Two similar geometry problems

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Abstract

In this note we give nice proofs of two similar geometry problems and their applications. The strong ties between them makes us suprised.

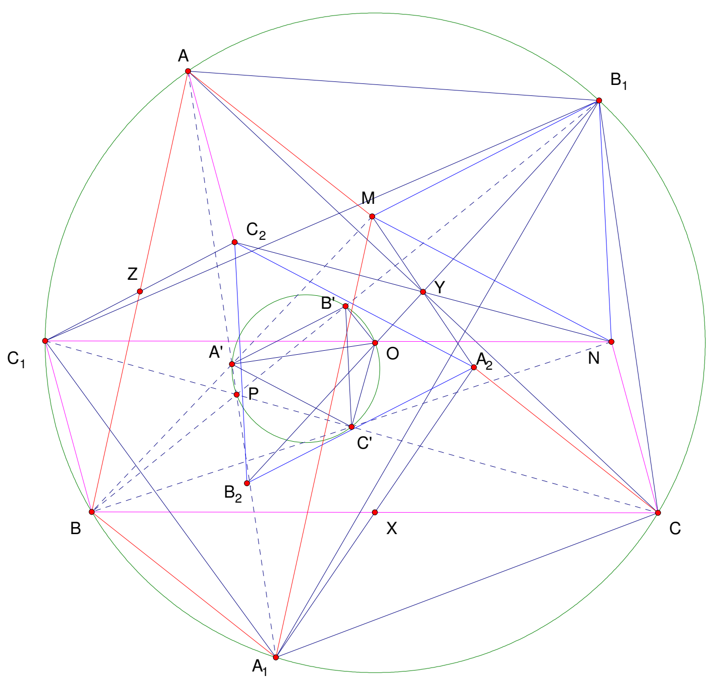
Let begin with problem which appeared in Vietnam IMO training 2009:

Problem 1:

Given triangle ABC and its circumcircle (O), its orthocenter H. P is an arbitrary point inside triangle ABC. Let A_1, B_1, C_1 be the second intersections of AP, BP, CP and $(O), A_2, B_2, C_2$ be the reflections of A_1, B_1, C_1 across the midpoints of BC, CA, AB, respectively. Prove that H, A_2, B_2, C_2 are concylic.

Solution:

First we will express a lemma:
Lemma 1
$\angle C_2B_2A_2 = \angle APC$
Proof:



Denote M, N the reflections of A_2, C_2 across the midpoint of AC then ΔMB_1N is the reflection of $\Delta A_2B_2C_2$ across the midpoint of AC.

Since $AM//CA_2, AM = CA_2$ and $BA_1//CA_2, BA_1 = CA_2$ we get AMA_1B is a parallelogram. Similarly, BC_1NC is a parallelogram too.

