Real Analysis Exam: Part I (Fall 1999)

Do all five problems.

1. Suppose $f_n : [0,1] \to \mathbf{R}$ is a sequence of absolutely continuous functions such that

$$\int f_n = 0$$

and such that

$$\int |f_n'|^p \leq 1$$

for some $p > 1$. Prove that there is a uniformly converging subsequence.

2. Let K be a compact subset of $\mathbf{R}^n$ and $\mathcal{C}$ be a vector space of continuous functions on $\mathbf{R}^n$ such that for each $u \in \mathcal{C}$,

$$|u(0)| \leq \max_{x \in K} |u(x)|.$$

Prove that there is a measure μ on K such that

$$u(0) = \int_K u \, d\mu$$

for every $u \in \mathcal{C}$.

3. Suppose f and g are positive, measurable functions on $[0,1]$ such that $f(x)g(x) \geq 1$ for every x. Prove that

$$\int f(x)\,dx \cdot \int g(x)\,dx \geq 1.$$

4. Show that in the Banach space $C[0,1]$, the functions of class C^1 form a set of the first category.

5. Suppose $f : \mathbf{N} \to \mathbf{R}$ is a function such that

$$f(n+m) \leq f(n) + f(m)$$

for all $n, m \in \mathbf{N}$. Prove that

$$\lim_{n \to \infty} \frac{f(n)}{n}$$

exists and is equal to

$$\inf_{n \in \mathbf{N}} \frac{f(n)}{n}.$$

Real Analysis Exam: Part II (Fall 1999)

Do all five problems.

1. Prove or disprove:
   (a) There is a bounded linear functional $T : L^\infty(\mathbf{R}) \to \mathbf{R}$ such that
   $$T(u) = u(0)$$
   provided u is continuous at 0.
   (b) There is a bounded linear functional $S : L^\infty(\mathbf{R}) \to \mathbf{R}$ such that
   $$S(u) = u'(0)$$
   provided u is differentiable at 0.

2. Let S be an uncountable subset of $\mathcal{L}^2(\mathbf{R})$. Prove that there exists a sequence u_i in S that converges (in L^2) to a limit u.

3. Suppose g is an $\mathcal{L}^1$ function on $[0,1]$ such that

$$\int_0^1 f'g \le \left(\int_0^1 |f|^2\right)^{1/2}$$

for every smooth function $f : [0,1] \to \mathbf{R}$.

Prove g is equal almost everywhere to a function that is differentiable almost everywhere.

4. Suppose $F : \mathbf{R}^k \to \mathbf{R}^n$ is a continuous, surjective map. Prove that there is an $r > 0$ such that the image of $\mathbf{B}(0,r)$ contains a nonempty open subset of $\mathbf{R}^n$.

5. Let X be a compact metric space and μ be a finite, nonnegative Borel measure on X. Suppose $\mu\{x\} = 0$ for every $x \in X$. Prove that for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\mu(E) < \epsilon$$

whenever E is a Borel set in X having diameter $\le \delta$.