b	$a+b$
1	4,028,050	2,008	4,030,057	4,032,065
2	2,014,025	2,009	2,016,032	2,018,041
5	805,610	2,012	807,617	809,629
10	402,805	2,017	404,812	406,829
13	309,850	2,020	311,857	313,877
25	161,122	2,032	163,129	165,161
26	154,925	2,033	156,932	158,965
50	80,561	2,057	82,568	84,625
65	61,970	2,072	63,977	66,049
130	30,985	2,137	32,992	35,129
325	12,394	2,332	14,401	16,733
650	6,197	2,657	8,204	10,861

Thus, the minimum value is 10,861, and the maximum value is 4,032,065.

Also solved by Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; John Nord, Spokane, WA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4943: *Proposed by Kenneth Korbin, New York, NY.*

  Given quadrilateral $ABCD$ with $\overline{AB} = 19$, $\overline{BC} = 8$, $\overline{CD} = 6$, and $\overline{AD} = 17$. Find the area of the quadrilateral if both $\overline{AC}$ and $\overline{BD}$ also have integer lengths.

  Solution by Brian D. Beasley, Clinton, SC.

  Let $x = \overline{AC}$ and $y = \overline{BD}$, where both x and y are positive integers. Let A_1 be the area of triangle ABC, A_2 be the area of triangle of ADC, A_3 be the area of triangle BAD, and A_4 be the area of triangle BCD. Then by Heron's formula, we have

  $$A_1 = \sqrt{s(s-19)(s-8)(s-x)} \qquad A_2 = \sqrt{t(t-17)(t-6)(t-x)},$$

  where $s = (19+8+x)/2$ and $t = (17+6+x)/2$. Similarly,

  $$A_3 = \sqrt{u(u-19)(u-17)(u-y)} \qquad A_4 = \sqrt{v(v-8)(v-6)(v-y)},$$

  where $u = (19+17+y)/2$ and $v = (8+6+y)/2$. Also, the lengths of the various triangle sides imply $x \in \{12, 13, \cdots, 22\}$ and $y \in \{3, 4, \cdots, 13\}$. We consider three cases for the area T of $ABCD$:

  Case 1: Assume $ABCD$ is convex. Then $T = A_1 + A_2 = A_3 + A_4$. But a search among the possible values for x and y yields no solutions in this case.

  Case 2: Assume $ABCD$ is not convex, with triangle BAD containing triangle BCD (i.e., C is interior to ABD). Then $T = A_1 + A_2 = A_3 - A_4$. Again, a search among the possible values for x and y yields no solutions in this case.

  Case 3: Assume $ABCD$ is not convex, with triangle ABC containing triangle ADC (i.e., D is interior to ABC). Then $T = A_1 - A_2 = A_3 + A_4$. In this case, a search among the possible values for x and y yields the unique solution $x = 22$ and $y = 4$; this produces $T = \sqrt{1815} = 11\sqrt{15}$.

  Due to the lengths of the quadrilateral, these are the only three cases for $ABCD$. Thus the unique value for its area is $11\sqrt{15}$.

  Also solved by Paul M. Harms, North Newton, KS; David Stone and John Hawkins, Statesboro, GA, and the proposer.

- 4944: *Proposed by James Bush, Waynesburg, PA.*

  Independent random numbers a and b are generated from the interval $[-1, 1]$ to fill the matrix $A = \begin{pmatrix} a^2 & a^2+b \\ a^2-b & a^2 \end{pmatrix}$. Find the probability that the matrix A has two real eigenvalues.

  Solution by Paul M. Harms, North Newton, KS.

  The characteristic equation is $(a^2-\lambda)^2-(a^4-b^2) = 0$. The solutions for λ are $a^2+\sqrt{a^4-b^2}$ and $a^2-\sqrt{a^4-b^2}$. There are two real eigenvalues when $a^4-b^2 > 0$ or $a^2 > |b|$. The region in the ab coordinate system which satisfies the inequality is between the parabolas $b = a^2$ and $b = -a^2$ and inside the square where a and b are both in $[-1, 1]$. From the symmetry of the region we see that the probability is the area in the first quadrant between the a-axis and $b = a^2$ from $a = 0$ to $a = 1$. Integrating gives a probability of $\frac{1}{3}$.

  Also solved by Tom Leong, Scotrun, PA; John Nord, Spokane, WA; David Stone and John Hawkins (jointly), Statesboro, GA; Boris Rays, Chesapeake, VA, and the proposer.

- 4945: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Prove that

$$17+\sqrt{2}\sum_{k=1}^{n}\Big(L_k^4+L_{k+1}^4+L_{k+2}^4\Big)^{1/2}=L_n^2+3L_{n+1}^2+5L_nL_{n+1}$$

where L_n is the n^{th} Lucas number defined by $L_0=2, L_1=1$ and for all $n\geq 2, L_n=L_{n-1}+L_{n-2}$.

Solution by Tom Leong, Scotrun, PA.

Using the identity $a^4+b^4+(a+b)^4=2(a^2+ab+b^2)^2$ we have

$$\begin{aligned}
17+\sqrt{2}\sum_{k=1}^{n}\Big(L_k^4+L_{k+1}^4+L_{k+2}^4\Big)^{1/2} &= 17+\sqrt{2}\sum_{k=1}^{n}\Big(L_k^4+L_{k+1}^4+(L_k+L_{k+1})^4\Big)^{1/2}\\
&= 17+2\sum_{k=1}^{n}\Big(L_k^2+L_kL_{k+1}+L_{k+1}^2\Big)\\
&= 17+\sum_{k=1}^{n}L_k^2+\sum_{k=1}^{n}L_{k+1}^2+\sum_{k=1}^{n}(L_k+L_{k+1})^2\\
&= 17+\sum_{k=1}^{n}L_k^2+\sum_{k=1}^{n}L_{k+1}^2+\sum_{k=1}^{n}L_{k+2}^2\\
&= 17+L_{n+2}^2+2L_{n+1}^2-L_2^2-2L_1^2+3\sum_{k=1}^{n}L_k^2\\
&= 17+(L_n+L_{n+1})^2+2L_{n+1}^2-3^2-2\cdot 1^2+3\sum_{k=1}^{n}L_k^2\\
&= L_n^2+3L_{n+1}^2+2L_nL_{n+1}+6+3\sum_{k=1}^{n}L_k^2\\
&= L_n^2+3L_{n+1}^2+2L_nL_{n+1}+6+3\left(L_nL_{n+1}-2\right)\\
&= L_n^2+3L_{n+1}^2+5L_nL_{n+1}
\end{aligned}$$

where we used the identity $\sum_{k=1}^{n}L_k^2=L_nL_{n+1}-2$ which is easily proved via induction.

Comment: Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie started off their solution with

$$2(L_k^4+L_{k+1}^4+L_{k+2}^4)=(L_k^2+L_{k+1}^2+L_{k+2}^2)^2$$

and noted that this is a special case of Candido's Identity $2(x^4+y^4+(x+y)^4)=(x^2+y^2+(x+y)^2)^2$, for which Roger Nelsen gave a proof without words in *Mathematics Magazine* (vol. 78,no. 2). Candido used this identity to establish that $2(F_n^4+F_{n+1}^4+F_{n+2}^4)=(F_n^2+F_{n+1}^2+F_{n+2}^2)$, where F_n denotes the n^{th} Fibonacci number.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS, and the proposer.

- 4946: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Let z_1, z_2 be nonzero complex numbers. Prove that

$$\left(\frac{1}{|z_1|}+\frac{1}{|z_2|}\right)\left(\left|\frac{z_1+z_2}{2}+\sqrt{z_1z_2}\right|+\left|\frac{z_1+z_2}{2}-\sqrt{z_1z_2}\right|\right)\geq 4.$$

Solution by David Stone and John Hawkins (jointly), Statesboro, GA.

We note that for $a, b > 0$,

$$\begin{aligned} a^2-2ab+b^2 &= (a-b)^2 \geq 0\\ \text{so } a^2+2ab+b^2 &\geq 4ab\\ \text{so } (a+b)(a+b) &\geq 4ab\\ \text{so } \frac{(a+b)}{ab}(a+b) &\geq 4\\ \text{or } \left(\frac{1}{a}+\frac{1}{b}\right)(a+b) &\geq 4 \end{aligned}$$

Therefore, **(1)** $\left(\frac{1}{|z_1|}+\frac{1}{|z_2|}\right)(|z_1|+|z_2|)\geq 4.$

For two complex numbers $w=a+bi$ and $v=c+di$, we have

$$\begin{aligned} |(w-v)^2|+|(w+v)^2| &= |w-v|^2+|w+v|^2=(a-c)^2+(b-d)^2+(a+c)^2+(b+d)^2\\ &= 2(a^2+b^2+c^2+d^2)=2(|w|^2+|v|^2) \end{aligned}$$

so, **(2)** $|(w-v)^2|+|(w+v)|^2=2(|w^2|+|v^2|)$.

Let w be such that $w^2=z_1$ and v be such that $v^2=z_2$. Substituting this into **(2),** we get $|w^2-2wv+v^2|+|w^2+2wv+v^2|=2(|z_1|+|z_2|)$, hence

$$\left|\frac{z_1+z_2}{2}-wv\right|+\left|\frac{z_1+z_2}{2}+wv\right|=|z_1|+|z_2|.$$

Since $(wv)^2=z_1z_2$, wv must equal $\sqrt{z_1z_2}$ or $-\sqrt{z_1z_2}$. Thus the preceding equation becomes

$$\left|\frac{z_1+z_2}{2}-\sqrt{z_1z_2}\right|+\left|\frac{z_1+z_2}{2}+\sqrt{z_1z_2}\right|=|z_1|+|z_2|.$$

Multiplying by $\frac{1}{|z_1|}+\frac{1}{|z_2|}$, we get

$$\left(\frac{1}{|z_1|}+\frac{1}{|z_2|}\right)\left(\left|\frac{z_1+z_2}{2}-\sqrt{z_1z_2}\right|+\left|\frac{z_1+z_2}{2}+\sqrt{z_1z_2}\right|\right)=\left(\frac{1}{|z_1|}+\frac{1}{|z_2|}\right)(|z_1|+|z_2|)\geq 4$$

by inequality **(1).**

Also solved by Tom Leong Scotrun, PA, and the proposers.

- 4947: *Proposed by Tom Leong, Brooklyn, NY.*

Define a set S of positive integers to be *among composites* if for any positive integer n, there exists an $x \in S$ such that all of the $2n$ integers $x \pm 1, x \pm 2, \ldots, x \pm n$ are composite. Which of the following sets are among composites? (a) The set $\{a + dk | k \in N\}$ of terms of any given arithmetic progression with $a, d \in N, d > 0$. (b) The set of squares. (c) The set of primes. (d)* The set of factorials.

Remarks and solution by the proposer, (with a few slight changes made in the comments by the editor).

This proposal arose after working Richard L. Francis's problems 4904 and 4905; it can be considered a variation on the idea in problem 4904. My original intention was to propose parts (c) and (d) only; however, I couldn't solve part (d) and, after searching the MAA journals, I later found that the question posed by part (c) is not original at all. An article in (*The Two-Year College Mathematics Journal*, Vol. 12, No. 1, Jan 1981, p. 36) solves part (c). However it appears that the appealing result of part (c) is not well-known and the solution I offer differs from the published one. Parts (a) and (b), as far as I know, are original.

Solution. The sets in (a), (b) and (c) are all among composites. In the solutions below, let n be any positive integer.

(a) Choose $m \geq n$ and $m > d$. Clearly the consecutive integers $(3m)! + 2, (3m)! + 3, \ldots, (3m)! + 3m$ are all composite. Furthermore since $d \leq m - 1$, one of the integers $(3m)! + m + 2, (3m)! + m + 3, \ldots, (3m)! + 2m$ belongs to the arithmetic progression and we are done.

(b) By Dirichlet's theorem on primes in arithmetic progressions, there are infinitely many primes congruent to 1 mod 4. Let $p > n$ be prime with $p \equiv 1 \pmod 4$. From the theory of quadratic residues, we know -1 is a quadratic residue mod p, that is, there is a positive integer r such that $r^2 \equiv -1 \pmod p$. Also by Wilson's theorem, $(p-1)! \equiv -1 \pmod p$. Put $x = [r(p-1)!]^2$. Then $x \pm 2, x \pm 3, \ldots, x \pm (p-1)$ are all composite. Furthermore, $x - 1 = [r(p-1)!]^2 - 1 = [r(p-1)! + 1][r(p-1)! - 1]$ is composite and $x \equiv r^2[(p-1)!]^2 \equiv -1(-1)^2 \equiv -1 \pmod p$, that is, $x + 1$ is composite.

(c) Let $p > n + 1$ be an odd prime. First note $p!$ and $(p-1)! - 1$ are relatively prime. Indeed, the prime divisors of $p!$ are all primes not exceeding p while none of those primes divide $(p-1)!-1$ (clearly primes less than p do not divide $(p-1)!-1$, while $(p-1)!-1 \equiv -2 \pmod p$ by Wilson's theorem). Appealing to Dirichlet's theorem again, there are infinitely many primes x of the form $x = kp! + (p-1)! - 1$. So $x - 1, x - 2, \ldots, x - (p-2)$ and $x+1, x+3, x+4, \ldots, x+p$ are all composite. By Wilson's theorem, $(p-1)!+1$ is divisible by p; hence $x + 2$ is divisible by p, that is, composite.

Remarks. (b) In fact, it can similarly be shown that the set of nth powers for any positive integer n is among composites.

(d) For any prime p, let $x = (p-1)!$. Then $x \pm 2, x \pm 3, \ldots, x \pm (p-1)$ are all composite and by Wilson's theorem, $x + 1$ is also composite. It remains: is $x - 1 = (p-1)! - 1$ composite? I don't know; however it's unlikely to be prime for all primes p.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
December 15, 2007

- 4978: *Proposed by Kenneth Korbin, New York, NY.*

  Given equilateral triangle ABC with side $\overline{\mathrm{AB}} = 9$ and with cevian $\overline{\mathrm{CD}}$. Find the length of $\overline{\mathrm{AD}}$ if $\triangle$ADC can be inscribed in a circle with diameter equal to 10.

- 4979: *Proposed by Kenneth Korbin, New York, NY.*

  Part I: Find two pairs of positive numbers (x, y) such that

  $$\frac{x}{\sqrt{1+y}-\sqrt{1-y}} = \frac{\sqrt{65}}{2},$$

  where x is an integer.

  Part II: Find four pairs of positive numbers (x, y) such that

  $$\frac{x}{\sqrt{1+y}-\sqrt{1-y}} = \frac{65}{2},$$

  where x is an integer.

- 4980: *J.P. Shiwalkar and M.N. Deshpande, Nagpur, India.*

  An unbiased coin is sequentially tossed until $(r + 1)$ heads are obtained. The resulting sequence of heads (H) and tails (T) is observed in a linear array. Let the random variable X denote the number of double heads (HH's, where overlapping is allowed) in the resulting sequence. For example: Let $r = 6$ so the unbiased coin is tossed till 7 heads are obtained and suppose the resulting sequence of H's and T's is as follows:

  HHTTT**H**TTTT**HHH**TT**H**

Now in the above sequence, there are three double heads (HH's) at toss number $(1, 2), (11, 12)$ and $(12, 13)$. So the random variable X takes the value 3 for the above observed sequence. In general, what is the expected value of X?

- 4981: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all real solutions of the equation

$$5^x + 3^x + 2^x - 28x + 18 = 0.$$

- 4982: *Proposed by Juan José Egozcue and José Luis Díaz-Barrero, Barcelona, Spain.*
  Calculate

$$\lim_{n\to\infty} \frac{1}{n+1} \left(\sum_{1\le i_1 \le n+1} \frac{1}{i_1} + \sum_{1\le i_1 < i_2 \le n+1} \frac{1}{i_1 i_2} + \cdots + \sum_{1 \le i_1 < \ldots < i_n \le n+1} \frac{1}{i_1 i_2 \ldots i_n} \right).$$

- 4983: *Proposed by Ovidiu Furdui, Kalamazoo, MI.*
  Let k be a positive integer. Evaluate

$$\int_0^1 \left\{ \frac{k}{x} \right\} dx,$$

  where $\{a\}$ is the *fractional part* of a.

Solutions

- 4948: *Proposed by Kenneth Korbin, New York, NY.*
  The sides of a triangle have lengths x_1, x_2, and x_3 respectively. Find the area of the triangle if

$$(x - x_1)(x - x_2)(x - x_3) = x^3 - 12x^2 + 47x - 60.$$

Solution by Jahangeer Kholdi and Robert Anderson (jointly), Portsmouth, VA.

The given equation implies that

$$\begin{aligned} x_1 + x_2 + x_3 &= 12 \\ x_1x_2 + x_1x_3 + x_2x_3 &= 47 \\ x_1x_2x_3 &= 60 \end{aligned}$$

from which by inspection, $x_1 = 3, x_2 = 4$ and $x_3 = 5$.

Editor's comment: At the time this problem was sent to the technical editor, the Journal was in a state of transition. A new editor- in-chief was coming on board and there was some question as to the future of the problem solving column. As such, I sent an advanced copy of the problem solving column to many of the regular contributors. In that advanced copy this polynomial was listed as $(x-x_1)(x-x_2)(x-x_3) = x^3-12x^2+47x-59$, and not with the constant term as listed above. Well, many of those who sent in solutions solved the problem in one of two ways: as above, obtaining the perimeter $x_1 + x_2 + x_3 = 12$; and then finding the area with Heron's formula. $A = \sqrt{6(6 - x_1)(6 - x_2)(6 - x_3)}$.

Substituting 6 into $(x-x_1)(x-x_2)(x-x_3) = x^3-12x^2+47x-59$ gives $(6-x_1)(6-x_2)(6-x_3) = 7$. So, $A = \sqrt{(6)(7)} = \sqrt{42}$. But others noted that the equation $x^3-12x^2+47x-59$ has only one real root, and this gives the impossible situation of having a triangle with the lengths of two of its sides being complex numbers. The intention of the problem was that a solution should exist, and so the version of this problem that was posted on the internet had a constant term of -60. In the end I counted a solution as being correct if the solution path was correct, with special kudos going to those who recognized that the advanced copy version of this problem was not solvable.

Also solved by Brian D. Beasley, Clinton, SC; Mark Cassell (student, St. George's School), Spokane, WA; Pat Costello, Richmond, KY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; Grant Evans (student, St. George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Charles McCracken (two solutions as outlined above), Dayton, OH; John Nord (two solutions as outlined above), Spokane, WA; Boris Rays, Chesapeake, VA; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **4949:** *Proposed by Kenneth Korbin, New York, NY.*

A convex pentagon is inscribed in a circle with diameter d. Find positive integers a, b, and d if the sides of the pentagon have lengths a, a, a, b, and b respectively and if $a > b$. Express the area of the pentagon in terms of a, b, and d.

Solution by David Stone and John Hawkins, Statesboro, GA.

Note, that any solution can be scaled upward by any integer factor to produce infinitely many similar solutions.

We have three isosceles triangles with base a and equal sides $\frac{d}{2}$, and two isosceles triangles with base b and equal sides $\frac{d}{2}$. Let α be the measure of the angle opposite base a, and let β be the measure of the angle opposite the base b. Then $3\alpha + 2\beta = 2\pi$.

For each triangle with base a, the perimeter is $d + a$, and Heron's formula gives

$$A_n = \sqrt{\left(\frac{d+a}{2}\right)\left(\frac{d-a}{2}\right)\left(\frac{a}{2}\right)} = \frac{a}{4}\sqrt{d^2-a^2}.$$

We can also use the Law of Cosines to express the cosine of α as $\cos\alpha = \dfrac{a^2 - 2\left(\dfrac{d}{2}\right)^2}{-2\left(\dfrac{d}{2}\right)^2} = \dfrac{d^2-2a^2}{d^2}$.

From the Pythagorean Identity, it follows that

$$\sin\alpha = \sqrt{1-\left(\frac{d^2-2a^2}{d^2}\right)^2} = \frac{1}{d^2}\sqrt{d^4-d^4+4a^2d^2-4a^4} = \frac{2a}{d^2}\sqrt{d^2-a^2}.$$

Because the triangle is isosceles, with equal sides forming the angle α, an altitude through angle α divides the triangle into two equal right triangles. Therefore, $\cos\frac{\alpha}{2} = \frac{1}{d}\sqrt{d^2-a^2}$ and $\sin\frac{\alpha}{2} = \frac{a}{d}$.

For the triangles with base b, we can similarly obtain $A_b = \frac{b}{4}\sqrt{d^2-b^2}$ and $\cos\beta = \frac{d^2-2b^2}{d^2}$.

The area for the convex polygon is then

$$\begin{aligned} A_{polygon} &= 3A_a + 2A_b \\ &= \frac{3a}{4}\sqrt{d^2-a^2} + \frac{b}{2}\sqrt{d^2-b^2} \\ &= \frac{1}{4}\left(3a\sqrt{d^2-a^2} + 2b\sqrt{d^2-b^2}\right) \end{aligned}$$

in terms of a, b, and d.

Solving $3\alpha + 2\beta = 2\pi$, we find $\beta = \frac{2\pi - 3\alpha}{2} = \pi - \frac{3\alpha}{2}$.

Therefore,

$$\cos\beta = \cos\left(\pi - \frac{3\alpha}{2}\right) = -\cos\left(\frac{3\alpha}{2}\right) = -\cos\left(\alpha + \frac{\alpha}{2}\right) = -\cos\frac{\alpha}{2}\cos\alpha + \sin\frac{\alpha}{2}\sin\alpha.$$

Replacing the trig functions in this formula with the values computed above gives

$$\frac{d^2-2b^2}{d^2} = -\frac{\sqrt{d^2-a^2}}{d}\left(\frac{d^2-2a^2}{d^2}\right) + \frac{a}{d}\left(\frac{2a}{d^2}\right)\sqrt{d^2-a^2} = \frac{\sqrt{d^2-a^2}}{d}\left(4a^2-d^2\right).$$

Solving for b^2 in terms of a and d gives

$$b^2 = \frac{d^3 - \sqrt{d^2-a^2}\left(4a^2-d^2\right)}{2d}, \text{ or } b = \sqrt{\frac{d^3 - \sqrt{d^2-a^2}\left(4a^2-d^2\right)}{2d}}.$$

Note also that **(1)** $2b^2 = d^2 - \frac{\sqrt{d^2-a^2}\left(4a^2-d^2\right)}{d}$.

We can use this expression for b to compute the area of the polygon solely in terms of a and d.

$$A_{polygon} = \frac{3a}{4}\sqrt{d^2-a^2} + \frac{b}{2}\sqrt{d^2-b^2} = \frac{3a}{4}\sqrt{d^2-a^2} + \frac{a|3d^2-4a^2|}{4d}.$$

To find specific values which satisfy the problem, we use equation **(1).**

If $d^2 - a^2 = m^2$, then **(1)** becomes **(2)** $2b^2 = d^2 - \frac{m\left(4a^2-d^2\right)}{d} = d^2 - \frac{m\left(3a^2-m^2\right)}{d}$**.**

Then(a, m, d) is a Pythagorean triple, and thus a scalar multiple of a primitive Pythagorean triple (A, B, C). Using the standard technique, this triple is generated by two parameters, s and t:

$$\begin{cases} A = 2st \\ B = s^2 - t^2 \\ C = s^2 + t^2 \end{cases},$$

where $s > t$, s and t are relatively prime and have opposite parity. There are the two possibilities, where k is some scalar:

$$a = kA = 2kst,\ m = kB = k\left(s^2 - t^2\right), \text{and } d = kC = k\left(s^2 + t^2\right)$$

or

$$m = kA = 2kst,\ a = kB = k\left(s^2 - t^2\right), \text{and } d = kC = d\left(s^2 + t^2\right).$$

We'll find solutions satisfying the first set of conditions, recognizing that this will probably not produce all solutions of the problem. Substituting these in **(2)**,we find

$$2b^2 = d^2 - \frac{m(3a^2 - m^2)}{d} = k\left(s^2 + t^2\right)^2 - \frac{k(s^2 - t^2)\left(3(2ks)^2 - k^2\left(s^2 - t^2\right)^2\right)}{k(s^2 + t^2)}.$$

Simplifying, we find that $b^2 = \dfrac{k^2 s^2\left(s^2 - 3t^2\right)^2}{s^2 + t^2}$, and we want this b to be an integer.

The simplest possible choice is to let $k^2 = s^2 + t^2$ (so that (s, t, k) is itself a Pythagorean triple); this forces $b = s\left(s^2 - 3t^2\right)$. We then have

$$a = 2kst = 2st\sqrt{s^2 + t^2},\ \ m = \sqrt{s^2 + t^2}\left(s^2 - t^2\right),\ \ d = k(s^2 + t^2) = k^3 = \left(s^2 + t^2\right)^{3/2} \text{ and}$$

$$b = s\left(s^2 - 3t^2\right).$$

That is, if (s, t, k) is a Pythagorean triple with $s^2 - 3t^2 > 0$, we have

$$\begin{cases} a = 2kst \\ b = s\left(s^2 - 3t^2\right) \\ d = k^3. \end{cases}$$

The restriction that $a > b$ imposes further conditions on s and t (roughly, $s < 3.08t$).

Some results, due to Excel:

s	t	k	b	a	d	$Area$
12	5	13	828	1,560	2,197	1,024,576
15	8	17	495	4,080	4,913	3,396,630
35	12	37	27,755	31,080	50,653	604,785,405
80	39	89	146,960	555,360	704,969	85,620,163,980
140	51	149	1,651,580	2,127,720	3,307,949	2,530,718,023,785
117	44	125	922,077	1,287,000	1,953,125	829,590,714,707
168	95	193	193,032	6,160,560	7,189,057	6,053,649,964,950
208	105	233	2,119,312	10,177,440	12,649,337	25,719,674,553,300
187	84	205	2,580,787	6,440,280	8,615,125	14,516,270,565,027
252	115	277	6,004,908	16,054,920	21,253,933	86,507,377,177,725
209	120	241	100,529	12,088,560	13,997,521	21,678,178,927,350
247	96	265	8,240,167	12,567,360	18,609,625	77,495,769,561,288
352	135	377	24,368,608	35,830,080	53,582,633	647,598,434,135,400

Also solved by the proposer

- 4950: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{a+b}{\sqrt[4]{a^3}+\sqrt[4]{b^3}}+\frac{b+c}{\sqrt[4]{b^3}+\sqrt[4]{c^3}}+\frac{c+a}{\sqrt[4]{c^3}+\sqrt[4]{a^3}} \geq 3.$$

Solution by Kee-Wai Lau, Hong Kong, China

Since

$$\begin{aligned} a+b &= \frac{(\sqrt[4]{a}+\sqrt[4]{b})(\sqrt[4]{a^3}+\sqrt[4]{b^3}) \ +(\sqrt[4]{a}-\sqrt[4]{b})^2(\sqrt{a}+\sqrt[4]{a}\sqrt[4]{b}+\sqrt{b})}{2} \\ &\geq \frac{(\sqrt[4]{a}+\sqrt[4]{b})(\sqrt[4]{a^3}+\sqrt[4]{b^3})}{2} \end{aligned}$$

with similar results for $b{+}c$ and $c{+}a$, so by the arithmetic mean-geometric mean inequality, we have

$$\begin{aligned} &\frac{a+b}{\sqrt[4]{a^3}+\sqrt[4]{b^3}}+\frac{b+c}{\sqrt[4]{b^3}+\sqrt[4]{c^3}}+\frac{c+a}{\sqrt[4]{c^3}+\sqrt[4]{a^3}} \\ \geq\ & \sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c} \\ \geq\ & 3\sqrt[12]{abc} \\ =\ & 3 \text{ as required.} \end{aligned}$$

Also solved by Michael Brozinsky (two solutions), Central Islip, NY; Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX, and the proposer.

- 4951: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let α, β, and γ be the angles of an acute triangle ABC. Prove that

$$\pi \sin\sqrt{\frac{\alpha^2+\beta^2+\gamma^2}{\pi}} \geq \alpha\sin\sqrt{\alpha}+\beta\sin\sqrt{\beta}+\gamma\sin\sqrt{\gamma}.$$

Solution by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo, TX.

Since α, β, and γ are the angles of an acute triangle,

$$\alpha,\beta,\gamma \in (0,\frac{\pi}{2}) \text{ and } \frac{\alpha}{\pi}+\frac{\beta}{\pi}+\frac{\gamma}{\pi}=1$$

Let $f(x) = \sin\sqrt{x}$ on $(0, \frac{\pi}{2})$. Then, since

$$f''(x) = -\frac{\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}}{4x^{3/2}} < 0$$

on $(0,\frac{\pi}{2})$, it follows that $f(x)$ is concave down on $(0, \frac{\pi}{2})$. Hence, by Jensen's Inequality and (1)

$$\frac{\alpha}{\pi}\sin\sqrt{\alpha}+\frac{\beta}{\pi}\sin\sqrt{\beta}+\frac{\gamma}{\pi}\sin\sqrt{\gamma} \quad \leq \quad \sin\sqrt{\frac{\alpha}{\pi}\cdot\alpha+\frac{\beta}{\pi}\cdot\beta+\frac{\gamma}{\pi}\cdot\gamma}$$

$$= \quad \sin\sqrt{\frac{\alpha^2+\beta^2+\gamma^2}{\pi}},$$

with equality if and only if $\alpha = \beta = \gamma = \dfrac{\pi}{3}$.

Also solved by the proposer

- 4952: *Proposed by Michael Brozinsky, Central Islip, NY & Robert Holt, Scotch Plains, NJ.*

  An archeological expedition discovered all dwellings in an ancient civilization had $1, 2$, or 3 of each of k independent features. Each plot of land contained three of these houses such that the k sums of the number of each of these features were all divisible by 3. Furthermore, no plot contained two houses with identical configurations of features and no two plots had the same configurations of three houses. Find **a)** the maximum number of plots that a house with a given configuration might be located on, and **b)** the maximum number of distinct possible plots.

  Solution by Paul M. Harms, North Newton, KS

  Let $\binom{n}{r}$ be the combination of n things taken r at a time. With k independent features there are $\binom{k}{1} = k$ number of different "groups" containing one feature, $\binom{k}{2}$ different "groups" containing two features, etc. To have the sum of independent features in a plot of three houses be divisible by three, there are four possibilities. **I.** Each house in a plot has one feature. **II.** Each house in a plot has two features. **III.** Each house in a plot has three features. **IV.** One house in a plot has one feature, another house has two features, and the third house has three features.

  The maximum number of distinct plots can be found by summing the number of plots for each of the four possibilities above. The sum is

$$\binom{\binom{k}{1}}{3} + \binom{\binom{k}{2}}{3} + \binom{\binom{k}{3}}{3} + \binom{k}{1}\binom{k}{2}\binom{k}{3}$$

  This is the result for part **b).**

  For part **a),** first consider a house with one fixed feature. There are plots in possibilities I and IV. In possibility I the other two houses can have any combination of the other $(k-1)$ single features so there are $\binom{k-1}{2}$ plots. In possibility IV the number of plots with a house with one fixed feature is $\binom{k}{1}\binom{k}{2}\binom{k}{3}$. The number of plots with houses with different features is the following: For a house with one fixed feature there are $\binom{k-1}{2} + \binom{k}{2}\binom{k}{3}$ plots. For a house with two fixed features there are $\binom{\binom{k}{2}-1}{2} + \binom{k}{1}\binom{k}{3}$ plots. For a house with three fixed features there are $\binom{\binom{k}{2}-1}{2} + \binom{k}{1}\binom{k}{2}$ plots.

Also solved by the proposer.