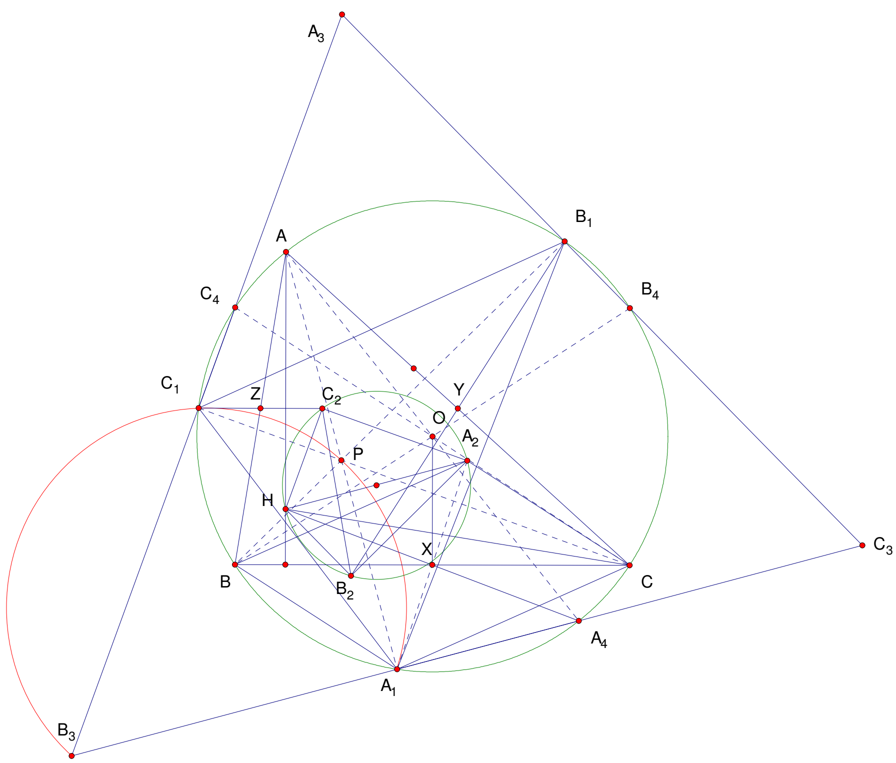
Let A', B', C' be the midpoints of AA_1, BB_1, CC_1 then A', C' are the midpoints of BM, BN, respectively. A homothety $H_B^{\frac{1}{2}} : M \to A', B_1 \to B', N \to C'$ therefore $\Delta MB_1N \to \Delta A'B'C'$.

Thus $\angle MB_1N = \angle A'B'C'$.

On the other side, $\angle OA'P = \angle OB'P = \angle OC'P = 90^o$ hence O, A', B', C', P are concyclic, which follows that $\angle A_2B_2C_2 = \angle MB_1N = \angle A'B'C' = \angle A'PC'$.

We are done.

Return to problem 1:



Construct three lines through A_1, B_1, C_1 and perpendicular to AA_1, BB_1, CC_1, respectively. They intersect each other and make triangle $A_3B_3C_3$, intersect (O) again at A_4, B_4, C_4.

Note that AA_4, BB_4, CC_4 are diameters of (O) therefore A_4, C_4 are the reflections of H across the midpoints of BC, AB. We get $HA_2A_4A_1, HC_2C_4C_1$ are parallelograms, which follows that $HA_2//A_1A_4, HC_2//C_1C_4$.

But $PC_1B_3A_1$ is a cyclic quadrilateral and applying lemma 1 we have $\angle C_2HA_2 = \angle C_1B_3A_1 = \angle C_1PA_1 = \angle CAP = \angle C_2B_2A_2$ or H, A_2, B_2, C_2 are concyclic. Our proof is completed.

Corollary 1:

Let A_5, B_5, C_5 be the second intersections of AH, BH, CH and $(A_2B_2C_2)$, respectively. Then $\Delta A_5B_5C_5 \sim \Delta ABC$.

Proof:

Since A_5, B_5, C_5, H are concyclic then $\angle C_5A_5B_5 = \angle C_5HB_5 = \angle BAC$. Similarly we get the result.

Note:

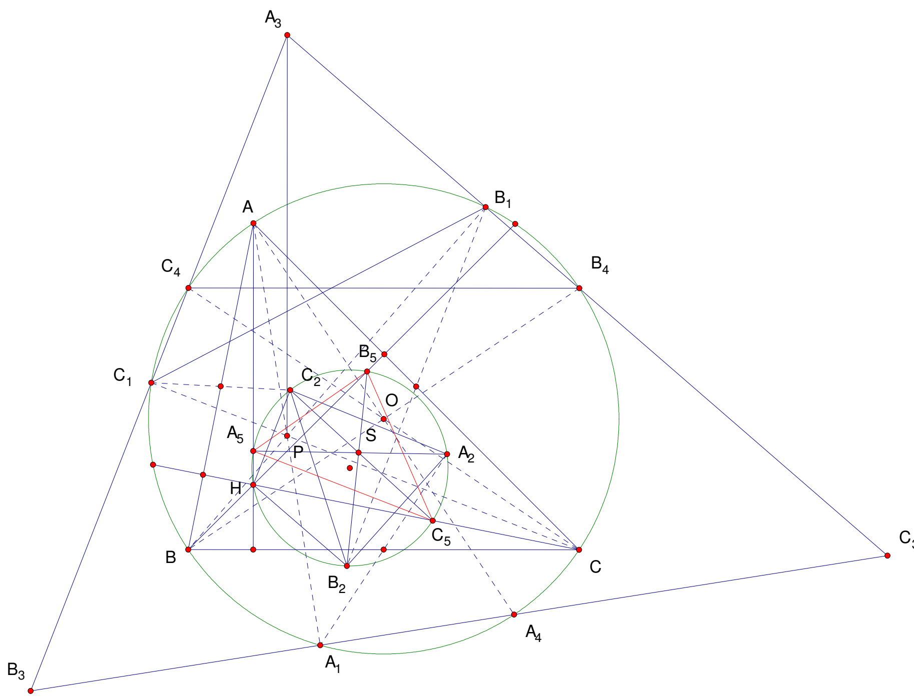
Corollary 1 is only a special case of this problem:

Given triangle ABC and an arbitrary P inside. Let $A_1B_1C_1$ is the pedal triangle of triangle ABC wrt P. Q is another arbitrary point inside triangle ABC. Let A_2, B_2, C_2 be the projections of Q on PA_1, PB_1, PC_1 then $\Delta A_2B_2C_2 \sim \Delta ABC$

Corollary 2:

A_2A_5, B_2B_5, C_2C_5 are concurrent.

Proof:



First we will show that $A_3P//AH$.

Since C_4B_4CB is a rectangle we get $C_4B_4//BC$. But $A_3C_1PB_1$ is cyclic quadrilateral thus $\angle PA_3B_1 = \angle PC_1B_1 = 90^o - \angle A_3C_1B_1 = 90^o - \angle A_3B_4C_4$

Therefore $A_3P \perp C_4B_4$ or $A_3P//AH$.

On the other side, $H, A_2, B_2, C_2, A_5, B_5, C_5$ are concyclic then $\angle C_2A_2A_5 = \angle C_2HA_5 = \angle(C_1C_4, AH) = \angle(C_1C_4, A_3P) = \angle B_3A_3P$

Similarly, $\angle A_5A_2B_2 = \angle C_3A_3P, \angle C_5C_2A_2 = \angle B3C_3P, \angle C_5C_2B_2 = \angle A_3C_3P, \angle B_5B_2C_2 = \angle A_3B_3P, \angle B_5B_2A_2 = \angle C_3B_3P$

Applying Ceva-sine theorem we obtain:

$$\frac{\sin\angle C_2A_2A_5}{\sin\angle A_5A_2B_2}.\frac{\sin\angle B_5B_2A_2}{\sin\angle B_5B_2C_2}.\frac{\sin\angle C_5C_2B_2}{\sin\angle C_5C_2A_2} = \frac{\sin\angle B_3A_3P}{\sin\angle C_3A_3P}.\frac{\sin\angle C_3B_3P}{\sin\angle A_3B_3P}.\frac{\sin\angle A_3C_3P}{\sin\angle B3C_3P} = 1$$

Our proof is completed then.

Next, we come to another problem :

Problem 2:

Let ABC be a triangle and (O) its circumcircle, P be an arbitrary point inside triangle $ABC.AP, BP, CP$ intersects (O) again at A_1, B_1, C_1. Let A_2, B_2, C_2 be the reflections of A_1, B_1, C_1 across the lines BC, CA, AB, respectively, H be the orthocenter of triangle ABC. Prove that H lies on the circumcircle of triangle $A_2B_2C_2$.

