The 12th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 10th April 2002
Category I

Problem 1. Differentiable functions $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}$ are linearly independent. Prove that there exist at least $n-1$ linearly independent functions among $f_1', \dots, f_n'$. [10 points]

Problem 2. Let $p > 3$ be a prime number and $n = (2^{2p} - 1)/3$. Show that n divides $2^n - 2$. [10 points]

Problem 3. Positive numbers $x_1, \dots, x_n$ satisfy

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n} = 1\,.$$

Prove that

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \geq (n-1)\left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right).$$

[10 points]

Problem 4. The numbers 1, 2, ..., n are assigned to the vertices of a regular n-gon in an arbitrary order. For each edge compute the product of the two numbers at the endpoints and sum up these products. What is the smallest possible value of this sum?

[10 points]

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Category II

Problem 1. Find all complex solutions to the system

$$\begin{aligned}(a+\mathrm{i}c)^3+(\mathrm{i}a+b)^3+(-b+\mathrm{i}c)^3&=-6\,,\\(a+\mathrm{i}c)^2+(\mathrm{i}a+b)^2+(-b+\mathrm{i}c)^2&=6\,,\\(1+\mathrm{i})a+2\mathrm{i}c&=0\,.\end{aligned}$$

[10 points]

Problem 2. A ring R (not necessarily commutative) contains at least one zero divisor and the number of zero divisors is finite. Prove that R is finite. [10 points]

Problem 3. Let E be the set of all continuous functions $u\colon[0,1]\to\mathbb{R}$ satisfying

$$u^2(t)\le 1+4\int_0^t s\big|u(s)\big|\,\mathrm{d}s\,,\qquad \forall t\in[0,1]\,.$$

Let $\varphi\colon E\to\mathbb{R}$ be defined by

$$\varphi(u)=\int_0^1\big(u^2(x)-u(x)\big)\,\mathrm{d}x\,.$$

Prove that φ has a maximum value and find it. [10 points]

Problem 4. Prove that

$$\lim_{n\to\infty} n^2\left(\int_0^1\sqrt[n]{1+x^n}\,\mathrm{d}x-1\right)=\frac{\pi^2}{12}\,.$$

[10 points]

Problem j12-I-1/j12-I-4. *Differentiable functions $f_1, \ldots, f_n \colon \mathbb{R} \to \mathbb{R}$ are linearly independent. Prove that there exist at least $n-1$ linearly independent functions among $f_1', \ldots, f_n'$.*

(Eötvös Loránd University, Budapest)

Solution. Select a maximal independent set from the derivatives. Without loss of generality, it can be assumed that this set is $f_1', \ldots, f_m'$, where $m \le n$. If $m \le n-2$, then f_{n-1}' and f_n' can be expressed as a linear combination of $f_1', \ldots, f_m'$; hence, there exist real numbers $a_1, \ldots, a_m$, $b_1, \ldots, b_m$ such that

$$\sum_{i=1}^{m} a_i f_i' - f_{n-1}' = \left(\sum_{i=1}^{m} a_i f_i - f_{n-1} \right)' = 0$$

and

$$\sum_{i=1}^{m} b_i f_i' - f_n' = \left(\sum_{i=1}^{m} b_i f_i - f_n \right)' = 0.$$

This implies that functions $\sum_{i=1}^{m} a_i f_i - f_{n-1}$ and $\sum_{i=1}^{m} b_i f_i - f_n$ are constant. Eliminating these constants, a linear combination of $f_1, \ldots, f_n$ is found which vanishes. $\square$

Problem j12-I-2/j12-I-9. *Let $p > 3$ be a prime number and $n = \frac{2^{2p}-1}{3}$. Show that n divides $2^n - 2$.* (Jagiellonian University in Kraków)

Solution. $n = \frac{2^{2p}-1}{3} = 4^{p-1}+4^{p-2}+\cdots+1$. Hence, in the binary system, $n = 1010\ldots101$ (number of 1's is p). Therefore, in the binary system,

$$(*) \qquad 2^n - 2 = 1111\ldots110 \qquad \text{(number of 1's is } n-1\text{)},$$

$$(**) \qquad 3n = 1111\ldots111 \qquad \text{(number of 1's is } 2p\text{)}.$$

Now if we prove that $2p$ divides $n-1$, then by $(*)$, $(**)$ and by the rules of multiplication in the binary system, we will get that $3n$ divides $2^n - 2$ — just what we need. But now observe:

$$\begin{aligned} 2p \mid (n-1) \iff (n \text{ is odd}\dagger) \iff p \mid (n-1) \iff& \\ \iff p \,\Big|\, \left(\frac{2^{2p}-1}{3} - 1\right) \iff p \,\Big|\, \left(\frac{2^{2p}-4}{3}\right) \iff& \\ \iff (p > 3 \text{ and prime}) \iff p \mid \left(2^{2p}-4\right) \iff& \\ \iff (p > 3 \text{ and prime}) \iff p \,\Big|\, \left(\frac{2^{2p}-4}{4}\right) \iff& \\ \iff p \mid \left(2^{2p-2}-1\right).& \end{aligned}$$

But now from Fermat's small theorem (p prime and p does not divide a, then $a^{p-1} - 1 \equiv 0 \pmod{p}$), we have $2^{p-1} \equiv 1 \pmod{p}$, hence $(2^{p-1})^2 \equiv 1^2 \pmod{p}$ and finally $2^{2p-2} \equiv 1 \pmod{p}$. $\square$

† The sentences in parentheses serve only as justifications of the stated equivalences here. Thus, e.g., $2p \mid (n-1) \Leftrightarrow (n \text{ is odd}) \Leftrightarrow p \mid (n-1)$ should be read as "$2p$ divides $(n-1)$ if and only if p divides $(n-1)$ because n is odd" and so on.

Problem j12-I-3/j12-II-59. *Positive numbers* $x_1, \ldots, x_n$ *satisfy*

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n} = 1. \tag{1}$$

Prove that

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \geq (n-1)\left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right).$$

(University of Ostrava)

Solution. It is sufficient to prove that

$$\left(\sqrt{x_1} + \frac{1}{\sqrt{x_1}}\right) + \left(\sqrt{x_2} + \frac{1}{\sqrt{x_2}}\right) + \cdots + \left(\sqrt{x_n} + \frac{1}{\sqrt{x_n}}\right) \geq n\left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right)$$

or equivalently (see (1))

$$\left(\frac{1+x_1}{\sqrt{x_1}} + \cdots + \frac{1+x_n}{\sqrt{x_n}}\right)\left(\frac{1}{1+x_1} + \cdots + \frac{1}{1+x_n}\right) \geq n\left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right). \tag{2}$$

Consider the function $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}} = \frac{x+1}{\sqrt{x}}$, $x \in (0, +\infty)$. It is easy to check that the function f is non-decreasing on $[1, +\infty)$ and that

$$f(x) = f(\tfrac{1}{x}) \tag{3}$$

holds for each $x > 0$.

Further, it follows from (1) that only x_1 can be less than 1 (i.e. $x_k \geq 1$, $k = 2, 3, \ldots$) and $\frac{1}{1+x_2} \leq 1 - \frac{1}{1+x_1} = \frac{x_1}{1+x_1}$. Hence

$$x_2 \geq \frac{1}{x_1} \tag{4}$$

(a contradiction otherwise). It is now apparent directly (if $x_1 \geq 1$) or from (3) and (4) (if $x_1 < 1$) that

$$f(x_1) = f(\tfrac{1}{x_1}) \leq f(x_2) \leq \cdots \leq f(x_n).$$

This means that the sequence $\left\{\frac{1+x_k}{\sqrt{x_k}}\right\}_{k=1}^{n}$ is non-decreasing. Thus (2) holds according to the well-known Chebyshev's inequality since the sequence $\left\{\frac{1}{1+x_k}\right\}_{k=1}^{n}$ is decreasing.

