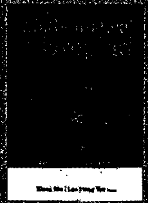

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2009 Chinese Mathematical Olympiad

2008-2009

CONTENTS

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Mathematical Association of China

2009 Chinese Mathematical Olympiad

Qiong hai, Hai nan

[redacted]

1. Given an acute triangle PBC, $PB \neq PC$. Let points A, D be on sides PB and PC, respectively. Let M, N be the midpoints of segments BC and AD, respectively. Lines AC and BD intersect at point O. Draw $OE \perp AB$ at point E and $OF \perp CD$ at point F.

(1) Prove that, if A, B, C, D are concyclic, then $EM \cdot FN = EN \cdot FM$.

(2) Are the four points A, B, C, D always concyclic if $EM \cdot FN = EN \cdot FM$? Prove your answer.

2. Find all pairs (p, q) of prime numbers such that $pq \mid 5^p + 5^q$.

3. Let m, n be integers with $4 < m < n$, and $A_1A_2 \cdots A_{2n+1}$ be a regular $2n+1$-polygon. Let $P = \{A_1, A_2, \cdots, A_{2n+1}\}$. Find the number of convex m-polygons with exactly two acute internal angles whose vertices are all in P.

Second Day 8:00-12:30
January 10, 2009

4. Let $n \geq 3$ be a given integer, and $a_1, a_2, \cdots, a_n$ be real numbers satisfying $\min\limits_{1 \leq i < j \leq n} |a_i - a_j| = 1$. Find the minimum value of $\sum\limits_{k=1}^{n} |a_k|^3$.

5. Find all integers n such that we can colour the all edges and diagonals of a convex n-polygon by n given colours satisfying the following conditions:

(1) Every one of edges or diagonals is coloured by only one colour;

(2) For any three distinct colours, there exists a triangle whose vertices are vertices of the n-polygon and three edges are coloured by the three colours, respectively.

6. Given an integer $n \geq 3$. Prove that there exists a set S of n distinct positive integers such that for any two distinct non-empty subsets A and B of S, the numbers

$$\frac{\sum\limits_{x \in A} x}{|A|} \quad \text{and} \quad \frac{\sum\limits_{x \in B} x}{|B|}$$

are two coprime composite integers, where $\sum\limits_{x \in X} x$ denotes the sum of all elements of a finite set X, and $|X|$ denotes the cardinality of X.

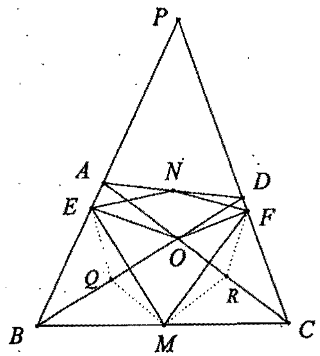
Solution

1. (1) Denote by Q, R the midpoints of OB, OC, respectively. It is easily to see that

$$EQ = \frac{1}{2}OB = RM, \quad MQ = \frac{1}{2}OC = RF,$$

and

$$\angle EQM = \angle EQO + \angle OQM = 2\angle EBO + \angle OQM,$$
$$\angle MRF = \angle FRO + \angle ORM = 2\angle FCO + \angle ORM.$$



Because of that A, B, C, D are concyclic, and Q, R are the midpoints of OB, OC, we have $\angle EBO = \angle FCO$, $\angle OQM = \angle ORM$.

So $\angle EQM = \angle MRF$, which follows that $\triangle EQM \cong \triangle MRF$, and $EM = FM$.

Similarly, we have $EN = FN$, so $EM \cdot FN = EN \cdot FM$ holds.

(2) Suppose that $OA = 2a, OB = 2b, OC = 2c, OD = 2d$ and $\angle OAB = \alpha, \angle OBA = \beta, \angle ODC = \gamma, \angle OCD = \theta$.

Then

$$\cos\angle EQM = \cos(\angle EQO + \angle OQM)$$
$$= \cos(2\beta + \angle AOB) = -\cos(\alpha - \beta).$$

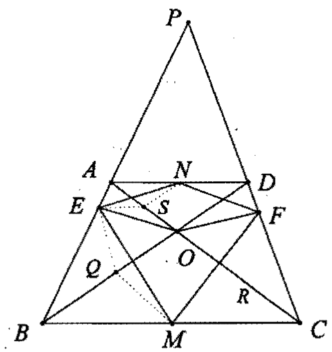
So

$$EM^2 = EQ^2 + QM^2 - 2EQ \cdot QM \cdot \cos\angle EQM$$
$$= b^2 + c^2 + 2bc\cos(\alpha - \beta).$$

Make the similar equations for EN, FN, FM, we have

$$EN \cdot FM = EM \cdot FN$$
$$\Leftrightarrow EN^2 \cdot FM^2 = EM^2 \cdot FN^2$$
$$\Leftrightarrow (a^2 + d^2 + 2ad\cos(\alpha - \beta))(b^2 + c^2 + 2bc\cos(\gamma - \theta))$$
$$= (a^2 + d^2 + 2ad\cos(\gamma - \theta))(b^2 + c^2 + 2bc\cos(\alpha - \beta))$$
$$\Leftrightarrow (\cos(\gamma - \theta) - \cos(\alpha - \beta))(ab - cd)(ac - bd) = 0.$$

Because of $\alpha + \beta = \gamma + \theta$, $\cos(\gamma - \theta) - \cos(\alpha - \beta) = 0$ holds if and only if $\alpha = \gamma, \beta = \theta$ (that is concyclic) or $\alpha = \theta, \beta = \gamma$ (that follows $AB // CD$, contradiction.); $ab - cd = 0$ holds if and only if $AD // BC$; $ac - bd = 0$ holds if and only if A, B, C, D are concyclic.



So, when $AD // BC$ holds, we also have $EM \cdot FN = EN \cdot FM$. We know that A, B, C, D are not concyclic in this case because of $PB \neq PC$, so the answer is "false".

2. If $2 \mid pq$, we suppose that $p = 2$ without loss of generality, then $q \mid 5^q + 25$. By Fermat theorem we have

$q \mid 5^q - 5$, so $q \mid 30$, here $(2,3)$ and $(2,5)$ are solutions. ($(2,2)$ don't fit)

If $5 \mid pq$, we suppose that $p = 5$ without loss of generality, then $q \mid 5^q + 5^5$. By Fermat theorem we have $q \mid 5^q - 5$, so $q \mid 3130$, here $(5,5)$ and $(5,313)$ are solutions.