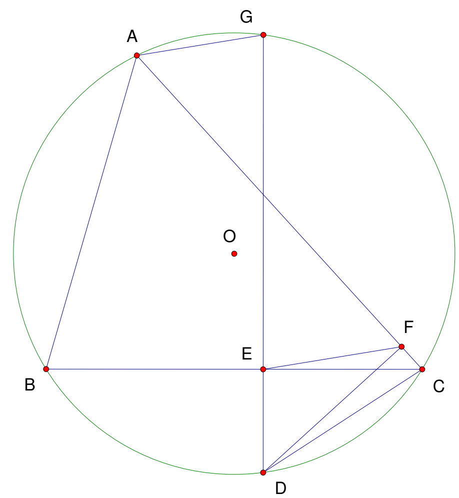
Solution:

First we will express three lemmas:

Lemma 2:

Given triangle ABC and its circumcircle (O). Let D be an arbitrary point on (O). A line d through D and $d \perp BC.d \cap (O) = \{G\}$ then AG is parallel to the Simson line of point D.

Proof:



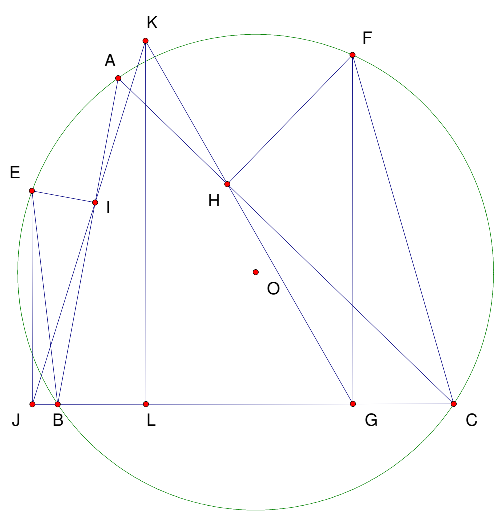
Denote $d \cap BC = \{E\}, F$ the projection of D on AC then EF is the Simson line of D.

Since $DEFC$ is a cyclic quadrilateral we get $\angle GAC = \angle GDC = \angle EFA$. Therefore $AG//EF$.

Lemma 3:

Given triangle ABC and its circumcircle (O). Let E, F be arbitrary points on (O). Then the angle between the Simson lines of two points E and F is half the measure of the arc EF.

Proof:



See on the figure, IJ, HG are the Simson lines of two points E and F, respectively. $IJ \cap HG = \{K\}$

Denote L the projection of K on BC.

We have $KL//FG$ and $FHGC$ is a cyclic quadrilateral thus $\angle GKL = \angle HGF = \angle ACF(1)$

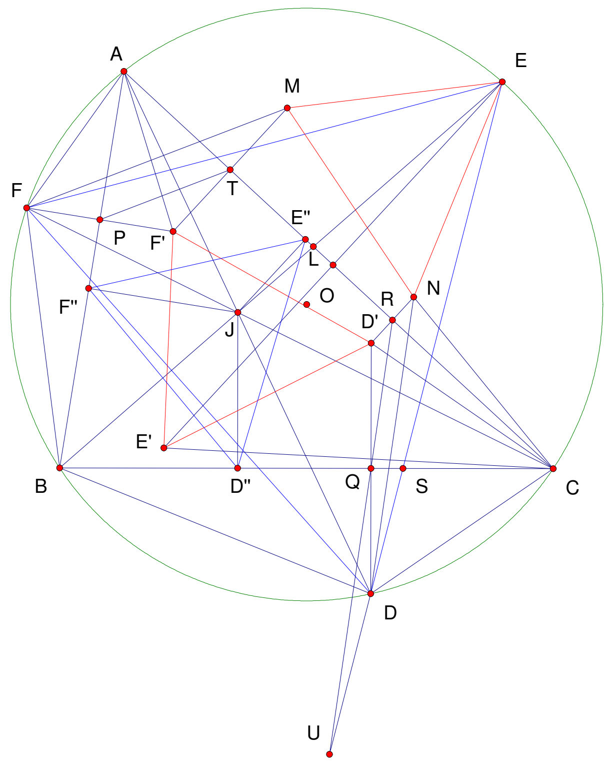
Similarly, $\angle LKJ = \angle EBA(2)$

From (1) and (2) we are done.