The equality in (2) holds if and only if

$$\frac{1}{1+x_1} = \frac{1}{1+x_2} = \cdots = \frac{1}{1+x_n} \qquad \text{or} \qquad \frac{1+x_1}{\sqrt{x_1}} = \frac{1+x_2}{\sqrt{x_2}} = \cdots = \frac{1+x_n}{\sqrt{x_n}},$$

which implies $x_1 = x_2 = \cdots = x_n$. Then we obtain from (1) that $x_1 = x_2 = \cdots = x_n = n-1$. $\square$

Problem j12-I-4/j12-I-5. *The numbers $1, 2, \ldots, n$ are assigned to the vertices of a regular n-gon in an arbitrary order. For each edge compute the product of the two numbers at the endpoints and sum up these products. What is the smallest possible value of this sum?*

(Babeş-Bolyai University, Cluj-Napoca)

Solution. Due to the $(a-b)^2 = a^2 - 2ab + b^2$ identity, it is sufficient to find the maximum of the sum

$$\sum_{k=1}^{n} \big(\sigma(k+1) - \sigma(k)\big)^2$$

where $\sigma(k)$ denotes the number from the k^{th} vertex and $\sigma(n+1) = \sigma(1)$. We will give an inductive algorithm to find an optimal arrangement and so we can find the maximal sum (or the minimal for the initial problem). Suppose we have an arbitrary arrangement with n numbers and construct an arrangement with $n+2$ numbers in the following way:

- Find the maximum of $\big|\sigma(k+1) - \sigma(k)\big|$. For such a k, denote $x = \min\big\{\sigma(k+1), \sigma(k)\big\}$ and $y = \max\big\{\sigma(k+1), \sigma(k)\big\}$.
- Increase each number by 1.
- Insert the numbers 1 and $n+2$ as in figure 1.

If we denote by s_{n+2} and s_n the corresponding distance sums, we have:

$$\begin{aligned} s_{n+2} &= s_n - (x-y)^2 + \big((n+1) - x\big)^2 + (n+1)^2 + y^2 \\ &= s_n + 2(n+1)^2 + 2xy - 2x - 2nx. \end{aligned}$$

On the other hand, from the obvious inequalities $x \geq 1$ and $n + 1 - y \geq 1$, we have $x(n+1-y) \geq 1$ and this implies $2xy - 2x - 2nx \leq -2$. Hence

$$s_{n+2} = s_n + 2(n+1)^2 - 2n = 2n(n+2).$$

If y_n is the maximal sum, we have $y_{n+2} = y_n + 2n(n+2)$ (because for $n = 3$ in the maximal arrangement $x = 1$, $y = 3$ and in each step the maximal distance $\big|\sigma(k+1) - \sigma(k)\big|$ occurs at $x = 1$ and $y = n$). For $n = 2$ and $n = 3$, we have $y_2 = 2$ and $y_3 = 6$ so from the obtained recurrence relation we can deduce $y_{2n} = 2 + \frac{8}{3}(n-1)n(n+1)$ and thus

$$x_{2n} = \frac{2\sum_{k=1}^{n} k^2 - y_{2n}}{2} = \frac{4n^3 + 6n^2 + 5n - 3}{3}$$

where x_n denotes the minimal sum for the initial problem. Analogously we have

$$x_{2n+1} = \frac{4n^3 + 12n^2 + 14n + 3}{3}.$$

For $n \in \{6, 7, 8, 9, 10\}$, we have illustrated the optimal arrangements on figures 2, 3, 4, 5 (in the exterior we have written the arrangement's numbers, inside the circle the product of any two adjacent number and in the inside circle the sum of these products).

Remark. For $p > 1$, the above arrangements will give the maximum of the sum $\sum_{k=1}^{n} \big(\sigma(k+1) - \sigma(k)\big)^p$, and this can be proved by the same method using the inequality

$$(n+1)^p + (n+1-x)^p + y^p - (y-x)^p \leq n^p + (n+1)^p + n^p - (n-1)^p.$$

$\square$

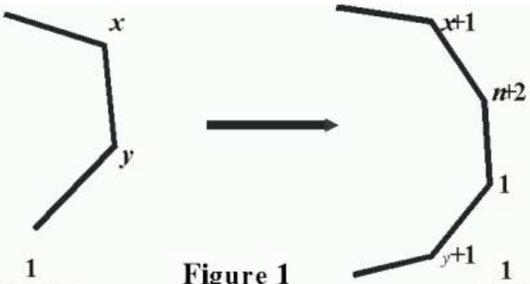

Figure 1

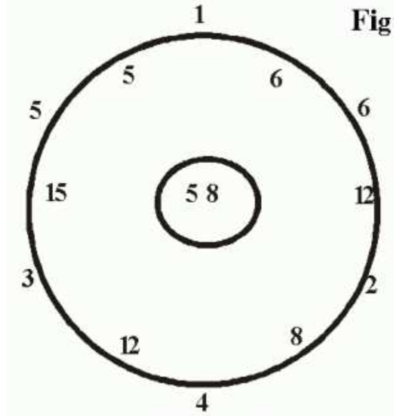

Figure 2

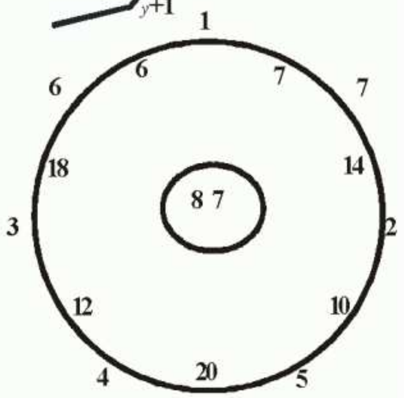

Figure 3

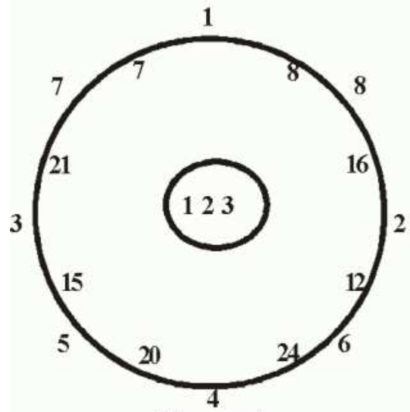

Figure 4

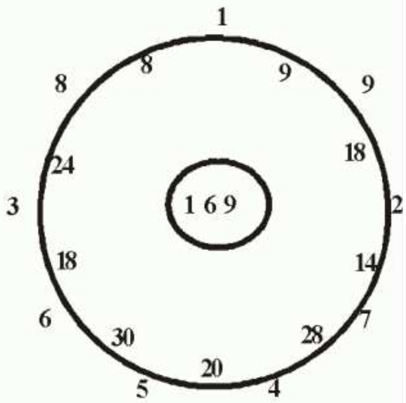

Figure 5

Problem j12-II-1/j12-II-56. *Find all complex solutions of the system*

$$\begin{aligned}(a+\mathrm{i}c)^3+(\mathrm{i}a+b)^3+(-b+\mathrm{i}c)^3&=-6\,,\\(a+\mathrm{i}c)^2+(\mathrm{i}a+b)^2+(-b+\mathrm{i}c)^2&=6\,,\\(1+\mathrm{i})a+2\mathrm{i}c&=0\,.\end{aligned}$$

(P. J. Šafárik University in Košice)

Solution. Let us notice that the third equation can be written as

$$(a+\mathrm{i}c)+(\mathrm{i}a+b)+(-b+\mathrm{i}c)=0;$$

that is why a natural substitution is

$$x=a+\mathrm{i}c,\qquad y=\mathrm{i}a+b,\qquad z=-b+\mathrm{i}c.$$

Then, our system is

$$\begin{aligned}x^3+y^3+z^3&=-6\\x^2+y^2+z^2&=6\\x+y+z&=0\end{aligned}$$

Using symmetric polynomials, we get

$$\begin{gathered}x+y+z=\sigma_1,\\x^2+y^2+z^2=(x+y+z)^2-2(xy+yz+xz)=\sigma_1^2-2\sigma_2,\\x^3+y^3+z^3=(x+y+z)^3-3(xy+yz+xz)(x+y+z)+3xyz=\sigma_1^3-3\sigma_1\sigma_2+3\sigma_3.\end{gathered}$$

It is a well-known fact that x, y, z must be roots of the cubic polynomial

$$f(t)=t^3-\sigma_1t^2+\sigma_2t-\sigma_3.$$

Since $\sigma_1=0$, $\sigma_1^2-2\sigma_2=6$, $\sigma_1^3-3\sigma_1\sigma_2+3\sigma_3=-6$, we have

$$\sigma_1=0,\qquad\sigma_2=-3,\qquad\sigma_3=-2.$$

Rational roots of the polynomial $f(t)=t^3-3t+2$ can only be from the set $\{-2,-1,1,2\}$. Trying these, it turns out that $t=1$ and $t=-2$ are roots. Decomposition of the polynomial then reveals that 1 is a double root.