Otherwise, we have $pq \mid 5^{p-1} + 5^{q-1}$, so

$$5^{p-1} + 5^{q-1} \equiv 0 \pmod{p}. \qquad ①$$

By Fermat theorem we have

$$5^{p-1} \equiv 1 \pmod{p}, \qquad ②$$

Because of ① and ②,

$$5^{q-1} \equiv -1 \pmod{p}. \qquad ③$$

Denote by $p - 1 = 2^k(2r - 1)$, $q - 1 = 2^l(2s - 1)$, where k, l, r, s are positive integers.

If $k \leq l$, then we get

$$1 = 1^{2^{l-k}(2s-1)} \equiv (5^{p-1})^{2^{l-k}(2s-1)} = 5^{2^l(2r-1)(2s-1)}$$
$$= (5^{q-1})^{2r-1} \equiv (-1)^{2r-1} \equiv -1 \pmod{p},$$

which is contradictory with $p \neq 2$. So $k > l$.

But we have $k < l$ by a similar reason, contradiction.

So, all the pairs of primes (p,q) are $(2,3)$, $(3,2)$, $(2,5)$, $(5,2)$, $(5,5)$, $(5,313)$ and $(313,5)$.

3. Notice that if a regular m polygon has exactly two acute angles, they must be at consecutive vertices: for otherwise there would be two disjoint pairs of sides that take up more than half of the circle each. Now assume that the last vertex, clockwise, of these four vertices that make up two acute angles is fixed; this reduces the total number of regular m polygons $2n+1$ times and we will later multiply by this factor.

Suppose the larger arc that the first and the last of these four vertices make contains k points, and the other arc contains $2n-1-k$ points. For each k, the vertices of the m polygon on the smaller arc may be arranged in $\binom{2n-1-k}{m-4}$ ways, and the two vertices on the larger arc may be arranged in $(k-n-1)^2$ ways (so that the two angles cut off more than half of the circle).

The total number of polygons given by k is thus $(k-n-1)^2\times\binom{2n-1-k}{m-4}$. Summation over all k and change of variable gives that the total number of polygons (divided by a factor of $2n+1$) is

$$\sum_{k\geq 0}k^2\cdot\binom{n-k-2}{m-4}.$$

This can be proven to be exactly $\binom{n}{m-1}+\binom{n+1}{m-1}$ by double induction on $n>m$ and $m>4$. The base cases $n=m+1$ and $m=5$ are readily calculated. The induction step is

$$\sum_{k\geq 0}k^2\cdot\binom{n-k-2}{m-4}=\sum_{k\geq 0}k^2\cdot\binom{(n-1)-k-2}{m-4}+\sum_{k\geq 0}k^2\cdot\binom{(n-1)-k-2}{(m-1)-4}$$
$$=\binom{n-1}{m-1}+\binom{n}{m-1}+\binom{n-1}{m-2}+\binom{n}{m-2}=\binom{n}{m-1}+\binom{n+1}{m-1}.$$

So the total number of $2n+1$ polygons is

$$(2n+1)\left(\binom{n}{m-1}+\binom{n+1}{m-1}\right).$$

4. Without loss of generality, let $a_1<a_2<\cdots<a_n$, note that

$$|a_k|+|a_{n-k+1}|\geq|a_{n-k+1}-a_k|\geq|n+1-2k|$$

for $1\leq k\leq n$.

So

$$\sum_{k=1}^{n}|a_k|^3=\frac{1}{2}\sum_{k=1}^{n}\left(|a_k|^3+|a_{n+1-k}|^3\right)$$
$$=\frac{1}{2}\sum_{k=1}^{n}\left(|a_k|+|a_{n+1-k}|\right)\left(\frac{3}{4}\left(|a_k|-|a_{n+1-k}|\right)^2+\frac{1}{4}\left(|a_k|+|a_{n+1-k}|\right)^2\right)$$
$$\geq\frac{1}{8}\sum_{k=1}^{n}\left(|a_k|+|a_{n+1-k}|\right)^3\geq\frac{1}{8}\sum_{k=1}^{n}|n+1-2k|^3.$$

When n is odd,

$$\sum_{k=1}^{n}|n+1-2k|^3=2\cdot 2^3\cdot\sum_{i=1}^{\frac{n-1}{2}}i^3=\frac{1}{4}(n^2-1)^2.$$

When n is even,

$$\sum_{k=1}^{n}|n+1-2k|^3=2\sum_{i=1}^{\frac{n}{2}}(2i-1)^3$$

$$=2\left(\sum_{j=1}^{n}j^3-\sum_{i=1}^{\frac{n}{2}}(2i)^3\right)=\frac{1}{4}n^2(n^2-2).$$

So, $\sum_{k=1}^{n}|a_k|^3\geq\frac{1}{32}(n^2-1)^2$ for odd n, and

$$\sum_{k=1}^{n}|a_k|^3\geq\frac{1}{32}n^2(n^2-2)$$

for even n. The equality holds at $a_i=i-\frac{n+1}{2},i=1,2,\cdots,n$.

5. For all odd number n that $n>1$.

First of all since there are $\binom{n}{3}$ ways to choose 3 among n colors, and $\binom{n}{3}$ ways to choose 3 vertices to form a triangle, so if the question's condition is fulfilled, all the triangles should have different color combination among each other. (a 1-1 correspondence) Note that each two line with the same color cannot have a P ' s vertex as a common point.

As each color combination is used in exactly one triangle, for each color, there should be exactly $\binom{n-1}{2}$ triangle which has

one side in this color, so there should be exactly $\frac{n-1}{2}$ lines in this color. So n is odd.

Now gives a construction method for all odd n.

As the orientation of the vertices doesn't really matters, we assume that the polygon is a regular n polygon. First color the n sides of the polygon in the n distinct colors. Then for each side, color those diagonals that are parallel to this side into the same color.

In this way, for each color, there are n diagonals colored in this color, notice that each of these diagonals are of different length. ①

Besides, for any two triangles with all vertices in, we shall prove that they should have different color combination. Suppose the contrary, they have exactly the same three colors as their sides. Because of that all the sides with the same line are parallel, the two triangles must be similar. For their vertices are in the same circle, they must be same, but it is the contradiction of ①. This completes the proof.

6. Let $f(X)$ be the average of elements of finite number set X.

First of all, make n different primes $p_1,p_2,\cdots\cdots,p_n$ which are all bigger than n, we prove that for any different non-empty subset A, B of set $S_1 = \{\frac{\prod_{i=1}^{n} p_i}{p_j} : 1 \le j \le n\}$, $f(A) \ne f(B)$ always holds.

In fact, we can suppose $\frac{\prod_{i=1}^{n} p_i}{p_1} \in A$ and $\frac{\prod_{i=1}^{n} p_i}{p_1} \notin B$ without loss of generality. Every element of B can be divided by p_1, so $p_1 \mid n!f(B)$. But A have exactly one element which cannot be divide by p_1, so we get $n!f(A)$ cannot divided by p_1 (Note that $p_1 > n$), so $n!f(A) \ne n!f(B)$, which follows $f(A) \ne f(B)$.