- 4953: *Proposed by Tom Leong, Brooklyn, NY.*

  Let$\pi(x)$ denote the number of primes not exceeding x. Fix a positive integer n and define sequences by $a_1 = b_1 = n$ and

  $$a_{k+1} = a_k - \pi(a_k) + n, \qquad b_{k+1} = \pi(b_k) + n + 1 \quad \text{for k} \geq 1.$$

  a) Show that $\lim\limits_{k\to\infty} a_k$ is the n^{th} prime.
  b) Show that $\lim\limits_{k\to\infty} b_k$ is the n^{th} composite.

  Solution by Paul M. Harms, North Newton, KS.

  Any positive integer m is less than the m^{th} prime since 1 is not a prime. In part **a)** with $a_1 = n$, we have $\pi(n)$ primes less than or equal to n. We need $n - \pi(n)$ more primes than n has in order to get to the n^{th} prime. Note that a_2 is greater than a_1 by $n - \pi(n)$. If **all** of the integers from $a_1 + 1$ to a_2 are prime, then a_2 is the n^{th} prime. If not all of the integers indicated in the last sentence are primes, we see that a_3 is greater than a_2 by the number of non-primes from $a_1 + 1$ to a_2. This is true in general from a_k to a_{k+1} since $a_{k+1} = a_k + (n - \pi(a_k))$. If a_k is not the n^{th} prime, then a_{k+1} will increase by the quantity of integers to get to the n^{th} prime provided all integers a_{k+1} will increase by the quantity of integers to get to the n^{th} prime provided all integers $a_k + 1$,to a_{k+1}. We see that the sequence increases until some $a_m = N$, the n^{th} prime. Then $a_{m+1} = a_m + (n - \pi(a_m)) = a_m + 0 = a_m$. In this same way it is seen that $a_k = a_m$ for all k greater that m. Thus the limit for the sequence in part **a)** is the n^{th} prime.

  For part **b)** note that n is less than the n^{th} composite. Since the integer 1 and integers $\pi(n)$ are not composite, the n^{th} composite must be at least $1+\pi(n)$ greater than n. With $b_1 = n$ we see that $b_2 = n + (1 + \pi(n))$. Then b_2 will be the n^{th} composte provided all integers $n + 1, n + 2, \cdots, n + 1 + \pi(n)$ are composites. If some of the integers in the last sentence are prime, then b_3 is greater than b_2 by the number of primes in the integers from $b_1 + 1$ to b_2. In general, b_{k+1} is greater than b_k by the number of primes in the integers from $b_{k-1} + 1$ to b_k and the sequence will be an increasing sequence until the n^{th} composite is reached. If $b_m = N$, the n^{th} composite, then all integers from $b_{m-1} + 1$ to b_m are composite. Then $\pi(b_{m-1}) = \pi(b_m)$ and $b_{m+1} = \pi(b_{m-1}) + 1 + n = b_m = N$. We see that $b_k = N$ for all k at least as great as m. Thus the limit of the sequence in part **b)** is the n^{th} composite.

  Also solved by David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4954: *Proposed by Kenneth Korbin, New York, NY.*

  Find four pairs of positive integers (a, b) that satisfy

  $$\frac{a+i}{a-i} \cdot \frac{b+i}{b-i} = \frac{111+i}{111-i}$$

  with $a < b$.

  Solution by David E. Manes, Oneonta, NY.

  The only solutions (a, b) with $a < b$ are $(112, 12433), (113, 6272), (172, 313)$, and $(212, 233)$.

  Expanding the given equation and clearing fractions, one obtains $[2(111)(a + b) - 2(ab - 1)]i = 0$. Therefore, $\dfrac{ab-1}{a+b} = 111$. Let $b = a + k$ for some positive integer k. Then the above equation reduces to a quadratic in a; namely $a^2 + (k - 222)a - (111k + 1) = 0$ with roots given by

$$a = \frac{(222 - k) \pm \sqrt{k^2 + 49288}}{2}.$$

Since a is a positive integer, it follows that $k^2 + 49288 = n^2$ or

$$n^2 - k^2 = (n + k)(n - k) = 49288 = 2^3 \cdot 61 \cdot 101.$$

Therefore, $n + k$ and $n - k$ are positive divisors of 49288. The only such divisors yielding solutions are

$n+k$	$n-k$
24644	2
12322	4
404	122
244	202

Solving these equations simultaneously gives the following values for(n, k) :

$$(12323, 12321), (6163, 6159), (263, 141), \text{ and } (223, 21)$$

from which the above cited solutions for a and b are found.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Daniel Copeland (student at St. George's School), Spokane, WA; Jeremy Erickson, Matthew Russell, and Chad Mangum (jointly; students at Taylor University), Upland, IN; Grant Evans (student at St. George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; John Nord, Spokane, WA; Homeira Pajoohesh, David Stone, and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4955: *Proposed by Kenneth Korbin, New York, NY.*

Between 100 and 200 pairs of red sox are mixed together with between 100 and 200 pairs of blue sox. If three sox are selected at random, then the probability that all three are the same color is 0.25. How many pairs of sox were there altogether?

Solution by Brian D. Beasley, Clinton, SC.

Let R be the number of pairs of red sox and B be the number of pairs of blue sox. Then $200 \le R + B \le 400$ and

$$\frac{2R(2R-1)(2R-2) + 2B(2B-1)(2B-2)}{(2R+2B)(2R+2B-1)(2R+2B-2)} = \frac{1}{4}.$$

Thus $4[R(2R-1)(R-1) + B(2B-1)(B-1)] = (R+B)(2R+2B-1)(R+B-1)$, or equivalently

$$4(2R^2 + 2B^2 - R - B - 2RB)(R + B - 1) = (2R^2 + 2B^2 - R - B + 4RB)(R + B - 1).$$

This yields $6R^2 + 6B^2 - 3R - 3B - 12RB = 0$ and hence $2(R - B)^2 = R + B$. Letting $x = R - B$, we obtain $R = x^2 + \frac{1}{2}x$ and $B = x^2 - \frac{1}{2}x$, so x is even. In addition, the size of $R + B$ forces $|x| \in \{10, 12, 14\}$. A quick check shows that only $|x| = 12$ produces values for R and B between 100 and 200, giving the unique solution $\{R, B\} = \{138, 150\}$. Thus $R + B = 288$.

Also solved by Pat Costello, Richmond, KY; Paul M. Harms, North Newton, KS, and the proposer.

- 4956: *Proposed by Kenneth Korbin, New York, NY.*

  A circle with radius $3\sqrt{2}$ is inscribed in a trapezoid having legs with lengths of 10 and 11. Find the lengths of the bases.

  Solution by Eric Malm, Stanford, CA.

  There are two different solutions: one when the trapezoid is shaped like /O\, and the other when it is configured like /O/. In fact, by reflecting the right-hand half of the plane about the x-axis, we can interchange between these two cases. Anyway, in the first case, the lengths of the bases are $7 - \sqrt{7}$ and $14 + \sqrt{7}$, and in the second case they are $7 + \sqrt{7}$ and $14 - \sqrt{7}$.

  Also solved by Michael Brozinsky, Central Islip, NY; Daniel Copeland (student at St. George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; Nate Wynn (student at St. George's School), Spokane, WA, and the proposer.

- 4957: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $\{a_n\}_{n\geq 0}$ be the sequence defined by $a_0 = 1, a_1 = 2, a_2 = 1$ and for all $n \geq 3$, $a_n^3 = a_{n-1}a_{n-2}a_{n-3}$. Find $\lim\limits_{n\to\infty} a_n$.

  Solution by Michael Brozinsky, Central Islip, NY.

  If we write $a_n = 2^{b_n}$ we have $b_n = \dfrac{b_{n-1} + b_{n-2} + b_{n-3}}{3}$ where $b_0 = 0, b_1 = 1$, and $b_2 = 0$. The characteristic equation is

$$\begin{aligned} x^3 &= \frac{x^2}{3} + \frac{x}{3} + \frac{1}{3} \text{ with roots} \\ r_1 &= 1,\ r_2 = \frac{-1 + i\sqrt{2}}{3}, \text{ and } r_3 = \frac{-1 - i\sqrt{2}}{3}. \end{aligned}$$

  The generating function $f(n)$ for $\{b_n\}$ is (using the initial conditions) found to be

$$\begin{aligned} f(n) &= A + B\left(\frac{-1 + i\sqrt{2}}{3}\right)^n + C\left(\frac{-1 - i\sqrt{2}}{3}\right)^n \text{ where} \\ A &= \frac{1}{3},\ B = -\frac{1}{6} - \frac{5i\sqrt{2}}{12}, \text{ and } C = -\frac{1}{6} + \frac{5i\sqrt{2}}{12}. \end{aligned}$$

  Since $|r_2| = |r_3| = \dfrac{\sqrt{6}}{4} < 1$ we have the last two terms in the expression for $f(n)$ approach 0 as n approaches infinity, and hence $\lim\limits_{n\to\infty} b_n = \dfrac{1}{3}$ and so $\lim\limits_{n\to\infty} a_n = \sqrt[3]{2}$.

  Also solved by Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Boris Rays and Jahangeer Khold (jointly), Chesapeake, VA & Portsmouth, VA; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins, Statesboro, GA, and the proposer.

- 4958: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let $f : [a,b] \to R$ $(0 < a < b)$ be a continuous function on $[a,b]$ and derivable in (a,b). Prove that there exists a $c \in (a,b)$ such that

$$f'(c) = \frac{1}{c\sqrt{ab}} \cdot \frac{\ln(ab/c^2)}{\ln(c/a) \cdot \ln(c/b)}.$$

Solution by the proposer.
Consider the function $F : [a,b] \to R$ defined by

$$F(x) = (\ln x - \ln a)(\ln x - \ln b) \exp\left[\sqrt{ab}\, f(x)\right]$$

Since F is continuous function on $[a,b]$, derivable in (a,b) and $F(a) = F(b) = 0$, then by Rolle's theorem there exists $c \in (a,b)$ such that $F'(c) = 0$. We have

$$F'(x) = \left[\frac{1}{x}(\ln x - \ln b) + \frac{1}{x}(\ln x - \ln a)\right.$$

$$\left. + \sqrt{ab}(\ln x - \ln a)(\ln x - \ln b) f'(x)\right] \exp\left[\sqrt{ab}\, f(x)\right]$$

and

$$\frac{1}{c}\ln\left(\frac{c^2}{ab}\right) + \sqrt{ab}\ln\left(\frac{c}{a}\right)\ln\left(\frac{c}{b}\right)\, f'(c) = 0$$

From the preceding immediately follows

$$\sqrt{ab}\ln(c/a)\ln(c/b)\; f'(c) = \frac{1}{c}\ln(ab/c^2)$$

and we are done.

- 4959: *Proposed by Juan-Bosco Romero Márquez, Valladolid, Spain.*

Find all numbers $N = ab$, were $a, b = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, such that

$$[S(N)]^2 = S(N^2),$$

where S(N)=a+b is the sum of the digits. For example:

$$\begin{array}{rcll} N & = & 12 & N^2 = 144 \\ S(N) & = & 3 & S(N^2) = 9 \quad \text{and } [\text{S(N)}]^2 = \text{S(N}^2). \end{array}$$

Solution by Jeremy Erickson, Matthew Russell, and Chad Mangum (jointly, students at Taylor University), Upland, IN.
We start by considering the possibilities that exist for N. Since there are 10 possibilities for a and for b, there are 100 possibilities for N. It would not be incorrect to check all 100 cases, however we need not do so.
We can eliminate the majority of these 100 cases without directly checking them. If we assume that $S(N) \geq 6$, then $[S(N)] \geq 36$, which means that for the property to hold, $S(N^2) \geq 36$ as well. This would require $N^2 \geq 9999$. However, this leads us to a contradiction because the largest possible value for N by our definition is $99, and$N^2 in that case is only $N^2 = 99^2 = 9801 < 9999$. Therefore, we need not check any number N such $S(N) > 6$. More precisely, any number N in the intervals$[6,9]; [15,19]; [24,29]; [33,39]; [42,49]; [51,99]$ need not be checked. This leaves us with 21 cases that can easily be checked.

After checking each of these cases separately, we find that for 13 of them, the property $[S(N)]^2 = S(N^2)$ does in fact hold. These 13 solutions are

$$N = 00, 01, 02, 03, 10, 11, 12, 13, 20, 21, 22, 30, 31.$$

We show the computation for $N = 31$ as an example:

$$\begin{aligned} N &= 31 & N^2 = 31^2 = 961 \\ S(N) &= 3 + 1 = 4 & S(N^2) = 9 + 6 + 1 = 16 \\ [S(N)]^2 &= 4^2 = 16 & \\ [S(N)]^2 &= S(N^2) = 16 \text{ for } N = 31. & \end{aligned}$$

The other 12 solutions can be checked similarly.

Also solved by Paul M. Harms, North Newton, KS; Jahangeer Kholdi, Robert Anderson and Boris Rays (jointly), Portsmouth, Portsmouth, & Chesapeake, VA; Peter E. Liley, Lafayette, IN; Jim Moore, Seth Bird and Jonathan Schrock (jointly, students at Taylor University), Upland, IN; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Late Solutions

Late solutions by **David E. Manes of Oneonta, NY** were received for problems 4942 and 4944.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
January 15, 2008

- 4984: *Proposed by Kenneth Korbin, New York, NY.*

  Prove that
  $$\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{5}+\sqrt{7}}+\cdots+\frac{1}{\sqrt{2009}+\sqrt{2011}}>\sqrt{120}.$$

- 4985: *Proposed by Kenneth Korbin, New York, NY.*

  A Heron triangle is one that has both integer length sides and integer area. Assume Heron triangle ABC is such that $\angle B = 2\angle A$ and with (a,b,c)=1.

  PartI : Find the dimensions of the triangle if side a = 25.
  PartII : Find the dimensions of the triangle if 100 < a < 200.

- 4986: *Michael Brozinsky, Central Islip, NY.*

  Show that if $0 < a < b$ and $c > 0$, that
  $$\sqrt{(a+c)^2+d^2}+\sqrt{(b-c)^2+d^2}\le\sqrt{(a-c)^2+d^2}+\sqrt{(b+c)^2+d^2}.$$

- 4987: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be the sides of a triangle ABC with area S. Prove that
  $$(a^2+b^2)(b^2+c^2)(c^2+a^2)\le 64S^3\csc 2A\csc 2B\csc 2C.$$

- 4988: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all real solutions of the equation

$$3^{x^2-x-z}+3^{y^2-y-x}+3^{z^2-z-y}=1.$$

- 4989: *Proposed by Tom Leong, Scotrun, PA.*

  The numbers $1, 2, 3, \cdots, 2n$ are randomly arranged onto $2n$ distinct points on a circle. For a chord joining two of these points, define its *value* to be the absolute value of the difference of the numbers on its endpoints. Show that we can connect the $2n$ points in disjoint pairs with n chords such that no two chords intersect inside the circle and the sum of the values of the chords is exactly n^2.

Solutions

- 4960: *Proposed by Kenneth Korbin, New York, NY.*

  Equilateral triangle ABC has an interior point P such that

$$\overline{AP}=\sqrt{5},\ \overline{BP}=\sqrt{12},\ \text{and}\ \overline{\text{CP}}=\sqrt{17}.$$

  Find the area of $\triangle APB$.

Solution by Scott H. Brown, Montgomery, AL.

First rotate $\triangle ABC$ about point C through a counter clockwise angle of 60^o. This will create equilateral triangle CBB' and interior point P'. Since triangle ABC is equialteral and $m\angle ACB = 60^o$, $\overline{AC}$ falls on $\overline{BC}$, and $\overline{CP'}=\sqrt{17}$, $\overline{B'P'}=\sqrt{12}$, $\overline{BP'}=\sqrt{5}$. Now $\triangle CPA \cong \triangle CP'B$ and $m\angle ACP = m\angle BCP'$, so $m\angle PCP' = 60^o$.

Second, draw $\overline{PP'}$, forming isosceles triangle PCP'. Since $m\angle PCP' = 60^o$, triangle PCP' is equilateral. We find $\overline{PP'}=\sqrt{17}$, $\overline{PA}=\overline{P'B}=\sqrt{5}$ and $\overline{PB}=\sqrt{12}$. So triangle PBP' is a right triangle.

Third, $m\angle APB' = 120^o$ and $m\angle PBP' = 90^o$. We find $m\angle PBA + m\angle P'BB' = 30^o$. Since $m\angle P'BB' = m\angle PAB$, then by substitution, $m\angle PBA + m\angle PAB = 30^o$. Thus $m\angle APB = 150^o$.

Finally, we find the area of triangle APB$=\dfrac{1}{2}(\sqrt{5})(\sqrt{12})\sin(150^o)=\dfrac{\sqrt{15}}{2}$ square units.
(Reference: Challenging Problems in Geometry 2, Posamentier & Salkind, p. 39.)

Also solved by Mark Cassell (student, Saint George's School), Spokane, WA; Matt DeLong, Upland, IN; Grant Evans (student, Saint George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Boris Rays and Jahangeer Kholdi (jointly), Chesapeake and Portsmouth, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4961: *Proposed by Kenneth Korbin, New York, NY.*

  A convex hexagon is inscribed in a circle with diameter d. Find the area of the hexagon if its sides are $3, 3, 3, 4, 4$ and 4.

Solution 1 by John Nord, Spokane, WA.

For cyclic quadrilateral ABCD with sides a, b, c, and d, two different formulations of the area are given, Brahmagupta's formula and Bretschneider's formula.

$$\begin{aligned}
A &= \sqrt{(s-a)(s-b)(s-c)(s-d)} \text{ where } s = \frac{a+b+c+d}{2} && (1)\\
A &= \frac{\sqrt{(ac+bd)(ad+bc)(ab+cd)}}{4R} \text{ where R is the circumradius} && (2)
\end{aligned}$$

In order to employ the cyclic quadrilateral theorems, place a diagonal into the hexagon to obtain two inscribed quadrilaterals. The first has side lengths of 3,3,3, and x and the second has side lengths of 4,4,4 and x.

Equating (1) and (2) and solving for R yields

$$R = \frac{1}{4}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}} \qquad (3)$$

Both quadrilaterals are inscribed in the same circle so (3) can be used for both quadrilaterals and they can be set equal to each other. Solving for x is surprisingly simple and the area computations can be calculated using (1) directly. The area of the inscribed hexagon with sides 3,3,3,4,4, and 4 is $\dfrac{73\sqrt{3}}{4}$.

Solution 2 by Jonathan Schrock, Seth Bird, and Jim Moore (jointly, students at Taylor University), Upland, IN.

Since the hexagon is convex and cyclic, a radius of the circumscribing circle can be drawn to each vertex producing six isosceles triangles. The formula for the height of one of these triangles is $\dfrac{1}{2}\sqrt{4r^2-c^2}$ where c is the length of the base of the triangle and r is the radius of the circle. Since $2r = d$ (the diameter of the circle), the area of any one of these triangles will therefore be $\dfrac{c}{4}\sqrt{d^2-c^2}$. The total area of the hexagon is the sum of the areas of the triangles. There are three triangles for which $c = 3$ and three for which $c = 4$. So the total area of the hexagon in terms of d is $3\sqrt{d^2-16}+\dfrac{9}{4}\sqrt{d^2-9}$.

We can determine d by rearranging the hexagon so that the side lengths alternate as 3,4,3,4,3,4. This creates three congruent quadrilaterals. Consider just one of these quadrilaterals and label it ABCO, where A, B, and C lie on the circle and O is the center of the circle. Since the interior angle for a circle is 360^o and there are three quadrilaterals, $\angle AOC = 120^o$. By constructing a line from A to C we can see by the symmetry of the rearranged hexagon, that $\angle ABC = 120^o$. Using the law of cosines,

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2\Big(\overline{AB}\Big)\Big(\overline{BC}\Big)\cos(120^o),$$

which can be written as $\overline{AC}^2 = 3^2+4^2-2(3)(4)\cos(120^o)$. That is, $\overline{AC} = \sqrt{37}$.
To determine d we use the law of cosines again. Here,

$$\overline{AC}^2 = \overline{AO}^2 + \overline{CO}^2 - 2\Big(\overline{AO}\Big)\Big(\overline{CO}\Big)\cos(120^o),$$

which can be written as $37 = \dfrac{d^2}{2} - \dfrac{d^2}{2}\cos(120^o)$. Solving for d gives $d = 2\sqrt{\dfrac{37}{3}}$.
Substituting this value of d into the formula $3\sqrt{d^2-16}+\dfrac{9}{4}\sqrt{d^2-9}$ gives the area of the hexagon as $\dfrac{73\sqrt{3}}{4}$.

Comment by editor: David Stone and John Hawkins of Statesboro GA generalized the problem for any convex, cyclic hexagon with side lengths a, a, a, b, b, b (with $0 < a \le b$) and with d as the diameter of the circumscribing circle. They showed that d is uniquely determined by the values of a and b, $d = \sqrt{\frac{4}{3}(a^2 + ab + b^2)}$. Then they asked the question: What fraction of the circle's area is covered by the hexagon? They found that in general, the fraction of the circle's area covered by the hexagon is:

$$\frac{\frac{\sqrt{3}}{4}(a^2+4ab+b^2)}{\frac{\pi}{3}(a^2+ab+b^2)} = \frac{3\sqrt{3}(a^2+4ab+b^2)}{4\pi(a^2+ab+b^2)} = \frac{3\sqrt{3}}{4\pi}\frac{(a+b)^2+2ab}{(a+b)^2-ab} = \left(\frac{3\sqrt{3}}{4\pi}\right)\frac{1+2c}{1-c}$$

where $c = \dfrac{ab}{(a+b)^2}$.

They continued on by stating that in fact, c takes on the values $0 < c \le 1/4$, thus forcing $1 < \dfrac{1+2c}{1-c} \le 2$. So by appropriate choices of a and b, the hexagon can cover from $\dfrac{3\sqrt{3}}{4\pi} \approx 0.4135$ of the circle up to $\dfrac{3\sqrt{3}}{4\pi} \cdot 2 \approx 0.827$ of the circle. A regular hexagon, where $a = b$ and $c = 1/4$, would achieve the upper bound and cover the largest possible fraction of the circle.

For instance, we can force the hexagon to cover exactly one half the circle by making $\left(\dfrac{3\sqrt{3}}{4\pi}\right)\dfrac{1+2c}{1-c} = \dfrac{1}{2}$. This would require $c = \dfrac{2\pi - 3\sqrt{3}}{2\left(3\sqrt{3}+\pi\right)} \approx 0.0651875$. Setting this equal to $\dfrac{ab}{(a+b)^2}$, we find that $\dfrac{a}{b} = \dfrac{\left(6\sqrt{3}-\pi\right) \pm \sqrt{3(27-\pi^2)}}{2\pi - 3\sqrt{3}}$.

That is, if $b = 13.2649868a$, the hexagon will cover half of the circle.

Also solved by Matt DeLong, Upland, IN; Peter E. Liley, Lafayette, IN; Mandy Isaacson, Julia Temple, and Adrienne Ramsay (jointly, students at Taylor University), Upland, IN; Paul M. Harms, North Newton, KS; Boris Rays and Jahangeer Kholdi (jointly), Chesapeake and Portsmouth, VA , and the proposer.

- 4962: *Proposed by Kenneth Korbin, New York, NY.*

  Find the area of quadrilateral $ABCD$ if the midpoints of the sides are the vertices of a square and if $AB = \sqrt{29}$ and $CD = \sqrt{65}$.

Solution by proposer.

Conclude that $AC \perp BD$ and that $AC = BD$. Then, there are positive numbers (w, x, y, z) such that

$$\begin{aligned} w+x &= AC, \\ y+z &= BD, \\ w^2+y^2 &= 29, \text{ and} \\ x^2+z^2 &= 65. \end{aligned}$$

Then, $(w, x, y, z) = (\frac{11}{\sqrt{10}}, \frac{19}{\sqrt{10}}, \frac{13}{\sqrt{10}}, \frac{17}{\sqrt{10}})$ and $AC = BD = \frac{30}{\sqrt{10}}$. The area of the quadrilateral then equals $\frac{1}{2}(AC)(BD) = \dfrac{1}{2}\left(\dfrac{30}{\sqrt{10}}\right)\left(\dfrac{30}{\sqrt{10}}\right) = 45.$

Also solved by Peter E. Liley, Lafayette, IN, and by Boris Rays and Jahangeer Kholdi (jointly), Chesapeake and Portsmouth, VA.

- 4963: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Calculate

$$\lim_{n\to\infty} \sum_{1\le i<j\le n} \frac{1}{3^{i+j}}.$$

Solution 1 by Ken Korbin, New York, NY.

$$\begin{aligned}
\sum_{1\le i<j\le n} \frac{1}{3^{i+j}} &= \left(\frac{1}{3^3}+\frac{1}{3^4}\right)+\left(\frac{2}{3^5}+\frac{2}{3^6}\right)+\left(\frac{3}{3^7}+\frac{3}{3^8}\right)+\left(\frac{4}{3^9}+\frac{4}{3^{10}}\right)+\cdots \\
&= \frac{4}{3^4}+\frac{8}{3^6}+\frac{12}{3^8}+\frac{16}{3^{10}}+\cdots \\
&= \frac{4}{3^4}\left[1+\frac{2}{3^2}+\frac{3}{3^4}+\frac{4}{3^6}+\cdots\right] \\
&= \frac{4}{3^4}\left[1+\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\cdots\right]^2 \\
&= \frac{4}{3^4}\left[\frac{1}{1-\frac{1}{3^2}}\right]^2 \\
&= \frac{4}{3^4}\left[\frac{9}{8}\right]^2 = \frac{1}{16}.
\end{aligned}$$

Solutions 2 and 3 by Pat Costello, Richmond, KY.

2) When $n=2$ we have $\dfrac{1}{3^{1+2}}$.

When $n=3$ we have $\dfrac{1}{3^{1+3}}+\dfrac{1}{3^{2+3}}$.

When $n=4$ we have $\dfrac{1}{3^{1+4}}+\dfrac{1}{3^{2+4}}+\dfrac{1}{3^{3+4}}$.

Adding down the columns we obtain:

$$\begin{aligned}
&\sum_{k=3}^{\infty}\frac{1}{3^k}+\sum_{k=5}^{\infty}\frac{1}{3^k}+\sum_{k=7}^{\infty}\frac{1}{3^k}+\cdots \\
&\qquad = \frac{(1/3)^3}{1-1/3}+\frac{(1/3)^5}{1-1/3}+\frac{(1/3)^7}{1-1/3}+\cdots \\
&\qquad = \frac{3}{2}\left(\frac{1}{3}\right)^3\left(1+(1/3)^2+(1/3)^4+\cdots\right) \\
&\qquad = \frac{3}{2}\left(\frac{1}{3}\right)^3\Big(1+(1/9)+(1/9)^2+\cdots\Big) \\
&\qquad = \frac{3}{2}\left(\frac{1}{3}\right)^3\left(\frac{1}{1-1/9}\right) = \frac{1}{16}.
\end{aligned}$$

3) Another way to see that the value is 1/16 is to write the limit as the double sum

$$\begin{aligned}
\sum_{n=2}^{\infty}\sum_{i=2}^{n-1}\frac{1}{3^{n+i}} &= \sum_{n=2}^{\infty}\frac{1}{3^n}\sum_{i=2}^{n-1}\frac{1}{3^i} = \sum_{n=2}^{\infty}\frac{1}{3^n}\left(\frac{(1/3)-(1/3)^n}{1-(1/3)}\right)\\
&= \frac{3}{2}\sum_{n=2}^{\infty}\frac{1}{3^n}\Big((1/3)-(1/3)^n\Big)\\
&= \frac{3}{2}\left((1/3)\sum_{n=2}^{\infty}\frac{1}{3^n}-\sum_{n=2}^{\infty}\frac{1}{9^n}\right)\\
&= \frac{3}{2}\left((\frac{1}{3})\frac{1/9}{1-1/3}-\frac{1/(81)}{1-1/9}\right)\\
&= \frac{3}{2}\left(\frac{1}{18}-\frac{1}{72}\right)=\frac{1}{16}.
\end{aligned}$$

Also solved by Bethany Ballard, Nicole Gottier, Jessica Heil (jointly, students at Taylor University), Upland, IN; Matt DeLong, Upland, IN; Paul M. Harms, North Newton, KS; Carl Libis, Kingston, RI; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4964: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let x, y be real numbers and we define the law of composition

$$x \perp y = x\sqrt{1+y^2} + y\sqrt{1+x^2}.$$

Prove that $(R,+)$ and $(R,\perp)$ are isomorphic and solve the equation $x \perp a = b$.

Solution by R. P. Sealy, Sackville, New Brunswick, Canada

Define $f:(R,+)\to(R,\perp)$ by $f(x)=\sinh x$.
Then f is one-to-one and onto, and

$$\begin{aligned}
f(a+b) &= \sinh(a+b)\\
&= \sinh a\cosh b + \cosh a \sinh b\\
&= \sinh a\sqrt{1+\sinh^2 b} + \sinh b\sqrt{1+\sinh^2 a}\\
&= f(a)\perp f(b)
\end{aligned}$$

Therefore $(R,+)$ and $(R,\perp)$ are isomorphic abelian groups.
Note that:

$$\begin{cases}
\textbf{i)}\ \ \text{f}(0)=0 \text{ and that } \text{f}(-\text{a})=-\text{f}(\text{a}).\\
\textbf{ii)}\ \ \text{In } (\text{R},\perp)\\
\qquad 0\perp a = 0\sqrt{1+a^2}+a\sqrt{1+0^2}=a \text{ and}\\
\qquad a\perp(-a) = a\sqrt{1+a^2}-a\sqrt{1+a^2}=0.
\end{cases}$$

If $x\perp a=b$, then $x=b\perp(-a)=b\sqrt{1+a^2}-a\sqrt{1+b^2}$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY, and the proposer.

- 4965: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let h_a, h_b, h_c be the heights of triangle ABC. Let P be any point inside $\triangle ABC$. Prove that

$$(a) \quad \frac{h_a}{d_a} + \frac{h_b}{d_b} + \frac{h_c}{d_c} \ge 9, \quad (b) \quad \frac{d_a^2}{h_a^2} + \frac{d_b^2}{h_b^2} + \frac{d_c^2}{h_c^2} \ge \frac{1}{3},$$

where d_a, d_b, d_c are the distances from P to the sides BC, CA and AB respectively.

Solution to part (a) by Scott H. Brown, Montgomery, AL.

Suppose P is any point inside triangle ABC. Let AP, BP, and CP be the line segments whose distances from the vertices are x, y, and z respectively. Let AP, BP, and CP intersect the sides BC, CA, and AB, at points L, M, and N respectively. Denote PL, PM, and PN by u, v, and w respectively.
In reference [1] it is shown that

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} \ge 6, \qquad (1)$$

with equality holding only if P in the centroid of triangle ABC.
Considering the heights h_a, h_b, and h_c, and the distances respectively to the sides from P as d_a, d_b, and d_c in terms of u, v, w, x, y, and z gives:

$$\frac{h_a}{d_a} = \frac{x+u}{u}, \qquad \frac{h_b}{d_b} = \frac{y+v}{v}, \qquad \frac{h_c}{d_c} = \frac{z+w}{w}. \qquad (2)$$

Applying inequality (1) gives:

$$\frac{h_a}{d_a} + \frac{h_b}{d_b} + \frac{h_c}{d_c} \ge 9,$$

with equality holding only if P is the centroid of triangle ABC.

Reference [1]. Some Inequalities For A Triangle, L. Carlitz,
American Mathematical Monthly, 1964, pp. 881-885.

Solution to part (b) by the proposers.