Thus, we have

$$(x,y,z)\in\big\{(1,1,-2),(1,-2,1),(-2,1,1)\big\}.$$

Returning back, we solve the system

$$\begin{aligned}a\phantom{{}+b}+\mathrm{i}c&=x\\\mathrm{i}a+b\phantom{{}+\mathrm{i}c}&=y\\-b+\mathrm{i}c&=z\end{aligned}$$

Its determinant is

$$|A|=\begin{vmatrix}1&0&\mathrm{i}\\\mathrm{i}&1&0\\0&-1&\mathrm{i}\end{vmatrix}=\mathrm{i}+1\neq0,$$

so for each (x,y,z) there is exactly one solution. It is easy to get the inverse matrix:

$$A^{-1}=\frac12\begin{pmatrix}1+\mathrm{i}&-1-\mathrm{i}&-1-\mathrm{i}\\1-\mathrm{i}&1+\mathrm{i}&-1+\mathrm{i}\\-1-\mathrm{i}&1-\mathrm{i}&1-\mathrm{i}\end{pmatrix}.$$

Multiplying this matrix by the vectors (x,y,z) gives three solutions (a,b,c):

$$(1+\mathrm{i},2-\mathrm{i},-1),\qquad(1+\mathrm{i},-1-\mathrm{i},-1),\qquad(-2-2\mathrm{i},-1+2\mathrm{i},2).$$

One can easily verify that all three satisfy the system. $\square$

Problem j12-II-2/j12-II-52. *A ring R (not necessarily commutative) contains at least one zero divisor and the number of zero divisors is finite. Prove that R is finite.*

(Eötvös Loránd University, Budapest)

Solution. Let m be the number of zero divisors and $u, v \in R$ two non-zero elements such that $uv = 0$.

We generate more zero divisors in the following way. For an arbitrary $x \in R$, the element xu is either 0 or also a zero divisor, since $(xu)v = x(uv) = 0$.†

If $xu = yu$ for some different elements $x, y \in R$, then $(x - y)u = 0$, and $x - y$ is a zero divisor. This implies that 0 or an arbitrary zero divisor can be obtained at most $m+1$ times in the form xu.‡

Thus, each of 0 and the m zero divisors is obtained at most m times and the number of elements of R cannot exceed $(m+1)^2$. $\square$

† The set $\{xu; x \in R\}$ is finite, its cardinality being $\leq m+1$.

‡ Define an equivalence relation: $x \sim y$ iff $xu = yu$. In each class of equivalence, there are $(m+1)$ elements at most. Finally, the number of the classes of equivalence is equal to the cardinality of the set $\{xu; x \in R\}$, which is finite.

Problem j12-II-3/j12-II-53. *Let E be the set of all continuous functions $u\colon [0,1] \to \mathbb{R}$ satisfying*

$$u^2(t) \le 1 + 4\int_0^t s\big|u(s)\big|\,\mathrm{d}s, \quad \forall t \in [0,1].$$

Let $\varphi\colon E \to \mathbb{R}$ be defined by

$$\varphi(u) = \int_0^1 \big(u^2(x) - u(x)\big)\,\mathrm{d}x.$$

Prove that φ has a maximum value and find it. (Babeş-Bolyai University, Cluj-Napoca)

Solution. Let

$$v(t) = 1 + 4\int_0^t s\big|u(s)\big|\,\mathrm{d}s, \quad \forall t \in [0,1].$$

We have

$$v'(t) = 4t\big|u(t)\big| \le 4t\sqrt{1 + 4\int_0^t s\big|u(s)\big|\,\mathrm{d}s} \le 4t\sqrt{v(t)}$$

so

$$\sqrt{v(t)} - 1 = \int_0^t \frac{v'(s)}{2\sqrt{v(s)}}\,\mathrm{d}s \le \int_0^t 2s\,\mathrm{d}s = t^2$$

therefore

$$\big|u(t)\big| \le \sqrt{v(t)} \le t^2 + 1.$$

If we consider φ, we have

$$\big|u^2(t) - u(t)\big| = \big|u(t)\big|\big|u(t) - 1\big| \le (t^2+1)(t^2+2),$$
$$\big|\varphi(u)\big| \le \int_0^1 \big|u^2(t) - u(t)\big|\,\mathrm{d}t \le \int_0^1 (t^2+1)(t^2+2)\,\mathrm{d}t = \frac{16}{5}.$$

Equality can be achieved if

$$\big|u(t)\big| = t^2 + 1 \qquad \text{and} \qquad \big|u(t) - 1\big| = t^2 + 2.$$

This is the case of $u(t) = -t^2 - 1$, which belongs to E. $\square$

Problem j12-II-4/j12-II-62. *Prove that*

$$\lim_{n\to\infty} n^2\left(\int_0^1 \sqrt[n]{1+x^n}\,\mathrm{d}x - 1\right) = \frac{\pi^2}{12}.$$

(Sofia University St. Kliment Ohridski)

Solution. We will prove that

$$\lim_{n\to\infty} n^2\left(\int_0^1 \sqrt[n]{1+x^n}\,\mathrm{d}x - 1\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{12}.$$

Let $a_n = n^2\big(\int_0^1 \sqrt[n]{1+x^n}\,\mathrm{d}x - 1\big)$. It is widely known that $(1+t)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}t^k$ for any $t\in[0,1]$ and $\alpha\in(0,1)$. Moreover, $\big|\binom{\alpha}{k}t^k\big| = (-1)^{k-1}\binom{\alpha}{k}t^k$ and $\big|\binom{\alpha}{k}t^k\big| \geq \big|\binom{\alpha}{k}t^{k+1}\big|$ for $k\geq 1$, $t\geq 0$ and $\alpha\in(0,1)$. Thus, the following inequalities hold:

$$\sum_{k=0}^{2p}\binom{\alpha}{k}t^k \leq (1+t)^\alpha \leq \sum_{k=0}^{2p+1}\binom{\alpha}{k}t^k.$$

Let us put $t = x^n$ and $\alpha = \frac{1}{n}$. Integrating on $[0,1]$, we obtain

$$\sum_{k=0}^{2p}\binom{1/n}{k}\frac{1}{nk+1} \leq \int_0^1 \sqrt[n]{1+x^n}\,\mathrm{d}x \leq \sum_{k=0}^{2p+1}\binom{1/n}{k}\frac{1}{nk+1}.$$

Hence,

$$0 \leq a_n - n^2\sum_{k=1}^{2p}\binom{1/n}{k}\frac{1}{nk+1} \leq n^2\binom{1/n}{2p+1}\frac{1}{n(2p+1)+1}.$$

A simple calculation gives the following estimation:

$$n^2\binom{1/n}{2p+1}\frac{1}{n(2p+1)+1} \leq \frac{1}{(2p+1)^2}.$$

Consequently, as n tends to infinity,

$$0 \leq \limsup_{n\to\infty}\left(a_n - \sum_{k=1}^{2p}\frac{(-1)^k}{k^2}\right) \leq \frac{1}{(2p+1)^2}$$

and

$$0 \leq \liminf_{n\to\infty}\left(a_n - \sum_{k=1}^{2p}\frac{(-1)^k}{k^2}\right) \leq \frac{1}{(2p+1)^2}.$$