Second, let $S_2 = \{n!x : x \in S_1\}$, then $f(A)$ and $f(B)$ are different positive integers when A, B are different non-empty subsets of S_2.

In fact, it is easily to see that there exists two sets A_1, B_1 which are different non-empty subsets of S_1, and $f(A) = n!f(A_1), f(B) = n!f(B_1)$ holds. We get $f(A) \ne f(B)$ from $f(A_1) \ne f(B_1)$, and $f(A), f(B)$ are positive integers from $|A|, |B| \le n$ and the elements of them are all positive.

Then, let K be the largest element of S_2. We prove that for every two distinct subset A, B of set $S_3 = \{K!x + 1 : x \in S_2\}$, $f(A)$ and $f(B)$ are coprime integers which are both larger than 1.

In fact, it is easily to see that there exists two sets A_1, B_1 which are different non-empty subsets of S_2, and $f(A) = K!f(A_1) + 1, f(B) = K!f(B_1) + 1$ holds. Obviously, $f(A)$ and $f(B)$ are different integers which are both larger than 1. If they have common divisors, let p be a prime common divisor of them without loss of generality. Clearly we have $p \mid (K! \cdot |f(A_1) - f(B_1)|)$. We get $1 \le |f(A_1) - f(B_1)| \le K$ by $0 < f(A_1), f(B_1) \le K$ and $f(A_1) \ne f(B_1)$, so $p \le K$, which follows $p \mid K!f(A_1)$, then $p \mid 1$, contradiction.

Last, let L be the largest element of S_3. We prove that for every two distinct non-empty subset A, B of set $S_4 = \{L! + x : x \in S_3\}$, $f(A)$ and $f(B)$ are two composites which share no common divisors.

In fact, it is easily to see that there exists two sets A_1, B_1 which are different non-empty subsets of S_3, and $f(A) = L! + f(A_1)$, $f(B) = L! + f(B_1)$ holds. Obviously, $f(A)$ and $f(B)$ are different integers which are both larger than 1. Because of that L is the largest element of S_3, we have $f(A_1) \mid L!$, and $f(A_1) \mid f(A)$. We get $f(A)$ is composite by $f(A_1) < f(A)$. By a similar reason, $f(B)$ is composite too. If they have common divisors, let p be a prime common divisor of them without loss of generality. It is obviously that $p \mid (L! \cdot |f(A_1) - f(B_1)|)$. We get $1 \le |f(A_1) - f(B_1)| \le L$ by $0 < f(A_1), f(B_1) \le L$ and $f(A_1) \ne f(B_1)$, so $p \le L$, which follows $p \mid f(A_1)$ and $p \mid f(B_1)$, which is a contradiction of the fact that $f(A_1)$ and $f(B_1)$ are coprime. That complete the proof.

2009 Chinese Girls' Mathematics Olympiad

Zhong shan, China

[redacted]

1. (a) Determine if the set $\{1,2,\ldots,96\}$ can be be partitioned into 32 sets of equal size and equal sum.

(b) Determine if the set $\{1,2,\ldots,99\}$ can be be partitioned Into 33 sets of equal size and equal sum.

2. Let $\phi(x)=ax^3+bx^2+cx+d$ be a polynomial with real coefficients. Given that $\phi(x)$ has three positive real roots and that $\phi(0)<0$, prove that $2b^3+9a^2d-7abc\le 0$.

3. Determine the least real number *a* greater than 1 such that for any point *P* in the interior of square *ABCD*, the area ratio between some two of the triangles *PAB*, *PBC*, *PCD*, *PDA* lies in the interval $\left[\frac{1}{a}, a\right]$.

4. Equilateral triangles *ABQ*, *BCR*, *CDS*, *DAP* are erected out-Side of the(convex) quadrilateral *ABCD*. Let *X*, *Y*, *Z*, *W* be the midpoints of the segments *PQ*, *QR*, *RS*, *SP*, respectively. Determine the maximum value of

$$\frac{XZ+YW}{AC+BD}.$$

Second Day
August 10, 2008

5. In (convex) quadrilateral $ABCD$, $AB=BC$ and $AD=DC$. Point E lies on segment AB and point F lies on segment AD such that B, E, F, D lie on a circle. Point P is such that triangles DPE and ADC are similar and the corresponding vertices are in the same orientation (clockwise or counterclockwise). Point Q is such that triangles BQF and ABC are similar and the corresponding vertices are in the same orientation. Prove that points A, P, Q are collinear.

6. Let $(x_1, x_2, \ldots)$ be a sequence of positive numbers such that $(8x_2-7x_1)x_1^7=8$ and

$$x_{k+1}x_{k-1}-x_k^2=\frac{x_{k-1}^8-x_k^8}{x_k^7x_{k-1}^7} \quad \text{for} \quad k=2,3,\ldots.$$

Determine the real number a such that if $x_1>a$, then the sequence is monotonically decreasing, and if $0<x_1<a$, then the sequence is not monotonic.

7. On a given 2008×2008 chessboard, each unit square is colored in a different color. Every unit square is filled with one of the letters C, G, M, O. The resulting board is called *harmonic* if every 2×2 subsquare contains all four different letters. How many harmonic boards are there?

8. For positive integers n, $f_n=[2^n\cdot\sqrt{2008}]+[2^n\cdot\sqrt{2009}]$. Prove that there are infinitely many odd numbers and in-finitely many even numbers in the sequence $f_1, f_2, \ldots$.

Solutions

1. The answer is no for part (a) and yes for part (b).

(a) Since $1+2+\cdots+96=48\cdot97$ is not divisible by 32, we cannot partition the the set $\{1,2,\ldots,96\}$ into 32 sets of equal sum.

(b) Since $1+2+\cdots+99=50\cdot99$, each of the 33 subsets must have sum 150. We partition the numbers in the set $\{1,2,\ldots,66\}$ into 33 pairs with their sums forming an arithmetic progression of common difference 1:

$$\{1,50\},\{3,49\},\{5,48\},\ldots,\{33,34\},$$
$$\{2,66\},\{4,65\},\ldots,\{32,51\}.$$

It is then easy to see that

$$\{1,50,99\},\{3,49,98\},\{5,48,97\},\ldots,\{33,34,83\},$$
$$\{2,66,82\},\{4,65,81\},\ldots,\{32,51,67\}$$

is a partition satisfying the conditions of the problem.

Remark: In general, the set $\{1,2,\ldots,3n\}$ can be be parti-tioned into n sets of equal size and equal sum if and only if n is odd.