For the triangles BPC, APC, APB we have,

$$\begin{aligned}
[BPC] &= d_a \times \frac{BC}{2} = \frac{d_a}{h_a} \times \frac{h_a\, BC}{2} = \frac{d_a}{h_a} \times [ABC] \\
[APC] &= d_b \times \frac{AC}{2} = \frac{d_b}{h_b} \times \frac{h_b\, AC}{2} = \frac{d_b}{h_b} \times [ABC] \\
[APB] &= d_c \times \frac{AB}{2} = \frac{d_c}{h_c} \times \frac{h_c\, ABC}{2} = \frac{d_c}{h_c} \times [ABC]
\end{aligned}$$

Adding up the preceding expressions yields,

$$\left(\frac{d_a}{h_a} + \frac{d_b}{h_b} + \frac{d_c}{h_c}\right)[ABC] = [ABC]$$

and

$$\frac{d_a}{h_a} + \frac{d_b}{h_b} + \frac{d_c}{h_c} = 1$$

Applying AM-QM inequality, we get

$$\sqrt{\frac{\frac{d_a^2}{h_a^2} + \frac{d_b^2}{h_b^2} + \frac{d_c^2}{h_c^2}}{3}} \geq \frac{1}{3}\left(\frac{d_a}{h_a} + \frac{d_b}{h_b} + \frac{d_c}{h_c}\right) = \frac{1}{3}$$

from which the inequality claimed immediately follows. Finally, notice that equality holds when $d_a/h_a = d_b/h_b = d_c/h_c = 1/3$. That is, when $\triangle ABC$ is equilateral and P is its centroid.

- 4966: *Proposed by Kenneth Korbin, New York, NY.*

Solve:

$$16x + 30\sqrt{1-x^2} = 17\sqrt{1+x} + 17\sqrt{1-x}$$

with $0 < x < 1$.

Solution 1 by Elsie Campbell, Dionne Bailey, & Charles Diminnie, San Angelo, TX.

Let $x = \cos\theta$ where $\theta \in (0, \frac{\pi}{2})$. Then,

$$16x + 30\sqrt{1-x^2} = 17\sqrt{1+x} + 17\sqrt{1-x}$$

becomes

$$\begin{aligned}
16\cos\theta + 30\sqrt{1-\cos^2\theta} &= 17\sqrt{1+\cos\theta} + 17\sqrt{1-\cos\theta} \\
&= 17\sqrt{2}\left(\sqrt{\frac{1+\cos\theta}{2}} + \sqrt{\frac{1-\cos\theta}{2}}\right) \\
&= 34\left(\frac{1}{\sqrt{2}}\cos\frac{\theta}{2} + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\right) \\
&= 34\left(\cos\frac{\pi}{4}\cos\frac{\theta}{2} + \sin\frac{\pi}{4}\sin\frac{\theta}{2}\right) \\
&= 34\cos(\frac{\pi}{4} - \frac{\theta}{2}). \qquad (1)
\end{aligned}$$

Let $\cos\theta_0 = \frac{8}{17}$. Then by (1),

$$\begin{aligned}
\cos(\frac{\pi}{4} - \frac{\theta}{2}) &= \frac{8}{17}\cos\theta + \frac{15}{17}\sin\theta \\
&= \cos\theta_0\cos\theta + \sin\theta_0\sin\theta \\
&= \cos(\theta_0 - \theta).
\end{aligned}$$

Therefore,

$$\begin{array}{rrcl}
 & \theta_0 - \theta & = & \frac{\pi}{4} - \frac{\theta}{2} \\
\Rightarrow & \theta & = & 2\theta_0 - \frac{\pi}{2} \\
\Rightarrow & x & = & \frac{240}{289}
\end{array}
\quad \text{or} \quad
\begin{array}{rrcl}
 & \theta_0 - \theta & = & -(\frac{\pi}{4} - \frac{\theta}{2}) \\
\Rightarrow & \theta & = & \frac{2}{3}\theta_0 + \frac{\pi}{6} \\
\Rightarrow & x & = & \cos(\frac{2}{3}\cos^{-1}\frac{8}{17} + \frac{\pi}{6}).
\end{array}$$

Remark: This solution is an adaptation of the solution on pp.13-14 from *Mathematical Miniatures* by Savchev and Andreescu.

Solution 2 by Brian D. Beasley, Clinton, SC.

Since $0 < x < 1$, each side of the given equation will be positive, so we may square both sides without introducing any extraneous solutions. After simplifying, this yields

$$(480x - 289)\sqrt{1-x^2} = 161(2x^2 - 1).$$

For each side of this equation to have the same sign (or zero), we require $x \in (0, 289/480] \cup [\sqrt{2}/2, 1)$. We now square again, checking for actual as well as extraneous solutions. This produces

$$(1156x^3 - 867x + 240)(289x - 240) = 0,$$

so one potential solution is $x = 240/289$. The cubic formula yields three more, namely

$$x \in \{-\cos(\frac{1}{3}\cos^{-1}(\frac{240}{289})), \sin(\frac{1}{3}\sin^{-1}(\frac{240}{289})), \cos(\frac{1}{3}\cos^{-1}(-\frac{240}{289}))\}.$$

Of these four values, only two are in $x \in (0, 289/480] \cup [\sqrt{2}/2, 1)$:

$$x = \frac{240}{289} \quad \text{and} \quad x = \sin(\frac{1}{3}\sin^{-1}(\frac{240}{289})).$$

Addendum. The given equation generalizes nicely to

$$2ax + 2b\sqrt{1-x^2} = c\sqrt{1+x} + c\sqrt{1-x},$$

where $a^2 + b^2 = c^2$ with $a < b$. The technique outlined above produces

$$(4c^2x^3 - 3c^2x + 2ab)(c^2x - 2ab) = 0,$$

so one solution (which checks in the original equation) is $x = 2ab/c^2$. Another solution (does it always check in the original equation?) is $x = \sin(\frac{1}{3}\sin^{-1}(\frac{2ab}{c^2}))$, which is connected to the right triangle with side lengths $(b^2 - a^2, 2ab, c^2)$ in the following way:

If we let 3θ be the angle opposite the side of length $2ab$ in this triangle, then we have $2ab/c^2 = \sin(3\theta) = -4\sin^3\theta + 3\sin\theta$, which brings us right back to $4c^2x^3 - 3c^2x + 2ab = 0$ for $x = \sin\theta$.

Similarly, we may show that the other two solutions are $x = -\cos(\frac{1}{3}\cos^{-1}(\frac{2ab}{c^2}))$ and $x = \cos(\frac{1}{3}\cos^{-1}(-\frac{2ab}{c^2}))$; the first of these is never in $(0,1)$, but will the second ever be a solution of the original equation?

Also solved by John Boncek, Montgomery, AL; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4967: *Proposed by Kenneth Korbin, New York, NY.*

Given equilateral triangle ABC with an interior point P such that $\overline{\text{AP}}^2 + \overline{\text{BP}}^2 = \overline{\text{CP}}^2$, and with an exterior point Q such that $\overline{\text{AQ}}^2 + \overline{\text{BQ}}^2 = \overline{\text{CQ}}^2$, where points C, P, and Q are in a line. Find the lengths of $\overline{\text{AQ}}$ and $\overline{\text{BQ}}$ if $\overline{\text{AP}} = \sqrt{21}$ and $\overline{\text{BP}} = \sqrt{28}$.

Solution by Paul M. Harms, North Newton, KS.

Put the equilateral triangle on a coordinate system with A at $(0,0)$, B at $(a, \sqrt{3}a)$ and C at $(2a, 0)$ where $a > 0$. The point P is at the intersection of the circles

$$x^2 + y^2 = 21$$

$$
\begin{aligned}
(x-a)^2+(y-\sqrt{3}a)^2 &= 28 \text{ and} \\
(x-2a)^2+y^2 &= 28+21=49.
\end{aligned}
$$

Using $x^2+y^2=21$ in the last two circles we obtain

$$
\begin{aligned}
-2ax-2\sqrt{3}ay+4a^2=28-21 &= 7 \text{ and} \\
-4ax+4a^2 &= 49-21=28.
\end{aligned}
$$

From the last equation $x=\dfrac{a^2-7}{a}$ and, using the linear equation, we get $y=\dfrac{2a^2+7}{2\sqrt{3}a}$. Putting these x, y values into $x^2+y^2=21$ yields the quadratic in a^2, $16a^4-392a^2+637=0$. From this equation $a^2=22.75$ or $a^2=1.75$. From the distances given in the problem, a^2 must be 22.75. The coordinates of P are $x=3.3021$ and $y=3.1774$. The line through C and P is $y=-0.5094x+4.85965$.

Let Q have coordinates (x_1, y_1). An equation for $\overline{AQ}^2+\overline{BQ}^2=\overline{CQ}^2$ can be found using the coordinates $Q(x_1,y_1), A(0,0), B(4.7697, 8.2614)$, and $C(9.5394, 0)$. An equation is

$$x_1^2+y_1^2+(x_1-4.7697)^2+(y_1-8.2614)^2=(x_1-9.5394)^2+y_1^2.$$

Simplifying and replacing y_1 by $-0.5094x_1+4.85965$ yields the quadratic equation $1.2595x_1^2+13.0052x_1-56.6783=0$. In order that Q is exterior to the triangle we need the solution $x_1=-13.6277$. Then $y_1=-0.5094x_1+4.85965=11.8020$. The distance from A to Q is $\sqrt{325}=18.0278$ and the distance from B to Q is $\sqrt{351}=18.7350$.

Also solved by Zhonghong Jiang, New York, NY, and the proposer.

- **4968:** *Proposed by Kenneth Korbin, New York, NY.*

Find two quadruples of positive integers (a, b, c, d) such that

$$\frac{a+i}{a-i}\cdot\frac{b+i}{b-i}\cdot\frac{c+i}{c-i}\cdot\frac{d+i}{d-i}=\frac{a-i}{a+i}\cdot\frac{b-i}{b+i}\cdot\frac{c-i}{c+i}\cdot\frac{d-i}{d+i}$$

with $a<b<c<d$ and $i=\sqrt{-1}$.

Solution 1 by Brian D. Beasley, Clinton, SC.

We need $((a+i)(b+i)(c+i)(d+i))^2=((a-i)(b-i)(c-i)(d-i))^2$, so

$$(a+i)(b+i)(c+i)(d+i)=\pm(a-i)(b-i)(c-i)(d-i).$$

Then either

$$(ab-1)(c+d)+(a+b)(cd-1)=0 \quad \text{or} \quad (ab-1)(cd-1)-(a+b)(c+d)=0.$$

But $(ab-1)(c+d)>0$ and $(a+b)(cd-1)>0$, so the first case cannot occur. In the second case, since $d=(ab+ac+bc-1)/(abc-a-b-c)>0$, we have $abc>a+b+c$. Then $d\geq 4$ implies

$$3\leq c\leq\frac{ab+4a+4b-1}{4ab-a-b-4},$$

where we note that $1\leq a<b$ implies $4ab>a+b+4$. Thus $2\leq b\leq(7a+11)/(11a-7)$, so $a\leq 5/3$. Thus $a=1$, which yields $b\in\{2,3,4\}$.

If $(a,b)=(1,2)$, then $d=(3c+1)/(c-3)$, so $c<d$ forces $c\in\{4,5,6\}$. Only $c\in\{4,5\}$ will yield integral values for d, producing the two solutions $(1,2,4,13)$ and $(1,2,5,8)$ for (a,b,c,d).

If $(a,b)=(1,3)$, then $d=(2c+1)/(c-2)$, so $3<c<d$ forces $c=4$. But this yields $d=9/2$.

If $(a,b)=(1,4)$, then $d=(5c+3)/(3c-5)$, but $4<c<d$ forces the contradiction $c\le 3$.

Hence the only two solutions for (a,b,c,d) are $(1,2,4,13)$ and $(1,2,5,8)$.

Solution 2 by Dionne Bailey, Elsie Campbell, & Charles Diminnie, San Angelo, TX.

By using the following properties of complex numbers,

$$(\overline{z_1 z_2})=\bar{z}_1\bar{z}_2,\quad \overline{\left(\frac{z_1}{z_2}\right)}=\frac{\overline{z_1}}{\overline{z_2}},\quad \overline{\overline{z}}=z,$$

we see that the left and right sides of the equation are conjugates and hence, the equation reduces to

$$Im\left(\frac{a+i}{a-i}\cdot\frac{b+i}{b-i}\cdot\frac{c+i}{c-i}\cdot\frac{d+i}{d-i}\right)=0. \tag{1}$$

If $z=(a+i)(b+i)(c+i)(d+i)=A+Bi$, then (1) becomes

$$Im\left(\frac{z}{\overline{z}}\right)=0,$$

which reduces to $AB=0$ or equivalently, $A=0$ or $B=0$. With some labor, we get

$$\begin{array}{rcl} A &=& 1-(ab+ac+ad+bc+bd+cd)+abcd \\ &=& (ab-1)(cd-1)-(a+b)(c+d) \text{ and} \\ B &=& (abc+abd+acd+bcd)-(a+b+c+d) \\ &=& (a+d)(bc-1)+(b+c)(ad-1). \end{array}$$

Therefore, a,b,c,d must satisfy

$$(ab-1)(cd-1)=(a+b)(c+d) \tag{2}$$

or

$$(a+d)(bc-1)+(b+c)(ad-1)=0. \tag{3}$$

Immediately, the condition $1\le a<b<c<d$ rules out equation (3) and we may restrict our attention to equation (2).

Since $c\ge 3$ and $d\ge 4$, we obtain

$$(cd-1)-(c+d)=(c-1)(d-1)-2>0$$

and hence,

$$c+d<cd-1.$$

Using this and the fact that $(ab-1)>0$, equation (2) implies that

$$(ab-1)(c+d)<(ab-1)(cd-1)=(a+b)(c+d),$$

or

$$ab-1<a+b.$$

This in turn implies that

$$0\le(a-1)(b-1)<2.$$

Then, since $1 \leq a < b$, we must have $a = 1$ and equation (2) becomes

$$(b-1)(cd-1) = (b+1)(c+d). \tag{4}$$

Finally, $b \geq 2$ implies that

$$cd - 1 = \frac{b+1}{b-1}(c+d) = \left(1 + \frac{2}{b-1}\right)(c+d) \leq 3(c+d)$$

or

$$0 \leq (c-3)(d-3) \leq 10. \tag{5}$$

To complete the solution, we solve each of the 11 possibilities presented by (5) and then substitute back into (4) to solve for the remaining variable. It turns out that the only situation which yields feasible answers for b, c, d is the case where $(c-3)(d-3) = 10$. We show this case and two others to indicate the reasoning applied.

Case 1. If

$$(c-3)(d-3) = 0,$$

then since $1 = a < b < c < d$, we must have $c = 3$ and $b = 2$. When these are substituted into (4), we get

$$3d - 1 = 3(3+d)$$

which is impossible.

Case 2. If

$$(c-3)(d-3) = 6,$$

then since $c < d$, we must have $c - 3 = 1$, $d - 3 = 6$ or $c - 3 = 2$, $d - 3 = 3$. These yield $c = 4$, $d = 9$ or $c = 5$, $d = 6$. However, neither pair gives an integral answer for b when these are substituted into (4).

Case 3. If

$$(c-3)(d-3) = 10,$$

then since $c < d$, we must have $c - 3 = 1$, $d - 3 = 10$ or $c - 3 = 2$, $d - 3 = 5$. These yield $c = 4$, $d = 13$ or $c = 5$, $d = 8$. When substituted into (4), both pairs give the answer $b = 2$.

Therefore, the only solutions for which a, b, c, d are integers, with $1 \leq a < b < c < d$, are $(a, b, c, d) = (1, 2, 4, 13)$ or $(1, 2, 5, 8)$.

Also solved by Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Raul A. Simon, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

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- 4969: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{1}{a^2\left(\dfrac{1}{a} + \dfrac{1}{c}\right)} + \frac{1}{b^2\left(\dfrac{1}{b} + \dfrac{1}{a}\right)} + \frac{1}{c^2\left(\dfrac{1}{c} + \dfrac{1}{b}\right)} \geq \frac{3}{2}$$

Solution by Kenneth Korbin, New York, NY.

Let $x = \dfrac{1}{a}, y = \dfrac{1}{b}, z = \dfrac{1}{c}$. Then, $K = \dfrac{x^2}{x+z} + \dfrac{y^2}{y+x} + \dfrac{z^2}{z+y}$.
Let $U_1 = \dfrac{x}{\sqrt{x+z}}, U_2 = \dfrac{y}{\sqrt{y+x}}, U_3 = \dfrac{z}{\sqrt{z+y}}$. Then, $K = (U_1)^2 + (U_2)^2 + (U_3)^2$.
Let $V_1 = \sqrt{x+z}, V_2 = \sqrt{y+x}, V_3 = \sqrt{z+y}$. Then, by the Cauchy inequality,

$$\begin{aligned} K &= (U_1)^2 + (U_2)^2 + (U_3)^2 \\ &\geq \frac{(U_1V_1 + U_2V_2 + U_3V_3)^2}{(V_1)^2 + (V_2)^2 + (V_3)^2} \\ &= \frac{(x+y+z+)^2}{2(x+y+z)} = \frac{x+y+z}{2} \end{aligned}$$

Then, by the AM-GM inequality,

$$\begin{aligned} K &\geq \frac{x+y+z}{2} \\ &\geq \frac{1}{2}(3)(\sqrt[3]{xyz}) \\ &= \frac{3}{2}(1) = \frac{3}{2}. \end{aligned}$$

Note: $abc = 1$ implies $xyz = 1$.

Comment by editor: John Boncek of Montgomery, AL noted that this problem is a variant of an exercise given in Andreescu and Enescu's *Mathematical Olympiad Treasures,* (Birkhauser, 2004, problem 6, page 108.)

Also solved by John Boncek; David E. Manes, Oneonta, NY, and the proposer.

- 4970: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*
Let $f : [0,1] \longrightarrow \mathbf{R}$ be a contintuous convex function. Prove that

$$\frac{3}{4}\int_0^{1/5} f(t)dt + \frac{1}{8}\int_0^{2/5} f(t)dt \geq \frac{4}{5}\int_0^{1/4} f(t)dt.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China.

By the convexity of f we have

$$\frac{3}{4}f\left(\frac{s}{5}\right) + \frac{1}{4}f\left(\frac{2s}{5}\right) \geq f\left((\frac{3}{4})(\frac{s}{5}) + (\frac{1}{4})(\frac{2s}{5})\right) = f\left(\frac{s}{4}\right)$$

for $0 \leq s \leq 1$. Hence,

$$\frac{3}{4}\int_0^1 f\left(\frac{s}{5}\right)ds + \frac{1}{4}\int_0^1 f\left(\frac{2s}{5}\right)ds \;\geq\; \int_0^1 f\left(\frac{s}{4}\right)ds.$$

By substituting $s = 5t$ in the first integral, $s = \dfrac{5t}{2}$ in the second at the left and $s = 4t$ in the integral at the right, we obtain the inequality of the problem.

Solution 2 by David Stone and John Hawkins, Statesboro, GA.

Note 1. Consider the behavior in the extreme case: if f is a linear function, then equality holds:

$$\frac{3}{4}\int_0^{1/5}(mt+b)dt+\frac{1}{8}\int_0^{2/5}(mt+b)dt=\frac{3}{4}\left[\frac{m}{2}\left(\frac{1}{5}\right)^2+b\frac{1}{5}\right]+\frac{1}{8}\left[\frac{m}{2}\left(\frac{2}{5}\right)^2+b\frac{2}{5}\right]=\frac{1}{40}m+\frac{1}{5}b,$$

and

$$\frac{4}{5}\int_0^{1/4}(mt+b)dt=\frac{4}{5}\left[\frac{m}{2}\left(\frac{1}{4}\right)^2+b\frac{1}{4}\right]=\frac{1}{40}m+\frac{1}{5}b.$$

We rewrite the inequality in an equivalent form by clearing fractions and splitting the integrals so that they are taken over non-overlapping intervals:

$$\begin{aligned}
&\frac{3}{4}\int_0^{1/5}f(t)dt+\frac{1}{8}\int_0^{2/5}f(t)dt\ge\frac{4}{5}\int_0^{1/4}f(t)dt \iff\\
&30\int_0^{1/5}f(t)dt+5\left[\int_0^{1/5}f(t)dt+\int_{1/5}^{1/4}f(t)dt+\int_{1/4}^{2/5}f(t)dt\right]\ge 32\left[\int_0^{1/5}f(t)dt+\int_{1/5}^{1/4}f(t)dt\right] \iff\\
&3\int_0^{1/5}f(t)dt+5\int_{1/4}^{2/5}f(t)dt\ge 27\int_{1/5}^{1/4}f(t)dt. \qquad (1)
\end{aligned}$$

So we see that the interval of interest, $\left[0,\frac{2}{5}\right]$, has been partitioned into three subintervals $\left[0,\frac{1}{5}\right]$, $\left[\frac{1}{5},\frac{1}{4}\right]$ and $\left[\frac{1}{4},\frac{2}{5}\right]$.

Consider the secant line through the two points $\left(\frac{1}{5},f\left(\frac{1}{5}\right)\right)$ and $\left(\frac{1}{4},f\left(\frac{1}{4}\right)\right)$. The linear function giving this line is $s(t)=20\left[f\left(\frac{1}{4}\right)-f\left(\frac{1}{5}\right)\right]t+\left[5f\left(\frac{1}{5}\right)-4f\left(\frac{1}{4}\right)\right]$. It is straightforward to use the convexity condition to show that this line lies above $f(t)$ on the middle interval$\left[\frac{1}{5}\right]$, and lies below $f(t)$ on the outside intervals $\left[0,\frac{1}{5}\right]$ and $\left[\frac{1}{4},\frac{2}{5}\right]$. That is

$$\begin{aligned}
s(t) &\ge f(t) \text{ on } \left[\frac{1}{5},\frac{1}{4}\right] \text{ and} \qquad (2)\\
s(t) &\le f(t) \text{ on } \left[0,\frac{1}{5}\right], \text{ and } \left[\frac{1}{4},\frac{2}{5}\right] \qquad (3).
\end{aligned}$$

Considering the sides of (1),

$$3\int_0^{1/5}f(t)dt+5\int_{1/4}^{2/5}f(t)dt\ge 3\int_0^{1/5}s(t)dt+5\int_{1/4}^{2/5}s(t)dt \text{ by (3).}$$

and

$$3\int_0^{1/5}s(t)dt+5\int_{1/4}^{2/5}s(t)dt=27\int_{1/5}^{1/4}s(t)dt \text{ by (Note 1),}$$

and

$$27\int_{1/5}^{1/4}s(t)dt\ge 27\int_{1/5}^{1/4}f(t)dt \text{ by (2).}$$

Therefore (1) is true.

Also solved by John Boncek, Montgomery, AL and the proposers.

- 4971: *Proposed by Howard Sporn, Great Neck, NY and Michael Brozinsky, Central Islip, NY.*

  Let $m \geq 2$ be a positive integer and let $1 \leq x < y$. Prove:

$$x^m - (x-1)^m < y^m - (y-1)^m.$$

Solution 1 by Brian D. Beasley, Clinton, SC.

We let $f(x) = x^m - (x-1)^m$ for $x \geq 1$ and show that f is strictly increasing on $[1, \infty)$. Since $f'(x) = mx^{m-1} - m(x-1)^{m-1}$, we have $f'(x) > 0$ if and only if $x^{m-1} > (x-1)^{m-1}$. Since $x \geq 1$ and $m \geq 2$, this latter inequality holds, so we are done.

Solution 2 by Matt DeLong, Upland, IN.

Let $X = x-1$ and $Y = y-1$. Then $0 \leq X < Y$, $x = X+1$, and $y = Y+1$. Expanding $(X+1)^m - X^m$ and $(Y+1)^m - Y^m$ we see that

$$(X+1)^m - X^m = mX^{m-1} + \frac{m(m-1)}{2}X^{m-2} + \cdots + mX + 1$$

and

$$(Y+1)^m - Y^m = mY^{m-1} + \frac{m(m-1)}{2}Y^{m-2} + \cdots + mY + 1.$$

Since $0 \leq X < Y$, we can compare these two sums term-by-term and conclude that each term involving Y is larger than the corresponding term involving X. Therefore,

$$(X+1)^m - X^m < (Y+1)^m - Y^m.$$

Since $x = X+1$ and $y = Y+1$, we have shown that

$$x^m - (x-1)^m < y^m - (y-1)^m.$$

.

Solution 3 by José Luis Díaz-Barrero, Barcelona, Spain.

We will argue by induction. The case when $m = 2$ trivially holds because $x^2 - (x-1)^2 = 2x - 1 < 2y - 1 = y^2 - (y-1)^2$. Suppose that

$$x^m - (x-1)^m < y^m - (y-1)^m$$

holds and we have to see that

$$x^{m+1} - (x-1)^{m+1} < y^{m+1} - (y-1)^{m+1}$$

holds. In fact, multiplying by $m+1$ both sides of $x^m - (x-1)^m < y^m - (y-1)^m$ yields

$$(m+1)(x^m - (x-1)^m) < (m+1)(y^m - (y-1)^m)$$

and

$$\int_1^x (m+1)(x^m - (x-1)^m)\, dx < \int_1^y (m+1)(y^m - (y-1)^m)\, dy$$

from which immediately follows

$$x^{m+1} - (x-1)^{m+1} < y^{m+1} - (y-1)^{m+1}$$

Therefore, by the PMI the statement is proved and we are done.

Solution 4 by Kenneth Korbin, New York, NY.

Let $m \geq 2$ be a positive integer, and let $1 \leq x < y$. Then,

$$\begin{array}{rcl} (y-1)^m & < & y^m, \text{ and} \\ (y-1)^{m-1}(x-1) & < & y^{m-1}(x), \text{ and} \\ (y-1)^{m-2}(x-1)^2 & < & y^{m-2}(x^2), \text{ and} \\ & . & \\ & . & \\ & . & \\ y^0 = 1 & \leq & x^m. \end{array}$$

Adding gives

$$\left[(y-1)^m + (y-1)^{m-1}(x-1) + \cdots + 1\right] < \left[y^m + y^{m-1}x + y^{m-2}x^2 + \cdots + x^m\right].$$

Multiplying both sides by $[(y-1)-(x-1)] = [y-x]$ gives

$$(y-1)^m - (x-1)^m < y^m - x^m.$$

Therefore

$$x^m - (x-1)^m < y^m - (y-1)^m.$$

Also solved by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; Raul A. Simon, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA; various teams of students at Taylor University in Upland, IN:

Bethany Ballard, Nicole Gottier, and Jessica Heil;
Mandy Isaacson, Julia Temple, and Adrienne Ramsay;
Jeremy Erickson, Matthew Russell, and Chad Mangum;
Seth Bird, Jim Moore, and Jonathan Schrock;

and the proposers.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before February 15, 2008

- 4990: *Proposed by Kenneth Korbin, New York, NY.*

  Solve
  $$40x + 42\sqrt{1-x^2} = 29\sqrt{1+x} + 29\sqrt{1-x}$$
  with $0 < x < 1$.

- 4991: *Proposed by Kenneth Korbin, New York, NY.*

  Find six triples of positive integers (a, b, c) such that
  $$\frac{9}{a} + \frac{a}{b} + \frac{b}{9} = c.$$

- 4992: *Proposed by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie, San Angelo, TX.*

  A closed circular cone has integral values for its height and base radius. Find all possible values for its dimensions if its volume V and its total area (including its circular base) A satisfy $V = 2A$.

- 4993: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all real solutions of the equation
  $$126x^7 - 127x^6 + 1 = 0.$$

- 4994: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b, c be three nonzero complex numbers lying on the circle $C = \{z \in \mathbf{C} : |\mathrm{z}| = \mathrm{r}\}$. Prove that the roots of the equation $az^2 + bz + c = 0$ lie in the ring shaped region $D = \left\{ z \in \mathbf{C} : \dfrac{1-\sqrt{5}}{2} \leq |\mathrm{z}| \leq \dfrac{1+\sqrt{5}}{2} \right\}$.

- 4995: *Proposed by K. S. Bhanu and M. N. Deshpande, Nagpur, India.*

  Let A be a triangular array $a_{i,j}$ where $i = 1, 2, \cdots,$ and $j = 0, 1, 2, \cdots, i$. Let

$$a_{1,0} = 1, \ a_{1,1} = 2, \text{ and } a_{i,0} = T(i+1) - 2 \text{ for } i = 2, 3, 4, \cdots,$$

where $T(i+1) = (i+1)(i+2)/2$, the usual triangular numbers. Furthermore, let $a_{i,j+1} - a_{i,j} = j + 1$ for all j. Thus, the array will look like this:

$$\begin{array}{c} 1 \quad 2 \\ 4 \quad 5 \quad 7 \\ 8 \quad 9 \quad 11 \quad 14 \\ 13 \quad 14 \quad 16 \quad 19 \quad 23 \\ 19 \quad 20 \quad 22 \quad 25 \quad 29 \quad 34 \end{array}$$

Show that for every pair (i, j), $4a_{i,j} + 9$ is the sum of two perfect squares.

Solutions

- 4972:*Proposed by Kenneth Korbin, New York, NY.*

  Find the length of the side of equilateral triangle ABC if it has a cevian $\overline{\mathrm{CD}}$ such that

$$\overline{\mathrm{AD}} = x, \quad \overline{\mathrm{BD}} = x + 1 \quad \overline{\mathrm{CD}} = \sqrt{y}$$

where x and y are positive integers with $20 < x < 120$.

Solution by Kee-Wai Lau, Hong Kong, China.

Applying the cosine formula to triangle CAD, we obtain

$$\overline{CD}^2 = \overline{AD}^2 + \overline{AC}^2 - 2\overline{AD} \cdot \overline{AC} \cos 60^o,$$

or

$$\begin{aligned} \left(\sqrt{y}\right)^2 &= x^2 + (2x+1)^2 - 2x(2x+1)\cos 60^o \\ y &= 3x^2 + 3x + 1. \end{aligned}$$

For $20 < x < 120$, we find using a calculator that y is the square of a positive integer if $x = 104, y = 32761$. Hence the length of the side of equilateral triangle ABC is 209.

Comments:

1) Scott H. Brown, Montgomery, AL.

The list of pairs (x, y) that satisfy the equation $y = 3x^2 + 3x + 1$ is so large I will not attempt to name each pair...numerous triangles with the given conditions can be found.

2) David Stone and John Hawkins, Statesboro, GA.

The restriction on x seems artificial–every x produces a triangle. In fact, if we require the cevian length to be an integer, this becomes a Pell's Equation problem and we can generate nice solutions recursively in the usual fashion. The first few for $x,\ s = 2x+1,\ y = 3x^2+3x+1,\ \&$ cevian $= \sqrt{\text{y}}$ are:

7	15	169	13
104	209	32761	181
1455	2911	6355441	2521
20272	40545	1232922769	35113

Also solved by Peter E. Liley, Lafayette, IN, and the proposer.

- 4973: *Proposed by Kenneth Korbin, New York, NY.*

Find the area of trapezoid ABCD if it is inscribed in a circle with radius R=2, and if it has base $\overline{\text{AB}} = 1$ and $\angle\text{ACD} = 60^o$.

Solution by David E. Manes, Oneonta, NY.

The area A of the trapezoid is given by $A = \dfrac{3\sqrt{3}}{8}\left(15+\sqrt{5}\right)$.