Lemma 4:

Let ABC be a triangle and (O) its circumcircle, J be an arbitrary point inside triangle $ABC. AJ, BJ, CJ$ intersects (O) again at D, E, F. Let D', E', F' be the reflections of D, E, F across the lines BC, CA, AB, respectively. Then $\Delta DEF \sim \Delta D'E'F'$.

Proof:



Let M, N be the reflections of F', D' across the line AC. It is easy to see that $EN = E'D', EM = E'F', MN = F'D'$. So we only need to prove that $\Delta MEN \sim \Delta FED$.

Now denote $R = D'N \cap AC, Q = D'D \cap BC, U = RQ \cap ED, S = BC \cap ED, L = BE \cap AC$.

We have $RQ//ND$ hence $\angle NDE = \angle RUE = \angle QDS - \angle D'QR = 90^o - \angle BSD - \angle D'CR = 90^o - \angle BSD - (\angle ACB - \angle BCD) = 90^o - \angle ACB - EDC = 90^o - \angle ACB - \angle EBC = 90^o - \angle(AC, BE)$.

Similarly, $\angle MFE = 90^o - \angle(AC, BE)$ thus $\angle NDE = \angle MFE (3)$

On the other side, $\dfrac{ND}{MF} = \dfrac{2RQ}{2TP} = \dfrac{D'C.\sin C}{F'A.\sin A} = \dfrac{DC.\sin C}{FA.\sin A} = \dfrac{JC.\sin C}{JA.\sin A}(4)$

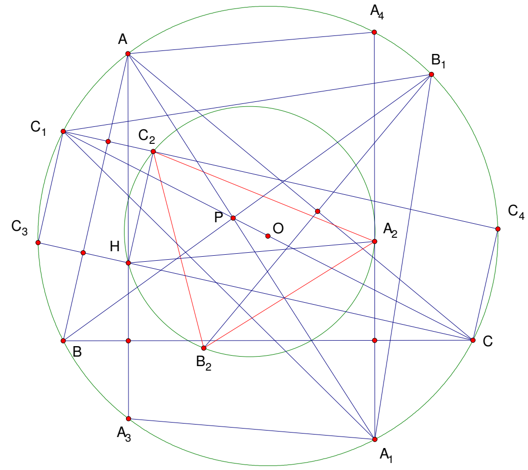
Let $D''E''F''$ be the pedal triangle of triangle ABC wrt J. This result is well-known (it also appeared in **Mathvn** magazine No.3): $\Delta D''E''F'' \sim \Delta DEF$.

Therefore $\dfrac{DE}{FE} = \dfrac{D''E''}{F''E''} = \dfrac{JC.\sin C}{JA.\sin A}(5)$

From $(3), (4)$ and (5) we get $\Delta DNE \sim \Delta FME$. Then $\dfrac{DE}{NE} = \dfrac{FE}{ME}$ or $\dfrac{DE}{D'E'} = \dfrac{FE}{F'E'}$

Similarly we claim $\dfrac{DE}{D'E'} = \dfrac{FE}{F'E'} = \dfrac{EF}{E'F'}$ and the result follows.

Return to our problem:



Denote A_3, C_3 the reflections of H across the lines BC, AB, respectively then A_3, C_3 lie on (O). Let $A_1A_2 \cap (O) = \{A_4, A_1\}, C_1C_2 \cap (O) = \{C_4, C_1\}$

Since $AA_3//A_1A_4$ and A, A_3, A_1, A_4 are concyclic we get $AA_3A_1A_4$ is an isoceles trapezoid. But $HA_3A_1A_2$ is also an isoceles trapezoid too hence $HA_2//AA_4$.