2. Solution 1. Denote by x_1, x_2, x_3 the roots of $\phi(x)$. By Vieta's relation, we have

$$x_1+x_2+x_3=-\frac{b}{a},\ x_1x_2+x_2x_3+x_3x_1=\frac{c}{a},\ x_1x_2x_3=-\frac{d}{a}.$$

Since $\phi(0) = d < 0$ and $x_1x_2x_3 > 0$, it follows that $a > 0$. Dividing both sides of the desired inequality by a^3 gives

$$2\left(\frac{b}{a}\right)^3 + 9\frac{d}{a} - 7\cdot\frac{b}{a}\cdot\frac{c}{a} \le 0$$

or

$$-2\ x_1 + x_2 + x()^3 - 9x_1x_2x_3 + 7(x_1 + x_2 + x_3)(x_1x_2 + x_2x_3 + x_3x_1) \le 0.$$

Expanding the terms on the left-hand side of the last inequality and simplifying gives

$$x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + x_2x_3^2 + x_3^2x_1 + x_3x_1^2 \le 2(x_1^3 + x_2^3 + x_3^3), \quad (1)$$

which is true by Schur's inequality and the AM-GM inequality. Indeed,

$$\begin{aligned} &x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + x_2x_3^2 + x_3^2x_1 + x_3x_1^2 \\ &\le x_1^3 + x_2^3 + x_3^3 + 3x_1x_2x_3 \le 2(x_1^3 + x_2^3 + x_3^3). \end{aligned}$$

Solution 2. One can also establish (1) by noting that

$$(x_1 - x_2)(x_1^2 - x_2^2) = (x_1 - x_2)^2(x_1 + x_2) > 2$$

or

$$x_1^2x_2 + x_1x_2^2 \le x_1^3 + x_2^3.$$

Adding the last inequality with its cyclic analogous forms yields (1).

Solution 3. One can also establish (*) by adding inequalities $x_1^2x_2 + x_2^2x_3 + x_3^2x_1 \le x_1^3 + x_2^3 + x_3^3$ and $x_2^2x_1 + x_3^2x_2 + x_1^2x_3 \le x_1^3 + x_2^3 + x_3^3$. (Both inequalities follow directly from the Re-arrangement inequality.)

3. The answer is the golden ratio $\phi = \dfrac{1+\sqrt{5}}{2}$.

We first show that there must always be two triangles whose ratio lies in the interval $\left[\frac{1}{\phi}, \phi\right]$.

By scaling, we may assume without loss of generality that $ABCD$ has area 2. Then $[PAB]+[PCD]=\frac{1}{2}[ABCD]=1$, and likewise $[PBC]+[PDA]=1$. Let $[PAB] = x$, so that $[PCD] = 1 - x$; by symmetry we may assume without loss of generality that $x \ge 1 - x$, or equivalently, that $x \ge \frac{1}{2}$. Likewise, let $[PBC] = y$ and $[PDA] = 1 - y$, and again we may assume $y \ge 1 - y$. Finally, we may also assume without loss of generality that $x \le y$.

We know that $\frac{1}{2} \le x < 1$: we now divide into cases based on the value of x.

Case 1: $x \le \frac{1}{\phi}$. In this case, $1 - x \ge 1 - \frac{1}{\phi} > 0$, so

$$\frac{x}{1-x} \le \frac{\frac{1}{\phi}}{1 - \frac{1}{\phi}} = \frac{1}{\phi - 1} = \frac{1}{\frac{1}{\phi}} = \phi$$

by the well-known identity that $\phi - 1 = \frac{1}{\phi}$. Since $x \ge 1 - x$, $\frac{x}{1-x} \ge 1 \ge \frac{1}{\phi}$, and so $\frac{x}{1-x} \in \left[\frac{1}{\phi}, \phi\right]$ as desired.

Case 2: $x \ge \frac{1}{\phi}$. Now, since $\frac{1}{\phi} < x \le y < 1$, we conclude that

$$1 \le \frac{y}{x} < \frac{1}{\frac{1}{\phi}} = \phi$$

as desired.

We now show that no smaller value of a works.

We first note that for any x and y strictly between 0 and 1, we can find a point P such that $[PAB]=x$, $[PCD]=1-x$, $[PBC]=y$, $[PDA]=1-y$: just take a point P which is distance $\frac{x}{\sqrt{2}}$ away from side AB and distance $\frac{y}{\sqrt{2}}$ away from side BC.

We now set $x=\frac{1}{\phi}$ and consider the limiting case as y approaches 0. By inspection, we see that the smallest triangle area ratio greater than 1 is $\frac{[PDA]}{[PCD]}=(1-y)\phi$, which approaches ϕ in the limit as $y \to 0$. Likewise, the largest area ratio less than 1 is $\frac{[PCD]}{[PDA]}=\frac{1}{(1-y)\phi}$, which approaches $\frac{1}{\phi}$ in the limiting case. Hence any a that satisfies the conditions of the problem must satisfy either $(1-y)\phi \le a$ or $\frac{1}{(1-y)\phi} \ge \frac{1}{a}$: either way, a must be at least $(1-y)\phi$. Taking the limiting case as $y \to 0$, we see that $a=\phi$ is the minimal value of a that works.

4. Solution 1. The answer is $\frac{1+\sqrt{3}}{2}$.

We consider the configuration shown above. (Our proofs can be adjusted slightly for other configurations.) Let P_1, Q_1, R_1, S_1 be the midpoints of segments DA, AB, BC, CA, respectively. It is well known that $P_1Q_1R_1S_1$ is a parallelogram (since $P_1Q_1 \parallel BD \parallel R_1S_1$ and $Q_1R_1 \parallel AC \parallel S_1P_1$). Let M and N be the midpoints of segments DP and DS, respectively. Note that

$$DS_1 = S_1N = DN = WN$$

and

$$DP_1 = P_1M = MD = WN.$$

Note also that

$$\angle P_1DS_1 = \angle PDS - 120^\circ = 180^\circ - \angle DNW - 120^\circ$$
$$= 60^\circ - \angle DNW = \angle WNS_1.$$

By SAS ($WN = DP_1$, $\angle P_1DS_1 = \angle WNS_1$, and $NS_1 = DS_1$), we conclude that triangles WNS_1 and P_DS_1 are congruent to each other. Likewise, we can show that triangles P_1MW and P_1DS are congruent to each other. It follows that $P_1S_1 = WS_1 = P_1W$; that is, triangle WS_1P_1 is equilateral. In exactly the same way, we can show that YQ_1R_1 is also equilateral.