Since the trapezoid is cyclic, it is isosceles so that $AD = BC$. Note that $\angle ACD = 60^o \Rightarrow \angle CAB = 60^o$ since alternate interior angles of a transversal intersecting two parallel lines are congruent. Therefore, from the law of sines in triangle ABC, $\dfrac{BC}{\sin 60^o} = 2R$ or $BC = 2\sqrt{3}$. Using the law of cosines in triangle ABC,

$$BC^2 = 1 + AC^2 - 2AC\cdot\cos 60^o,\ \text{ or } \text{AC}^2 - \text{AC} - 11 = 0.$$

Thus, AC is the positive root of this equation so that $AC = \dfrac{1+3\sqrt{5}}{2}$. Similarly, using the law of cosines in triangle ACD and recalling that $AD = BC$, one obtains

$$AD^2 = AC^2 + DC^2 - 2\cdot AC\cdot DC\cdot\cos 60^o$$

or $\text{DC}^2 - \left(\dfrac{1+3\sqrt{5}}{2}\right)DC + \dfrac{-1+3\sqrt{5}}{2} = 0$. Noting that $DC > 2$ and $\sqrt{6-2\sqrt{5}} = \sqrt{(1-\sqrt{5})^2} = \sqrt{5}-1$, it follows that $DC = 3\sqrt{5}-1$. Finally, let H be the point on line segment $\overline{DC}$ such that $\overline{AH}$ is perpendicular to $\overline{DC}$. Then the height h of the trapezoid is given by $h = AC\cdot\sin 60^o = \dfrac{\sqrt{3}}{4}\left(1+3\sqrt{5}\right)$. Hence,

$$A = \frac{1}{2}\left(AB+DC\right)\cdot h = \frac{1}{2}\left(1+3\sqrt{5}-1\right)\frac{\sqrt{3}}{4}\left(1+3\sqrt{5}\right) = \frac{3\sqrt{3}}{8}\left(15+\sqrt{5}\right).$$

Also solved by Robert Anderson, Gino Mizusawa, and Jahangeer Kholdi (jointly), Portsmouth, VA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Zhonghong Jiang, NY, NY; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4974: *Proposed by Kenneth Korbin, New York, NY.*

  A convex cyclic hexagon has sides a, a, a, b, b, and b. Express the values of the circumradius and the area of the hexagon in terms of a and b.

  Solution by Kee-Wai Lau, Hong Cong, China.

  We show that the circumradius R is $\sqrt{\dfrac{a^2+ab+b^2}{3}}$ and the area A of the hexagon is $\dfrac{\sqrt{3}(a^2+4ab+b^2)}{4}$.

  Denote the angle subtended by side a and side b at the center of the circumcircle respectively by θ and ϕ. Since $3\theta + 3\phi = 360^o$ so $\theta = 120 - \phi$ and

  $$\cos\theta = \cos(120^o - \phi) = \frac{-\cos\phi + \sqrt{3}\sin\phi}{2}. \text{ Hence,}$$

  $$(2\cos\theta + \cos\phi)^2 = 3(1-\cos^2\phi) \text{ or } 4\cos^2\theta + 4\cos\theta\cos\phi + 4\cos^2\phi - 3 = 0.$$

  Now by the cosine formula $\cos\theta = \dfrac{2R^2 - a^2}{2R^2}$ and $\cos\phi = \dfrac{2R^2-b^2}{2R^2}$.

  Therefore,

  $$\begin{aligned}(2R^2-a^2)^2 + (2R^2-a^2)(2R^2-b^2) + (2R^2-b^2)^2 - 3R^4 &= 0 \text{ or}\\ 9R^4 - 6(a^2+b^2)R^2 + a^4 + a^2b^2 + b^4 &= 0.\end{aligned}$$

  Solving the equation we obtain $R^2 = \dfrac{a^2+ab+b^2}{3}$ or $R^2 = \dfrac{a^2-ab+b^2}{3}$. The latter result is rejected because if not, then for $a = b$, we have $\cos\theta = \cos\phi < 0$ so that $\theta + \phi > 180^o$, which is not true. Hence, $R = \sqrt{\dfrac{a^2+ab+b^2}{3}}$.

  To find A, we need to find the area of the triangles with sides R, R, a and R, R, b. The heights to bases a and b are respectively $\dfrac{\sqrt{4R^2-a^2}}{2} = \dfrac{\sqrt{3}(a+2b)}{6}$ and $\dfrac{\sqrt{4R^2-b^2}}{2} = \dfrac{\sqrt{3}(2a+b)}{6}$. Hence the area of the hexagon equals

  $$3\left(\frac{\sqrt{3}a(a+2b)}{12} + \frac{\sqrt{3}b(2a+b)}{12}\right) = \frac{\sqrt{3}}{4}\left(a^2+4ab+b^2\right) \text{ as claimed.}$$

  Also solved by Matt DeLong, Upland, IN; Paul M. Harms, North Newton, KS; Zhonghong Jiang, NY, NY; David E. Manes, Oneonta, NY; M. N. Deshpande, Nagpur, India; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA; Jonathan Schrock, Seth Bird, and Jim Moore (jointly, students at Taylor University), Upland, IN; David Wilson, Winston-Salem, NC, and the proposer.

- 4975: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Solve in R the following system of equations

$$\left.\begin{array}{c} 2x_1 = 3x_2\ \sqrt{1+x_3^2} \\ 2x_2 = 3x_3\ \sqrt{1+x_4^2} \\ \cdots\cdots \\ 2x_n = 3x_1\ \sqrt{1+x_2^2} \end{array}\right\}$$

Solution by David Stone and John Hawkins, Statesboro, GA.

Squaring each equation and summing, we have

$$4(x_1^2+x_2^2+x_3^2+\cdots+x_n^2) = 9(x_1^2+x_2^2+x_3^2+\cdots+x_n^2)+9(x_1^2x_2^2+x_2^2x_3^2+x_3^2x_4^2+\cdots+x_{n-1}^2x_n^2).$$

So

$$0 = 5(x_1^2+x_2^2+x_3^2+\cdots+x_n^2)+9(x_1^2x_2^2+x_2^2x_3^2+x_3^2x_4^2+\cdots+x_{n-1}^2x_n^2).$$

Because these squares are non-negative and the sum is zero, each term on the right-hand side must indeed equal 0. Therefore $x_1 = x_2 = x_3 = \cdots = x_n = 0$.

Alternatively, we could multiply the equations to obtain

$$2^n x_1x_2x_3x_4\cdots x_n = 3^n x_1x_2x_3x_{4n}\sqrt{1+x_1^2}\sqrt{1+x_2^2}\cdots\sqrt{1+x_n^2}.$$

If all x_k are non-zero, we'll have $\left(\dfrac{2}{3}\right)^n = \sqrt{1+x_1^2}\sqrt{1+x_2^2}\cdots\sqrt{1+x_n^2}$. The term on the left is < 1, while each term on the right is > 1, so the product is > 1. Thus we have reached a contradiction, forcing all x_k to be zero.

Also solved by Bethany Ballard, Nicole Gottier, and Jessica Heil (jointly, students, Taylor University), Upland, IN; Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie, San Angelo, TX; Matt DeLong, Upland, IN; Paul M. Harms, North Newton, KS; Mandy Isaacson, Julia Temple, and Adrienne Ramsay (jointly, students, Taylor University), Upland, IN; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA, and the proposer.

- 4976: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers. Prove that

$$\frac{a^2+3b^2+9c^2}{bc}+\frac{b^2+3c^2+9a^2}{ca}+\frac{c^2+3a^2+9b^2}{ab} \ge 27.$$

Solution by Matt DeLong, Upland, IN.

In fact, I will prove that the sum is at least 39. Rewrite the sum

$$\frac{a^2+3b^2+9c^2}{bc}+\frac{b^2+3c^2+9a^2}{ca}+\frac{c^2+3a^2+9b^2}{ab} \text{ as}$$

$$\frac{a^2}{bc}+3\frac{b}{c}+9\frac{c}{b}+\frac{b^2}{ca}+3\frac{c}{a}+9\frac{a}{c}+\frac{c^2}{ab}+3\frac{a}{b}+9\frac{b}{a}.$$

Rearranging the terms gives

$$\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)+3\left(\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+\frac{a}{b}+\frac{b}{a}\right)+6\left(\frac{c}{b}+\frac{a}{c}+\frac{b}{a}\right)$$

Now, repeatedly apply the Arithmetic Mean-Geometric Mean inequality.

$$\begin{aligned}
\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab} &\geq 3\left(\frac{a^2b^2c^2}{bccaab}\right)^{1/3}=3\\
\frac{b}{c}+\frac{c}{b} &\geq 2\left(\frac{bc}{cb}\right)^{1/2}=2\\
\frac{c}{a}+\frac{a}{c} &\geq 2\left(\frac{ac}{ca}\right)^{1/2}=2\\
\frac{a}{b}+\frac{b}{a} &\geq 2\left(\frac{ab}{ba}\right)^{1/2}=2\\
\frac{c}{b}+\frac{a}{c}+\frac{b}{a} &\geq 3\left(\frac{cab}{bca}\right)^{1/3}=3.
\end{aligned}$$

Thus, we have

$$\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)+3\left(\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+\frac{a}{b}+\frac{b}{a}\right)+6\left(\frac{c}{b}+\frac{a}{c}+\frac{b}{a}\right)\geq 3+3(2+2+2)+6(3).$$

In other words

$$\frac{a^2+3b^2+9c^2}{bc}+\frac{b^2+3c^2+9a^2}{ca}+\frac{c^2+3a^2+9b^2}{ab}\geq 39$$

.

Also solved by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie, San Angelo, TX; Jeremy Erickson, Matthew Russell, and Chad Mangum (jointly, students, Taylor University), Upland, IN; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4977: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let $1<a<b$ be real numbers. Prove that for any $x_1,x_2,x_3\in[a,b]$ there exist $c\in(a,b)$ such that

$$\frac{1}{\log x_1}+\frac{1}{\log x_2}+\frac{1}{\log x_3}+\frac{3}{\log x_1x_2x_3}=\frac{4}{\log c}.$$

Solution by Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX .

Strictly speaking, the conclusion is incorrect as stated. If $a=x_1=x_2=x_3$, then

$$\frac{1}{\log x_1}+\frac{1}{\log x_2}+\frac{1}{\log x_3}+\frac{3}{\log x_1x_2x_3}=\frac{4}{\log a}.$$

Similarly,

$$\frac{1}{\log x_1}+\frac{1}{\log x_2}+\frac{1}{\log x_3}+\frac{3}{\log x_1x_2x_3}=\frac{4}{\log b}$$

when $b=x_1=x_2=x_3$.
The statement is true when $1<a\leq x_1\leq x_2\leq x_3\leq b$ with $x_1\neq x_3$. Since

$$\frac{3}{\log x_1x_2x_3}=\frac{3}{\log x_1+\log x_2+\log x_3},$$

then
$$\frac{4}{\log x_3} < \frac{1}{\log x_1} + \frac{1}{\log x_2} + \frac{1}{\log x_3} + \frac{3}{\log x_1x_2x_3} < \frac{4}{\log x_1}.$$

By the Intermediate Value Theorem, there exists $c \in (a, b)$ such that
$$\frac{1}{\log x_1} + \frac{1}{\log x_2} + \frac{1}{\log x_3} + \frac{3}{\log x_1x_2x_3} = \frac{4}{\log c}.$$

Solution 2 by Paul M. Harms, North Newton, KS.

Assume $x_1 < x_3$ with x_2 in the interval $[x_1, x_3]$. For $x > 1$, we note that $f(x) = \log(x)$ and $g(x) = 1/\log(x)$ are both continuous, one-to-one, positive functions with $f(x)$ strictly increasing and $g(x)$ strictly decreasing.

Consider
$$\frac{3}{\log(x_1x_2x_3)} = \frac{1}{\dfrac{\log(x_1) + \log(x_2) + \log(x_3)}{3}}.$$

The denominator is the average of the 3 log values which means this average value is between the extremes $\log x_1$ and $\log x_3$. Since $f(x)$ is one-to-one and continuous there is a value x_4 where $x_1 < x_4 < x_3$ and $\log x_4 = \dfrac{(\log x_1 + \log x_2 + \log x_3)}{3}$ with $\log x_4$ between $\log x_1$ and $\log x_3$.

The equation in the problem can now be written
$$\frac{\frac{1}{\log x_1} + \frac{1}{\log x_2} + \frac{1}{\log x_3} + \frac{1}{\log x_4}}{4} = \frac{1}{\log c} \text{ or}$$
$$\frac{g(x_1) + g(x_2) + g(x_3) + g(x_4)}{4} = \frac{1}{\log c}.$$

The average of the four $g(x)$ values is between the extremes $g(x_1)$ and $g(x_3)$. Since $g(x)$ is continuous and one-to-one there is a value $x = c$ such that
$$g(c) = \frac{1}{\log c} = \frac{g(x_1) + g(x_2) + g(x_3) + g(x_4)}{4}$$

where $x_1 < c < x_3$ and, thus, $a < c < b$.

Note that if $x_1 = x_2 = x_3$, then we obtain $c = x_1 = x_2 = x_3$. If we want $a < c < b$, then it appears that we need to keep x_1, x_2 and x_3 away from a and b when these three x-values are equal to each other.

Also solved by Michael Brozinsky, Central Islip, NY; Matt DeLong, Upland, IN; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
March 15, 2008

- 4996: *Proposed by Kenneth Korbin, New York, NY.*

  Simplify:
  $$\sum_{i=1}^{N}\binom{N}{i}\left(2^{i-1}\right)\left(1+3^{N-i}\right).$$

- 4997: *Proposed by Kenneth Korbin, New York, NY.*

  Three different triangles with integer-length sides all have the same perimeter P and all have the same area K.
  Find the dimensions of these triangles if $K = 420$.

- 4998: *Proposed by Jyoti P. Shiwalkar & M.N. Deshpande, Nagpur, India.*

  Let $A = [a_{i,j}]$, $i = 1, 2, \cdots$ and $j = 1, 2, \cdots, i$ be a triangular array satisfying the following conditions:

  $$\begin{aligned} 1)\ a_{i,1} &= L(i) \text{ for all } i \\ 2)\ a_{i,i} &= i \text{ for all } i \\ 3)\ a_{i,j} &= a_{i-1,j} + a_{i-2,j} + a_{i-1,j-1} - a_{i-2,j-1} \text{ for } 2 \le j \le (i-1). \end{aligned}$$

  If $T(i) = \displaystyle\sum_{j=1}^{i} a_{i,j}$ for all $i \ge 2$, then find a closed form for $T(i)$, where $L(i)$ are the Lucas numbers, $L(1) = 1, L(2) = 3$, and $L(i) = L(i-1) + L(i-2)$ for $i \ge 3$.

- 4999: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all real triplets (x, y, z) such that

$$\begin{aligned} x + y + z &= 2 \\ 2^{x+y^2} + 2^{y+z^2} + 2^{z+x^2} &= 6\sqrt[9]{2} \end{aligned}$$

- 5000: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

  Of all the right triangles inscribed in the unit circle, which has the Morley triangle of greatest area?

- 5001: *Proposed by Ovidiu Furdui, Toledo, OH.*

  Evaluate:

$$\int_0^\infty \ln^2\left(\frac{x^2}{x^2+3x+2}\right)dx.$$

Solutions

- 4978: *Proposed by Kenneth Korbin, New York, NY.*

  Given equilateral triangle ABC with side $\overline{\text{AB}} = 9$ and with cevian $\overline{\text{CD}}$. Find the length of $\overline{\text{AD}}$ if $\triangle$ADC can be inscribed in a circle with diameter equal to 10.

Solution by Dionne Bailey, Elsie Campbell, Charles Diminnie, Karl Havlak, and Paula Koca (jointly), San Angelo, TX.

Let $x = \overline{AD}$ and $y = \overline{CD}$. If A is the area of $\triangle ADC$, then

$$A = \frac{1}{2}(9)\,x\sin 60^o = \frac{9}{4}\sqrt{3}x.$$

Since the circumradius of $\triangle ADC$ is 5, we have

$$5 = \frac{9xy}{4A} = \frac{y}{\sqrt{3}}$$

and hence,

$$y = 5\sqrt{3}.$$

Then, by the Law of Cosines,

$$75 = y^2 = x^2 + 81 - 2\,(9)\,x\cos 60^o = x^2 - 9x + 81$$

which reduces to

$$x^2 - 9x + 6 = 0.$$

Therefore, there are two possible solutions:

$$\overline{AD} = x = \frac{9 \pm \sqrt{57}}{2}.$$

Also solved by Scott H. Brown, Montgomery, AL; Daniel Copeland, Portland, OR; M.N. Deshpande, Nagpur, India; Paul M. Harms, North

Newton, KS; Jahangeer Kholdi, Portsmouth, VA; Xiezhang Li, David Stone & John Hawkins (jointly), Statesboro, GA; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Charles, McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 4979: *Proposed by Kenneth Korbin, New York, NY.*

Part I: Find two pairs of positive numbers (x, y) such that

$$\frac{x}{\sqrt{1+y}-\sqrt{1-y}} = \frac{\sqrt{65}}{2},$$

where x is an integer.

Part II: Find four pairs of positive numbers (x, y) such that

$$\frac{x}{\sqrt{1+y}-\sqrt{1-y}} = \frac{65}{2},$$

where x is an integer.

Solution 1 by Brian D. Beasley, Clinton, SC.

(I) We need $0 < y \le 1$, so requiring x to be an integer yields

$$x = \frac{\sqrt{65}}{2}\left(\sqrt{1+y}-\sqrt{1-y}\right) \in \{1, 2, 3, 4, 5\}.$$

We solve for y to obtain $y = 2x\sqrt{65-x^2}/65$. Substituting $x \in \{1, 2, 3, 4, 5\}$ yields five solutions for (x, y), with two of these also having y rational, namely

$$(x, y) = (1, 16/65) \quad \text{and} \quad (x, y) = (4, 56/65).$$

(II) We again need $0 < y \le 1$, so requiring x to be an integer yields

$$x = \frac{65}{2}\left(\sqrt{1+y}-\sqrt{1-y}\right) \in \{1, 2, \ldots, 45\}.$$

We solve for y to obtain $y = 2x\sqrt{4225-x^2}/4225$. Substituting $x \in \{1, 2, \ldots, 45\}$ yields 45 solutions for (x, y), with four of these also having y rational, namely

$$(x, y) = (16, 2016/4225); \quad (x, y) = (25, 120/169);$$
$$(x, y) = (33, 3696/4225); \quad (x, y) = (39, 24/25).$$

Solution 2 by James Colin Hill, Cambridge, MA.

Part I: The given equation yields $4x^2 = 130(1+\sqrt{1-y^2})$. Let $y = \cos\theta$. Then

$$\sin\theta = \frac{4x^2}{130} - 1.$$

For $x \in Z^+$, we find several solutions, including the following (rational) pair:

$$\begin{aligned} x &= 1,\ y = 16/65 \\ x &= 4,\ y = 56/64. \end{aligned}$$

Part II: The given equation yields $\sin\theta = \dfrac{4x^2}{8450} - 1$, where $y = \cos\theta$ as before. For $x \in Z^+$, we find many solutions, including the following (rational) four:

$$x = 16,\ y = 2016/4225$$

$$\begin{aligned} x &= 25,\ y = 120/169 \\ x &= 33,\ y = 3696/4225 \\ x &= 39,\ y = 24/25 \end{aligned}$$

Also solved by John Boncek, Montgomery, AL; Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, David C. Wilson, Winston-Salem, NC, and the proposer.

- 4980: *J.P. Shiwalkar and M.N. Deshpande, Nagpur, India.*

  An unbiased coin is sequentially tossed until $(r+1)$ heads are obtained. The resulting sequence of heads (H) and tails (T) is observed in a linear array. Let the random variable X denote the number of double heads (HH's, where overlapping is allowed) in the resulting sequence. For example: Let $r = 6$ so the unbiased coin is tossed till 7 heads are obtained and suppose the resulting sequence of H's and T's is as follows:

  HHTTT**H**TTTT**HHH**TT**H**

  Now in the above sequence, there are three double heads (HH's) at toss number $(1,2), (11,12)$ and $(12,13)$. So the random variable X takes the value 3 for the above observed sequence.

  In general, what is the expected value of X?

  Solution by N. J. Kuenzi, Oshkosh,WI.

  Let $X(r)$ be the number of double heads (HH) in the resulting sequence.
  First consider the case $r = 1$. Since the resulting sequence of heads (H) and tails (T) ends in either TH or HH, $P[X(1) = 0] = \dfrac{1}{2}$ and $P[X(1) = 1] = \dfrac{1}{2}$. So $E[X(1)] = \dfrac{1}{2}$.

  Next let $r > 1$, an unbiased coin is tossed until $(r+1)$ heads are obtained. If the resulting sequence of $H's$ and $T's$ ends in TH then $X(r) = X(r-1)$. And if the resulting sequence of $H's$ and $T's$ ends in HH then $X(r) = X(r-1)$. So

  $$P[X(r) = X(r-1)] = \frac{1}{2} \text{ and } P[X(r) = X(r-1)+1] = \frac{1}{2}.$$

  It follows that

  $$E[X(r)] = \frac{1}{2}E[X(r-1)] + \frac{1}{2}E[X(r-1)+1] = E[X(r-1)] + \frac{1}{2}.$$

  Finally, the Principle of Mathematical Induction can be used to show that $E[X(r)] = \dfrac{r}{2}$.

  Also solved by Kee-Wai Lau, Hong Kong, China; Harry Sedinger, St. Bonvatenture, NY, and the proposers.

- 4981: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all real solutions of the equation

  $$5^x + 3^x + 2^x - 28x + 18 = 0.$$

Solution by Paolo Perfetti, Dept. of Mathematics, University of Rome, Italy.

Let $f(x) = 5^x + 3^x + 2^x - 28x + 18$. The values for $x \le 0$ are excluded from being solutions because for these values $f(x) > 0$. It is immediately seen that $f(x) = 0$ for $x = 1, 2$. Moreover, the derivative $f'(x) = 5^x \ln 5 + 3^x \ln 3 + 2^x \ln 2 - 28$ is an increasing continuous function such that:

$$\begin{aligned}
1)\ f'(0) &= \ln 30 - 28 < 0, \quad \lim_{x + \infty} f'(x) = +\infty \\
2)\ f'(1) &= 5\ln 5 + 3\ln 3 + 2\ln 2 - 28 < 10 + 6 + 2 - 28 = -10 \\
3)\ f'(2) &= 25\ln 5 + 9\ln 3 + 4\ln 2 - 28 \ge 34 - 28 > 0.
\end{aligned}$$

By continuity this implies that $f'(x) = 0$ for just one point x_o between 1 and 2, and that the graph of $f(x)$ has a minimum only at $x = x_o$. It follows that there are no values of x other than $x = 1, 2$ satisfying $f(x) = 0$.

Also solved by Brain D. Beasley, Clinton, SC; Pat Costello, Richmond, KY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; M.N. Deshpande, Nagpur, India; Paul M. Harms, North Newton, KS; Jahangeer Kholdi, Portsmouth, VA; Kee-Wai Lau, Hong Kong, China; Kenneth Korbin, NY, NY; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins, Statesboro, GA, and the proposers.

- 4982: *Proposed by Juan José Egozcue and José Luis Díaz-Barrero, Barcelona, Spain.*

  Calculate

$$\lim_{n\to\infty} \frac{1}{n+1}\left(\sum_{1\le i_1\le n+1} \frac{1}{i_1} + \sum_{1\le i_1<i_2\le n+1} \frac{1}{i_1 i_2} + \cdots + \sum_{1\le i_1<\ldots<i_n\le n+1} \frac{1}{i_1 i_2 \ldots i_n}\right).$$

Solution 1 by Paul M. Harms, North Newton, KS.

Let $S(n)$ be the addition of the summations inside the parentheses of the expression in the problem. When $n = 1$. The expression in the problem is

$$\frac{1}{2}\left(\left[\frac{1}{1} + \frac{1}{2}\right] + \left[\frac{1}{1(2)}\right]\right) = (\frac{1}{2})2 = 1, \text{ where } S(1) = 2.$$

When $n = 2$ the expression is

$$\begin{aligned}
&= \frac{1}{3}\left(\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right] + \left[\frac{1}{1(2)} + \frac{1}{1(3)} + \frac{1}{(2)(3)}\right] + \left[\frac{1}{1(2)(3)}\right]\right) \\
&= \frac{1}{3}\left(S(1) + \frac{1}{3}\left[1 + S(1)\right]\right) \\
&= \frac{1}{3}\left(2 + \frac{1}{3}3\right) = 1, \text{ where } S(2) = 3.
\end{aligned}$$

When $n = 3$ the expression is

$$\frac{1}{4}\left(S(2) + \frac{1}{4}\left[1 + S(2)\right]\right) = \frac{1}{4}\left(3 + \frac{1}{4}\left[1 + 3\right]\right) = 1, \text{ where } S(3) = 4.$$

When $n = k + 1$ the expression becomes

$$\frac{1}{k+2}\left(S(k) + \frac{1}{k+2}\left[1 + S(k)\right]\right) = 1, \text{ where } S(k) = k + 1.$$

The limit in the problem is one.

Solution 2 by David E. Manes,Oneonta, NY.

Let

$$a_n = \frac{1}{n+1}\left(\sum_{1\le i_1\le n+1}\frac{1}{i_1} + \sum_{1\le i_1<i_2\le n+1}\frac{1}{i_1 i_2} + \cdots + \sum_{1\le i_1<\ldots<i_n\le n+1}\frac{1}{i_1 i_2\ldots i_n}\right).$$

Then $a_1 = 3/4, a_2 = 17/18, a_3 = 95/96$, and $a_4 = 599/600$.

We will show that

$$a_n = 1 - \frac{1}{(n+1)(n+1)!}$$

Note that the equation is true for $n = 1$ and assume inductively that it is true for some integer $n \geq 1$. Then

$$\begin{aligned}
a_n &= \frac{1}{n+1}\left(\sum_{1\le i_1\le n+1}\frac{1}{i_1} + \sum_{1\le i_1<i_2\le n+1}\frac{1}{i_1 i_2} + \cdots + \sum_{1\le i_1<\ldots<i_n\le n+1}\frac{1}{i_1 i_2\ldots i_n}\right)\\
&= \frac{1}{n+2}\left[(n+1)a_n + \frac{1}{n+2} + \left(\frac{n+1}{n+2}\right)a_n + \frac{1}{(n+1)!}\right]\\
&= \frac{1}{n+2}\left[(n+1)\left(1-\frac{1}{(n+1)(n+1)!}\right) + \frac{1}{n+2} + \left(\frac{n+1}{n+2}\right)\left(1-\frac{1}{(n+1)(n+1)!}\right) + \frac{1}{(n+1)!}\right]\\
&= \frac{1}{n+2}\left[(n+1) - \frac{1}{(n+1)!} + 1 - \frac{1}{(n+2)(n+1)!} + \frac{1}{(n+1)!}\right]\\
&= \frac{1}{n+2}\left[n+2-\frac{1}{(n+2)!}\right] = 1 - \frac{1}{(n+2)(n+2)!}.
\end{aligned}$$

Therefore, the result is true for $n+1$. By induction $a_n = 1 - \dfrac{1}{(n+1)(n+1)!}$ is valid for all integers $n \geq 1$. Hence $\lim\limits_{n\to\infty} a_n = 1$.

Also solved by Carl Libis, Kingston, RI; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- 4983: *Proposed by Ovidiu Furdui, Kalamazoo, MI.*

Let k be a positive integer. Evaluate

$$\int_0^1 \left\{\frac{k}{x}\right\} dx,$$

where $\{a\}$ is the *fractional part* of a.

Solution by Kee-Wai Lau, Hong Kong, China.

We show that

$$\int_0^1 \left\{\frac{k}{x}\right\} dx = k\left(\sum_{n=1}^{k}\frac{1}{n} - \ln k - \gamma\right),$$

where γ is Euler's constant. By substituting $x = ky$, we obtain

$$\int_0^1 \left\{\frac{k}{x}\right\} dx = k\int_0^{1/k} \left\{\frac{1}{y}\right\} dy.$$ For any integer $M > k$, we have

$$\begin{aligned}
\int_{1/M}^{1/k} \left\{\frac{1}{y}\right\} dy &= \sum_{n=k}^{M-1} \int_{1/(n+1)}^{1/n} \left\{\frac{1}{y}\right\} dy \\
&= \sum_{n=k}^{M-1} \int_{1/(n+1)}^{1/n} \left\{\frac{1}{y} - n\right\} dy \\
&= \sum_{n=k}^{M-1} \left(\ln(\frac{n+1}{n}) - \frac{1}{n+1} \right) \\
&= \ln\left(\frac{M}{k}\right) - \sum_{n=k+1}^{M} \frac{1}{n} \\
&= \sum_{n=1}^{k} \frac{1}{n} - \ln k - \left(\sum_{n=1}^{M} \frac{1}{n} - \ln M \right).
\end{aligned}$$

Since $\lim\limits_{M\to\infty} \left(\sum_{n=1}^{M} \frac{1}{n} - \ln M \right) = \gamma$, we obtain our result.

Also solved by Brian D. Beasley, Clinton, SC; Jahangeer Kholdi, Portsmouth, VA; David E. Manes, Oneonta, NY; Paolo Perfetti, Dept. of Mathematics, University of Rome, Italy; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Acknowledgments

The name of **Dr. Peter E. Liley of Lafayette, IN** should have been listed as having solved problems 4966, 4973 and 4974. His name was inadvertently omitted from the listing; mea culpa.

Problem 4952 was posted in the January 07 issue of this column. It was proposed by **Michael Brozinsky of Central Islip, NY & Robert Holt of Scotch Plains, NJ.** I received one solution to this problem; it was from **Paul M. Harms of North Newton, KS.** His solution, which was different from the one presented by proposers, made a lot of sense to me and it was published in the October 07 issue of this column. Michael then wrote to me stating that he thinks Paul misinterpreted the problem. For the sake of completeness, here is the proposers' solution to their problem.

- 4952: An archeological expedition discovered all dwellings in an ancient civilization had $1, 2$, or 3 of each of k independent features. Each plot of land contained three of these houses such that the k sums of the number of each of these features were all divisible by 3. Furthermore, no plot contained two houses with identical configurations of features and no two plots had the same configurations of three houses. Find **a)** the maximum number of plots that a house with a given configuration might be located on, and **b)** the maximum number of distinct possible plots.

Solution by the proposers: a) Clearly these maximum numbers will be attained using the 3^k possible configurations for a house.

Note: For any two houses on a plot:
1) if they have the same number of any given feature then the third house will necessarily have this same number of that feature since the sum must be divisible by three, and
2) if they have a different number of a given feature then the third house will have a different number of that feature than the first two houses since the sum must be divisible by three.

It follows then that any fixed house can be adjoined with $\dfrac{3^k-1}{2}$ possible pairs of houses to be placed on a plot since the second house can be any of the remaining 3^k-1 house configurations but the third configuration is uniquely determined by the above note and the fact that no two houses on a plot can be identically configured. These 3^k-1 permutations of the second and third house thus must have arisen from the $\dfrac{3^k-1}{2}$ possible pairs claimed above. The answer is thus $\dfrac{3^k-1}{2}$.
b) The above note shows that for any two differently configured houses only one of the remaining 3^k-2 configurations will form a plot with these two. Hence, the probability that 3 configurations chosen randomly from the 3^k configurations are suitable for a plot is $\dfrac{1}{3^k-2}$. Since there are $\dbinom{3^k}{3}$ subsets of size three that can be formed from the 3^k configurations, it follows that the maximum number of distinct possible plots is

$$\frac{\dbinom{3^k}{3}}{3^k-2}=\frac{3^{k-1}(3^k-1)}{2}.$$

Problems

Ted Eisenberg, Section Editor

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Solutions to the problems stated in this issue should be posted before April 15, 2008

- 5002: *Proposed by Kenneth Korbin, New York, NY.*

  A convex hexagon with sides $3x, 3x, 3x, 5x, 5x$ and $5x$ is inscribed in a unit circle. Find the value of x.

- 5003: *Proposed by Kenneth Korbin, New York, NY.*

  Find positive numbers x and y such that

$$\begin{aligned} \sqrt[3]{x+\sqrt{x^2-1}}+\sqrt[3]{x-\sqrt{x^2-1}} &= \frac{7}{2} \text{ and} \\ \sqrt[3]{y+\sqrt{y^2-1}}+\sqrt[3]{y-\sqrt{y^2-1}} &= \sqrt{10} \end{aligned}$$

- 5004: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be nonnegative real numbers. Prove that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c} \ge \frac{\sqrt{ab}}{1+a+b}+\frac{\sqrt{bc}}{1+b+c}+\frac{\sqrt{ca}}{1+c+a}$$

- 5005: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{\sqrt{3}}{2}\left(a+b+c\right)^{1/2} \ge \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}.$$

- 5006: *Proposed by Ovidiu Furdui, Toledo, OH.*

  Find the sum
$$\sum_{k=2}^{\infty}(-1)^k \ln\left(1-\frac{1}{k^2}\right).$$

- 5007: *Richard L. Francis, Cape Girardeau, MO.*

  Is the centroid of a triangle the same as the centroid of its Morley triangle?