Letting $p\to\infty$, we obtain the desired result. $\square$

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Category I

Problem 1. Let $d(k)$ denote the number of all natural divisors of a natural number k. Prove that for any natural number n_0 the sequence $\left\{d(n^2+1)\right\}_{n=n_0}^{\infty}$ is not strictly monotone. [10 points]

Problem 2. Let $A = (a_{ij})$ be an $m \times n$ real matrix with at least one non-zero element. For each $i \in \{1, \dots, m\}$, let $R_i = \sum_{j=1}^{n} a_{ij}$ be the sum of the i-th row of the matrix A, and for each $j \in \{1, \dots, n\}$, let $C_j = \sum_{i=1}^{m} a_{ij}$ be the sum of the j-th column of the matrix A. Prove that there exist indices $k \in \{1, \dots, m\}$ and $l \in \{1, \dots, n\}$ such that

$$a_{kl} > 0\,, \qquad R_k \geq 0\,, \qquad C_l \geq 0\,,$$

or

$$a_{kl} < 0\,, \qquad R_k \leq 0\,, \qquad C_l \leq 0\,.$$

[10 points]

Problem 3. Find the limit

$$\lim_{n\to\infty} \sqrt{1 + 2\sqrt{1 + 3\sqrt{\cdots + (n-1)\sqrt{1+n}}}}\;.$$

[10 points]

Problem 4. Let A and B be complex Hermitian 2×2 matrices having the pairs of eigenvalues (α_1, α_2) and (β_1, β_2), respectively. Determine all possible pairs of eigenvalues (γ_1, γ_2) of the matrix $C = A + B$. (We recall that a matrix $A = (a_{ij})$ is Hermitian if and only if $a_{ij} = \overline{a_{ji}}$ for all i and j.) [10 points]

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Category II

Problem 1. Two real square matrices A and B satisfy the conditions $A^{2002} = B^{2003} = I$ and $AB = BA$. Prove that $A + B + I$ is invertible. (The symbol I denotes the identity matrix.) [10 points]

Problem 2. Let $\{D_1, D_2, \ldots, D_n\}$ be a set of disks in the Euclidean plane. (A disk is a set of points whose distance from the given centre is less than or equal to the given radius.) Let $a_{ij} = S(D_i \cap D_j)$ be the area of $D_i \cap D_j$. Prove that the inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \geq 0$$

holds for any real numbers $x_1, x_2, \ldots, x_n$. [10 points]

Problem 3. Let $\{a_n\}_{n=0}^{\infty}$ be the sequence of real numbers satisfying $a_0 = 0, \ \ a_1 = 1$ and

$$a_{n+2} = a_{n+1} + \frac{a_n}{2^n}$$

for every $n \geq 0$. Prove that

$$\lim_{n\to\infty} a_n = 1 + \sum_{n=1}^{\infty} \frac{1}{2^{n(n-1)/2} \prod_{k=1}^{n} (2^k - 1)} \, .$$

[10 points]

Problem 4. Let $f, g \colon [0,1] \to (0, +\infty)$ be two continuous functions such that f and $\frac{g}{f}$ are increasing. Prove that

$$\int_0^1 \frac{\int_0^x f(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t} \,\mathrm{d}x \leq 2 \int_0^1 \frac{f(t)}{g(t)} \,\mathrm{d}t \, .$$

[10 points]

Problem j13-I-1/j13-I-15. *Let $d(k)$ be the number of all natural divisors of a number $k \in \mathbb{N}$. Prove that for any $n_0 \in \mathbb{N}$ the sequence $\big(d(n^2+1)\big)_{n=n_0}^{\infty}$ is not strictly monotone.*

(Vilnius University)

Solution. Note that $d(n^2+1) < n$ for all even n. Indeed, the number n^2+1 is not square and so it is possible to split the set of all its divisors into pairs $\{\, d,\ (n^2+1)/d \,\}$ where $d < n$ and d is odd. The number of divisors in all such pairs does not exceed n.

Let us assume that starting from some $n_0 \in \mathbb{N}$, the sequence is strictly monotone. For $d(n^2+1)$ is always even, we get

$$d\big((n+1)^2+1\big) \geq d\big(n^2+1\big) + 2$$

or, in general,

$$d\big((n+k)^2+1\big) \geq d\big(n^2+1\big) + 2k$$

for any natural numbers $n \geq n_0$ and $k \geq 1$. Let $N \geq n_0$ (e.g., $N = n_0$). Taking any $s \geq N - d(N^2+1)$ (such that $N+s$ is even), we get

$$d\big((N+s)^2+1\big) \geq d\big(N^2+1\big) + 2s \geq N+s,$$

which is a contradiction with $d\big((N+s)^2+1\big) < N+s$. $\square$

Problem j13-I-2/j13-I-19. *Let $A = [a_{i,j}]$ be an $m \times n$ real matrix with at least one non-zero element. For each $i \in \{1, \ldots, m\}$ let $R_i := \sum_{j=1}^{n} a_{i,j}$ (the sum of the i-th row of A) and for each $j \in \{1, \ldots, n\}$ let $C_j := \sum_{i=1}^{m} a_{i,j}$ (the sum of the j-th column of A). Prove that there exist indices $k \in \{1, \ldots, m\}$ and $l \in \{1, \ldots, n\}$ such that*

$$a_{k,l} > 0\,, \qquad R_k \geq 0\,, \qquad C_l \geq 0\,,$$

or

$$a_{k,l} < 0\,, \qquad R_k \leq 0\,, \qquad C_l \leq 0\,.$$

(University of Zagreb)

Solution. Consider the following sets of indices (some of them may be empty):

$$\begin{aligned}
I^+ &:= \{\, i \in \{1, \ldots, m\} \mid R_i \geq 0 \,\}\,,\\
I^- &:= \{\, i \in \{1, \ldots, m\} \mid R_i < 0 \,\}\,,\\
J^+ &:= \{\, j \in \{1, \ldots, n\} \mid C_j > 0 \,\}\,,\\
J^- &:= \{\, j \in \{1, \ldots, n\} \mid C_j \leq 0 \,\}\,.
\end{aligned}$$

Suppose that the statement of the problem does not hold. Then (but not equivalently) we have $a_{i,j} \leq 0$ for every $(i,j) \in I^+ \times J^+$ and we have $a_{i,j} \geq 0$ for every $(i,j) \in I^- \times J^-$. Let us write the sum $\sum_{(i,j) \in I^- \times J^+} a_{i,j}$ in two different ways:

$$\begin{aligned}
\sum_{(i,j) \in I^- \times J^+} a_{i,j} &= \sum_{i \in I^-} \Big(\sum_{j=1}^{n} a_{i,j} - \sum_{j \in J^-} a_{i,j} \Big) = \sum_{i \in I^-} R_i - \sum_{(i,j) \in I^- \times J^-} a_{i,j} \leq 0\,,\\
\sum_{(i,j) \in I^- \times J^+} a_{i,j} &= \sum_{j \in J^+} \Big(\sum_{i=1}^{m} a_{i,j} - \sum_{i \in I^+} a_{i,j} \Big) = \sum_{j \in J^+} C_j - \sum_{(i,j) \in I^+ \times J^+} a_{i,j} \geq 0\,.
\end{aligned}$$

Therefore, $\sum_{(i,j) \in I^- \times J^+} a_{i,j} = 0$ and we have only equalities in the two formulae above. This is only possible if $\sum_{i \in I^-} R_i = 0$ and $\sum_{j \in J^+} C_j = 0$, so $I^- = \emptyset$ and $J^+ = \emptyset$,† which means $R_i \geq 0$ for all $i = 1, \ldots, m$ and $C_j \leq 0$ for all $j = 1, \ldots, n$. Moreover, from

$$0 \leq \sum_{i=1}^{m} R_i = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{i,j} = \sum_{j=1}^{n} C_j \leq 0,$$

we conclude $R_i = 0$ for $i = 1, \ldots, m$ and $C_j = 0$ for $j = 1, \ldots, n$. Since A is a non-zero matrix, there are indices k and l such that $a_{k,l} \neq 0$, but $R_k = 0$ and $C_l = 0$, which leads to a contradiction with the assumption that the statement of the problem is false. $\square$

† If $I^- \neq \emptyset$, then $\sum_{(i,j) \in I^- \times J^+} a_{i,j} \leq \sum_{i \in I^-} R_i < 0$ — a contradiction. We can argue similarly to show $J^+ = \emptyset$.