Let U and V be the midpoints of segments S_1P_1 and Q_1R_1. Then in parallelogram $P_1Q_1R_1S_1$, $UV = P_1Q_1 = \frac{BD}{2}$. By the triangle inequality, we have

$$YW \le YV + VU + UW = \frac{WP_1\sqrt{3}}{2} + P_1Q_1 + \frac{Q_1Y\sqrt{3}}{2}$$
$$= \frac{S_1P_1\sqrt{3} + BD + Q_1R_1\sqrt{3}}{2} = \frac{BD + AC\sqrt{3}}{2}.$$

Similarly, we can show that

$$XZ \le \frac{AC + BD\sqrt{3}}{2}.$$

Adding the last two inequalities yields

$$XZ + YW \le \frac{1+\sqrt{3}}{2} \cdot (AC + BD) \quad \text{or} \quad \frac{XZ+YW}{AC+BD} \le \frac{1+\sqrt{3}}{2}.$$

Equality holds when W, U, V, Y are collinear; that is, $AC \perp BD$.

Solution 2. Let the lower case letter denote the complex number associated with the point labeled by the upper case letter. Let $\omega = \frac{1+\sqrt{3}}{2}$ be a cube root of unity, so that $\omega^2 + \omega + 1 = 0$.

The statement that ABQ is an equilateral triangle can be restated as $(b-q) = \omega(a-b)$, or, expanding and using $\omega + 1 = -\omega^2$, $q + \omega a + \omega^2 b = 0$. Solving for q, we obtain $q = -\omega a - \omega^2 b$. Likewise, we obtain the formulas

$$r = -\omega b - \omega^2 c, \quad s = -\omega c - \omega^2 d, \quad p = -\omega d - \omega^2 a.$$

Since X is the midpoint of QR, it follows that

$$x = \frac{q+r}{2} = -\frac{1}{2}(\omega(a+b)) + \omega^2(b+c)). \qquad (2)$$

And likewise

$$y = -\frac{1}{2}(\omega(b+c) + \omega^2(c+d)), \qquad (3)$$

$$z = -\frac{1}{2}(\omega(c+d) + \omega^2(d+a)), \qquad (4)$$

$$w = -\frac{1}{2}(\omega(d+a) + \omega^2(a+b)). \qquad (5)$$

Now, the length XZ is equal to the absolute value $|x-z|$, and using the above we can write

$$x - z = \frac{1}{2}(\omega(c+d-a-b) + \omega^2(d+a-b-c))$$

or

$$x - z = \left(\frac{\omega - \omega^2}{2}\right)(c-a) + \left(\frac{\omega^2 + \omega}{2}\right)(d-b). \qquad (6)$$

Note that $\omega - \omega^2 = \frac{1+\sqrt{-3}}{2} - \frac{1-\sqrt{-3}}{2} = \sqrt{-3}$ and $\omega + \omega^2 = -1$.

Using this to apply the triangle inequality to (6) yields

$$XZ = |x-z| = \left|\frac{\sqrt{-3}}{2}(c-a) + \frac{1}{2}(b-d)\right|,$$

from which it follows that

$$XZ \le \frac{\sqrt{3}}{2}|c-a| + \frac{1}{2}|b-d| = \frac{\sqrt{3}}{2}AC + \frac{1}{2}BD. \qquad (7)$$

Similarly, we can show that

$$YW \le \frac{\sqrt{3}}{2}BD + \frac{1}{2}AC. \qquad (8)$$

We now complete the proof as in the first solution.

5. Let O be the center of the circle through B, E, F, and D. Because O is the circumcenter of triangle BDF, $\angle BOF = 2\angle BDF = 2\angle BDA$. Also, $\angle CDA = 2\angle BDA$ because triangles ADB and ADC are congruent. Since triangles BOF and CDA are both isosceles and $\angle BOF = \angle CDA$, triangles BOF and CDA are similar. But we are given that CDA is similar to EPD, so we conclude that triangles BOF and EPD are similar. Also, triangles BAF and ADE are similar because quadrilateral $BEFD$ is cyclic. Putting these facts together, we see that quadrilaterals *ABOF* and *ADPE* are similar. In particular, $\angle BAO = \angle DAQ$.

By exactly the same argument as above, quadrilaterals *AEOD* and *AFQB* are also similar, and likewise $\angle BAO = \angle DAP$.

We conclude that $\angle DAQ = \angle DAP$ and the three points *A*, *P*, *Q* are collinear, as desired.

6. Dividing the equation $x_{k+1}x_{k-1} - x_k^2 = \dfrac{x_{k-1}^8 - x_k^8}{x_k^7 x_{k-1}^7}$ by $x_k x_{k-1}$ and manipulating yields

$$\frac{x_{k+1}}{x_k} - \frac{x_k}{x_{k-1}} = \frac{1}{x_k^8} - \frac{1}{x_{k-1}^8}. \tag{9}$$

Rearranging, we find that

$$\frac{x_{k+1}}{x_k} - \frac{1}{x_k^8} = \frac{x_k}{x_{k-1}} - \frac{1}{x_{k-1}^8} = \cdots = \frac{x_2}{x_1} - \frac{1}{x_1^8} = \frac{7}{8}. \tag{10}$$

Using the condition $(8x_{20} - 7x_1)x_1^7 = 8$. We can rewrite this as a recurrence

$$x_{k+1} = \frac{7}{8}x_k + x_k^{-7} \tag{11}$$

for all $k \ge 1$. We now see inductively that the starting condition $x_1 > 0$ implies that x_k is positive for all *k*.

We can rewrite the recurrence (11) as $x_{k+1} - x_k = x_k\left(x_k^{-8} - \dfrac{1}{8}\right)$.

Since x_k is positive, we see that if $x_k > 8^{\frac{1}{8}}$, then $x_k > x_{k+1}$, and if $x_k < 8^{\frac{1}{8}}$, then $x_k < x_{k+1}$.

For all $k \ge 1$, $x_{k+1} = \frac{7}{8}x_k + \frac{1}{8}(8x_k^{-7}) \ge x_k^{\frac{7}{8}} \cdot (8x_k)^{\frac{1}{8}} = 8^{\frac{1}{8}}$ by weighted AM-GM, with equality in the case that $x_k = 8^{\frac{1}{8}}$.

We now proceed to show that $a = 8^{\frac{1}{8}}$. We must show that if $x_1 > 8^{\frac{1}{8}}$, then the sequence $\{x_k\}$ is monotone decreasing, and that if $0 < x_1 < 8^{\frac{1}{8}}$, the sequence is not monotone. First suppose that $x_1 > 8^{\frac{1}{8}}$. By the above, we have $x_1 > x_2 > 8^{\frac{1}{8}}$, and repeating this argument iteratively yields $x_1 > x_2 > x_3 > \cdots > 8^{\frac{1}{8}}$, so the sequence $\{x_k\}$ is monotone increasing as desired.

On the other hand, if $x_1 < 8^{\frac{1}{8}}$, we have seen above that $x_1 < x_2$; however, by the AM-GM above, it is the case that $x_2 > 8^{\frac{1}{8}}$, and so by the argument of the previous case, we see that $x_2 > x_3 > \cdots$. Since $x_1 < x_2 > x_3 > \cdots$, the sequence $\{x_k\}$ is not monotone, as desired.