Solutions

- 4984: *Proposed by Kenneth Korbin, New York, NY.*

  Prove that
$$\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{5}+\sqrt{7}}+\cdots+\frac{1}{\sqrt{2009}+\sqrt{2011}}>\sqrt{120}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China.

The sum

$$\begin{aligned}
& \sum_{k=1}^{503}\frac{1}{\sqrt{4k-3}+\sqrt{4k-1}} \\
> \ & \frac{1}{2}\sum_{k=1}^{503}\left(\frac{1}{\sqrt{4k-3}+\sqrt{4k-1}}+\frac{1}{\sqrt{4k-1}+\sqrt{4k+1}}\right) \\
= \ & \frac{1}{2}\sum_{k=1}^{503}\left(\frac{\sqrt{4k-1}-\sqrt{4k-3}}{2}+\frac{\sqrt{4k+1}-\sqrt{4k-1}}{2}\right) \\
= \ & \frac{1}{4}\sum_{k=1}^{503}\left(\sqrt{4k+1}-\sqrt{4k-3}\right) \\
= \ & \frac{1}{4}\left(\sqrt{2013}-1\right) \\
= \ & \frac{1}{4}\sqrt{2013-2\sqrt{2013}+1} \\
> \ & \frac{1}{4}\left(\sqrt{2013-2(45)}+1\right) \\
> \ & \frac{1}{4}\sqrt{1920} \\
= \ & \sqrt{120}
\end{aligned}$$

as required.

Solution 2 by Kenneth Korbin, the proposer.

Let $K = \dfrac{1}{\sqrt{1}+\sqrt{3}} + \dfrac{1}{\sqrt{5}+\sqrt{7}} + \cdots + \dfrac{1}{\sqrt{2009}+\sqrt{2011}}$.
Then, $K > \dfrac{1}{\sqrt{3}+\sqrt{5}} + \dfrac{1}{\sqrt{7}+\sqrt{9}} + \cdots + \dfrac{1}{\sqrt{2011}+\sqrt{2013}}$
and,

$$\begin{aligned}
2K &> \frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \cdots + \frac{1}{\sqrt{2011}+\sqrt{2013}} \\
&= \frac{\sqrt{3}-\sqrt{1}}{2} + \frac{\sqrt{5}-\sqrt{3}}{2} + \frac{\sqrt{7}-\sqrt{5}}{2} + \cdots + \frac{\sqrt{2013}-\sqrt{2011}}{2} \\
&= \frac{\sqrt{2013}-1}{2}. \text{ So,} \\
K &> \frac{\sqrt{2013}-1}{4} > \sqrt{120}.
\end{aligned}$$

Also solved by Brian D. Beasley, Clinton, SC; Charles R. Diminnie, San Angelo, TX; Paul M. Harms, North Newton, KS; Paolo Perfetti, Mathematics Department, U. of Rome, Italy, and David Stone & John Hawkins (jointly), Statesboro, GA.

- 4985: *Proposed by Kenneth Korbin, New York, NY.*

  A Heron triangle is one that has both integer length sides and integer area. Assume Heron triangle ABC is such that $\angle B = 2\angle A$ and with (a,b,c)=1.

PartI :	Find the dimensions of the triangle if side a = 25.
PartII :	Find the dimensions of the triangle if 100 < a < 200.

Solution by Brian D. Beasely, Clinton, SC.

Using the Law of Sines, we obtain
$$\frac{\sin A}{a} = \frac{\sin(2A)}{b} = \frac{\sin(180^\circ - 3A)}{c} = \frac{\sin(3A)}{c},$$
where $\angle B = 2\angle A$ forces $0^\circ < A < 60^\circ$. Since $\sin(2A) = 2\sin A\cos A$ and $\sin(3A) = 3\sin A - 4\sin^3 A$, we have $b = 2a\cos A$ and $c = a(3 - 4\sin^2 A)$. In particular, $a < b < 2a$, and using $A = \cos^{-1}\left(\dfrac{b}{2a}\right)$ implies
$$c = 3a - 4a\left(1 - \left(\frac{b}{2a}\right)^2\right) = -a + \frac{b^2}{a}.$$
Then a divides b^2, so we claim that a must be a perfect square: Otherwise, if a prime p divides a but p^2 does not, then p divides b^2; thus p divides b, yet p^2 does not divide a, which would imply that p divides b^2/a and hence p divides c, a contradiction of $(a, b, c) = 1$.

Next, we note that the area of the triangle is $(1/2)bc\sin A$, which becomes
$$\frac{b(b+a)(b-a)}{2a}\sqrt{1 - \left(\frac{b}{2a}\right)^2} = \frac{b(b+a)(b-a)}{4a^2}\sqrt{4a^2 - b^2}.$$

I. Let $a = 25$. Then $25 < b < 50$ and $c = -25 + b^2/25$, so 5 divides b. Checking $b \in \{30, 35, 40, 45\}$ yields two solutions for which the area of the triangle is an integer:

$$(a, b, c) = (25, 30, 11) \text{ with area} = 132; \qquad (a, b, c) = (25, 40, 39) \text{ with area} = 468.$$

II. Let $100 < a < 200$. Then $a \in \{121, 144, 169, 196\}$.

If $a = 121$, then 11 divides b, so $b = 11d$ for $d \in \{12, 13, \ldots, 21\}$. Since the area formula requires $4a^2 - b^2 = 11^2(22^2 - d^2)$ to be a perfect square, we check that no such d produces a perfect square $22^2 - d^2$.

If $a = 144$, then 12 divides b, so $b = 12d$ for $d \in \{13, 14, \ldots, 23\}$. Since $4a^2 - b^2 = 12^2(24^2 - d^2)$ must be a perfect square, we check that no such d produces a perfect square $24^2 - d^2$.

If $a = 169$, then 13 divides b, so $b = 13d$ for $d \in \{14, 15, \ldots, 25\}$. Since $4a^2 - b^2 = 13^2(26^2 - d^2)$ must be a perfect square, we check that the only such d to produce a perfect square $26^2 - d^2$ is $d = 24$. This yields the triangle

$$(a, b, c) = (169, 312, 407) \text{ with area 24,420.}$$

If $a = 196$, then 14 divides b, so $b = 14d$ for $d \in \{15, 16, \ldots, 27\}$. Since $4a^2 - b^2 = 14^2(28^2 - d^2)$ must be a perfect square, we check that no such d produces a perfect square $28^2 - d^2$.

Comment: David Stone and John Hawkins of Statesboro, GA conjectured that in order to meet the conditions of the problem, a must equal p^2, where p is an odd prime congruent to 1 mod 4. With $p = m^2 + n^2$, there are one or two triangles, according to the ratio of m and n. If $\sqrt{3}n < m < (2 + \sqrt{3})n$, there are two solutions; if $m > (2 + \sqrt{3})n$, there is one solution; and if $n < m < \sqrt{3}n$, there is one solution.

Also solved by M.N. Deshpande, Nagpur, India; Grant Evans (student, Saint George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; John Nord, Spokane, WA; David Stone & John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4986: *Michael Brozinsky, Central Islip, NY.*

  Show that if $0 < a < b$ and $c > 0$, that

$$\sqrt{(a+c)^2 + d^2} + \sqrt{(b-c)^2 + d^2} \le \sqrt{(a-c)^2 + d^2} + \sqrt{(b+c)^2 + d^2}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China.

Squaring both sides and simplifying, we reduce the desired inequality to

$$2c(b-a) + \sqrt{(a-c)^2 + d^2}\sqrt{(b+c)^2 + d^2} \ge \sqrt{(a+c)^2 + d^2}\sqrt{b-c)^2 + d^2}.$$

Squaring the last inequality and simplifying we obtain

$$\sqrt{(a-c)^2 + d^2}\sqrt{(b+c)^2 + d^2} \ge ab + ac - bc - c^2 - d^2. \qquad (1)$$

If $ab + ac - bc - c^2 - d^2 \le 0$, (1) is certainly true. If $ab + ac - bc - c^2 - d^2 > 0$, we square both sides of (1) and the resulting inequality simplifies to the trivial inequality $(a+b)^2 d^2 \ge 0$. This completes the solution.

Solution 2 by Paolo Perfetti, Mathematics Department, U. of Rome, Italy.
The inequality is

$$\sqrt{(b-c)^2+d^2}-\sqrt{(a-c)^2+d^2}\le\sqrt{(b+c)^2+d^2}-\sqrt{(a+c)^2+d^2}.$$

Defining $f(x)=\sqrt{(b+x)^2+d^2}-\sqrt{(a+x)^2+d^2},\ -c\le x\le c$, the inequality becomes $f(-c)\le f(c)$ so we prove that

$$f'(x)=\frac{b+x}{\sqrt{(b+x)^2+d^2}}-\frac{a+x}{\sqrt{(a+x)^2+d^2}}>0.$$

There are three possibilities: 1) $b+x>a+x\ge 0$, 2) $a+x<b+x<0$, and 3) $b+x>0,\ a+x<0$. It is evident that 3) implies $f'(x)>0$. With the condition 1), after squaring, we obtain

$$\begin{aligned}(b+x)^2((a+x)^2+d^2) &> (a+x)^2((b+x)^2+d^2)\text{ or}\\ (b+x)^2 &> (a+x)^2\text{which is true.}\end{aligned}$$

As for 2) we have

$$\begin{aligned}\frac{|b+x|}{\sqrt{(b+x)^2+d^2}} &< \frac{|a+x|}{\sqrt{(a+x)^2+d^2}}\quad\text{or}\\ (b+x)^2 &< (a+x)^2\end{aligned}$$

and making the square root $-(b+x)<-(a+x)$ which is true as well.

Also solved by Angelo State University Problem Solving Group, San Angelo, TX; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY, and the proposer.

- 4987: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let a,b,c be the sides of a triangle ABC with area S. Prove that

$$(a^2+b^2)(b^2+c^2)(c^2+a^2)\le 64S^3\csc 2A\csc 2B\csc 2C.$$

Solution by José Luis Díaz-Barrero, the proposer.

Let $A'\in BC$ be the foot of h_a. We have,

$$h_a=c\sin B\quad\text{and}\quad BA'=c\cos B \tag{1}$$

and

$$h_a=b\sin C\quad\text{and}\quad A'C=b\cos C \tag{2}$$

Multiplying up and adding the resulting expressions yields

$$h_a(BA'+A'C)=\frac{b^2\sin 2C}{2}+\frac{c^2\sin 2B}{2}$$

or

$$c^2\sin 2B+b^2\sin 2C=4S$$

Likewise, we have

$$a^2 \sin 2C + c^2 \sin 2A = 4S,$$
$$a^2 \sin 2B + b^2 \sin 2A = 4S.$$

Adding up the above expressions yields

$$(a^2 + b^2) \sin 2C + (b^2 + c^2) \sin 2A + (c^2 + a^2) \sin 2B = 12S$$

Applying the AM-GM inequality yields

$$\sqrt[3]{(a^2 + b^2) \sin 2C(b^2 + c^2) \sin 2A(c^2 + a^2) \sin 2B} \leq 4S$$

from which the statement follows. Equality holds when $\triangle ABC$ is equilateral and we are done.

- 4988: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all real solutions of the equation

$$3^{x^2-x-z} + 3^{y^2-y-x} + 3^{z^2-z-y} = 1.$$

Solution by Dionne Bailey, Elsie Campbell, Charles Diminnie, Karl Havlak, and Paula Koca (jointly), San Angelo, TX.

By the Arithmetic - Geometric Mean Inequality,

$$\begin{aligned} 1 &= 3^{x^2-x-z} + 3^{y^2-y-x} + 3^{z^2-z-y} \\ &\geq 3\sqrt[3]{3^{x^2-2x+y^2-2y+z^2-2z}} \\ &= \sqrt[3]{3^{(x-1)^2+(y-1)^2+(z-1)^2}} \end{aligned}$$

and hence,

$$3^{(x-1)^2+(y-1)^2+(z-1)^2} \leq 1.$$

It follows that

$$(x-1)^2 + (y-1)^2 + (z-1)^2 = 0,$$

i.e.,

$$x = y = z = 1.$$

Since it is easily checked that these values satisfy the original equation, the solution is complete.

Also solved by Kee-Wai Lau, Hong Kong, China; Charles McCracken, Dayton, OH; Paolo Perfetti, Mathematics Department, U. of Rome, Italy; Boris Rays, Chesapeake, VA, and the proposer.

- 4989: *Proposed by Tom Leong, Scotrun, PA.*

  The numbers $1, 2, 3, \cdots, 2n$ are randomly arranged onto $2n$ distinct points on a circle. For a chord joining two of these points, define its *value* to be the absolute value of the difference of the numbers on its endpoints. Show that we can connect the $2n$ points in disjoint pairs with n chords such that no two chords intersect inside the circle and the sum of the values of the chords is exactly n^2.

Solution 1 by Harry Sedinger, St. Bonaventure, NY.

First we show by induction that if there are n red points and n blue points (all distinct) on the circle, then there exist n nonintersecting chords, each connecting a read point an a blue point (with each point being used exactly once). This is obvious for $n = 1$.
Assume it is true for n and consider the case for $n + 1$. There obviously is a pair of adjacent points (no other points between them on one arc), one read and one blue. Clearly they can be connected by a chord which does not intersect any chord connecting two other points. Removing this chord and the two end points then reduces the problem to the case for n, which can be done according to the induction hypothesis. The desired result is then true for $n + 1$ and by induction true for all n.
Now for the given problem, color the points numbered $1, 2, \cdots, n$ red and color the ones numbered $n + 1, n + 2, \cdots, 2n$ blue. From above there exists n nonintersecting chords and the sum of their values is

$$\sum_{k=n+1}^{2n} k - \sum_{k=1}^{n} k = \sum_{k=1}^{2n} k - 2\sum_{k=1}^{n} k = \frac{2n(2n+1)}{2} - 2\frac{n(n+1)}{2} = n^2.$$

Solution 2 by Kenneth Korbin, New York, NY.

Arrange the numbers $1, 2, 3, \cdots, 2n$ randomly on points of a circle. Place a red checker on each point from 1 through n. Let

$$\sum R = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Place a black checker on each point numbered from $n + 1$ through $2n$. Let

$$\sum B = (n+1) + (n+2) + \cdots + (2n) = n^2 + \frac{n(n+1)}{2}.$$

Remove a pair of adjacent checkers that have different colors. Connect the two points with a chord. The value of this chord is $(B_1 - R_1)$.
Remove another pair of adjacent checkers with different colors. The chord between these two points will have value $(B_2 - R_2)$.
Continue this procedure until the last checkers are removed and the last chord will have value $(B_n - R_n)$.
The sum of the value of these n chords is

$$(B_1 - R_1) + (B_2 - R_2) + \cdots + (B_n - R_n) = \sum B - \sum R = n^2.$$

Also solved by N.J. Kuenzi, Oshkosh, WI; David Stone & John Hawkins (jointly), Statesboro,GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
May 15, 2008

- 5008: *Proposed by Kenneth Korbin, New York, NY.*

  Given isosceles trapezoid $ABCD$ with $\angle ABD = 60^o$, and with legs $\overline{BC} = \overline{AD} = 31$.

  Find the perimeter of the trapezoid if each of the bases has positive integer length with $\overline{AB} > \overline{CD}$.

- 5009: *Proposed by Kenneth Korbin, New York, NY.*

  Given equilateral triangle ABC with a cevian $\overline{CD}$ such that $\overline{AD}$ and $\overline{BD}$ have integer lengths. Find the side of the triangle $\overline{AB}$ if $\overline{CD} = 1729$ and if $(\overline{AB}, 1729) = 1$.

- 5010: *Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let α, β, and γ be real numbers such that $0 < \alpha \le \beta \le \gamma < \pi/2$. Prove that

  $$\frac{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}{(\sin\alpha + \sin\beta + \sin\gamma)(\cos\alpha + \cos\beta + \cos\gamma)} \le \frac{2}{3}.$$

- 5011: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $\{a_n\}_{n\ge 0}$ be the sequence defined by $a_0 = a_1 = 2$ and for $n \ge 2,\ a_n = 2a_{n-1} - \dfrac{1}{2}a_{n-2}$. Prove that

  $$2^p a_{p+q} + a_{q-p} = 2^p a_p a_q$$

  where $p \le q$ are nonnegative integers.

- 5012 *Richard L. Francis, Cape Girardeau, MO.*

  Is the incenter of a triangle the same as the incenter of its Morley triangle?

- 5013: *Proposed by Ovidiu Furdui, Toledo, OH.*

  Let $k \geq 2$ be a natural number. Find the sum

$$\sum_{n_1,n_2,\cdots,n_k \geq 1} \frac{(-1)^{n_1+n_2+\cdots+n_k}}{n_1+n_2+\cdots+n_k}.$$

Solutions

- 4990: *Proposed by Kenneth Korbin, New York, NY.*

  Solve

$$40x + 42\sqrt{1-x^2} = 29\sqrt{1+x} + 29\sqrt{1-x}$$

with $0 < x < 1$.

Solution by Boris Rays, Chesapeake, VA.

Let $x = \cos\alpha$, where $\alpha \in (0, \pi/2)$. Then

$$\begin{aligned}
40\cos\alpha + 42\sqrt{1-\cos^2\alpha} &= 29\sqrt{1+\cos} + 29\sqrt{1-\cos\alpha} \\
&= 29\sqrt{2}\left(\sqrt{\frac{1+\cos\alpha}{2}} + \sqrt{\frac{1-\cos\alpha}{2}}\right) \\
&= 29\cdot\frac{2}{\sqrt{2}}\left(\sqrt{\frac{1+\cos\alpha}{2}} + \sqrt{\frac{1-\cos\alpha}{2}}\right) \\
&= 29\cdot 2\left(\frac{1}{\sqrt{2}}\cos\frac{\alpha}{2} + \frac{1}{\sqrt{2}}\sin\frac{\alpha}{2}\right) \\
&= 58\left(\cos\frac{\pi}{4}\cos\frac{\alpha}{2} + \sin\frac{\pi}{4}\sin\frac{\alpha}{2}\right) = 58\cos(\frac{\pi}{4} - \frac{\alpha}{2}). \quad \text{Therefore,}
\end{aligned}$$

$$\begin{aligned}
40\cos\alpha + 42\sin\alpha &= 58\cos(\frac{\pi}{4} - \frac{\alpha}{2}). \\
\frac{40}{58}\cos\alpha + \frac{42}{58}\sin\alpha &= \cos(\frac{\pi}{4} - \frac{\alpha}{2}) \\
\frac{20}{29}\cos\alpha + \frac{21}{29}\sin\alpha &= \cos(\frac{\pi}{4} - \frac{\alpha}{2}).
\end{aligned}$$

Let $\cos\alpha_0 = \dfrac{20}{29}$. Then $\sin\alpha_0 = \sqrt{1 - \left(\dfrac{20}{29}\right)^2} = \dfrac{21}{29}$.

$$\cos\alpha_0\cos\alpha + \sin\alpha_0\sin\alpha = \cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$$

$$\cos(\alpha_0 - \alpha) \quad = \quad \cos\left(\frac{\pi}{4} - \frac{\alpha}{2}\right).$$

Therefore we obtain from the above,

$$\begin{aligned}
1) \quad \alpha_0 - \alpha_1 &= \frac{\pi}{4} - \frac{\alpha_1}{2} \\
\alpha_1 &= 2\alpha_0 - \frac{\pi}{2}, \text{ where } \alpha_0 = \arccos\frac{20}{29}. \\
2) \quad \alpha_0 - \alpha_2 &= -\left(\frac{\pi}{4} - \frac{\alpha_2}{2}\right) = \frac{\alpha_2}{2} - \frac{\pi}{4} \\
\frac{3}{2}\alpha_2 &= \alpha_0 + \frac{\pi}{4} \\
\alpha_2 &= \frac{2}{3}\alpha_0 + \frac{\pi}{6}, \text{ where } \alpha_0 = \arccos\frac{20}{29}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
1) \quad x_1 &= \cos\left(2\alpha_0 - \frac{\pi}{2}\right) = \cos(2\alpha_0)\cos\frac{\pi}{2} + \sin(2\alpha_0)\sin\frac{\pi}{2} \\
&= 2\sin\alpha_0\cos\alpha_0 \cdot 1 = 2 \cdot \frac{21}{29} \cdot \frac{20}{29} = \frac{840}{841}. \\
2) \quad x_2 &= \cos\left(\frac{2}{3}\alpha_0 + \frac{\pi}{6}\right) = \cos\left(\frac{2}{3}\arccos\left(\frac{20}{29}\right) + \frac{\pi}{6}\right).
\end{aligned}$$

The solution is:

$$x_1 = \frac{840}{841} \qquad x_2 = \cos\left(\frac{2}{3}\arccos\left(\frac{20}{29}\right) + \frac{\pi}{6}\right).$$

Remark: This solution is an adaptation of the solution to SSM problem 4966, which is an adaptation of the solution on pages 13-14 of *Mathematical Miniatures* by Savchev and Andreescu.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; José Hernández Santiago (student at UTM), Oaxaca, México; Kee-Wai Lau, Hong Kong, China; Peter E. Liley, Lafayette, IN; John Nord, Spokane, WA; Paolo Perfetti, Math Dept., U. of Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4991: *Proposed by Kenneth Korbin, New York, NY.*

Find six triples of positive integers (a, b, c) such that

$$\frac{9}{a} + \frac{a}{b} + \frac{b}{9} = c.$$

Solution by David Stone and John Hawkins, Statesboro, GA,(with comments by editor).

David Stone and John Hawkins submitted a six page densely packed analysis of the problem, but it is too long to include here. Listed below is their solution and the gist of their analysis as to how they solved it. (Interested readers may obtain their full analysis by writing to David at <dstone@georgiasouthern.edu> or to me at <eisenbt@013.net>. Others who solved the problem programmed a computer.

David and John began by listing what they believe to be all ten solutions to the problem.

$$\begin{pmatrix} a & b & c \\ 2 & 12 & 6 \\ 9 & 9 & 3 \\ 14 & 588 & 66 \\ 18 & 36 & 5 \\ 54 & 12 & 6 \\ 162 & 4 & 41 \\ 378 & 588 & 66 \\ 405 & 25 & 19 \\ 11826 & 21316 & 2369 \\ 29565 & 133225 & 14803 \end{pmatrix}$$

The analysis in their words:
Rewriting the equation, we seek positive integer solutions to

$$(1) \qquad 81b + 9a^2 + ab^2 = 9abc.$$

Theorem. A solution must have the form $a = 3^i A, b = 3^j A^2$, where $(A, 3) = 1,\ i, j \geq 0$. At least one of i, j must be ≥ 1.

Proof. From equation (1), we see that 9 divides all terms but ab^2, so 9 divides ab^2, so 3 divides a or b so at least one of i, j must be ≥ 1.

Also from equation (1), it is clear that if p is a prime different from 3, then p divides a if and only if p divides b.

Suppose p is such a prime and $a = 3^i p^m C, b = 3^j p^n D$, where $m, n \geq 1$, and C and D are not divisible by 3 or p. Then equation (1) becomes

$$81\left(3^j p^n D\right) + 9\left(3^i p^m C\right)^2 + \left(3^i p^m C\right)\left(3^j p^n D\right)^2 = 9\left(3^i p^m C\right)\left(3^j p^n D\right)c,$$

or

$$(\#)\ \ 3^{j+4}p^n D + 3^{2i+2}p^{2m}C^2 + 3^{i+2j}p^{m+2n}CD^2 = 3^{i+j+2}p^{m+n}CDc.$$

If $n < 2m$, we can divide equation (#) by p^n to obtain

$$3^{j+4}D + 3^{2i+2}p^{2m-n}C^2 + 3^{i+2j}p^{m+n}CD^2 = 3^{i+j+2}p^m CDc.$$

But then p divides each term after the first, so p divides $3^{j+4}D$, which is impossible.

If $n > 2m$, we can divide through equation (#) by p^{2m} to obtain

$$\begin{aligned} 3^{j+4}p^{n-2m}D + 3^{2i+2}C^2 + 3^{i+2j}p^{2n-m}CD^2 &= 3^{i+j+2}p^{n-m}CDc \\ 81p^{m-2n}D + 9C^2 + p^m CD^2 &= 9p^{m-n}CDc. \end{aligned}$$

Noting that $2n > 4m > m$ and $n > 2m > m$, we see that p divides each term except $3^{2i+2}C^2$, so p divides $3^{2i+2}C^2$, which is impossible.

Therefore $n = 2m$.

That is, a and b have the same prime divisors, and in b, the power on each such prime is twice the corresponding power in a; therefore, in b, the product of all divisors other than 3 is the square of the analogous product in a. So the proof is complete.

They then substituted this result into equation (1) obtaining

$$81\left(3^jA^2\right)+9\left(3^iA^2\right)+\left(3^iA\right)\left(3^jA^2\right)^2=9\left(3^iA\right)\left(3^jA^2\right)c,$$

or

$$(2)\qquad \left(2^{j+4}+3^{2i+2}\right)+3^{i+2j}A^3=3^{i+j+2}Ac$$

and started looking for values of i, j, A and c satisfying this equation.

Analyzing the cases **(1)** where 3 divides b but not a; **(2)** where 3 divides a but not b; and **(3)** where 3 divides a and b led to the solutions listed above.

They ended their submission with comments about the patterns they observed in solving analogous equations of the form $\dfrac{N}{a}+b+\dfrac{c}{N}=c$ for various integral values of N.

Also solved by Charles Ashbacher, Marion, IA; Britton Stamper (student at Saint George's School), Spokane, WA, and the proposer.

- 4992: *Proposed by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie, San Angelo, TX.*

A closed circular cone has integral values for its height and base radius. Find all possible values for its dimensions if its volume V and its total area (including its circular base) A satisfy $V=2A$.

Solution by R. P. Sealy, Sackville, New Brunswick, Canada.

$$\begin{aligned}\frac{1}{3}\pi r^2h &= 2(\pi r^2+\pi r\sqrt{r^2+h^2}) \text{ or}\\ rh &= 6r+6\sqrt{r^2+h^2}.\end{aligned}$$

Squaring and simplifying gives $r^2=36\dfrac{h}{h-12}$. Therefore, $\dfrac{h}{h-12}$ is a square, and $\dfrac{h}{h-12}\in\{1,4,9,16,\ldots\}$. Note that $f(h)=\dfrac{h}{h-12}$ is a decreasing function of h for $h>12$ and that $h(16)=4$. Note also that $f(13)$, $f(14)$ and $f(15)$ are not squares of integers. Therefore $(h,r)=(16,24)$ is the only solution.

Also solved by Paul M.Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; Britton Stamper (student at Saint George's School), Spokane, WA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4993: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Find all real solutions of the equation

$$126x^7-127x^6+1=0.$$

Solution by N. J. Kuenzi, Oshkosh, WI.

Both 1 and 1/2 are easily seen to be positive rational roots of the given equation. So $(x-1)$ and $(2x-1)$ are both factors of the polynomial $126x^7 - 127x^6 + 1$. Factoring yields

$$126x^2 - 127x^6 + 1 = (x-1)(2x-1)(63x^5 + 31x^4 + 15c^3 + 7x^2 + 3x + 1).$$

The equation $(63x^5 + 31x^4 + 15c^3 + 7x^2 + 3x + 1)$ does not have any rational roots (Rational Roots Theorem) nor any positive real roots (Descartes' Rule of Signs).
Using numerical techniques one can find that -0.420834167 is the approximate value of a real root.
The four other roots are complex with approximate values:

$$0.1956354060 + 0.4093830251i \qquad 0.1956354060 - 0.4093830251i$$

$$-0.2312499936 + 0.3601917120i \quad -0.2312499936 - 0.3601917120i$$

So the real solutions of the equation $126x^7 - 127x^6 + 1 = 0$ are 1, 1/2 and -0.420834167.

Also solved by Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro GA, and the proposer.

- 4994: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b, c be three nonzero complex numbers lying on the circle $C = \{z \in \mathbf{C} : |\mathrm{z}| = \mathrm{r}\}$. Prove that the roots of the equation $az^2 + bz + c = 0$ lie in the ring shaped region $D = \left\{ z \in \mathbf{C} : \dfrac{1-\sqrt{5}}{2} \le |\mathrm{z}| \le \dfrac{1+\sqrt{5}}{2} \right\}$.

Solution by Kee-Wai Lau, Hong Kong, China.

By rewriting the equation as $az^2 = -bz - c$, we obtain

$$\begin{aligned} |a||z|^2 &= |az^2| = |bz + c| \le |b||z| + |c| \text{ or } |z|^2 - |z| - 1 \le 0 \\ &\text{or } \left(|z| + \frac{\sqrt{5}-1}{2}\right)\left(|z| - \frac{\sqrt{5}+1}{2}\right) \le 0 \text{ so that } |z| \le \frac{1+\sqrt{5}}{2}. \end{aligned}$$

By rewriting the equation as $c = -az^2 - bz$, we obtain

$$\begin{aligned} |c| &= |-az^2 - bz| \le |a||z|^2 + |b||z| \text{ or } |z|^2 + |z| - 1 \ge 0 \\ &\text{or } \left(|z| + \frac{\sqrt{5}+1}{2}\right)\left(|z| - \frac{\sqrt{5}-1}{2}\right) \ge 0 \text{ so that } |z| \ge \frac{\sqrt{5}-1}{2}. \end{aligned}$$

This finishes the solution.

Also solved by Michael Brozinsky, Central Islip, NY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Russell Euler and Jawad Sadek (jointly), Maryville, MO; Boris Rays, Chesapeake, VA; José Hernández Santiago (student at UTM) Oaxaca, México; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- **4995:** *Proposed by K. S. Bhanu and M. N. Deshpande, Nagpur, India.*

  Let A be a triangular array $a_{i,j}$ where $i = 1, 2, \cdots,$ and $j = 0, 1, 2, \cdots, i$. Let

$$a_{1,0} = 1, \ a_{1,1} = 2, \text{ and } a_{i,0} = T(i+1) - 2 \text{ for } i = 2, 3, 4, \cdots,$$

where $T(i+1) = (i+1)(i+2)/2$, the usual triangular numbers. Furthermore, let $a_{i,j+1} - a_{i,j} = j + 1$ for all j. Thus, the array will look like this:

$$\begin{array}{c} 1 \quad 2 \\ 4 \quad 5 \quad 7 \\ 8 \quad 9 \quad 11 \quad 14 \\ 13 \quad 14 \quad 16 \quad 19 \quad 23 \\ 19 \quad 20 \quad 22 \quad 25 \quad 29 \quad 34 \end{array}$$

Show that for every pair (i, j), $4a_{i,j} + 9$ is the sum of two perfect squares.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX.

If we allow $T(0) = 0$, then for $i \geq 1$ and $j = 0, 1, \ldots, i$, it's clear from the definition of $a_{i,j}$ that

$$\begin{aligned} a_{i,j} &= a_{i,0} + T(j) \\ &= T(i+1) - 2 + T(j) \\ &= \frac{i^2 + 3i - 2 + j^2 + j}{2}. \end{aligned}$$

Therefore, for every pair (i, j),

$$\begin{aligned} 4a_{i,j} + 9 &= 2\left(i^2 + 3i - 2 + j^2 + j\right) + 9 \\ &= 2\left(i^2 + 3i + j^2 + j\right) + 5 \\ &= (i + j + 2)^2 + (i - j + 1)^2. \end{aligned}$$

Solution 2 by Carl Libis, Kingston, RI.