Similarly, $HC_2//CC_4$.

Therefore $\angle C_2HA_2 = \angle(AA_4, CC_4)$. Applying lemma 1, if we denote d_a, d_c the Simson lines of A_1, C_1 then $\angle(AA_4, CC_4) = \angle(d_1, d_2)$.

Applying lemma 2, $\angle(d_1, d_2) = \angle C_1B_1A_1$. So $\angle C_2HA_2 = \angle C_1B_1A_1$.

Applying lemma 3, $\Delta A_1B_1C_1 \sim \Delta A_2B_2C_2$ then $\angle C_1B_1A_1 = \angle C_2B_2A_2$. Therefore $\angle C_2HA_2 = \angle C_2B_2A_2$ which follows that A_2, B_2, C_2, H are concylic.

We complete the proof.

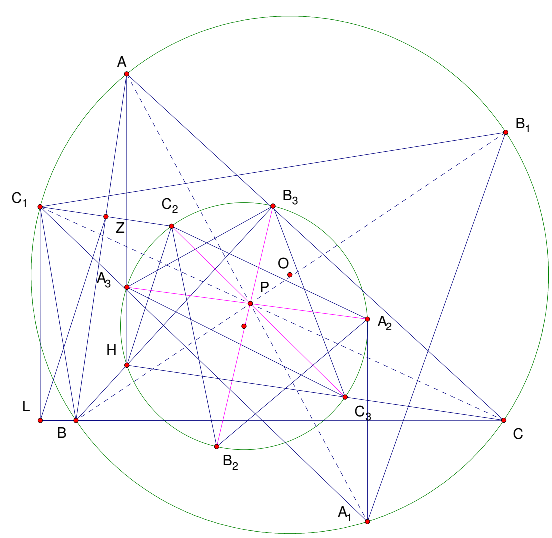
Note:

Problem 2 is the generalization of the Fuhmann circle.

Corollary 3:

Let A_3, B_3, C_3 be the second intersections of AH, BH, CH and $(A_2B_2C_2)$, respectively. Then A_2A_3, B_2B_3, C_2C_3 concur at P.

Proof:



$\angle C_2A_2A_3 = \angle A_3HC_2 = \angle(AH, C_2H) = 90^o - \angle(BC, C_2H)$. Let L, Z be the projections of C_1 on BC, AB, respectively then LZ is the Simson line of C_1. Applying lemma 2 we get $HC_2//LZ$

Therefore $\angle(BC, C_2H) = \angle BLZ = \angle BC_1Z$.

Hence $\angle C_2A_2A_3 = 90^o - \angle BC_1Z = \angle C_1BA = \angle C_1CA$.

From corollary 1, we have $\Delta A_3B_3C_3 \sim \Delta ABC$ and from lemma 4, $\Delta A_2B_2C_2 \sim \Delta A_1B_1C_1$.

Thus there exist a similarity transformation which takes $A_3C_2B_3A_2C_3$ to $AC_1BA_1CB_1$ and we have A_2A_3, B_2B_3, C_2C_3 concur at P'. Moreover, $\dfrac{AP}{A_1P} = \dfrac{A_3P'}{A_2P'}$.

But $AA_3//A_1A_2$ then applying Thales's theorem we get $P' \equiv P$.