7. The answer is $12 \cdot 2^{2008} - 24$.

Let a configuration of letters be called *legal* if it fulfills the condition of the problem. We first establish the following observation:

> In every legal configuration, at least one of the following must occur: every row alternates between some two letters, or every column alternates between some two letters.

Indeed, suppose that some row does not alternate; then this row must contain some three successive distinct letters. Suppose without loss of generality that these letters are C, G, M, as in the left-hand side figure shown below. It is then easy to conclude first that

$X_2 = X_5 = O$, and second that $X_1 = X_4 = M$ and $X_3 = X_6 = C$, as in the right-hand side figure shown below.

X_1	X_2	X_3
C	G	M
X_4	X_5	X_6

M	O	C
C	G	M
M	O	C

We can easily repeat this argument to show that the these three columns must alternate between two letters, and then it is easy to see that every column must alternate between two letters.

Now that our initial claim is proven, we can count the legal configurations. If the leftmost column alternates between some two letters (say C and M), then a straightforward induction shows that every odd-indexed column alternates between these two letters, and every even-indexed column alternates between the other two letters (e.g. G and O). Each column may begin with either of its two letters; it is easy to check that any configuration thus obtained is legal. Hence, we have $\binom{4}{2} = 6$ ways to choose which two letters occur in the first column, and 2^{2008} ways to decide what letter begins each column, overall, then, we have $6 \cdot 2^{2008}$ possible configurations in which each column alternates. Likewise, we also have $6 \cdot 2^{2008}$ possible configurations in which each row alternates.

All that remains is to subtract off the number of configurations that have been counted twice——that is, in which each row

alternates and each column alternates. Certainly, any arrangement of the four different letters in the upper-left-hand 2×2 corner can be extended to a legal configuration of the entire grid, simply by filling in the first two columns so that they alternate, then filling in all the rows so that they alternate, conversely, any such doubly-alternating configuration is uniquely determined by this upper-left-hand corner. There are 4!=24 ways to arrange the four letters in this upper-left- hand corner, so we get 24 configurations in which every row and every column alternates, and the above answer follows.

Remark: This problem is inspired by the following problem from the 1996 IMO shortlist:

A square of dimensions $(n-1)\times(n-1)$ is divided Into $(n-1)^2$ unit squares in the usual manner. Each of then n^2 vertices of these squares is to be colored red or blue. Find the number of different colorings such that each unit square has exactly two red vertices. (Two coloring schemes are regarded as different if at least one vertex is colored differently in the two schemes.)

8. We write $\sqrt{2008}$ and $\sqrt{2009}$ in base 2:

$$\sqrt{2008} = \overline{101100.a_1a_2\cdots}_{(2)} \quad \text{and} \quad \sqrt{2009} = \overline{101100.b_1b_2\cdots}_{(2)}$$

First, we show that there are infinitely many even numbers in the sequence. Assume on the contrary that there are only finitely many even numbers in the sequence. Then there exists a positive integer N

such that f_n is odd for every positive integer $n > N$. Adding the base 2 expansions, we find that for every positive integer i

$$f_{N+i} = \overline{101100.b_1b_2\ldots b_{N+i(2)}} + \overline{101100.a_1a_2\ldots a_{N+i(2)}},$$

and the right-hand side is congruent to $b_{N+i} + a_{N+i}$ modulo 2. Since f_{N+i} is odd, it follows that $\{b_{N+i}, a_{N+i}\} = \{0,1\}$ and, in particular, $a_{N+i} + b_{N+i} = 1$. Therefore,

$$\sqrt{2008} + \sqrt{2009} = \overline{1011001.c_1c_2\ldots c_N111\ldots_{(2)}}.$$

In particular, $\sqrt{2008}+\sqrt{2009}$ is rational in base 2, which is impossible, since $\sqrt{2008}+\sqrt{2009}$ is irrational. Thus, our assumption was wrong, and there are infinitely many even numbers in the sequence.

Similarly, we can show that there are infinitely many odd numbers in the sequence. Set $g_n = \lfloor n\sqrt{2009} \rfloor - \lfloor n\sqrt{2008} \rfloor$. It is clear that g_n and f_n have the same parity. Thus, for $n > N$, g_n is even. Note that, in base 2,

$$g_{N+i} = \overline{101100b_1b_2\ldots b_{N+i(2)}} - \overline{101100a_1a_2\ldots a_{N+i(2)}},$$

which is congruent to $b_{N+i} - a_{N+i}$ modulo 2. Since f_{N+i} is odd, $b_{N+i} = a_{N+i}$. Therefore,

$$\sqrt{2008} + \sqrt{2009} = \overline{0.d_1d_2\ldots d_N000\ldots_{(2)}}.$$

In particular, $\sqrt{2008}+\sqrt{2009}$ is rational in base 2,which is impossible, since $\sqrt{2008}+\sqrt{2009}$ is irrational. Thus, our

assumption was wrong, and there are infinitely many even numbers in the sequence.

Remark: The problem is inspired by the following classic problem on the Pigeonhole principle:

For positive integers n, $f_n = \lfloor n\sqrt{2008} \rfloor + \lfloor n\sqrt{2009} \rfloor$.

Prove that there are infinitely many odd numbers and infinitely many even numbers in the sequence $f_1, f_2, \ldots$.

2008 China Western Mathematical Olympiad

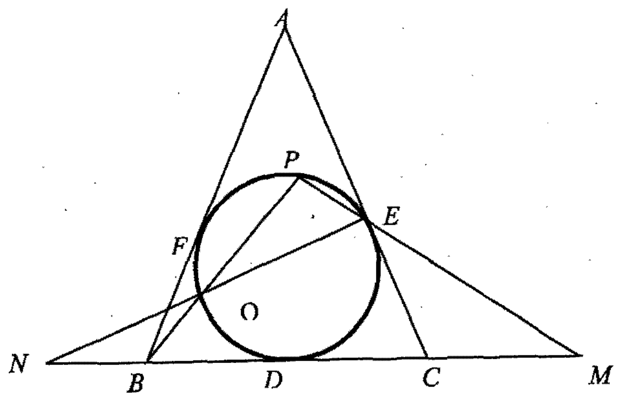
Gui yang, Gui zhou

[redacted]

(Attempt all problems; each carries 15 marks)

1. A sequence of real numbers $\{a_n\}$ is defined by $a_0 \neq 0, 1$, $a_1 = 1 - a_0$, $a_{n+1} = 1 - a_n(1 - a_n)$, $n = 1, 2, \cdots$. Prove that for any positive integer n, we have

$$a_0 a_1 \cdots a_n \left(\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n} \right) = 1.$$



2. In $\triangle ABC$, $AB = AC$, the inscribed circle I touches BC, CA, AB at points D, E and F respectively. P is a point on arc EF (not content D). Line BP intersects circle I at another point Q, lines EP, EQ meet line BC at M, N respectively. Prove that

(1) P, F, B, M are concyclic;

(2) $\dfrac{EM}{EN} = \dfrac{BD}{BP}$.