For every pair $(i, j), 4a(i, j) + 9 = (i - j + 1)^2 + (i + j + 2)^2$ since

$$\begin{aligned} 4a(i,j) + 9 &= 4\left[a(i,0) + \frac{j(j+1)}{2}\right] + 9 = 4\left[\frac{(i+1)(i+2)}{2} - 2 + \frac{j(j+1)}{2}\right] + 9 \\ &= 2(i+1)(i+2) - 8 + 2j(j+1) + 9 \\ &= 2i^2 + 6i + 4 + 2j^2 + 2j + 1 \\ &= (i - j + 1)^2 + (i + j + 2)^2. \end{aligned}$$

Also solved by Paul M. Harms, North Newton, KS; N. J. Kuenzi, Oshkosh, WI; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro GA; José Hernándz Santiago (student at UTM), Oaxaca, México, and the proposers.

Problems

Ted Eisenberg, Section Editor

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1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

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Solutions to the problems stated in this issue should be posted before June 15, 2008

- 5014: *Proposed by Kenneth Korbin, New York, NY.*

  Given triangle ABC with $a = 100, b = 105$, and with equal cevians $\overline{AD}$ and $\overline{BE}$. Find the perimeter of the triangle if $\overline{AE} \cdot \overline{BD} = \overline{CE} \cdot \overline{CD}$.

- 5015: *Proposed by Kenneth Korbin, New York, NY.*

  Part I: Find the value of
  $$\sum_{x=1}^{10} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right).$$
  Part II: Find the value of
  $$\sum_{x=1}^{\infty} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right).$$

- 5016: *Proposed by John Nord, Spokane, WA.*

  Locate a point (p, q) in the Cartesian plane with integral values, such that for any line through (p, q) expressed in the general form $ax + by = c$, the coefficients a, b, c form an arithmetic progression.

- 5017: *Proposed by M.N. Deshpande, Nagpur, India.*

  Let ABC be a triangle such that each angle is less than 90^0. Show that
  $$\frac{a}{c \cdot \sin B} + \frac{1}{\tan A} = \frac{b}{a \cdot \sin C} + \frac{1}{\tan B} = \frac{c}{b \cdot \sin A} + \frac{1}{\tan C}$$

where $a = l(\overline{BC}), b = l(\overline{AC})$, and $c = l(\overline{AB})$.

- 5018: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Write the polynomial $x^{5020} + x^{1004} + 1$ as a product of two polynomials with integer coefficients.

- 5019: *Michael Brozinsky, Central Islip, NY.*

  In a horse race with 10 horses the horse with the number one on its saddle is referred to as the number one horse, and so on for the other numbers. The outcome of the race showed the number one horse did not finish first, the number two horse did not finish second, the number three horse did not finish third and the number four horse did not finish fourth. However, the number five horse did finish fifth. How many possible orders of finish are there for the ten horses assuming no ties?

Solutions

- 4996: *Proposed by Kenneth Korbin, New York, NY.*

  Simplify:

$$\sum_{i=1}^{N}\binom{N}{i}\left(2^{i-1}\right)\left(1+3^{N-i}\right).$$

Solution by José Hernández Santiago, (student, UTM, Oaxaca, México.)

$$\begin{aligned}
\sum_{i=1}^{N}\binom{N}{i}\left(2^{i-1}\right)\left(1+3^{N-i}\right) &= \sum_{i=1}^{N}\binom{N}{i}2^{i-1}+\sum_{i=1}^{N}\binom{N}{i}2^{i-1}\cdot 3^{N-i} \\
&= \frac{1}{2}\sum_{i=1}^{N}\binom{N}{i}2^{i}+\frac{3^N}{2}\sum_{i=1}^{N}\binom{N}{i}\left(\frac{2}{3}\right)^{i} \\
&= \left(\frac{1}{2}\right)\left(\left(2+1\right)^{N}-1\right)+\left(\frac{3^N}{2}\right)\left(\left(\frac{2}{3}+1\right)^{N}-1\right) \\
&= \frac{3^N-1}{2}+\frac{3^N}{2}\left(\frac{5^N-3^N}{3^N}\right) \\
&= \frac{(3^N-1)3^N+3^N(5^N-3^N)}{2\cdot 3^N} \\
&= \frac{15^N-3^N}{2\cdot 3^N} \\
&= \frac{5^N-1}{2}
\end{aligned}$$

Also solved by Brian D. Beasley, Clinton, SC; Michael Brozinsky, Central

Islip, NY; John Boncek, Montgomery, AL; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; Paul M. Harms, North Newton, KS; N. J. Kuenzi, Oshkosh, WI; Kee-Wai Lau, Hong Kong, China; Carl Libis, Kingston, RI; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4997: *Proposed by Kenneth Korbin, New York, NY.*

Three different triangles with integer-length sides all have the same perimeter P and all have the same area K.
Find the dimensions of these triangles if $K = 420$.

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX.

Let a, b, c be the sides of the triangle and, for convenience, assume that $a \le b \le c$. By Heron's Formula,

$$(420)^2 = \left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{-a+b+c}{2}\right) \tag{1}$$

Since a, b, c are positive integers, it is easily demonstrated that the quantities $(a+b-c)$, $(a-b+c)$, and $(-a+b+c)$ are all odd or all even. By (1), it is clear that in this case, they are all even. Therefore, there are positive integers x, y, z such that $a+b-c = 2x$, $a-b+c = 2y$, and $-a+b+c = 2z$. Then, $a = x+y$, $b = x+z$, $c = y+z$, $a+b+c = 2(x+y+z)$, and $a \le b \le c$ implies that $x \le y \le z$. With this substitution, (1) becomes

$$(420)^2 = xyz(x+y+z) \tag{2}$$

Since $x \le y \le z < x+y+z$, (2) implies that

$$x^4 < xyz(x+y+z) = (420)^2$$

and hence,

$$1 \le x \le \left\lfloor \sqrt{420} \right\rfloor = 20,$$

where $\lfloor m \rfloor$ denotes the greatest integer $\le m$. Therefore, the possible values of x are $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20$ (since x must also be a factor of $(420)^2$). Further, for each x, (2) implies that

$$y^3 < yz(x+y+z) = \frac{(420)^2}{x},$$

i.e.,

$$x \le y \le \left\lfloor \sqrt[3]{\frac{(420)^2}{x}} \right\rfloor,$$

(and y is a factor of $(420)^2 x$). Once we have assigned values to x and y, (2) becomes

$$z(x+y+z) = \frac{(420)^2}{xy}, \tag{3}$$

which is a quadratic equation in z. If (3) yields an integral solution $\ge y$, we have found a viable solution for x, y, z and hence, for a, b, c also. By finding all such solutions, we can find all Heronian triangles (triangles with integral sides and integral area) whose area is 420. Then, we must find three of these with the same perimeter to complete our solution.

The following two cases illustrate the typical steps encountered in this approach.

Case 1. If $x = 1$ and $y = 18$, (3) becomes

$$z^2 + 19z - 9800 = 0.$$

Since this has no integral solutions, this case does not lead to feasible values for a, b, c.

Case 2. If $x = 2$ and $y = 24$, (3) becomes

$$z^2 + 26z - 3675 = 0,$$

which has $z = 49$ as its only positive integral solution. These values of x, y, z yield $a = 26$, $b = 51$, $c = 73$, and $P = 150$.

The results of our approach are summarized in the following table.

x	y	z	a	b	c	P
1	6	168	7	169	174	350
1	14	105	15	106	119	240
1	20	84	21	85	104	210
1	25	72	26	73	97	196
1	40	49	41	50	89	180
2	24	49	26	51	73	150
2	35	35	37	37	70	144
4	21	35	25	39	56	120
5	9	56	14	61	65	140
5	21	30	26	35	51	112
6	15	35	21	41	50	112
8	21	21	29	29	42	100
9	20	20	29	29	40	98
10	15	24	25	34	39	98
12	12	25	24	37	37	98

Now, it is obvious that the last three entires constitute the solution of this problem.

Also solved by Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4998: *Proposed by Jyoti P. Shiwalkar & M.N. Deshpande, Nagpur, India.*

Let $A = [a_{i,j}]$, $i = 1, 2, \cdots$ and $j = 1, 2, \cdots, i$ be a triangular array satisfying the following conditions:

$$\begin{aligned} 1)\ a_{i,1} &= L(i) \text{ for all } i \\ 2)\ a_{i,i} &= i \text{ for all } i \\ 3)\ a_{i,j} &= a_{i-1,j} + a_{i-2,j} + a_{i-1,j-1} - a_{i-2,j-1} \text{ for } 2 \le j \le (i-1). \end{aligned}$$

If $T(i) = \sum_{j=1}^{i} a_{i,j}$ for all $i \ge 2$, then find a closed form for $T(i)$, where $L(i)$ are the Lucas numbers, $L(1) = 1, L(2) = 3$, and $L(i) = L(i-1) + L(i-2)$ for $i \ge 3$.

Solution by Paul M. Harms, North Newton, KS.

Note that $a_{i-2,j}$ is not in the triangular array when $j = i-1$, so we set $a_{i-2,i-1} = 0$. From Lucas numbers $a_{i,1} = a_{i-1,1} + a_{i-2,1}$ for $i > 2$. For $i > 2$,

$$\begin{aligned} T(i) &= a_{i,1} + a_{i,2} + \cdots + a_{i,i-1} + i \\ &= (a_{i-1,1} + a_{i-2,1}) + (a_{i-1,2} + a_{i-2,2} + a_{i-1,1} - a_{i-2,1}) + \cdots \\ &\quad +(a_{i-1,i-1} + a_{i-2,i-1} + a_{i-1,i-2} - a_{i-2,i-2}) + i. \end{aligned}$$

Therefore we have

$$(a_{i-1,i-1} + a_{i-2,i-1} + a_{i-1,i-2} - a_{i-2,i-2}) = (i-1) + 0 + a_{i-1,i-2} - (i-2).$$

Note that in $T(i)$ each term of row $(i-2)$ appears twice and subtracts out and each term of row $(i-1)$ except for the last term $(i-1)$, is added to itself. The term $(i-1)$ appears once. If we write the last term, i, of $T(i)$ as $i = (i-1)+1$, then $T(i) = 2T(i-1) + 1$. The values of the row sums are:

$$\begin{aligned} T(1) &= 1 \\ T(2) &= 5 \\ T(3) &= 2(5) + 1 \\ T(4) &= 2(2(5)+1) + 1 = 2^2(5) + 2 + 1 \\ T(5) &= 2\Big(2[2(5)+1]+1\Big) + 1 = 2^3(5) + 2^2 + 2 + 1, \text{ and in general} \\ T(i) &= 2^{i-2}(5) + (2^{i-3} + 2^{i-4} + \cdots + 1) \\ &= 2^{i-2}(5) + (2^{i-2} - 1) \\ &= 2^{i-2}(6) - 1 \\ &= 2^{i-1}(3) \text{ for } i \geq 2. \end{aligned}$$

Also solved by Carl Libis, Kingston, RI; N. J. Kuenzi, Oshkosh, WI; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- 4999: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all real triplets (x, y, z) such that

$$\begin{aligned} x + y + z &= 2 \\ 2^{x+y^2} + 2^{y+z^2} + 2^{z+x^2} &= 6\sqrt[9]{2} \end{aligned}$$

Solution by David E. Manes, Oneonta, NY.

The only real solution is $(x, y, z) = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$. Note that these values do satisfy each of the equations.

By the Arithmetic-Geometric Mean Inequality,

$$6\sqrt[9]{2} = 2^{x+y^2} + 2^{y+z^2} + 2^{z+x^2}$$

$$\geq \; 3\sqrt[3]{2^{x+y+z}\cdot 2^{x^2+y^2+z^2}} = 2\cdot 2^{2/3}\sqrt[3]{2^{x^2+y^2+z^2}}.$$

Therefore, $2^{x^2+y^2+z^2} \leq 2^{4/3}$ so that $x^2+y^2+z^2 \leq 4/3 \quad (\mathbf{1})$. Note that

$$\begin{aligned} 4=(x+y+z)^2 &= x^2+y^2+z^2+2(xy+yz+zx), \text{ so that} \\ x^2+y^2+z^2 &= 4-2(xy+yz+zx). \end{aligned}$$

Substituting in $(\mathbf{1})$ yields the inequality $xy+yz+zx \geq \frac{4}{3}$. From $(x-y)^2+(y-z)^2+(z-x)^2 \geq 0$ with equality if and only if $x=y=z$, one now obtains the inequalities

$$\frac{4}{3} \geq x^2+y^2+z^2 \geq xy+yz+zx \geq \frac{4}{3}.$$

Hence

$$\begin{aligned} x^2+y^2+z^2 &= xy+yz+zx = \frac{4}{3} \\ (x-y)^2+(y-z)^2+(z-x)^2 &= 0, \text{ and } x=y=z=\frac{2}{3}. \end{aligned}$$

.

Also solved by Dionne Bailey, Elsie Campbell, Charles Diminnie and Karl Havlak (jointly), San Angelo, TX; Michael Brozinsky, Central Islip, NY; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Math Dept. U. of Rome, Italy; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- 5000: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

Of all the right triangles inscribed in the unit circle, which has the Morley triangle of greatest area?

Solution by Ken Korbin, New York, NY.

Given $\triangle ABC$ with circumradius $R=1$ and with $A+B=C=90^o$.
The side x of the Morley triangle is given by the formula

$$\begin{aligned} x &= 8\cdot R\cdot \sin(\frac{A}{3})\cdot\sin(\frac{B}{3})\cdot\sin(\frac{C}{3}) \\ &= 8\cdot 1\cdot \sin(\frac{A}{3})\cdot\sin(\frac{B}{3})\cdot\frac{1}{2} \\ &= 4\sin(\frac{A}{3})\sin(\frac{B}{3}). \end{aligned}$$

x will have a maximum value if

$$\frac{A}{3}=\frac{B}{3}=\frac{45^o}{3}=15^o.$$

Then,

$$x = 4\sin^2(15^o)$$

$$
\begin{aligned}
&= 4\Big(\frac{1-\cos 30^o}{2}\Big)\\
&= 2 - 2\cos 30^o\\
&= 2-\sqrt{3}.
\end{aligned}
$$

The area of this Morley triangle is

$$
\begin{aligned}
& \frac{1}{2}\cdot(2-\sqrt{3})^2\cdot\sin 60^o\\
= & \frac{1}{2}(7-4\sqrt{3})\cdot\frac{\sqrt{3}}{2} = \frac{7\sqrt{3}-12}{4}.
\end{aligned}
$$

Comment by David Stone and John Hawkins: "It may be the maximum, but it is pretty small!"

Also solved by Michael Brozinsky, Central Islip, NY; Kee-Wai Lau, Hong Kong, China; David Stone an John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5001:** *Proposed by Ovidiu Furdui, Toledo, OH.*

Evaluate:

$$\int_0^\infty \ln^2\left(\frac{x^2}{x^2+3x+2}\right)dx.$$

Solution by Kee-Wai Lau, Hong Kong, China.

We show that $\displaystyle\int_0^\infty \ln^2\left(\frac{x^2}{x^2+3x+2}\right)dx = 2\ln^2 2+\frac{11\pi^2}{6}$.

Denote the integral by I. Replacing x by $1/x$, we obtain

$$
\begin{aligned}
I &= \int_0^\infty \frac{\ln^2\Big((x+1)(2x+1)\Big)}{x^2}dx = \int_0^\infty \frac{\ln^2(x+1)}{x^2}dx + \int_0^\infty\frac{\ln^2(2x+1)}{x^2}dx\\
&\qquad +2\int_0^\infty \frac{\ln(x+1)\ln(2x+1)}{x}dx\\
&= I_1+I_2+2I_3, \text{ say.}
\end{aligned}
$$

Integrating by parts, we obtain

$$I_1 = \int_0^\infty \ln^2(x+1)d(\frac{-1}{x}) = 2\int_0^\infty\frac{\ln(x+1)}{x(x+1)}dx = 2\int_1^\infty \frac{\ln x}{x(x-1)}dx.$$

Replacing x by $/(1-x)$,we obtain $I_1 = -2\displaystyle\int_0^1\frac{\ln(1-x)}{x}dx = 2\sum_{n=1}^\infty\frac{1}{n^2} = \frac{\pi^2}{3}$.

Replacing x by $x/2$ in I_2, we see that $I_2 = 2I_1 = \dfrac{2\pi^2}{3}$. Next note that

$$I_3 = \int_0^\infty \ln(x+1)\ln(2x+1)d(\frac{-1}{x}) = \int_0^\infty\frac{\ln(2x+1)}{x(x+1)}dx + 2\int_0^\infty\frac{\ln(x+1)}{x(2x+1)}dx = J_1+2J_2, \text{ say.}$$

Replacing x by $x/2$, then x by $x-1$ and then x by $1/x$, we have

$$J_1 = 2\int_0^{\infty} \frac{\ln(x+1)}{x(x+2)}dx \;=\; 2\int_1^{\infty} \frac{\ln x}{(x-1)(x+1)}dx$$

$$= -2\int_0^1 \frac{\ln x}{(1-x)(1+x)}dx = -\int_0^1 \ln x\Big(\frac{1}{1-x}+\frac{1}{1+x}\Big)dx.$$

Integrating by parts, we have,

$$J_1 = \int_0^1 \frac{-\ln(1-x)+\ln(1+x)}{x}dx = \sum_{n=1}^{\infty}\frac{1+(-1)^{n-1}}{n^2} = \frac{\pi^2}{6}+\frac{\pi^2}{12} = \frac{\pi^2}{4}.$$

We now evaluate J_2. Replacing $x+1$ by x and then x by $1/x$, we have

$$J_2 = \int_1^{\infty}\frac{\ln x}{(x-1)(2x-1)}dx = -\int_0^1 \frac{\ln x}{(1-x)(2-x)}dx = -\int_0^1 \frac{\ln x}{1-x}+\int_0^1 \frac{\ln x}{2-x}dx = \frac{\pi^2}{6}+K, \text{ say.}$$

Replacing x by $1-x$

$$K = \int_0^1 \frac{\ln(1-x)}{1+x}dx \;=\; \int_0^1 \frac{\ln(1+x)}{1+x}dx + \int_0^1 \frac{\ln(1-x)-\ln(1+x)}{1+x}dx$$

$$= \frac{1}{2}\ln^2 2 + \int_0^1 \frac{\ln\left(\dfrac{1-x}{1+x}\right)}{1+x}dx.$$

By putting $y = \dfrac{1-x}{1+x}$, we see that the last integral reduces to $\displaystyle\int_0^1 \frac{\ln y}{1+y}dy = -\frac{\pi^2}{12}$.

Hence, $K = \dfrac{1}{2}\ln^2 2 - \dfrac{\pi^2}{12}$, $J_2 = \dfrac{1}{2}\ln^2 2 + \dfrac{\pi^2}{12}$, $I_3 = \ln^2 2 + \dfrac{5\pi^2}{12}$ and finally

$$I = I_1 + I_2 + 2I_3 = \frac{\pi^2}{3} + \frac{2\pi^2}{3} + 2\Big(\ln^2 2 + \frac{5\pi^2}{12}\Big) = 2\ln^2 2 + \frac{11\pi^2}{6} \text{ as desired.}$$

Also solved by Paolo Perfetti, Math. Dept., U. of Rome, Italy; Worapol Rattanapan (student at Montfort College (high school)), Chiang Mai, Thailand, and the proposer.

Problems

Ted Eisenberg, Section Editor

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Solutions to the problems stated in this issue should be posted before
September 15, 2008

- 5020: *Proposed by Kenneth Korbin, New York, NY.*

  Find positive numbers x and y such that

$$\begin{cases} x^7 - 13y = 21 \\ 13x - y^7 = 21 \end{cases}$$

- 5021: *Proposed by Kenneth Korbin, New York, NY.*

  Given

$$\frac{x + x^2}{1 - 34x + x^2} = x + 35x^2 + \cdots + a_n x^n + \cdots$$

  Find an explicit formula for a_n.

- 5022: *Proposed by Michael Brozinsky, Central Islip, NY .*

  Show that

$$\sin\left(\frac{x}{3}\right)\sin\left(\frac{\pi + x}{3}\right)\sin\left(\frac{2\pi + x}{3}\right)$$

  is proportional to $\sin(x)$.

- 5023: *Proposed by M.N. Deshpande, Nagpur, India.*

  Let $A_1A_2A_3 \cdots A_n$ be a regular n-gon ($n \geq 4$) whose sides are of unit length. From A_k draw L_k parallel to $\mathrm{A}_{k+1}A_{k+2}$ and let L_k meet L_{k+1} at T_k. Then we have a "necklace" of congruent isosceles triangles bordering $A_1A_2A_3 \cdots A_n$ on the inside boundary. Find the total area of this necklace of triangles.

- 5024: *Proposed by José Luis Díaz-Barrero and Josep Rubió-Massegú, Barcelona, Spain.*

  Find all real solutions to the equation

$$\sqrt{1+\sqrt{1-x}}-2\sqrt{1-\sqrt{1-x}}=\sqrt[4]{x}.$$

- 5025: *Ovidiu Furdui, Toledo, OH.*

  Calculate the double integral

$$\int_0^1\int_0^1\{x-y\}dxdy,$$

  where $\{a\}=a-[a]$ denotes the fractional part of a.

Solutions

- 5002: *Proposed by Kenneth Korbin, New York, NY.*

  A convex hexagon with sides $3x, 3x, 3x, 5x, 5x$ and $5x$ is inscribed in a unit circle. Find the value of x.

  Solution by David E. Manes, Oneonta, NY.

  The value of x is $\dfrac{\sqrt{3}}{7}$.

  Note that each inscribed side of the hexagon subtends an angle at the center of the circle that is independent of its position in the circle The sides are subject to the constraint that the sum of the angles subtended at the center equals 360^o. Therefore the sides of the hexagon can be permuted from $3x, 3x, 3x, 5x, 5x, 5x$ to $3x, 5x, 3x, 5x, 3x, 5x$. In problem **4974** : (December 2007, Korbin, Lau) it is shown that the circumradius r is then given by

$$r=\sqrt{\frac{(3x)^2+(5x)^2+(3x)(5x)}{3}}.$$

  With $r=1$, one obtains $x=\dfrac{\sqrt{3}}{7}$.

  Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; John Boncek, Montgomery, AL; M.N. Deshpande, Nagpur, India; José Luis Díaz-Barrero, Barcelona, Spain; Grant Evans (student at St George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Minerva P. Harwell (student at Auburn University), Montgomery, AL; Kee-Wai Lau, Hong Kong, China; Peter E. Liley, Lafayette, IN; Amanda Miller (student at St. George's School), Spokane, WA; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5003: *Proposed by Kenneth Korbin, New York, NY.*

  Find positive numbers x and y such that

$$\sqrt[3]{x+\sqrt{x^2-1}}+\sqrt[3]{x-\sqrt{x^2-1}} \quad = \quad \frac{7}{2} \text{ and}$$

$$\sqrt[3]{y+\sqrt{y^2-1}}+\sqrt[3]{y-\sqrt{y^2-1}} \quad = \quad \sqrt{10}$$

Solution by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX.

Let $A = \sqrt[3]{x+\sqrt{x^2-1}}$ and $B = \sqrt[3]{x-\sqrt{x^2-1}}$. Note that

$$\begin{aligned} A^3+B^3 &= 2x \quad \text{and} \\ AB &= 1. \end{aligned}$$

Since $A+B = \dfrac{7}{2}$,

$$\begin{aligned} \frac{343}{8} &= (A+B)^3 \\ &= A^3+3A^2B+3AB^2+B^3 \\ &= A^3+B^3+3AB(A+B) \\ &= 2x+\frac{21}{2}. \end{aligned}$$

Thus, $x = \dfrac{259}{16}$.

Similarly,

$$2y+3\sqrt{10} = 10\sqrt{10}$$

and, thus, $y = \dfrac{7\sqrt{10}}{2}$.

Also solved by Brian D. Beasley, Clinton, SC; John Boncek, Montgomery, AL; M.N. Deshpande, Nagpur, India; José Luis Díaz-Barrero, Barcelona, Spain; Grant Evans (student at St. George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Amanda Miller (student at St. George's School), Spokane, WA; John Nord, Spokane, WA; Paolo Perfetti (Department of Mathematics, University of Rome), Italy; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5004: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b, c be nonnegative real numbers. Prove that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c} \geq \frac{\sqrt{ab}}{1+a+b}+\frac{\sqrt{bc}}{1+b+c}+\frac{\sqrt{ac}}{1+c+a}$$

Solution by John Boncek, Montgomery, AL.

We use the arithmetic-geometric inequality: If $x, y \geq 0$, then $x+y \geq 2\sqrt{xy}$. Now

$$\begin{aligned} \frac{a}{1+a} &\geq \frac{a}{1+a+b}, \quad \text{and} \\ \frac{b}{1+b} &\geq \frac{b}{1+a+b}, \quad \text{so} \end{aligned}$$

$$\frac{a}{1+a}+\frac{b}{1+b} \quad \geq \quad \frac{a+b}{1+a+b} \geq \frac{2\sqrt{ab}}{1+a+b}.$$

Similarly,

$$\frac{a}{1+a}+\frac{c}{1+c} \geq \frac{2\sqrt{ac}}{1+a+c}, \text{ and}$$

$$\frac{b}{1+b}+\frac{c}{1+c} \geq \frac{2\sqrt{bc}}{1+b+c}\ .$$

Summing up all three inequalities, we obtain

$$2\left(\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}\right) \geq \frac{2\sqrt{ab}}{1+a+b}+\frac{2\sqrt{ac}}{1+a+c}+\frac{2\sqrt{bc}}{1+b+c}.$$

Divide both sides of the inequality by 2 to obtain the result.

Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; M.N. Deshpande, Nagpur, India; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Paolo Perfetti (Department of Mathematics, University of Rome), Italy; Boris Rays, Chesapeake, VA, and the proposers.

- **5005:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{\sqrt{3}}{2}\left(a+b+c\right)^{1/2} \geq \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China.

Since $a+b \geq 2\sqrt{ab} = \dfrac{2}{\sqrt{c}}$ and so on, and by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \\
\leq\ & \frac{\sqrt{c}+\sqrt{a}+\sqrt{b}}{2} \\
=\ & \frac{1}{2}\Big((1)\sqrt{a}+(1)\sqrt{b}+(1)\sqrt{c}\Big) \\
\leq\ & \frac{1}{2}\sqrt{1+1+1}\sqrt{a+b+c} \\
=\ & \frac{\sqrt{3}}{2}\Big(a+b+c)^{1/2}
\end{aligned}$$

as required.

Solution 2 by Charles McCracken, Dayton, OH.

Suppose a=b=c=1. Then the original inequality reduces to $\dfrac{3}{2} \geq \dfrac{3}{2}$ which is certainly true.

Let L represent the left side of the original inequality and let R represent the right side. Allow a, b, and c to vary and take partial derivatives.

$$\frac{\partial L}{\partial a} = \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\Big(a+b+c\Big)^{-1/2} > 0. \text{ Similarly, } \frac{\partial L}{\partial b} > 0 \text{ and } \frac{\partial \mathrm{L}}{\partial \mathrm{c}} > 0.$$
$$\frac{\partial R}{\partial a} = -(a+b)^{-2} - c(a+b)^{-2} < 0. \text{ Similarly, } \frac{\partial R}{\partial b} < 0 \text{ and } \frac{\partial R}{\partial c} < 0.$$

So any change in a, b or c results in an increase in L and a decrease in R so that L is always greater than R.

Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Paolo Perfetti (Department of Mathematics, University of Rome), Italy, and the proposer.

- 5006: *Proposed by Ovidiu Furdui, Toledo, OH.*

Find the sum
$$\sum_{k=2}^{\infty}(-1)^k \ln\left(1-\frac{1}{k^2}\right).$$

Solution 1 by Paul M. Harms, North Newton, KS.

Using $\ln\left(1-\dfrac{1}{k^2}\right) = \ln\left(\dfrac{k-1}{k}\right) + \ln\left(\dfrac{k+1}{k}\right)$, the summation is

$$\begin{aligned}
&\left(\ln\frac{1}{2}+\ln\frac{3}{2}\right) - \left(\ln\frac{2}{3}+\ln\frac{4}{3}\right) + \left(\ln\frac{3}{4}+\ln\frac{5}{4}\right) - \ln\left(\frac{4}{5}+\ln\frac{6}{5}\right) + \cdots\\
= &\ln\left(\frac{1}{2}\right) + \ln\left(\frac{3}{2}\right)^2 + \ln\left(\frac{3}{4}\right)^2 + \ln\left(\frac{5}{4}\right)^2 + \cdots\\
= &\ln\left(\frac{1}{2}\right) + 2\Big[\ln\left(\frac{3}{2}\right) + \ln\left(\frac{3}{4}\right) + \ln\left(\frac{5}{4}\right) + \ln\left(\frac{5}{6}\right) + \cdots\Big]\\
= &\ln\left(\frac{1}{2}\right) + 2\ln\left(\frac{3}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\cdots.
\end{aligned}$$

Wallis' product for $\dfrac{\pi}{2}$ is

$$\frac{\pi}{2} = \left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{5}\right)\left(\frac{6}{7}\right)\cdots.$$

Dividing both sides by 2 and taking the reciprocal yields

$$\frac{4}{\pi} = \left(\frac{3}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{7}{8}\right)\cdots.$$

The summation in the problem is then

$$\ln\left(\frac{1}{2}\right) + 2\ln\left(\frac{4}{\pi}\right) = \ln\left[\left(\frac{1}{2}\right)\left(\frac{16}{\pi^2}\right)\right] = \ln\left(\frac{8}{\pi^2}\right).$$

Solution 2 by Kee-Wai Lau, Hong Kong, China.

It can be proved readily by induction that for positive intergers n,

$$\sum_{k=2}^{2n}(-1)^k \ln\left(1-\frac{1}{k^2}\right) = 4\Big(\ln((2n)!) - 2\ln(n!)\Big) + \ln n + \ln(2n+1) - 2(4n-1)\ln 2.$$

By using the Stirling approximation $\ln(n!) = n\ln n - n + \dfrac{1}{2}\ln(2\pi n) + O\Big(\dfrac{1}{n}\Big)$ as $n \to \infty$, we obtain

$$\ln((2n)!) - 2\ln(n!) = 2n\ln 2 - \frac{\ln n}{2} - \frac{\ln \pi}{2} + O\Big(\frac{1}{n}\Big).$$

It follows that

$$\sum_{k=2}^{2n}(-1)^k \ln\left(1-\frac{1}{k^2}\right) = 3\ln 2 - 2\ln\pi + \ln\left(1+\frac{1}{2n}\right) + O\Big(\frac{1}{n}\Big) = 3\ln 2 - 2\ln\pi + O\Big(\frac{1}{n}\Big)$$

and that $\displaystyle\sum_{k=2}^{2n+1}(-1)^k \ln\left(1-\frac{1}{k^2}\right) = 3\ln 2 - 2\ln\pi + O\Big(\frac{1}{n}\Big)$ as well.

This shows that the sum of the problem equal $3\ln 2 - 2\ln\pi = \ln\left(\dfrac{8}{\pi^2}\right)$.

Also solved by Brian D. Beasley, Clinton, SC; Worapol Rattanapan (student at Montfort College (high school)), Chiang Mai, Thailand; Paolo Perfetti (Department of Mathematics, University of Rome), Italy; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5007: *Richard L. Francis, Cape Girardeau, MO.*

  Is the centroid of a triangle the same as the centroid of its Morley triangle?

  Solution by Kenneth Korbin, New York, NY.

  The centroids are not the same unless the triangle is equilateral.

  For example, the isosceles right triangle with vertices at $(-6,0), (6,0)$ and $(0,6)$ has its centroid at $(0,2)$.