Problem j13-I-3/j13-I-9. *Find the limit*

$$\lim_{n\to\infty} \sqrt{1+2\sqrt{1+3\sqrt{\cdots+(n-1)\sqrt{1+n}}}}\ .$$

(Dr. Moubinool Omarjee, Paris†)

Solution. Let

$$u_{m,n} = \sqrt{1+m\sqrt{1+(m+1)\sqrt{\cdots+(n-1)\sqrt{1+n}}}}\ .$$

We have

$$\begin{aligned} u_{m,n}^2 &= 1 + m u_{m+1,n}\ , \\ u_{m,n}^2 - (m+1)^2 &= m\big(u_{m+1,n} - (m+2)\big). \end{aligned}$$

Using the equality $|a-b| = |a^2-b^2|/|a+b|$ and inequality $u_{m,n} + m + 1 \geq m+2$, we get

$$\big|u_{m,n} - m - 1\big| \leq \frac{m}{m+2}\big|u_{m+1,n} - (m+2)\big|.$$

We deduce that

$$\begin{aligned} |u_{2,n} - 3| &\leq \frac{2}{4}\cdot\frac{3}{5}\cdot\cdots\cdot\frac{n-1}{n+1}\cdot|u_{n-1,n} - n|, \\ |u_{2,n} - 3| &\leq \frac{6}{n(n+1)}\left(\sqrt{1+(n-1)\sqrt{1+n}} - n\right) = O\left(\frac{1}{n}\right). \end{aligned}$$

So we get

$$\lim_{n\to\infty} u_{2,n} = 3.$$

□

† This problem is formally proposed by the University of Ostrava.

Problem j13-I-4/j13-I-12. *Let A and B be complex hermitian 2×2 matrices with pairs of eigenvalues (α_1, α_2) and (β_1, β_2), respectively. Determine all possible pairs (γ_1, γ_2) of eigenvalues of the matrix $C = A + B$. (A matrix $A = [a_{i,j}]$ is hermitian if and only if $a_{i,j} = \overline{a_{j,i}}$ for all i, j.)* (Charles University in Prague)

Solution. Recall that all eigenvalues of a hermitian matrix are real numbers and that there exists an orthonormal basis consisting of eigenvectors of the matrix. As we can add a suffitiently large multiple of the identity matrix to both matrices A and B, we can suppose wlog that $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and also $\gamma_1, \gamma_2 > 0$.

Let us also wlog suppose $\alpha_1 \ge \alpha_2$, $\beta_1 \ge \beta_2$, $\gamma_1 \ge \gamma_2$ and $\alpha_1 - \alpha_2 \ge \beta_1 - \beta_2$. By easy arguments, we can see

$$\gamma_1 + \gamma_2 = \operatorname{Tr} C = \operatorname{Tr} A + \operatorname{Tr} B = \alpha_1 + \alpha_2 + \beta_1 + \beta_2.$$

Further, it holds that

$$\gamma_1 \le \alpha_1 + \beta_1, \qquad \gamma_2 \ge \alpha_2 + \beta_2.$$

(The first inequality can be seen if we rewrite it slightly: $\gamma_1 = \|C\| \le \|A\| + \|B\| = \alpha_1 + \beta_1$. The second inequality follows if we consider the equality above and the first inequality together. — Alternatively, $\gamma_1 = \max(Cx, x)/(x, x) \le \max(Ax, x)/(x, x) + \max(Bx, x)/(x, x) = \alpha_1 + \beta_1$ and $\gamma_2 = \min(Cx, x)/(x, x) \ge \min(Ax, x)/(x, x) + \min(Bx, x)/(x, x) = \alpha_2 + \beta_2$.) Later we will also prove the inequalities

$$\gamma_1 \ge \alpha_1 + \beta_2, \qquad \gamma_2 \le \beta_1 + \alpha_2$$

(in fact, it suffices to prove only the first one because the second one follows if we use the equality given above).

From these inequalities, we can see that $\gamma_1 \in [\alpha_1 + \beta_2, \alpha_1 + \beta_1]$. (The value of γ_2 has to be “complementary” to obtain the right value of the sum $\gamma_1 + \gamma_2$. It also worths noting that even if $\gamma_1 = \alpha_1 + \beta_2$, then still $\gamma_1 \ge \gamma_2 = \beta_1 + \alpha_2$. This follows from the assumption $\alpha_1 - \alpha_2 \ge \beta_1 - \beta_2$.) We will show that γ_1 can assume any value from the given interval $[\alpha_1 + \beta_2, \alpha_1 + \beta_1]$. Consequently, the set of all possible pairs (γ_1, γ_2) of eigenvalues of the matrix $C = A + B$ is

$$\big\{ (\gamma_1, \gamma_2) : \alpha_1 + \beta_2 \le \gamma_1 \le \alpha_1 + \beta_1,\ \gamma_1 + \gamma_2 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2 \big\}.$$

To see this, let us put

$$A = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \qquad B = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}, \qquad P(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

The matrix A obviously has eigenvalues (α_1, α_2). The matrix $B(t) = P^{-1}(t) B P(t)$ obviously has eigenvalues (β_1, β_2). If we note that $P^{-1}(t) = P^T(t)$ and define the matrix $C(t) = A + B(t)$, we have

$$C(0) = A + B = \begin{pmatrix} \alpha_1 + \beta_1 & 0 \\ 0 & \alpha_2 + \beta_2 \end{pmatrix}, \qquad C(\tfrac{\pi}{2}) = \begin{pmatrix} \alpha_1 + \beta_2 & 0 \\ 0 & \alpha_2 + \beta_1 \end{pmatrix}.$$

The matrix $C(0)$ has the eigenvalue $\gamma_1(0) = \alpha_1 + \beta_1$. (Note that $\gamma_1(0) \ge \gamma_2(0) = \alpha_2 + \beta_2$.) The matrix $C(\pi/2)$ has the eigenvalue $\gamma_1(\pi/2) = \alpha_1 + \beta_2$. (Note that $\gamma_1(\pi/2) \ge \gamma_2(\pi/2) = \alpha_2 + \beta_1$.) As both eigenvalues (γ_1, γ_2) of a matrix C depend continuously on the coefficients of the matrix, we deduce that $\gamma_1(t)$ is a continuous function. Consequently, it assumes every value from the interval $[\alpha_1 + \beta_2, \alpha_1 + \beta_1]$, which we wanted to demonstrate.