3. Given an integer $m \geq 2$, and m positive integers $a_1, a_2, \cdots, a_m$. Prove that there exist infinitely many positive integers n, such that $a_1 \cdot 1^n + a_2 \cdot 2^n + \cdots + a_m \cdot m^n$ is composite.

4. Given an integer $m \geq 2$, and two real numbers a, b with $a > 0$ and $b \neq 0$. The sequence $\{x_n\}$ is such that $x_1 = b$, and $x_{n+1} = a x_n^m + b, n = 1, 2, \cdots$. Prove that

(1) when $b < 0$ and m is even, the sequence $\{x_n\}$ is bounded if and only if $ab^{m-1} \geq -2$; and

(2) when $b < 0$ and m is odd, or when $b > 0$, the sequence $\{x_n\}$ is bounded if and only if $ab^{m-1} \leq \dfrac{(m-1)^{m-1}}{m^m}$.

Second Day 2008-12-10
[illegible]

(Attempt all problems; each carries 15 marks)

5. Four frogs are postioned at four points on a straight line such that the distance between any two neighbouring points is 1 unit length. Suppose that every frog can jump to its corresponding point of reflection, by taking any one of the other 3 frogs as the reference point. Prove that, there is no such case that the distance between any two neighbouring points, where the frogs stay, are all equal to 2008 unit length.

6. Given $x, y, z \in (0,1)$ satisfying that

$$\sqrt{\frac{1-x}{yz}}+\sqrt{\frac{1-y}{zx}}+\sqrt{\frac{1-z}{xy}}=2.$$

Find the maximum value of xyz.

7. For a given positive integer n, find the greatest positive integer k, such that there exist three sets of k distinct non-negative integers,

$$A=\{x_1, x_2, \cdots, x_k\},\quad B=\{y_1, y_2, \cdots, y_k\}$$

and $C=\{z_1, z_2, \cdots, z_k\}$ with $x_j+y_j+z_j=n$ for any $1 \le j \le k$.

8. Let P be an interior point of a regular n-gon $A_1A_2\cdots A_n$, the lines A_iP meet the regular n-gon $A_1A_2\cdots A_n$ at another point B_i, where $i=1, 2, \ldots, n$. Prove that $\sum_{i=1}^{n} PA_i \ge \sum_{i=1}^{n} PB_i$.

Solution

1. From the given condition, we have

$$\begin{aligned}1-a_{n+1}&=a_n(1-a_n)=a_na_{n-1}(1-a_{n-1})=\cdots\\&=a_n\cdots a_1(1-a_1)=a_n\cdots a_1a_0,\end{aligned}$$

i.e. $a_{n+1}=1-a_0a_1\cdots a_n$, $n=1,2,\cdots$.

By Mathematical Induction.

When $n=1$, the proposition holds. Assuming that it holds for $n=k$, then when $n=k+1$, we have

$$a_0a_1\cdots a_{k+1}\left(\frac{1}{a_0}+\frac{1}{a_1}+\cdots+\frac{1}{a_k}+\frac{1}{a_{k+1}}\right)$$

$$=a_0a_1a_2\cdots a_k\left(\frac{1}{a_0}+\frac{1}{a_1}+\cdots+\frac{1}{a_k}\right)a_{k+1}+a_0a_1a_2\cdots a_k$$

$$=a_{k+1}+a_0a_1a_2\cdots a_k=1.$$

So it also holds when $n=k+1$.

Hence, it holds for any positive integer n.

2. (1) From the given condition, $EF /\!/ BC$, so

$$\begin{aligned}\angle ABC=\angle AFE&=\angle AFP+\angle PFE\\&=\angle PEF+\angle PFE=180^\circ-\angle FPE.\end{aligned}$$

Thus P, F, B, M are concyclic.

(2) By Sine Law, $EF//BC$ and the fact that P, F, B, M are concyclic, we have

$$\frac{EM}{EN}=\frac{\sin\angle ENM}{\sin\angle EMN}=\frac{\sin\angle FEN}{\sin(\pi)-\angle PFB}=\frac{\sin\angle FPB}{\sin\angle PFB}=\frac{BF}{BP}.$$

Together with $BF=BD$, the proposition is proved.

3. Let p be a prime factor of $a_1+2a_2+\cdots+ma_m$, Fermat Theorem, we have for any k and m satisfying $1\le k\le m$, we have $k^p\equiv k(\bmod p)$. Thus, for any positive integer n, we have

$$a_1\cdot 1^{p^n}+a_2\cdot 2^{p^n}+\cdots+a_m\cdot m^{p^n}\equiv a_1+2a_2+\cdots ma_m\equiv 0(\bmod p),$$

Hence, $a_1\cdot 1^{p^n}+a_2\cdot 2^{p^n}+\cdots+a_m\cdot m^{p^n}(n=1,2,\cdots)$ is composite.

4. (1) When $b<0$ and m is even, in order that $ab^{m-1}<-2$, we should first have $ab^m+b>-b>0$, therefore $a(ab^m+b)^m+b>ab^m+b>0$, that is $x_3>x_2>0$. Using the fact that ax^m+b is monotonically increasing on $(0,+\infty)$, it can be established that each succeeding term of sequence $\{x_n\}$ is greater than its preceeding term, and is greater than $-b$ starting from the second term.

Considering any three consecutive terms of the sequence $x_n, x_{n+1}, x_{n+2}, n=2,3,\cdots$, we have

$$\begin{aligned}x_{n+2}-x_{n+1}&=a(x_{n+1}^m-x_n^m)=a(x_{n+1}-x_n)(x_{n+1}^{m-1}+x_{n+1}^{m-2}x_n+\cdots x_n^m)\\&>amx_n^{m-1}(x_{n+1}-x_n)>am(-b)^{m-1}(x_{n+1}-x_n)\\&>2m(x_{n+1}-x_n)>x_{n+1}-x_n,\end{aligned}$$

it is obvious that the difference of any two consecutive terms of sequence $\{x_n\}$ is increasing, and hence it is not bounded.

When $ab^{m-1}\ge -2$, Mathematical Induction is used to prove that each term of the sequence $\{x_n\}$ falls on the closed interval $[b,-b]$.