  Its Morley triangle has verticies at $(0, 12-6\sqrt{3}), (-6+3\sqrt{3}, 3)$, and $(6-3\sqrt{3}, 3)$ and has its centroid at $(0, 6-2\sqrt{3})$.

  Also solved by Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
December 15, 2008

- 5026: *Proposed by Kenneth Korbin, New York, NY.*

  Given quadrilateral $ABCD$ with coordinates $A(-3,0),\ B(12,0),\ C(4,15)$, and $D(0,4)$. Point P has coordinates $(x,3)$. Find the value of x if

  $$\text{area}\ \triangle\text{PAD} + \text{area}\ \triangle\text{PBC} = \text{area}\ \triangle\text{PAB} + \text{area}\ \triangle\text{PCD}.$$

- 5027: *Proposed by Kenneth Korbin, New York, NY.*

  Find the x and y intercepts of

  $$y = x^7 + x^6 + x^4 + x^3 + 1.$$

- 5028: *Proposed by Michael Brozinsky, Central Islip, NY .*

  If the ratio of the area of the square inscribed in an isosceles triangle with one side on the base to the area of the triangle uniquely determine the base angles, find the base angles.

- 5029: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $x > 1$ be a non-integer number. Prove that

  $$\left(\frac{x+\{x\}}{[x]} - \frac{[x]}{x+\{x\}}\right) + \left(\frac{x+[x]}{\{x\}} - \frac{\{x\}}{x+[x]}\right) > \frac{9}{2},$$

  where $[x]$ and $\{x\}$ represents the entire and fractional part of x.

- 5030: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $A_1, A_2, \cdots, A_n \in M_2(\mathbf{C}), (n \geq 2)$, be the solutions of the equation $X^n = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$.

  Prove that $\sum_{k=1}^{n} Tr(A_k) = 0$.

- 5031: *Ovidiu Furdui, Toledo, OH.*

  Let x be a real number. Find the sum

$$\sum_{n=1}^{\infty}(-1)^{n-1}n\left(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\right).$$

Solutions

- 5008: *Proposed by Kenneth Korbin, New York, NY.*

  Given isosceles trapezoid $ABCD$ with $\angle ABD = 60^o$, and with legs $\overline{BC} = \overline{AD} = 31$. Find the perimeter of the trapezoid if each of the bases has positive integer length with $\overline{AB} > \overline{CD}$.

  Solution by David C. Wilson, Winston-Salem, N.C.

  Let the side lengths of $\overline{\text{AB}}= x$, $\overline{\text{BC}}$=31, $\overline{CD}= y$, $\overline{\text{DA}}$=31, and $\overline{\text{BD}}$=z.
  By the law of cosines

$$\begin{aligned}
31^2 &= x^2 + z^2 - 2xz\cos 60^o \ \text{ and} \\
31^2 &= y^2 + z^2 - 2yz\cos 60^o \implies \\
961 &= z^2 + x^2 - xz \ \text{ and} \\
961 &= y^2 + z^2 - yz \implies \\
0 &= (y^2 - x^2) - yz + xz \implies \\
0 &= (y - x)(y + x) - z(y - x) = (y - x)(y + x - z) \implies \\
y - x &= 0 \text{ or } \ y + x - z = 0.
\end{aligned}$$

  But $\overline{AB} > \overline{CD} \implies x > y \implies y - x \neq 0$. Thus, $y + x - z = 0 \implies z = x + y$. Thus,

$$961 = (x + y)^2 + x^2 - x(x + y) = x^2 + 2xy + y^2 + x^2 - x^2 - xy = x^2 + xy + y^2.$$

  Consider $x = 30, 29, \cdots, 18$. After trial and error with a calculator, when $x = 24$ then $y = 11 \implies z = 35$ and these check. Thus, the perimeter of $ABCD$ is $35 + 31 + 31 = 97$.

  Also solved by Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie (jointly), San Angelo, TX; Matt DeLong, Upland, IN; Lauren Christenson, Taylor Brennan, Ross Hayden, and Meaghan Haynes (jointly; students at Taylor University), Upland, IN; Charles McCracken, Dayton, OH; Amanda Miller (student, St.George's School), Spokane, WA; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5009:** *Proposed by Kenneth Korbin, New York, NY.*

Given equilateral triangle ABC with a cevian $\overline{CD}$ such that $\overline{AD}$ and $\overline{BD}$ have integer lengths. Find the side of the triangle $\overline{AB}$ if $\overline{CD} = 1729$ and if $(\overline{AB}, 1729) = 1$.

Solution by David Stone and John Hawkins, Statesboro, GA.

The answer: $\overline{AB} = 1775, 1840, 1961, 1984$.

Let $x = \overline{AD}$ and $y = \overline{BD}$, with $s = x + y =$ the side length $\overline{AB}$. Applying the Law of cosines in each "subtriangle," we have

$$\begin{aligned} 1729^2 &= s^2 + x^2 - 2sx\cos\frac{\pi}{3} = s^2 + x^2 - sx \text{ and} \\ 1729^2 &= s^2 + y^2 - 2sy\cos\frac{\pi}{3} = s^2 + y^2 - sy. \end{aligned}$$

After adding equations and doing some algebra, we obtain the equation

$$y^2 + xy + x^2 = 1729^2.$$

Solving for y by the Quadratic Formula, we obtain

$$y = \frac{-x \pm \sqrt{4 \cdot 1729^2 - 3x^2}}{2} = \frac{-x \pm z}{2}$$

where $z = \sqrt{4 \cdot 1729^2 - 3x^2}$ must be an integer.

Because y must be positive, we have to choose $y = \dfrac{-x + z}{2}$.

Now we let Excel calculate, trying $x = 1, 2, \cdots, 1729$. We have 13 "solutions", but only four of them have $s = \overline{AB}$ relatively prime to 1729; hence only equilateral triangles of side length $AB = 1775, 1840, 1961$, and 1984 admit the cevian described in the problem.

x	$z = \sqrt{34586^2 - 3x^2}$	$y = (-x + z)/2$	$s = x + y$	$gcd(1729, s)$
96	3454	1679	1775	1
209	3439	1615	1824	19
249	3431	1591	1840	1
299	3419	1560	1859	13
361	3401	1520	1881	19
455	3367	1456	1911	91
504	3346	1421	1925	7
651	3269	1309	1960	7
656	3266	1305	1961	1
741	3211	1235	1976	247
799	3169	1185	1984	1
845	3133	1144	1989	13
931	3059	1064	1995	133

Note that we could let x run further, but the problem is symmetric in x and y, so we'd just recover these same solutions with x and y interchanged.

Comment by Kenneth Korbin, the proposer.

In the problem $\overline{CD} = (7)(13)(19)$ and there were exactly 4 possible answers. If $\overline{CD}$ would have been equal to $(7)(13)(19)(31)$then there would have been exactly 8 possible solutions.

Similarly, there are exactly 4 primitive Pythagorean triangles with hypotenuse (5)(13)(17) and there exactly 8 primitive Pythagorean triangles with hypotenuse (5)(13)(17)(29). And so on.

Also solved by Charles McCracken, Dayton, OH; David E. Manes, Oneonta, NY; David C. Wilson, Winston-Salem, NC, and the proposer.

- 5010: *Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let α, β, and γ be real numbers such that $0 < \alpha \leq \beta \leq \gamma < \pi/2$. Prove that

$$\frac{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}{(\sin\alpha + \sin\beta + \sin\gamma)(\cos\alpha + \cos\beta + \cos\gamma)} \leq \frac{2}{3}.$$

Solution by Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy.

Proof After some simple simplification the inequality is

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma \leq \sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha)$$

The concavity of $\sin(x)$ in the interval $[0, \pi]$ allows us to write $\sin(x+y) \geq (\sin(2x) + \sin(2y))/2$ thus

$$\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) \geq \sin 2\alpha + \sin 2\beta + \sin 2\gamma$$

concluding the proof.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Chesapeak, VA, and the proposers.

- 5011: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $\{a_n\}_{n\geq 0}$ be the sequence defined by $a_0 = a_1 = 2$ and for $n \geq 2,\ a_n = 2a_{n-1} - \dfrac{1}{2}a_{n-2}$. Prove that

$$2^p a_{p+q} + a_{q-p} = 2^p a_p a_q$$

where $p \leq q$ are nonnegative integers.

Solution 1 by R. P. Sealy, Sackville, New Brunswick, Canada.

Solving the characteristic equation

$$r^2 - 2r + \frac{1}{2} = 0$$

and using the intitial conditions, we obtain the solution

$$a_n = \left(\frac{2+\sqrt{2}}{2}\right)^n + \left(\frac{2-\sqrt{2}}{2}\right)^n.$$

Note that

$$2^p a_{p+q} \quad = \quad \frac{(2+\sqrt{2})^{p+q} + (2-\sqrt{2})^{p+q}}{2^q} \text{ and}$$

$$\begin{aligned}
a_{q-p} &= \frac{(2+\sqrt{2})^{q-p}+(2-\sqrt{2})^{q-p}}{2^{q-p}} \text{ while}\\
2^p a_p a_q &= \frac{(2+\sqrt{2})^{p+q}+(2-\sqrt{2})^{p+q}+2^p[(2+\sqrt{2})^{q-p}+(2-\sqrt{2})^{q-p}]}{2^q}\\
&= 2^p a_{p+q}+a_{q-p}.
\end{aligned}$$

Solution 2 by Kee-Wai Lau, Hong Kong, China.

By induction, we obtain readily that for $n \geq 0$,

$$a_n = \left(\frac{2+\sqrt{2}}{2}\right)^n + \left(\frac{2-\sqrt{2}}{2}\right)^n.$$

Hence

$$\begin{aligned}
a_p a_q &= \left(\left(\frac{2+\sqrt{2}}{2}\right)^p + \left(\frac{2-\sqrt{2}}{2}\right)^p\right)\left(\left(\frac{2+\sqrt{2}}{2}\right)^q + \left(\frac{2-\sqrt{2}}{2}\right)^q\right)\\
&= \left(\left(\frac{2+\sqrt{2}}{2}\right)^{p+q} + \left(\frac{2-\sqrt{2}}{2}\right)^{p+q}\right) + \left(\frac{2+\sqrt{2}}{2}\right)^p\left(\frac{2-\sqrt{2}}{2}\right)^q + \left(\frac{2-\sqrt{2}}{2}\right)^p\left(\frac{2+\sqrt{2}}{2}\right)^q\\
&= a_{p+q} + \left(\frac{2+\sqrt{2}}{2}\right)^p\left(\frac{2-\sqrt{2}}{2}\right)^q\left(\left(\frac{2-\sqrt{2}}{2}\right)^{q-p} + \left(\frac{2+\sqrt{2}}{2}\right)^q\right)\\
&= a_{p+q} + \frac{1}{2^p}a_{q-p},
\end{aligned}$$

and the identity of the problem follows.

Also solved by Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Jose Hernández Santiago (student, UTM), Oaxaca, México; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 5012: *Richard L. Francis, Cape Girardeau, MO.*

Is the incenter of a triangle the same as the incenter of its Morley triangle?

Solution 1 by Kenneth Korbin, New York, NY.

The incenters are not the same unless the triangle is equilateral. For example, the isosceles right triangle with vertices at $(-6,0),(6,0)$ and $(0,6)$ has its incenter at $(0,6\sqrt{2}-6)$.

Its Morely triangle has vertices at $(0,12-6\sqrt{3}),(-6+3\sqrt{3},3)$, and $(6-3\sqrt{3},3)$ and has its incenter at $(0,6-2\sqrt{3})$.

Solution 2 by Kee-Wai Lau, Hong-Kong, China.

We show that the incenter I of a triangle ABC is the same as the incenter I_M of its Morley triangle if and only if ABC is equilateral.

In homogeneous trilinear coordinates, I is $1:1:1$ and I_M is

$$\cos\left(\frac{A}{3}\right)+2\cos\left(\frac{B}{3}\right)\cos\left(\frac{C}{3}\right):\cos\left(\frac{B}{3}\right)+2\cos\left(\frac{C}{3}\right)\cos\left(\frac{A}{3}\right):\cos\left(\frac{C}{3}\right)+2\cos\left(\frac{A}{3}\right)\cos\left(\frac{B}{3}\right).$$

Clearly if ABC is equilateral, then $I=I_M$. Now suppose that $I=I_M$ so that

$$\cos\left(\frac{A}{3}\right)+2\cos\left(\frac{B}{3}\right)\cos\left(\frac{C}{3}\right) = \cos\left(\frac{B}{3}\right)+2\cos\left(\frac{C}{3}\right)\cos\left(\frac{A}{3}\right) \quad (1)$$

$$\cos\left(\frac{B}{3}\right)+2\cos\left(\frac{C}{3}\right)\cos\left(\frac{A}{3}\right) = \cos\left(\frac{C}{3}\right)+2\cos\left(\frac{A}{3}\right)\cos\left(\frac{B}{3}\right). \quad (2)$$

From (1) we obtain

$$\left(\cos\left(\frac{A}{3}\right)-\cos\left(\frac{B}{3}\right)\right)\left(1-2\cos\left(\frac{C}{3}\right)\right)=0.$$

Since $0<C<\pi$, so

$$1-2\cos\left(\frac{C}{3}\right)<0.$$

Thus,

$$\cos\left(\frac{A}{3}\right)=\cos\left(\frac{B}{3}\right) \text{ or } A=B.$$

Similarly from (2) we obtain $B=C$. It follows that ABC is equilateral and this completes the solution.

Also solved by David E. Manes, Oneonta, NY, and the proposer.

- 5013: *Proposed by Ovidiu Furdui, Toledo, OH.*

Let $k\geq 2$ be a natural number. Find the sum

$$\sum_{n_1,n_2,\cdots,n_k\geq 1}\frac{(-1)^{n_1+n_2+\cdots+n_k}}{n_1+n_2+\cdots+n_k}.$$

Solution by Kee-Wai Lau, Hong Kong, China.

For positive integers $M_1, M_2, \cdots, M_k$, we have

$$\sum_{n_1=1}^{M_1}\sum_{n_2=1}^{M_2}\cdots\sum_{n_k=1}^{M_k}\frac{(-1)^{n_1+n_2+\cdots+n_k}}{n_1+n_2\cdots+n_k}$$

$$= \sum_{n_1=1}^{M_1}\sum_{n_2=1}^{M_2}\cdots\sum_{n_k=1}^{M_k}(-1)^{n_1+n_2+\cdots+n_k}\int_0^1 x^{n_1+n_2+\cdots+n_k-1}dx$$

$$= \int_0^1\left(\sum_{n_1=1}^{M_1}(-1)^{n_1}x^{n_1}\right)\left(\sum_{n_2=1}^{M_2}(-1)^{n_2}x^{n_2}\right)\cdots\left(\sum_{n_k=1}^{M_k}(-1)^{n_k}x^{n_k}\right)x^{-1}dx$$

$$= \int_0^1\left(\frac{-x(1-(-x)^{M_1})}{1+x}\right)\left(\frac{-x(1-(-x)^{M_2})}{1+x}\right)\left(\frac{-x(1-(-x)^{M_k})}{1+x}\right)x^{-1}dx$$

$$\begin{aligned}
&= (-1)^k \int_0^1 \frac{x^{k-1}(1-(-x)^{M_1})(1-(-x)^{M_2})\cdots(1-(-x)^{M_k}))}{(1+x)^k}dx \\
&= (-1)^k \int_0^1 \frac{x^{k-1}}{(1+x)^k}dx + O\Big(\int_0^1 x^{M_1} + x^{M_2} + \cdots + x^{M_k}\Big)dx \\
&= (-1)^k \int_0^1 \frac{x^{k-1}}{(1+x)^k}dx + O\Big(\frac{1}{M_1} + \frac{1}{M_2} + \cdots + \frac{1}{M_k}\Big)
\end{aligned}$$

as $M_1, M_2, \cdots, M_k$ tend to infinity. Here the constants implied by the $O's$ depend at most on k.
It follows that the sum of the problem equals

$$(-1)^k \int_0^1 \frac{x^{k-1}}{1+x)^k}dx = (-1)^k I_k, \quad \text{say.}$$

Integrating by parts, we have for $k \geq 3$,

$$\begin{aligned}
I_k &= \frac{1}{1-k}\int_0^1 x^{k-1} d((1+x)^{1-k}) \\
&= \frac{-1}{(k-1)2^{k-1}} + I_{k-1}.
\end{aligned}$$

Since $I_2 = \ln 2 - \dfrac{1}{2}$, we obtain readily by induction that for $k \geq 2$.

$$I_k = \ln 2 - \sum_{j=2}^{k} \frac{1}{(j-1)2^{j-1}}.$$

we now conclude that for $k \geq 2$,

$$\sum_{n_1, n_2, \cdots, n_k \geq 1} \frac{(-1)^{n_1+n_2+\cdots+n_k}}{n_1+n_2+\cdots+n_k} = (-1)^k\Big(\ln 2 - \sum_{j=1}^{k-1}\frac{1}{j(2^j)}\Big).$$

Also solved by Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy; Paul M. Harms, North Newton, KS; Boris Rays, Chesapeake, VA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
January 15, 2009

- 5032: *Proposed by Kenneth Korbin, New York, NY.*

  Given positive acute angles A, B, C such that

  $$\tan A \cdot \tan B + \tan B \cdot \tan C + \tan C \cdot \tan A = 1.$$

  Find the value of
  $$\frac{\sin A}{\cos B \cdot \cos C} + \frac{\sin B}{\cos A \cdot \cos C} + \frac{\sin C}{\cos A \cdot \cos B}.$$

- 5033: *Proposed by Kenneth Korbin, New York, NY.*

  Given quadrilateral $ABCD$ with coordinates $A(-3,0), B(12,0), C(4,15)$, and $D(0,4)$. Point P is on side $\overline{AB}$ and point Q is on side $\overline{CD}$. Find the coordinates of P and Q if area $\triangle PCD =$ area $\triangle QAB = \frac{1}{2}$area quadrilateral $ABCD$.

- 5034: *Proposed by Roger Izard, Dallas, TX.*

  In rectangle $MDCB$, $MB \perp MD$. F is the midpoint of BC, and points N, E and G lie on line segments DC, DM, and MB respectively, such that $NC = GB$. Let the area of quadrilateral $MGFC$ be A_1 and let the area of quadrilateral $MGFE$ be A_2. Determine the area of quadrilateral $EDNF$ in terms of A_1 and A_2.

- 5035: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers. Prove that

  $$(a^a b^b c^c)^2 (a^{-(b+c)} + b^{-(c+a)} + c^{-(a+b)})^3 \geq 27.$$

- 5036: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all triples (x, y, z) of nonnegative numbers such that

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ 3^x + 3^y + 3^z = 5 \end{cases}$$

- 5037: *Ovidiu Furdui, Toledo, OH.*

  Let k, p be natural numbers. Prove that

$$1^k + 3^k + 5^k + \cdots + (2n+1)^k = (1 + 3 + \cdots + (2n+1))^p$$

  for all $n \geq 1$ if and only if $k = p = 1$.

Solutions

- 5014: *Proposed by Kenneth Korbin, New York, NY.*

  Given triangle ABC with $a = 100, b = 105$, and with equal cevians $\overline{AD}$ and $\overline{BE}$. Find the perimeter of the triangle if $\overline{AE} \cdot \overline{BD} = \overline{CE} \cdot \overline{CD}$.

  Solution by David Stone and John Hawkins, Statesboro, GA.

  The solution to this problem is more complex than expected. There are infinitely many triangles satisfying the given conditions, governed in a sense by two types of degeneracy. The nicest of these solutions is a right triangle with integer sides, dictated by the given data: $100 = 5(20)$ and $105 = 5(21)$ and $(20, 21, 29)$ is a Pythagorean triple.

  One type of degeneracy is the usual: if $AB = 5$ or $AB = 205$, we have a degenerate triangle which can be shown to satisfy the conditions of the problem.

  The other type of degeneracy is problem specific: when neither cevian intersects the interior of its targeted side, but lies along a side of the triangle. In these two situations, the problem's condition are also met.

  Let x = length of CE so $0 \leq x \leq 105$. The following table summarizes our results.

$x = CE$	$\cos(C)$	C	AB	BD	*Perimeter*	$AD = BE$	*note*
0	$\frac{21}{40}$	$\cos^{-1}\left(\frac{21}{40}\right)$	100	0	305	100	1
21	$\frac{194}{350}$	$\cos^{-1}\left(\frac{194}{350}\right)$	$\sqrt{9385}$	20	301.88	$\sqrt{8113}$	2
$\frac{1985}{41}$	1	0	5	$\frac{1900}{41}$	210	$\frac{2105}{41}$	3
excluded	values	*****	*****	*****	*****	*****	
$\frac{2205}{41}$	-1	180^o	205	$\frac{2100}{41}$	410	$\frac{6305}{41}$	3
$\frac{105}{41}(\sqrt{178081} - 400)$	0	90^o	145	≈ 53.65	350	≈ 114.775	4
105	$\frac{10}{21}$	$\cos^{-1}\left(\frac{10}{21}\right)$	105	100	310	105	5

Notes:

1. Cevian BE is side BC; Cevian AD is side AB. **2.** A "nice" value for x. **3.** Degenerate triangle. **4.** Right triangle. **5.** Cevian BE is side AB; Cevian AD is side AC.

In short, the perimeter assumes all values in $[210, 305] \cup [310, 410]$.

Now we support these assertions. Consider $\triangle ABC$ with cevians BE (from $\angle B$ to side AC) and AD (from $\angle A$ to side BC). Let $CE = x, AE = 105 - x$ and $CD = 100 - BD$. To find the perimeter, we only need to compute AB.

We have $AE = 105 - x$, so

$$\begin{aligned} AE \cdot BD &= CE \cdot CD \\ (105 - x)BD &= x(100 - BD) \\ 100x &= 105BD \\ \text{so } BD &= \frac{20}{21}x \text{ and } CD = \mathit{100} - \frac{\mathit{20}}{\mathit{21}}x. \end{aligned}$$

Applying the Law of Cosines three times, we have

$$\begin{array}{ll} (1) & BE^2 = x^2 + 100^2 - 2(100)x\cos C \\ (2) & AD^2 = CD^2 + 105^2 - 2(105)CD\cos(C) \text{ and} \\ (3) & AB^2 = 100^2 + 105^2 - 2 \cdot 100 \cdot 105\cos(C). \end{array}$$

Because we must have $AD = BE$, we combine (1) and (2) to get

$$\begin{aligned} CD^2 + 105^2 - 2(105)CD\cos(C) &= x^2 + 100^2 - 2(100)x\cos(C) \text{ or} \\ \left(100 - \frac{20}{21}x\right)^2 + 105^5 - 210\left(100 - \frac{20}{21}x\right)\cos(C) &= x^2 + 100^2 - 2(100)x\cos(C). \end{aligned}$$

Solving for $\cos(C)$, we obtain a rational expression in x:

$$(4) \quad \cos(C) = \frac{41x^2 + 84000x - 2205^2}{21^2 \cdot 200(2x - 105)}.$$

Substituting this value into (3) we have

$$\begin{aligned} AB^2 &= 21025 - 2100 \cdot \frac{41x^2 + 84000x - 2205^2}{21^2 \cdot 200(200x - 105)}, \text{ so} \\ (5) \;\; AB^2 &= \frac{5}{21}\frac{41x^2 + 92610x + 4410000}{105 - 2x}. \end{aligned}$$

Thus we can then calculate AB and the perimeter

$$P = 205 + \sqrt{\frac{5}{21}\frac{41x^2 + 92610x + 4410000}{105 - 2x}}.$$

The graphs of $\cos(C)$ and of AB have vertical asymptotes at $x = \dfrac{105}{2}$, in the center of our interval $[0, 105]$. Other than an interval bracketing this singularity, each value of x produces a solution to the problem.

We explore the endpoints and the "degenerate" solutions, obtaining the values exhibited in the table above.

I. $x = 0$: That is $CE = 0$, so $E = C$ and the cevian from vertex B is actually the side BC. Therefore, $BE = BC = 100$. Hence, the condition

$$\begin{aligned} AE \cdot BD &= CE \cdot CD \text{ becomes} \\ AC \cdot BD &= 0 \cdot CD \text{ or} \\ 100 \cdot BD &= 0. \end{aligned}$$

Thus $BD = 0$, so $D = B$ and the cevian from vertex A is actually the side AB; $AD = AB$.
Computing by (4) and (5): $\cos(C) = \dfrac{21}{40}$ and $AB = 100$. Thus $AD = AB = 100 = BC = BE$, so this triangle satisfies the required conditions. Its perimeter is 305.

II. $x = 105$**:** gives a similar result, a $(105, 105, 100)$ triangle with cevians lying along the sides and $P = 310$.

III. The degenerate case $C = 0$ occurs when $\cos(C) = 1$. By (4), this happens when $x = \dfrac{1985}{41}$. Also, $C = 0$ if and only if $AB = 5$, which is the smallest possible value (by the Triangle Inequality).

IV. The degenerate case $C = \pi$ occurs when $\cos(C) = -1$. By (4) this happens when $x = \dfrac{2205}{41}$. Also $C = \pi$ if and only if $AB = 205$, which is the largest possible value (by the Triangle Inequality).

The values of x appearing in III and IV are the endpoints of the interval of excluded values bracketing $\dfrac{105}{2}$.

V. The degenerate case $C = \pi/2$ occurs when $\cos(C) = 0$. By (4), this happens when x takes on the ugly irrational $\dfrac{105}{41}\left(\sqrt{178081} - 400\right)$. In this case, $AB = 145$ and our triangle is the $(20, 21, 29)$ Pythagorean triangle scaled up by a factor of 5. The common value of the cevians is $AD = BE = \dfrac{5}{41}\sqrt{49788121 - 352800\sqrt{178081}} \approx 114.775$.

VI. Because $BD = \dfrac{20}{21}x$, some nice results occur when x is a multiple of 21. The table shows the values for $x = 21$.

Excel has produced many values of these triangles, letting x range from 0 to 105, except for the excluded interval $\left(\dfrac{1985}{41}, \dfrac{2205}{41}\right)$, but in summary,

- the perimeter assumes all values in $[210, 305] \cup [310, 410]$.
- side AB assumes all values in $[5, 100] \cup [105, 205]$.
- $\angle C$ assumes all values in

$$\left[0,\cos^{-1}\left(\frac{21}{40}\right)\right]\cup\left[\cos^{-1}\left(\frac{10}{21},180^o\right)\right]=[0,58.33^o]\cup[61.56^o,180^o].$$

– The common cevians achieve the values
$$\left[\frac{2105}{41},100\right]\cup\left[105,\frac{6305}{41}\right]\approx[51.34,100]\cup[105,153.78].$$

Our final comment: AB assumes all integer values in $[5,100]\cup[105,205]$, so the right triangle described above is not the only solution with all sides integral. For any integer AB in $[5,100]\cup[105,205]$, we can use (5) to determine the appropriate value of x, C, etc. Of course, this raises another question: are any of these triangles Heronian?

Also solved by the proposer.

5015: *Proposed by Kenneth Korbin, New York, NY.*

Part I: Find the value of
$$\sum_{x=1}^{10}\text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right).$$

Part II: Find the value of
$$\sum_{x=1}^{\infty}\text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right).$$

Solution by David C. Wilson, Winston-Salem, N.C.

First, let's look for a pattern.

$\mathbf{x=1}:\quad \text{Arcsin}(\frac{4}{5}).$

$\mathbf{x=2}:\ \text{Arcsin}(\frac{4}{5})+\text{Arcsin}(\frac{16}{65})=\text{Arcsin}(\frac{12}{13}).$

Let $\theta=\text{Arcsin}(\frac{4}{5})$ and $\phi=\text{Arcsin}(\frac{16}{65})$.

$$\sin\theta=\frac{4}{5}\qquad\qquad \sin\phi=\frac{16}{65}$$
$$\cos\theta=\frac{3}{5}\qquad\qquad \cos\phi=\frac{63}{65}$$

$$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi=(\frac{4}{5})(\frac{63}{65})+(\frac{3}{5})(\frac{16}{65})=\frac{300}{325}=\frac{12}{13}=\text{Arcsin}(\frac{12}{13}).$$

$\mathbf{x=3}:\ \text{Arcsin}(\frac{12}{13})+\text{Arcsin}(\frac{36}{325})=\text{Arcsin}(\frac{24}{25}).$

Let $\theta=\text{Arcsin}(\frac{12}{13})$ and $\phi=\text{Arcsin}(\frac{36}{325})$.

$$\sin\theta=\frac{12}{13}\qquad\qquad \sin\phi=\frac{36}{325}$$
$$\cos\theta=\frac{5}{13}\qquad\qquad \cos\phi=\frac{323}{325}$$

$$\sin(\theta+\phi)=(\frac{12}{13})(\frac{323}{325})+(\frac{5}{13})(\frac{36}{325})=\frac{4056}{4225}=\frac{24}{25}=\text{Arcsin}(\frac{24}{25}).$$

$\mathbf{x=4}:\ \text{Arcsin}(\frac{24}{25})+\text{Arcsin}(\frac{64}{1025})=\text{Arcsin}(\frac{40}{41}).$

Let $\theta = \text{Arcsin}(\frac{24}{25})$ and $\phi = \text{Arcsin}(\frac{64}{1025})$.

$$\sin\theta = \frac{24}{25} \qquad \sin\phi = \frac{64}{1025}$$

$$\cos\theta = \frac{7}{25} \qquad \cos\phi = \frac{1023}{1025}$$

$$\sin(\theta+\phi) = (\frac{24}{25})(\frac{1023}{1025}) + (\frac{7}{25})(\frac{64}{1025}) = \frac{25000}{25625} = \frac{40}{41} = \text{Arcsin}(\frac{40}{41}).$$

Therefore, the conjecture is

$$\sum_{x=1}^{n} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) = \text{Arcsin}\left(\frac{2n^2+2n}{2n^2+2n+1}\right).$$

Proof is by induction.

1) For $n = 1$, we obtain $\text{Arcsin}(\frac{4}{5}) = \text{Arcsin}(\frac{4}{5})$.

2) Assume true for n; i.e.,

$$\sum_{x=1}^{n} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) = \text{Arcsin}\left(\frac{2n^2+2n}{2n^2+2n+1}\right).$$

3) For $n+1$, we have

$$\begin{aligned}\sum_{x=1}^{n+1} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) &= \sum_{x=1}^{n} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) + \text{Arcsin}\left(\frac{4(n+1)^2}{4(n+1)^4+1}\right) \\ &= \sum_{x=1}^{n} \text{Arcsin}\left(\frac{2n^2+2n}{2n^2+2n+1}\right) + \text{Arcsin}\left(\frac{4(n+1)^2}{4(n+1)^4+1}\right).\end{aligned}$$

Let $\theta = \text{Arcsin}\left(\frac{2\text{n}^2+2\text{n}}{2\text{n}^2+2\text{n}+1}\right)$ and $\phi = \text{Arcsin}\left(\frac{4(\text{n}+1)^2}{4(\text{n}+1)^4+1}\right)$.

$$\sin\theta = \frac{2n^2+2n}{2n^2+2n+1} \qquad \sin\phi = \frac{4(n+1)^2}{4(n+1)^4+1}$$

$$\cos\theta = \frac{2n+1}{2n^2+2n+1} \qquad \cos\phi = \frac{4(n+1)^4-1}{4(n+1)^4+1}$$

$$\begin{aligned}\sin(\theta+\phi) &= \sin\theta\cos\phi + \cos\theta\sin\phi \\ &= \left(\frac{2n^2+2n}{2n^2+2n+1}\right)\left[\frac{4(n+1)^4-1}{4(n+1)^4+1}\right] + \left(\frac{2n+1}{2n^2+2n+1}\right)\left[\frac{4(n+1)^2}{4(n+1)^4+1}\right] \\ &= \frac{8n^6+40n^5+80n^4+88n^3+58n^2+22n+4}{(2n^2+2n+1)(2n^2+6n+5)(2n^2+2n+1)} \\ &= \frac{(2n^2+2n+1)^2(2n^2+6n+4)}{(2n^2+2n+1)^2(2n^2+6n+5)} = \frac{2n^2+6n+4}{2n^2+6n+5} = \frac{2(n+1)^2+2(n+1)}{2(n+1)^2+2(n+1)+1}.\end{aligned}$$

Thus $\sum_{x=1}^{n+1} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) = \text{Arcsin}\left[\frac{2(n+1)^2+2(n+1)}{2(n+1)^2+2(n+1)+1}\right]$ and this proves the conjecture.