Now it only remains to prove the inequality $\gamma_1 \ge \alpha_1 + \beta_2$ for any two complex hermitian matrices A and B. Let us recall that we still wlog suppose $\alpha_1 \ge \alpha_2 > 0$, $\beta_1 \ge \beta_2 > 0$ and $\gamma_1 \ge \gamma_2 > 0$. Let v_1 and v_2 denote the eigenvectors of the matrix A corresponding to the eigenvalues α_1 and α_2, respectively, and let w_1 and w_2 denote the eigenvectors of B corresponding to the eigenvalues β_1 and β_2, respectively. We can suppose that the bases

$\{v_1, v_2\}$ and $\{w_1, w_2\}$ are orthonormal. So there exists some unitary matrix $U = \left(\begin{smallmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{smallmatrix}\right)$ such that

$$\begin{aligned} v_1 &= u_{11}w_1 + u_{12}w_2, \\ v_2 &= u_{21}w_1 + u_{22}w_2, \end{aligned} \qquad \text{and} \qquad \begin{aligned} w_1 &= \overline{u_{11}}v_1 + \overline{u_{21}}v_2, \\ w_2 &= \overline{u_{12}}v_1 + \overline{u_{22}}v_2. \end{aligned}$$

We will estimate γ_1 in the following way. First,

$$\gamma_1 = \sup\big\{\, \|Cx\| : \|x\| = 1 \,\big\} \geq \|Cv_1\|$$

where $\|\cdot\|$ denotes the Euclidean norm. (Let us justify the formula. Recall that $\gamma_1 = \max_{\|x\|=1}(Cx, x)$. Obviously, γ_1^2 is the greater eigenvalue of C^2. Consequently, it follows that $\gamma_1^2 = \max_{\|x\|=1}(C^2x, x)$. As C is hermitian, we have $(C^2x, x) = x^*CCx = x^*C^*Cx = (Cx, Cx) = \|Cx\|^2$.) Second,

$$\begin{aligned} Cv_1 = (A + B)v_1 &= \alpha_1 v_1 + \beta_1 u_{11}w_1 + \beta_2 u_{12}w_2 = (\alpha_1 + \beta_2)v_1 + (\beta_1 - \beta_2)u_{11}w_1 = \\ &= \big(\alpha_1 + \beta_2 + (\beta_1 - \beta_2)u_{11}\overline{u_{11}}\big)v_1 + (\beta_1 - \beta_2)u_{11}\overline{u_{21}}v_2. \end{aligned}$$

As the vectors v_1 and v_2 are orthonormal and $(\beta_1 - \beta_2)u_{11}\overline{u_{11}} \geq 0$, we conclude

$$\begin{aligned} \gamma_1 \geq \|Cv_1\| &= \sqrt{\big|\alpha_1 + \beta_2 + (\beta_1 - \beta_2)u_{11}\overline{u_{11}}\big|^2 + \big|(\beta_1 - \beta_2)u_{11}\overline{u_{21}}\big|^2} \geq \\ &\geq \sqrt{\big|\alpha_1 + \beta_2 + (\beta_1 - \beta_2)u_{11}\overline{u_{11}}\big|^2} \geq \alpha_1 + \beta_2. \end{aligned}$$

$\square$

Problem j13-II-1/j13-II-51. *Two real square matrices A and B satisfy the conditions $A^{2002} = B^{2003} = I$ and $AB = BA$. Prove that $A + B + I$ is invertible. (The symbol I denotes the identity matrix.)* (University of Belgrade)

Solution. Let $(A + B + I)x = 0$ for some vector x, i.e., $(B + I)x = -Ax$. Then we have $-A^2x = A(B + I)x = (B + I)Ax = -(B + I)^2x$, and, continuing in this way, $(B + I)^k x = (-1)^k A^k x$. As $A^{2002} = I$, we get $(B + I)^{2002}x = x$, i.e.,

$$\big((B + I)^{2002} - I\big)x = \big(B^{2003} - I\big)x = 0.$$

(Recall $B^{2003} = I$.) In other words, taking that $p(t) = (t + 1)^{2002} - 1$ and $q(t) = t^{2003} - 1$ are polynomials, we have just got

$$p(B)x = q(B)x = 0.$$

But, since 2003 is a prime, $q(t)/(t - 1)$ is a primitive polynomial for all its roots, and therefore none of them is a root of the another monic polynomial $p(t)$ of degree 2002; further, the remained root $t = 1$ of $q(t)$ is not a root of $p(t)$, which implies that $p(t)$ and $q(t)$ are coprime.†

Since there exist non-zero polynomials $r(t)$ and $s(t)$ such that $r(t)p(t) - s(t)q(t) = 1$ (recall the Euclidean algorithm), we can conclude that $x = r(B)p(B)x - s(B)q(B)x = 0$, and so $A + B + I$ must be invertible indeed. □

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† The polynomials $p(t)$ and $q(t)$ are really coprime (i.e. relatively prime). Here is another argument: Every polynomial (of degree ≥ 1) can be written as a product of factors of degree 1. In particular, $p(t) = (t + 1)^{2002} - 1 = \prod_{k=1}^{2002}(t - z_{p,k})$ and $q(t) = t^{2003} - 1 = \prod_{k=1}^{2003}(t - z_{q,k})$, where $z_{p,1}, \ldots, z_{p,2002}$ and $z_{q,1}, \ldots, z_{q,2003}$ are the roots of the polynomial p and q, respectively. Obviously, the polynomials p and q are relatively prime iff they have no root in common.

It is easy to see that the roots of q lie on the unit circle in the complex plane. Similarly, it is easy to see that all roots of p are on the circle with radius 1 and its centre at the point -1.

Thus, the intersections of the two circles,

$$\tfrac{\sqrt{2}}{2} \pm \mathrm{i}\tfrac{\sqrt{2}}{2} = \cos \pm\tfrac{3\pi}{2} + \mathrm{i}\sin \pm\tfrac{3\pi}{2} = \big(-1\big) + \big(\cos \pm\tfrac{\pi}{2} + \mathrm{i}\sin \pm\tfrac{\pi}{2}\big),$$

are the only possible common roots of q and p. But none of these two points is a root of q. It follows that p and q are coprime.

Problem j13-II-2/j13-I-17. *Let $\{D_1, D_2, \ldots, D_n\}$ be a set of disks (a disk is a circle with its interior) in the Euclidean plane and $a_{ij} = S(D_i \cap D_j)$ be the area of $D_i \cap D_j$. Prove that for any numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$ the following inequality holds:*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \geq 0.$$

(Warsaw University)

Solution. Let $\chi_{D_i} \colon \mathbb{R}^2 \to \{0, 1\}$ be the characteristic function of the set D_i:

$$\chi_{D_i}(x, y) = \begin{cases} 1, & \text{if } (x, y) \in D_i, \\ 0, & \text{if } (x, y) \notin D_i. \end{cases}$$

We have:

$$\chi_{D_i \cap Dj} = \chi_{D_i} \chi_{D_j},$$

$$S(D_i) = \int_{\mathbb{R}^2} \chi_{D_i}(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}^2} \chi^2_{D_i}(x, y) \, \mathrm{d}x \, \mathrm{d}y,$$

$$S(D_i \cap D_j) = \int_{\mathbb{R}^2} \chi_{D_i \cap D_j}(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}^2} \chi_{D_i}(x, y) \chi_{D_j}(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$

Thus,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = \int_{\mathbb{R}^2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \chi_{D_i}(x, y) x_j \chi_{D_j}(x, y) \, \mathrm{d}x \, \mathrm{d}y =$$

$$= \int_{\mathbb{R}^2} \big(x_1 \chi_{D_1}(x, y) + \cdots + x_n \chi_{D_n}(x, y) \big)^2 \, \mathrm{d}x \, \mathrm{d}y \geq 0.$$

□

Problem j13-II-3/j13-II-70. *A sequence $(a_n)_{n=0}^\infty$ of real numbers is defined recursively by*

$$a_0 := 0, \qquad a_1 := 1, \qquad a_{n+2} := a_{n+1} + \frac{a_n}{2^n}\,,\ n \geq 0.$$

Prove that

$$\lim_{n\to\infty} a_n = 1 + \sum_{n=1}^{\infty} \frac{1}{2^{\frac{n(n-1)}{2}} \cdot \prod_{k=1}^{n}(2^k-1)}\,.$$

(University of Zagreb)

Remark. In fact, we will prove the following:

(a) The sequence $(a_n)_{n=0}^\infty$ is convergent.
(b) $\lim_{n\to\infty} a_n = 1 + \sum_{n=1}^\infty 1/\big(2^{n(n-1)/2} \cdot \prod_{k=1}^n (2^k-1)\big)$.
(c) The limit $\lim_{n\to\infty} a_n$ is an irrational number.