The first term b falls on the interval $[b,-b]$. Suppose the term x_n satisfies the condition $b\le x_n\le -b$ for a particular n, then $0\le x_n^m\le b^m$, and hence

$$b=a\cdot 0^m+b\le x_{n+1}\le ab^m+b\le -b.$$

Thus, the sequence $\{x_n\}$ is bounded if and only if $ab^{m-1}\ge -2$.

(2) When $b>0$, each term of the sequence $\{x_n\}$ is positive. So, we first prove that, the sequence $\{x_n\}$ is bounded if and only if the equation $ax^m+b=x$ has positive real roots.

Suppose $ax^m+b=x$ has no positive real roots, in such case, the minimum value of the function $p(x)=ax^m+b-x$ on the interval $(0,+\infty)$ is greater than zero. Let t be the minimum value, it follows that for any two consecutive terms of the sequence x_n and x_{n+1}, we have $x_{n+1}-x_n=ax_n^m-x_n+b$. Thus each succeeding term of the sequence $\{x_n\}$ is greater than the preceeding term at least by t. Hence, it is not bounded.

If the equation $ax^m+b=x$ has positive real roots, let x_0 be one of the positive real roots, then by using Mathematical Induction to prove that each term of the sequence $\{x_n\}$ is less than x_0. Firstly, the first term b is less than x_0. Suppose $x_n<x_0$ for a particular n, by virtue of the fact that ax^m+b is increasing on the interval $[0,+\infty)$, it can be established that $x_{n+1}=ax_n^m+b<ax_0^m+b=x_0$. Therefore the sequence is bounded

Further the equation $ax^m+b=x$ has positve roots if and only if the minimum value of $ax^{m-1}+\frac{b}{x}$ on the interval $(0,+\infty)$ is not greater than1, whereas the minimum value of

$ax^{m-1}+\frac{b}{x}$ can be determined by mean inequality, that is

$$ax^{m-1}+\frac{b}{x}=ax^{m-1}+\frac{b}{(m-1)x}+\cdots+\frac{b}{(m-1)x}\geq m\sqrt[m]{\frac{ab^{m-1}}{(m-1)^{m-1}}}.$$

As such, the sequence $\{x_n\}$ is bounded if and only if

$m\sqrt[m]{\frac{ab^{m-1}}{(m-1)^{m-1}}}\leq 1$, that is, $ab^{m-1}\leq\frac{(m-1)^{m-1}}{m^m}$.

When $b<0$, and m is odd, let $y_n=-x_n$, then $y_1=-b>0$, $y_{n+1}=ay_n^m+(-b)$, giving that sequence $\{x_n\}$ is bounded if and only if sequence $\{y_n\}$ is bounded. Thus, by using the above reasoning, it can be proved that (2) holds.

5. Without loss of generality, we may think of the initial positioning of the four frogs are on the real number line at points 1, 2, 3, and 4. Further, it can be established that the frogs at odd number positions remain at odd number positions after each jumping, and likewise for frogs at even number positions. Thus, no matter after how many number of jumping, there are two frogs remain at odd number positions while the other two frogs remain at even number positions. Therefore, in order that the distances between any two neighbouring frogs are all equal to 2008, all the frogs need to stay at points which are either all odd or all even, which is contrary to the actual situation. Hence, the proposition is proved.

6. Denoting $u=\sqrt[6]{xyz}$, then by the given condition and Mean Inequality

$$2u^3=2\sqrt{xyz}=\frac{1}{\sqrt{3}}\sum\sqrt{x(3-3x)}$$

$$\le\frac{1}{\sqrt{3}}\sum\frac{x+(3-3x)}{2}=\frac{3\sqrt{3}}{2}-\frac{1}{\sqrt{3}}(x+y+z)$$

$$\le\frac{3\sqrt{3}}{2}-\sqrt{3}\cdot\sqrt[3]{xyz}=\frac{3\sqrt{3}}{2}-\sqrt{3}u^2.$$

Therefore, $4u^3+2\sqrt{3}u^2-3\sqrt{3}\le 0$, i.e.

$$\left(2u-\sqrt{3}\right)\left(2u^2+2\sqrt{3}u+3\right)\le 0,$$

and thus $u\le\frac{\sqrt{3}}{2}$. Following this, we have $xyz\le\frac{27}{64}$, and equality holds when $x=y=z=\frac{3}{4}$. Hence, maximum is $\frac{27}{64}$.

7. By the given condition, we have

$$kn\ge\sum_{i=1}^{k}(x_i+y_i+z_i)\ge 3\sum_{i=0}^{k-1}i=\frac{3k(k-1)}{2},$$

and then $k\le\left[\frac{2n}{3}\right]+1$.

The following illustrates the case of $k=\left[\frac{2n}{3}\right]+1$.

When $n=3m$, for $1\le j\le m+1$, let $x_j=j-1$, $y_j=m+j-1$, $z_j=2m-2j+2$; for $m+2\le j\le 2m+1$, let $x_j=j-1$, $y_j=j-m-2$, $z_j=4m-2j+3$, the result is obvious;

When $n=3m+1$, for $1\le j\le m$, let $x_j=j-1$, $y_j=m+j$, $z_j=2m-2j+2$; for $m+1\le j\le 2m$, let $x_j=j+1$, $y_j=j-m-1$, $z_j=4m+1-2j$; and $x_{2m+1}=m$, $y_{2m+1}=2m+1$, $z_{2m+1}=0$ will lead to the expected result;

When $n=3m+2$, for $1\le j\le m+1$, let $x_j=j-1$, $y_j=m+j$, $z_j=2m-2j+3$; for $m+2\le j\le 2m+1$, let $x_j=j$, $y_j=j-m-2$, $z_j=4m-2j+4$; and

$x_{2m+2} = 2m+2$, $y_{m+2} = m$, $z_{m+2} = 0$, the result follows.

In summary, the maximum value of k is $\left[\frac{2n}{3}\right]+1$.

8. Denoting $t = \left[\frac{n}{2}\right]+1$, and let $A_{n+j} = A_j$, $j = 1,2,\cdots,n$.

Noting that the distance between any vertex of a regular n-gon and a point on its side is not greater than its longest diagonal d, we therefore have, for any $1 \le i \le n$,

$$A_iP + PB_i = A_iB_i \le d. \qquad ①$$

Furthermore, using the fact that the sum of any two sides of a triangle is longer than the third side, we have, for any $1 \le i \le n$,

$$A_iP + PA_{i+t} \ge A_iA_{i+t} = d. \qquad ②$$

Summing up ①,② for $i = 1,2,\cdots,n$, we have

$$\sum_{i=1}^{n}(A_iP + PA_{i+t}) \ge nd \ge \sum_{i=1}^{n}(A_iP + PB_i),$$

i.e. $2\sum_{i=1}^{n} PA_i \ge \sum_{i=1}^{n} A_iP + \sum_{i=1}^{n} PB_i$, following which the proposition is proved.