Part I: $\displaystyle\sum_{x=1}^{10} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) = \text{Arcsin}\left[\frac{220}{221}\right].$

Part II:

$$\begin{aligned}
\sum_{x=1}^{\infty} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right) &= \lim_{n\to\infty} \sum_{x=1}^{n} \text{Arcsin}\left[\frac{4x^2}{4x^4+1}\right] \\
&= \lim_{n\to\infty} \text{Arcsin}\left(\frac{2n^2+2n}{2n^2+2n+1}\right) \\
&= \text{Arcsin}\left[\lim_{n\to\infty} \frac{2n^2+2n}{2n^2+2n+1}\right] = \text{Arcsin}(1) = \frac{\pi}{2}.
\end{aligned}$$

Also solved by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Roger Zarnowski (jointly), San Angelo, TX; Brian D. Beasley, Clinton, SC; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy; Boris Rays, Chesapeake, VA; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

5016: *Proposed by John Nord, Spokane, WA.*

Locate a point (p, q) in the Cartesian plane with integral values, such that for any line through (p, q) expressed in the general form $ax + by = c$, the coefficients a, b, c form an arithmetic progression.

Solution 1 by Nate Wynn (student at Saint George's School), Spokane, WA.

As $\{a, b, c\}$ is an arithmetic progression, b can be written as $a + n$ and c can be written as $a + 2n$. Then using a series of two equations:

$$\begin{cases} ap + (a+n)q & = a+2n \\ tp + (t+u)q & = t+2u \end{cases}$$

Solving this system gives

$$(tn - au)q = 2tn - 2au, \text{thus } q = 2.$$

Placing this value into the first equation and solving gives

$$\begin{aligned}
ap + 2a + 2n &= a + 2n \\
a(p+1) &= 0 \\
p &= -1.
\end{aligned}$$

Therefore the point is $(-1, 2)$.

Solution 2 by Eric Malm (graduate student at Stanford University, and an alumnus of Saint George's School in Spokane), Stanford, CA.

The only such point is $(-1, 2)$.

Suppose that each line through (p, q) is of the form $ax + by = c$ with (a, b, c) an arithmetic progression. Then $c = 2b - a$. Taking $a = 0$ yields the line $by = 2b$ or $y = 2$, so $q = 2$. Taking $a \neq 0$, p= must satisfy $ap + 2b = 2b - a$, so $p = -1$.

Conversely, any line through $(p, q) = (-1, 2)$ must be of the form
$ax + by = ap + bq = 2b - a$, in which case the coefficients $(a, b, 2b - a)$ form an arithmetic progression.

Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Matt DeLong, Upland, IN; Rachel Demeo, Matthew Hussey, Allison Reece, and Brian Tencher (jointly, students at Talyor University, Upland, IN); Michael N. Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; Raul A. Simon, Chile; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

5017: *Proposed by M.N. Deshpande, Nagpur, India.*

Let ABC be a triangle such that each angle is less than 90^0. Show that

$$\frac{a}{c \cdot \sin B} + \frac{1}{\tan A} = \frac{b}{a \cdot \sin C} + \frac{1}{\tan B} = \frac{c}{b \cdot \sin A} + \frac{1}{\tan C}$$

where $a = l(\overline{BC}), b = l(\overline{AC})$, and $c = l(\overline{AB})$.

Solution by John Boncek, Montgomery, AL.

From the Law of Sines:

$$\begin{aligned} a \sin B &= b \sin A \quad \rightarrow \quad \sin B = \frac{b \sin A}{a} \\ b \sin C &= c \sin B \quad \rightarrow \quad \sin C = \frac{c \sin B}{b} \\ c \sin A &= a \sin C \quad \rightarrow \quad \sin A = \frac{a \sin C}{c}, \end{aligned}$$

and from the Law of Cosines, we have

$$\begin{aligned} bc \cos A &= \frac{1}{2}(b^2 + c^2 - a^2) \\ ac \cos B &= \frac{1}{2}(a^2 + c^2 - b^2) \\ ab \cos C &= \frac{1}{2}(a^2 + b^2 - c^2). \end{aligned}$$

Thus,

$$\begin{aligned} \frac{a}{c \sin B} + \frac{1}{\tan A} &= \frac{a^2}{bc \sin A} + \frac{\cos A}{\sin A} \\ &= \frac{a^2 + bc \cos A}{bc \sin A} \\ &= \frac{a^2 + b^2 + c^2}{2bc \sin A}, \end{aligned}$$

$$\frac{b}{a \sin C} + \frac{1}{\tan B} = \frac{b^2}{ac \sin B} + \frac{\cos B}{\sin B}$$

$$
\begin{aligned}
&= \frac{b^2 + ac\cos B}{ac\sin B}\\
&= \frac{a^2+b^2+c^2}{2ac\sin B}\\
&= \frac{a^2+b^2+c^2}{2c(a\sin B)}\\
&= \frac{a^2+b^2+c^2}{2bc\sin A},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{c}{b\sin A} + \frac{1}{\tan C} &= \frac{c^2}{ab\sin C} + \frac{\cos C}{\sin C}\\
&= \frac{c^2 + ab\cos C}{ab\sin C}\\
&= \frac{a^2+b^2+c^2}{2ab\sin C}\\
&= \frac{a^2+b^2+c^2}{2b(a\sin C)}\\
&= \frac{a^2+b^2+c^2}{2bc\sin A}.
\end{aligned}
$$

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Michael C. Faleski, University Center, MI; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

5018: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Write the polynomial $x^{5020} + x^{1004} + 1$ as a product of two polynomials with integer coefficients.

Solution by Kee-Wai Lau, Hong Kong, China.

Clearly the polynomial $y^5 + y + 1$ has no linear factor with integer coefficients.
We suppose that for some integers a, b, c, d, e

$$
\begin{aligned}
y^5 + y + 1 &= (y^3 + ay^2 + by + c)(y^2 + dy + e)\\
&= y^5 + (a+d)y^4 + (b+e+ad)y^3 + (ae+bd+c)y^2 + (be+cd)y + ce.
\end{aligned}
$$

Hence

$$a + d = b + e + ad = ae + bd + c = 0, \quad be + cd = ce = 1.$$

It is easy to check that $a = -1, b = 0, c = 1, d = 1, e = 1$ so that

$$y^5 + y + 1 = (y^3 - y^2 + 1)(y^2 + y + 1)$$

and

$$x^{5020}+x^{1004}+1=\Big(x^{3012}-x^{2008}+1\Big)\Big(x^{2008}+x^{1004}+1\Big).$$

Comment by Kenneth Korbin, New York, NY. Note that (y^2+y+1) is a factor of (y^N+y+1) for all N congruent to 2(mod3) with $N>1$.

Also solved by Landon Anspach, Nicki Reishus, Jessi Byl, and Laura Schindler (jointly, students at Taylor University), Upland, IN; Brian D. Beasley, Clinton, SC; John Boncek, Montgomery, AL; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX; Matt DeLong, Upland, IN; Paul M. Harms, North Newton, KS; Matthew Hussey Rachel DeMeo, Brian Tencher, and Allison Reece (jointly, students at Taylor University), Upland IN; Kenneth Korbin, New York, NY; N. J. Kuenzi, Oshkosh, WI; Carl Libis, Kingston, RI; Eric Malm, Stanford, CA; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

5019: *Michael Brozinsky, Central Islip, NY.*

In a horse race with 10 horses the horse with the number one on its saddle is referred to as the number one horse, and so on for the other numbers. The outcome of the race showed the number one horse did not finish first, the number two horse did not finish second, the number three horse did not finish third and the number four horse did not finish fourth. However, the number five horse did finish fifth. How many possible orders of finish are there for the ten horses assuming no ties?

Solution 1 by R. P. Sealy, Sackville, New Brunswick, Canada.

There are 229,080 possible orders of finish.

For $k=0,1,2,3,4$ we perform the following calculations:

a) Choose the k horses numbered 1 through 4 which finish in places 1 through 4.

b) Arrange the k horses in places 1 through 4 and count the permutations with no "fixed points."

c) Arrange the remaining $(4-k)$ horses numbered 1 through 4 in places 6 through 10.

d) Arrange the 5 horses numbered 6 through 10 in the remaining 5 places.

Case 1: K=0.

${}_4C_0\cdot 5\cdot 4\cdot 3\cdot 2\cdot 5!=120\cdot 5!$

Case 2: K=1.

${}_4C_1\cdot 3\cdot 5\cdot 4\cdot 3\cdot 5!=720\cdot 5!$

Case 3: K=2.

${}_4C_2\cdot 7\cdot 5\cdot 4\cdot 5!=840\cdot 5!$

Case 4: K=3.

${}_4C_3\cdot 11\cdot 5\cdot 5!=220\cdot 5!$

Case 5: K=4.

${}_4C_4\cdot 9\cdot 5!=9\cdot 5!$

Solution 2 by Matt DeLong, Upland, IN.

We must count the total number of ways that 10 horses can be put in order subject to the given conditions. Since the number five horse always finishes fifth, we are essentially only counting the total number of way that 9 horses can be put in order subject to the other given conditions. Thus there are at most 9! possibilities.

However, this over counts, since it doesn't exclude the orderings with the number one horse finishing first, etc. By considering the number of ways to order the other eight horses, we can see that there are 8! ways in which the number one horse does finish first. Likewise, there are 8! ways in which each of the horses numbered two through four finish in the position corresponding to its saddle number. By eliminating these from consideration, we see that there are at least 9!-4(8!) possibilities.

However, this under counts, since we twice removed orderings in which both horse one finished first and horse two finished second etc. There are 6(7!) such orderings, since there are 6 ways to choose 2 horses from among 4, and once those are chosen the other 7 horses must be ordered. We can add these back in, but then we will again be over counting. We would need to subtract out those orderings in which three of the first four horses finish according to their saddle numbers. There are 4(6!) of these, since there are 4 ways to choose 3 horses from among 4, and once those are chosen the other 6 horses must be ordered. Finally, we would then need to add back in the number of orderings in which all four horses numbered one through four finish according to their saddle numbers. There are 5! such orderings.

In sum, we are applying the inclusion-exclusion principle, and the total that we are interested in is $9! - 4(8!) + 6(7!) - 4(6!) + 5! = 229,080$.

Also solved by Michael C. Faleski, University Center, MI; Paul M. Harms, North Newton, KS; Nate Kirsch and Isaac Bryan (students at Taylor University), Upland, IN; N. J. Kuenzi, Oshkosh, WI; Kee-Wai Lau, Hong Kong, China; Carl Libis, Kingston, RI; David E. Manes, Oneonta, NY; Harry Sedinger, St. Bonaventure, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before February 15, 2009

- 5038: *Proposed by Kenneth Korbin, New York, NY.*

  Given the equations

$$\begin{cases} \sqrt{1+\sqrt{1-x}} \quad -5\cdot\sqrt{1-\sqrt{1-x}} = 4\cdot\sqrt[4]{x} \quad \text{and} \\ 4\cdot\sqrt{1+\sqrt{1-y}} \quad -5\cdot\sqrt{1-\sqrt{1-y}} = \sqrt[4]{y}. \end{cases}$$

  Find the positive values of x and y.

- 5039: *Proposed by Kenneth Korbin, New York, NY.*

  Let d be equal to the product of the first N prime numbers which are congruent to $1 (\mathrm{mod}\ 4)$. That is

$$d = 5\cdot 13\cdot 17\cdot 29\cdots P_N.$$

  A convex polygon with integer length sides is inscribed in a circle with diameter d. Prove or disprove that the maximum possible number of sides of the polygon is the N^{th} term of the sequence $t = (4, 8, 20, 32, 80, \cdots, t_N, \cdots)$ where $t_N = 4t_{N-2}$ for $N > 3$.

  Examples: If $N = 1$, then $d = 5$, and the maximum polygon has 4 sides $(3, 3, 4, 4)$. If $N = 2$, then $d = 5\cdot 13 = 65$ and the maximum polygon has 8 sides $(16, 16, 25, 25, 25, 25, 33, 33)$.

  Editor's comment: In correspondence with Ken about this problem he wrote that he has been unable to prove the formula for $N > 5$; so it remains technically a conjecture.

- 5040: *Proposed by John Nord, Spokane, WA.*

  Two circles of equal radii overlap to form a lens. Find the distance between the centers if the area in circle A that is not covered by circle B is $\dfrac{1}{3}\left(2\pi + 3\sqrt{3}\right)r^2$.

- 5041: *Proposed by Michael Brozinsky, Central Islip, NY.*

  Quadrilateral $ABCD$ (with diagonals $AC = d_1$ and $BD = d_2$ and sides $AB = s_1, BC = s_2, CD = s_3$, and $DA = s_4$) is inscribed in a circle. Show that:

$$d_1^2 + d_2^2 + d_1 d_2 > \frac{s_1^2 + s_2^2 + s_3^2 + s_4^2}{2}.$$

- 5042: *Proposed by Miquel Grau-Sánchez and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ $(a_k \neq 0)$ and $B(z) = z^{n+1} + \sum_{k=0}^{n} b_k z^k$ $(b_k \neq 0)$ be two prime polynomials with roots $z_1, z_2, \ldots, z_n$ and $w_1, w_2, \ldots, w_{n+1}$ respectively. Prove that

  $$\frac{A(w_1)A(w_2)\ldots A(w_{n+1})}{B(z_1)B(z_2)\ldots B(z_n)}$$

  is an integer and determine its value.

- 5043: *Ovidiu Furdui, Toledo, OH.*

  Solve the following diophantine equation in positive integers k, m, and n

  $$k \cdot n! \cdot m! + m! + n! = (m+n)!.$$

Solutions

- 5020: *Proposed by Kenneth Korbin, New York, NY.*

  Find positive numbers x and y such that

  $$\begin{cases} x^7 - 13y = 21 \\ 13x - y^7 = 21 \end{cases}$$

Solution 1 by Brian D. Beasley, Clinton, SC.

Using the Fibonacci numbers $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-2} + F_{n-1}$ for each integer $n \geq 3$, we generalize the given problem by finding numbers x and y such that

$$\begin{cases} x^n - F_n y & = & F_{n+1} \\ F_n x + (-1)^n y^n & = & F_{n+1} \end{cases}$$

for each positive integer n. (The given problem is the case $n = 7$.)

We let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ and apply the Binet formula $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ for each positive integer n to show that we may take $x = \alpha > 0$ and $y = -\beta > 0$:

$$\alpha^n - F_n(-\beta) = \frac{\alpha^n(\beta+\sqrt{5}) - \beta^{n+1}}{\sqrt{5}} = \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{5}} = F_{n+1};$$

$$F_n(\alpha) + (-1)^n(-\beta)^n = \frac{\alpha^{n+1} - \beta^n(\alpha - \sqrt{5})}{\sqrt{5}} = \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{5}} = F_{n+1}.$$

Solution 2 by David Stone and John Hawkins, Statesboro, GA.

The solution anticipated by the poser is probably $(\alpha, -\beta)$, where $\alpha = \dfrac{1+\sqrt{5}}{2} \approx 1.618034$ is the Golden Ratio and $\beta = \dfrac{1-\sqrt{5}}{2} \approx -0.618034$ its companion (in the official terminology of *The Fibonacci Quarterly*).

Note that:

(#) if (x, y) is a solution to the system, then so is $(-y, x)$. Thus we may as well look for all solutions, not just positive solutions. We graph the system in the form

$$\begin{cases} y = \dfrac{x^7 - 21}{13} \\ y = (13x - 21)^{1/7} \end{cases}.$$

There are five points of intersection. Graphically conditional (#) appears as symmetry of intersections across the line $y = -x$ (even though the curves themselves have no such symmetry).

Although we cannot determine all solutions analytically, we have their approximate numerical values:

$$\begin{array}{rcl} (x & , & y) \\ (1.6418599 & , & 0.85866981) \\ (1.61803399 & , & 0.61803399) \\ (1.249536927 & , & -1.249536927) \\ (-0.61803399 & , & -1.61803399) \\ (-0.85866981 & , & -1.6418599) \end{array}$$

(1) The second and fourth solutions seem to lie on the line $y = x - 1$, suggesting that x satisfies $x^7 - 13(x-1) = 21$, so $x^7 - 13x - 8 = 0$. Factoring,

$$x^7 - 13x - 8 = (x^2 - x - 1)(x^5 + x^4 + 2x^3 + 3x^2 + 5x + 8)$$

and the quadratic factor has (well-known) roots

$$\alpha = \frac{1 + \sqrt{5}}{2} \approx 1.618034 \text{ and } \beta = \frac{1 - \sqrt{5}}{2} \approx -0.618034.$$

Thus we actually know the second and fourth solutions are $(\alpha, -\beta)$ and $(\beta, -\alpha)$.

We verify that $(\alpha, -\beta)$ is indeed a solution to the given system. Note that by the first well-known relationship to the Fibonacci numbers, $\alpha^n = \alpha F_n + F_{n-1}$, we have $\alpha^7 = \alpha F_7 + F_6 = 13\alpha + 8$.

Now, substituting into the first equation:

$$\begin{aligned} \alpha^7 - 13(-\beta) &= \alpha^7 - 13(\alpha - 1) \\ &= \alpha^7 - 13\alpha + 13 \\ &= 13\alpha + 8 - 13\alpha + 13 = 21, \text{as desired.} \end{aligned}$$

It is also straight forward to verify the second equation: $13\alpha - (-\beta)^7 = 21$, using $\alpha\beta = -1$.

(2) The third solution lies on the line $y = -x$, so x is the sole real zero of $x^7 + 13x - 21 = 0$. This polynomial equation is not solvable in radicals – according to Maple, the Galois group of $x^7 + 13x - 21$ is S_7, which is not a solvable group. Hence, an approximation is probably the best we can do (barring some ingenious treatment employing transcendental functions.)

Unfortunately, we do not have any analytic characterization of the first and fifth solutions.

A final comment: the problem involves the exponent 7 and the Fibonacci numbers $F_7 = 13$ and $F_8 = 21$, so there is almost certainly a more general version with solution $(\alpha, -\beta)$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 5021: *Proposed by Kenneth Korbin, New York, NY.*

Given
$$\frac{x+x^2}{1-34x+x^2} = x + 35x^2 + \cdots + a_n x^n + \cdots$$
Find an explicit formula for a_n.

Solution by David E. Manes, Oneonta, NY.

An explicit formula for a_n is given by
$$a_n = -\frac{1}{8}\Big[(4-3\sqrt{2})(17+12\sqrt{2})^n + (4+3\sqrt{2})(17-12\sqrt{2})^n\Big].$$
Let $F(x) = \dfrac{x+x^2}{1-34x+x^2}$ be the generating function for the sequence $\Big(a_n\Big)_{n\geq 1}$, where $a_1 = 1$ and $a_2 = 35$.

Then the characteristic equation is $\lambda^2 - 34\lambda + 1 = 0$, with roots $r_1 = 17 + 12\sqrt{2}$ and $r_2 = 17 - 12\sqrt{2}$.

Therefore,
$$a_n = \alpha\Big(17+12\sqrt{2}\Big)^n + \beta\Big(17-12\sqrt{2}\Big)^n$$
for some real numbers α and β. From the initial conditions one obtains
$$\begin{aligned} 1 &= \alpha\Big(17+12\sqrt{2}\Big) + \beta\Big(17-12\sqrt{2}\Big) \\ 35 &= \alpha\Big(17+12\sqrt{2}\Big)^2 + \beta\Big(17-12\sqrt{2}\Big)^2. \end{aligned}$$

The solution for this system of equations is
$$\begin{aligned} \alpha &= -\frac{1}{8}\Big(4-3\sqrt{2}\Big) \\ \beta &= -\frac{1}{8}\Big(4+3\sqrt{2}\Big). \end{aligned}$$

Hence, if $n \geq 1$, then
$$a_n = -\frac{1}{8}\Big[(4-3\sqrt{2})(17+12\sqrt{2})^n + (4+3\sqrt{2})(17-12\sqrt{2})^n\Big].$$

Also solved by Brian D. Beasley, Clinton, SC; Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie (jointly), San Angelo, TX; Bruno Salgueiro

Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Boris Rays, Chesapeake, VA; David Stone and John Hawkins, Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 5022: *Proposed by Michael Brozinsky, Central Islip, NY.*

Show that

$$\sin\left(\frac{x}{3}\right)\sin\left(\frac{\pi+x}{3}\right)\sin\left(\frac{2\pi+x}{3}\right)$$

is proportional to $\sin(x)$.

Solution 1 by José Hernández Santiago, (student, UTM), Oaxaca, México.

From the well-known identity $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$, we derive that

$$\begin{aligned}
\sin 3\theta &= 4\sin\theta\Big(\frac{3}{4}\cos^2\theta - \frac{1}{4}\sin^2\theta\Big)\\
&= 4\sin\theta\Big(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\Big)\Big(\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\Big)\\
&= 4\sin\theta\sin\left(\frac{\pi}{3}-\theta\right)\sin\left(\frac{\pi}{3}+\theta\right).
\end{aligned}$$

When we let $\theta = \dfrac{x}{3}$, the latter formula becomes:

$$\sin 3\Big(\frac{x}{3}\Big) = 4\sin\left(\frac{x}{3}\right)\sin\left(\frac{\pi-x}{3}\right)\sin\left(\frac{\pi+x}{3}\right) \qquad (1)$$

Now, the fact that

$$\begin{aligned}
\sin\left(\frac{x+2\pi}{3}\right) &= \sin\left(\frac{x-\pi}{3}+\pi\right)\\
&= \sin\left(\frac{x-\pi}{3}\right)\cos\pi\\
&= \sin\left(\frac{\pi-x}{3}\right)
\end{aligned}$$

allows us to put (1) in the form

$$\sin x = 4\sin\left(\frac{x}{3}\right)\sin\left(\frac{\pi+x}{3}\right)\sin\left(\frac{x+2\pi}{3}\right);$$

and clearly this is equivalent to what the problem asked us to demonstrate.

Solution 2 by Kee-Wai Lau, Hong Kong, China.

Since

$$\begin{aligned}
\sin\left(\frac{\pi+x}{3}\right)\sin\left(\frac{2\pi+x}{3}\right) &= \frac{1}{2}\Big(\cos\left(\frac{\pi}{3}\right)+\cos\left(\frac{2x}{3}\right)\Big)\\
&= \frac{1}{2}\Big(\frac{1}{2}+1-2\sin^2\left(\frac{x}{3}\right)\Big)\\
&= \frac{1}{4}\Big(3-4\sin^2\left(\frac{x}{3}\right)\Big),
\end{aligned}$$

so

$$\sin\left(\frac{x}{3}\right)\sin\left(\frac{\pi+x}{3}\right)\sin\left(\frac{2\pi+x}{3}\right)=\frac{1}{4}\left(3\sin\left(\frac{x}{3}\right)-4\sin^3\left(\frac{x}{3}\right)\right)=\frac{1}{4}\sin(x),$$

which is proportional to $\sin(x)$.

Also solved by Brian D. Beasley, Clinton, SC; John Boncek, Montgomery, AL; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Michael C. Faleski, University Center, MI; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Jahangeer Kholdi, Portsmouth, VA; Kenneth Korbin, NY, NY: Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Charles, McCracken, Dayton, OH; John Nord, Spokane, WA; Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 5023: *Proposed by M.N. Deshpande, Nagpur, India.*

Let $A_1A_2A_3\cdots A_n$ be a regular $n-$gon $(n\geq 4)$ whose sides are of unit length. From A_k draw L_k parallel to $\mathrm{A}_{k+1}A_{k+2}$ and let L_k meet L_{k+1} at T_k. Then we have a "necklace" of congruent isosceles triangles bordering $A_1A_2A_3\cdots A_n$ on the inside boundary. Find the total area of this necklace of triangles.

Solution 1 by Paul M. Harms, North Newton, KS.

In order that the "necklace" of triangles have the $n-$gon as an inside boundary, it appears that line L_k (through A_k) should be parallel to $A_{k-1}A_{k+1}$ rather than $A_{k+1}A_{k+2}$. With this interpretation in mind, we now consider the n isosceles triangles with a vertex at the center of the $n-$gon and the opposite side being a side of unit length. The measure of the central angles are $360^o/n$. The angle inside the n-gon at the intersection of 2 unit sides is twice one of the equal angles of the isosceles triangles with a vertex at the center of the $n-$gon, so it has a degree measure of $180^{\text{o}}-(360^{\text{o}}/n)$.

The isosceles triangle $A_{k-1}A_kA_{k+1}$ has two equal angles (opposite the sides of unit length) with a measure of

$$\frac{1}{2}\left(180^o-(180^o-(360^o/n))\right)=\frac{180^o}{n}.$$

A side of length one intersects the two parallel lines $(A_{k-1}A_{k+1})$ and the line parallel to it through A_k. Using equal angles for a line intersecting parallel lines, we see that the equal angles in one necklace isosceles triangle has a measure of $180^o/n$.

Using the side of length one as a base, the area of one necklace triangle is

$$\frac{1}{2}(\text{base})\cdot(\text{height})=\frac{1}{2}(1)\left(\frac{1}{2}\tan(180^{\text{o}}/n)\right)=\frac{1}{4}\tan(180^{\text{o}}/n).$$

The total area of n necklace triangles is $\dfrac{n}{4}\tan(180^o/n)$. It is interesting to note that the total area approaches $\pi/4$ as n gets large.

Solution 2 by David Stone and John Hawkins, Statesboro, GA.

David and John looked at the problem a bit differently than the other solvers. They wrote: "In order to get a clearer picture of what is going on, we introduce additional points that we will call B_k, where we define B_k to be the intersection of L_k and L_{k-2}, for $3 \le k \le n$ and the intersection of L_k and L_{k+n-2} for $k = 1$ or 2."

Doing this gave them a "necklace of isosceles triangles with bases along the interior boundary of the polygon: $\triangle A_1B_1A_2, \triangle A_2B_2A_3, \triangle A_3B_3A_4, \cdots, \triangle A_nB_nA_1$." *(Note that by doing this A_kT_k does pass through A_{k+3}.)*

They went on: "It is not clear that this was the intended necklace, because these triangles do not involve the points T_k. Let's call this the Perimeter Necklace."

There is a second necklace of isosceles triangle whose bases do involve the points $T_k : \triangle T_1B_3T_2,\ \triangle T_2B_4T_3,\ \triangle T_3B_5T_4,\ \cdots,\ \triangle T_{n-2}B_nTn_1,\ \triangle T_{n-1}B_1T_1,\ \triangle T_nB_2T_1$. Let's call this the Inner Necklace.

They then found the areas for both necklaces and summarized their results as follows:

$n = 4$: Area of Perimeter Necklace $= 0$. No Inner Necklace.
$n = 5$:

$$\begin{aligned}\text{Area of Perimeter Necklace} &= \frac{5}{4}\tan\left(\frac{\pi}{5}\right)\\ \text{Area of Inner Necklace} &= \frac{5}{4}\left(1-\tan^2\frac{\pi}{5}\right)\sin\left(\frac{\pi}{5}\right)\end{aligned}$$

$n = 6$

$$\begin{aligned}\text{Area of Perimeter Necklace} &= \frac{3}{2}\tan\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}\\ \text{Area of Inner Necklace} &= 0.\end{aligned}$$

$n > 5$

$$\begin{aligned}\text{Area of Perimeter Necklace} &= \frac{n}{4}\tan\left(\frac{2\pi}{n}\right)\\ \text{Area of Inner Necklace} &= \frac{\pi}{4}\left(4\sin\frac{2\pi}{n}\cos\frac{2\pi}{n} - 4\sin\frac{2\pi}{n} + \tan\frac{2\pi}{n}\right)\end{aligned}$$

Note that these give the correct results for n=6.

Then they used Excel to compute the areas of the necklaces for various values of n, and proved that for large values of n, the ratio of the areas approaches one.

n	$PerimeterNecklace$	$InnerNecklace$
6	2.59876211	0
10	1.81635632	0.693786379
100	1.572747657	1.561067973
500	1.570879015	1.570382935

$$\lim_{n\to\infty} \frac{\frac{n}{4}\left(4\sin\frac{2\pi}{n}\sin\frac{2\pi}{n} - 4\sin\frac{2\pi}{n} + \tan\frac{2\pi}{n}\right)}{\frac{n}{4}\tan\frac{2\pi}{n}} = 1.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Michael N. Fried, Kibbutz Revivim, Israel; Grant Evans (student, Saint George's School), Spokane, WA; Boris Rays, Chesapeake, VA, and the proposer.

- 5024: *Proposed by Luis Díaz-Barrero and Josep Rubió-Masségú, Barcelona, Spain.*

  Find all real solutions to the equation

$$\sqrt{1+\sqrt{1-x}}-2\sqrt{1-\sqrt{1-x}}=\sqrt[4]{x}.$$

Solution by Jahangeer Kholdi, Portsmouth, VA.

Square both sides of the equation, simplify, and then factor to obtain

$$5(1-\sqrt{x})=3\sqrt{1-x}.$$

Squaring again gives $17x-25\sqrt{x}+8=0$, and now using the quadratic formula gives $x=1$ and $x=\dfrac{64}{289}$. But $x=1$ does not satisfy the original equation. The only real solution to the original equation is $x=\dfrac{64}{289}$.

Also solved by Brian D. Beasley, Clinton, SC; John Boncek, Montgomery, AL; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Matt DeLong, Upland, IN; Grant Evans (student, Saint George's School), Spokane, WA; Michael C. Faleski, University Center, MI; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kenneth Korbin, NY, NY; Kee-Wai Lau, Hong Kong, China; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Wattana Namkaew (student, Nakhon Ratchasima Rajabhat University), Thailand; John Nord, Spokane, WA; Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposers.

- 5025: *Ovidiu Furdui, Toledo, OH.*

  Calculate the double integral

$$\int_0^1\int_0^1\{x-y\}dxdy,$$

where $\{a\}=a-[a]$ denotes the fractional part of a.

Solution by R. P. Sealy, Sackville, New Brunswick, Canada.

$$\begin{aligned}\int_0^1\int_0^1\{x-y\}dxdy &= \int_0^1\int_0^x\{x-y\}dydx+\int_0^1\int_0^y\{x-y\}dxdy\\ &= \int_0^1\int_0^x(x-y)dydx+\int_0^1\int_0^y(x-y+1)dxdy\\ &= \int_0^1\left(xy-\frac{y^2}{2}\right)\Bigg|_0^x dx+\int_0^1\left(\frac{x^2}{2}-xy+x\right)\Bigg|_0^y dy\\ &= \int_0^1\frac{x^2}{2}dx+\int_0^1\left(y-\frac{y^2}{2}\right)dy\end{aligned}$$

$$
\begin{aligned}
&= \left.\frac{x^3}{6}\right|_0^1 + \left.\left(\frac{y^2}{2} - \frac{y^3}{6}\right)\right|_0^1 \\
&= \frac{1}{2}.
\end{aligned}
$$

Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Matt DeLong, Upland, IN; Michael C. Faleski, University Center, MI; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Nate Kirsch and Isaac Bryan (jointly, students at Taylor University), Upland, IN; Kee-Wai Lau, Hong Kong, China; Matthew Hussey, Rachel DeMeo, Brian Tencher (jointly, students at Taylor University), Upland, IN; Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy; Nicki Reishus, Laura Schindler, Landon Anspach and Jessi Byl (jointly, students at Taylor University), Upland, IN; José Hernández Santiago (student, UTM), Oaxaca, México, Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.