Solution. (a) Obviously, $a_n \geq 0$ for every $n \geq 0$. The sequence $(a_n)_{n=0}^\infty$ is increasing since $a_{n+2} - a_{n+1} = a_n/2^n \geq 0$ for every $n \geq 0$. It suffices to show that $(a_n)_{n=0}^\infty$ is bounded from above. For each $n \geq 0$, we have $a_{n+2} \leq a_{n+1} + a_{n+1}/2^n = a_{n+1}(1 + 1/2^n)$. Using the inequality between geometric and arithmetic mean, for every $n \geq 1$ we obtain

$$a_{n+2} \leq \prod_{k=0}^{n}\left(1+\frac{1}{2^k}\right) = 2\prod_{k=1}^{n}\left(1+\frac{1}{2^k}\right) \leq 2\left(\frac{1}{n}\left(n+\sum_{k=1}^{n}\frac{1}{2^k}\right)\right)^n \leq 2\left(\frac{n+1}{n}\right)^n \leq 2\mathrm{e}.$$

(b) Consider the power series $\sum_{n=0}^\infty a_n z^n$. Since $\limsup_{n\to\infty}\sqrt[n]{|a_n|} \leq \lim_{n\to\infty}\sqrt[n]{2\mathrm{e}} = 1$, its radius of convergence is $R \geq 1$. Therefore, on the open unit disc, with center at the origin, it converges to a holomorphic function $f(z) := \sum_{n=0}^\infty a_n z^n$. Inductively, we obtain $a_{n+2} = 1 + \sum_{k=0}^n a_k/2^k$ for any $n \geq 0$. So $\lim_{n\to\infty} a_n = 1 + \sum_{k=0}^\infty a_k/2^k = 1 + f\big(\frac12\big)$ and we have to find $f\big(\frac12\big)$.

Now we use the recurrent relation for $(a_n)_{n=0}^\infty$ to obtain a functional equation for f. We multiply $a_{n+2} := a_{n+1} + a_n/2^n$ by z^{n+2} and sum over all $n \geq 0$ to get

$$\sum_{n=0}^\infty a_{n+2} z^{n+2} = z\sum_{n=0}^\infty a_{n+1} z^{n+1} + z^2 \sum_{n=0}^\infty a_n \big(\tfrac{z}{2}\big)^n,$$

that is

$$f(z) - z = zf(z) + z^2 f\big(\tfrac{z}{2}\big),$$

or

$$(1-z)f(z) = z^2 f\big(\tfrac{z}{2}\big) + z \qquad \text{for } |z| < 1. \tag{1}$$

We substitute $z = 1/2^n$ for $n = 1, \ldots, N$ (where $N \geq 1$ is a fixed number) into (1), then multiply the n-th equality by some constant $s_n > 0$ and finally sum up those N equalities:

$$\begin{array}{rcll}
\big(1-\frac12\big)f\big(\frac12\big) &=& \big(\frac12\big)^2 f\big(\frac14\big) + \frac12\,, & \quad |\cdot s_1\,,\\
\big(1-\frac14\big)f\big(\frac14\big) &=& \big(\frac14\big)^2 f\big(\frac18\big) + \frac14\,, & \quad |\cdot s_2\,,\\
&\vdots&&\\
\big(1-\frac{1}{2^n}\big)f\big(\frac{1}{2^n}\big) &=& \big(\frac{1}{2^n}\big)^2 f\big(\frac{1}{2^{n+1}}\big) + \frac{1}{2^n}\,, & \quad |\cdot s_n\,,\\
\big(1-\frac{1}{2^{n+1}}\big)f\big(\frac{1}{2^{n+1}}\big) &=& \big(\frac{1}{2^{n+1}}\big)^2 f\big(\frac{1}{2^{n+2}}\big) + \frac{1}{2^{n+1}}\,, & \quad |\cdot s_{n+1}\,,\\
&\vdots&&\\
\big(1-\frac{1}{2^N}\big)f\big(\frac{1}{2^N}\big) &=& \big(\frac{1}{2^N}\big)^2 f\big(\frac{1}{2^{N+1}}\big) + \frac{1}{2^N}\,, & \quad |\cdot s_N\,,\\
\hline
\frac{s_1}{2} f\big(\frac12\big) &=& \frac{s_N}{2^{2N}} f\big(\frac{1}{2^{N+1}}\big) + \sum_{n=1}^N \frac{s_n}{2^n}\,. &
\end{array}$$

To obtain the given result (namely, to achieve cancelling of the terms with $f\big(\frac{1}{2^n}\big)$ for $n = 2, \ldots, N$), we had to choose the numbers s_n so that

$$\big(1 - \tfrac{1}{2^{n+1}}\big)s_{n+1} = \big(\tfrac{1}{2^n}\big)^2 s_n, \qquad \text{for } n \geq 0. \tag{2a}$$

Let us put

$$s_0 := 1. \tag{2b}$$

It follows that $s_1 = 2$. Equalities (2b) and (2a) lead to

$$s_n = \prod_{k=0}^{n-1} \frac{s_{k+1}}{s_k} = \prod_{k=0}^{n-1} \frac{\left(\frac{1}{2^k}\right)^2}{1 - \frac{1}{2^{k+1}}} = \prod_{k=0}^{n-1} \frac{1}{2^{k-1}(2^{k+1}-1)} = \frac{1}{2^{\frac{n(n-1)}{2}-n} \prod_{k=1}^{n}(2^k-1)}$$

for every $n \geq 1$. Finally, we have

$$f\left(\tfrac{1}{2}\right) = \frac{s_N}{2^{2N}} f\left(\tfrac{1}{2^{N+1}}\right) + \sum_{n=1}^{N} \frac{s_n}{2^n} = \frac{f\left(\frac{1}{2^{N+1}}\right)}{2^{\frac{N(N-1)}{2}+N} \prod_{k=1}^{N}(2^k-1)} + \sum_{n=1}^{N} \frac{1}{2^{\frac{n(n-1)}{2}} \prod_{k=1}^{n}(2^k-1)}.$$

The first term tends to 0 when $N \to \infty$, so

$$f\left(\tfrac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{2^{\frac{n(n-1)}{2}} \prod_{k=1}^{n}(2^k-1)}. \tag{3}$$

(c) The proof of $\lim_{n\to\infty} a_n \in \mathbb{R} \setminus \mathbb{Q}$ is based on the fact that the series in (3) converges "very rapidly". Suppose that its sum equals $\frac{p}{q}$ for some positive integers p and q. For each integer $N \geq 1$, denote

$$q_N := 2^{\frac{N(N-1)}{2}} \prod_{k=1}^{N}(2^k-1), \qquad p_N := q_N \sum_{n=1}^{N} \frac{1}{2^{\frac{n(n-1)}{2}} \prod_{k=1}^{n}(2^k-1)}.$$

Obviously, p_N and q_N are positive integers. We manage to estimate $pq_N - qp_N$. We have

$$q_N = 2^{\frac{N(N-1)}{2}} \prod_{k=1}^{N}(2^k-1) < 2^{\frac{N(N-1)}{2}} \prod_{k=1}^{N} 2^k = 2^{N^2}$$

and

$$\begin{aligned} \frac{p}{q} - \frac{p_N}{q_N} &= \sum_{n=N+1}^{\infty} \frac{1}{2^{\frac{n(n-1)}{2}} \prod_{k=1}^{n}(2^k-1)} \leq \sum_{n=N+1}^{\infty} \frac{1}{2^{\frac{n(n-1)}{2}} \prod_{k=1}^{n} 2^{k-1}} = \\ &= \sum_{n=N+1}^{\infty} \frac{1}{2^{n(n-1)}} \leq \sum_{m=N(N+1)}^{\infty} \frac{1}{2^m} = \frac{1}{2^{N^2+N-1}} < \frac{1}{2^{N-1} q_N}. \end{aligned}$$

Thus, $0 < pq_N - qp_N < \frac{q}{2^{N-1}}$,† so $(pq_N - qp_N)_{N\geq 1}$ is a sequence of positive integers that converges to 0. This is a contradiction and we are done. □

† It is easy to see from the definition of the numbers p_N that the sequence $\left(\frac{p_N}{q_N}\right)$ is strictly increasing to the limit $\frac{p}{q}$. Hence $\frac{p_N}{q_N} < \frac{p}{q}$, $qp_n < pq_N$, and $0 < pq_N - qp_N$. As the difference is integer, we have even $1 \leq pq_N - qp_N$.