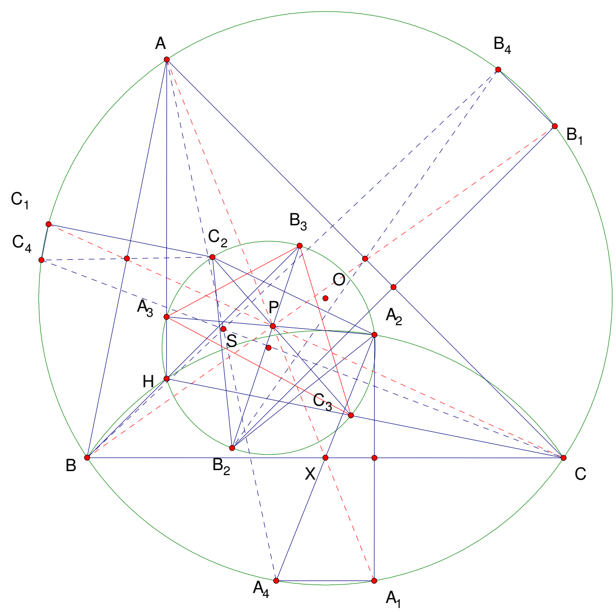
Our proof is completed then.

Now, what does make problem 1 and 2 be similar?

Let A_4, B_4, C_4 be the reflections of A_2, B_2, C_2 across the midpoints of BC, CA, AB. Since (BA_2C) is the reflection of (ABC) across the midpoint of BC then $A_4 \in (ABC)$

On the other side, BC is the midline and parallel to A_4A_1 of triangle $A_1A_2A_4$ thus $A_4A_1//BC$, which follows that $\angle CAA_1 = \angle BAA_4$. Similarly, $\angle CBB_1 = \angle ABB_4, \angle ACC_1 = \angle BCC_4$.

Thus AA_4, BB_4, CC_4 concur at an isogonal conjugate point Q of P wrt triangle ABC.



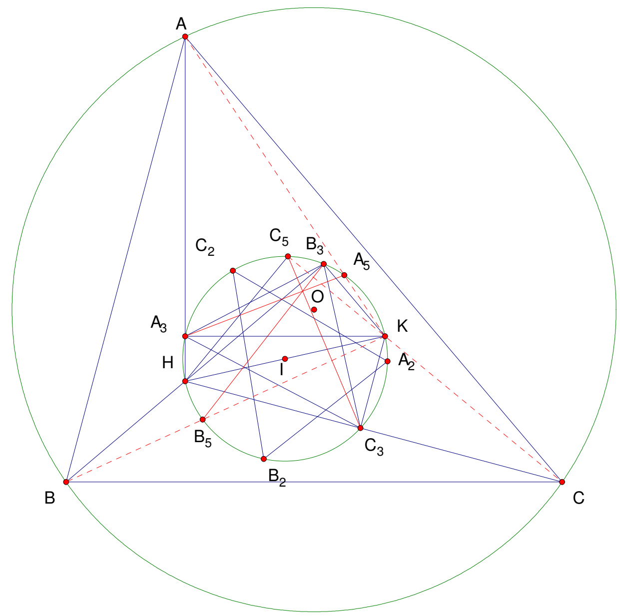
As that result above we came to the following conclusion: Problem 1 and 2 are equivalent. Therefore there are at least two solutions for each problem and its corollary!

At the end we will prove another result about those nice problems:

Corollary 4:

Let I be the circumcenter of triangle $A_2B_2C_2$. K is the reflection of H across I. AK, BK, CK cut (I) again at A_5, B_5, C_5. Then A_3A_5, B_3B_5, C_3C_5 are concurrent.

Proof:



Since $KA_3 \perp AH$ we have $\angle C_5C_3A_3 = \angle C_5KA_3 = \angle KCB$. Similarly, $\angle C_5C_3B_3 = \angle KCA, \angle A_3B_3B_5 = \angle KBC, \angle B_5B_3C_3 = \angle ABK, \angle B_3A_3A_5 = \angle CAK, \angle C_3A_3A_5 = \angle BAK$. Then applying Ceva-sine theorem for triangle $A_3B_3C_3$ we are done.

References:

[1] Ha Khuong Duy- Vietnam IMO training 2009, problem 142, 167.