Problem j13-II-4/j13-I-18. *Let $f, g\colon [0,1] \to (0,+\infty)$ be continuous functions such that f and $\frac{g}{f}$ are increasing. Prove that*

$$\int_0^1 \frac{\int_0^x f(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t}\,\mathrm{d}x \le 2\int_0^1 \frac{f(t)}{g(t)}\,\mathrm{d}t.$$

(University of Zagreb)

Solution. First, we estimate the expression inside the integral sign on the left side of the given inequality. By the Chebycheff's inequality for integrals applied to increasing functions f and $\frac{g}{f}$ on the segment $[0,x]$ (where $x \in (0,1]$ is fixed), we get

$$\left(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\right)\left(\frac{1}{x}\int_0^x \frac{g(t)}{f(t)}\,\mathrm{d}t\right) \le \frac{1}{x}\int_0^x g(t)\,\mathrm{d}t,$$

that is,

$$\frac{\int_0^x f(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t} \le \frac{x}{\int_0^x \frac{g(t)}{f(t)}\,\mathrm{d}t} \tag{1}$$

for every $x \in (0,1]$. From the integral form of the Cauchy-Schwarz inequality on the segment $[0,x]$, we have

$$\left(\int_0^x \frac{g(t)}{f(t)}\,\mathrm{d}t\right)\left(\int_0^x \frac{t^2 f(t)}{g(t)}\,\mathrm{d}t\right) \ge \left(\int_0^x t\,\mathrm{d}t\right)^2 = \frac{x^4}{4},$$

or

$$\frac{1}{\int_0^x \frac{g(t)}{f(t)}\,\mathrm{d}t} \le \frac{4}{x^4}\int_0^x \frac{t^2 f(t)}{g(t)}\,\mathrm{d}t. \tag{2}$$

From (1) and (2) we obtain

$$\frac{\int_0^x f(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t} \le \frac{4}{x^3}\int_0^x \frac{t^2 f(t)}{g(t)}\,\mathrm{d}t. \tag{3}$$

Finally, it remains to integrate (3) over $x \in (0,1]$ and to reverse the order of integration.

$$\begin{aligned}\int_0^1 \frac{\int_0^x f(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t}\,\mathrm{d}x &\le \int_0^1 \left(\int_0^x \frac{4t^2 f(t)}{x^3 g(t)}\,\mathrm{d}t\right)\mathrm{d}x = \int_0^1 \left(\int_t^1 \frac{4t^2 f(t)}{x^3 g(t)}\,\mathrm{d}x\right)\mathrm{d}t = \\ &= \int_0^1 \frac{4t^2 f(t)}{g(t)}\left(\int_t^1 \frac{\mathrm{d}x}{x^3}\right)\mathrm{d}t = \int_0^1 \frac{4t^2 f(t)}{g(t)}\left(\frac{1}{2t^2} - \frac{1}{2}\right)\mathrm{d}t = \\ &= 2\int_0^1 \frac{f(t)}{g(t)}(1-t^2)\,\mathrm{d}t \le 2\int_0^1 \frac{f(t)}{g(t)}\,\mathrm{d}t.\end{aligned}$$

(*Remark.* The constant 2 on the right hand side of the given inequality is optimal, i.e., the least possible. Consider $f(t) := 1$ and $g(t) := t + \varepsilon$ for some fixed $\varepsilon > 0$. Then

$$\int_0^1 \frac{\int_0^x f(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t}\,\mathrm{d}x = \int_0^1 \frac{x}{\frac{1}{2}x^2 + \varepsilon x}\,\mathrm{d}x = 2\int_0^1 \frac{\mathrm{d}x}{x+2\varepsilon} = 2\ln(1+2\varepsilon) - 2\ln 2 - 2\ln\varepsilon$$

and

$$\int_0^1 \frac{f(t)}{g(t)}\,\mathrm{d}t = \int_0^1 \frac{\mathrm{d}t}{t+\varepsilon} = \ln(1+\varepsilon) - \ln\varepsilon.$$

The quotient of these two expressions can be made arbitrarily close to 2 since

$$\lim_{\varepsilon\searrow 0} \frac{2\ln(1+2\varepsilon) - 2\ln 2 - 2\ln\varepsilon}{\ln(1+\varepsilon) - \ln\varepsilon} = 2\lim_{\varepsilon\searrow 0} \frac{-\frac{\ln(1+2\varepsilon)}{\ln\varepsilon} + \frac{\ln 2}{\ln\varepsilon} + 1}{-\frac{\ln(1+\varepsilon)}{\ln\varepsilon} + 1} = 2.$$

Therefore, the constant 2 is the best possible one.) □

The 17th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 28th March 2007
Category I

Problem 1. *Can the set of positive rationals be split into two nonempty disjoint subsets Q_1 and Q_2, such that both are closed under addition, i.e. $p+q \in Q_k$ for every $p, q \in Q_k$, $k = 1, 2$?*

Can it be done when addition is exchanged for multiplication, i.e. $p \cdot q \in Q_k$ for every $p, q \in Q_k$, $k = 1, 2$?

[10 points]

Problem 2. *Alice has got a circular key ring with n keys, $n \geq 3$. When she takes it out of her pocket, she does not know whether it got rotated and/or flipped. The only way she can distinguish the keys is by colouring them (a colour is assigned to each key). What is the minimum number of colours needed?* [10 points]

Problem 3. *A function $f\colon [0,\infty) \to \mathbb{R} \setminus \{0\}$ is called* slowly changing *if for any $t > 1$ the limit $\lim\limits_{x\to\infty} \frac{f(tx)}{f(x)}$ exists and is equal to 1. Is it true that every slowly changing function has for sufficiently large x a constant sign (i.e., is it true that for every slowly changing f there exists an N such that for every $x, y > N$ we have $f(x)f(y) > 0$?)* [10 points]

Problem 4. *Let $f\colon [0,1] \to [0,\infty)$ be an arbitrary function satisfying*

$$\frac{f(x)+f(y)}{2} \leq f\Big(\frac{x+y}{2}\Big) + 1$$

for all pairs $x, y \in [0,1]$. Prove that for all $0 \leq u < v < w \leq 1$,

$$\frac{w-v}{w-u} f(u) + \frac{v-u}{w-u} f(w) \leq f(v) + 2\,.$$

[10 points]

The 17th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 28th March 2007
Category II

Problem 1. *Construct a set $A \subset [0,1] \times [0,1]$ such that A is dense in $[0,1] \times [0,1]$ and every vertical and every horizontal line intersects A in at most one point.* [10 points]

Problem 2. *Let A be a real $n \times n$ matrix satisfying*

$$A + A^t = I\,,$$

where A^t denotes the transpose of A and I the $n \times n$ identity matrix. Show that $\det A > 0$. [10 points]

Problem 3. *Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1) = 0$. Prove that the set*

$$A := \big\{ h \in [0,1] : f(x+h) = f(x) \text{ for some } x \in [0,1] \big\}$$

is Lebesgue measureable and has Lebesgue measure at least $\frac{1}{2}$. [10 points]

Problem 4. *Let S be a finite set with n elements and $\mathcal{F}$ a family of subsets of S with the following property:*

$$A \in \mathcal{F},\ A \subseteq B \subseteq S \Longrightarrow B \in \mathcal{F}.$$

Prove that the function $f\colon [0,1] \to \mathbb{R}$ given by

$$f(t) := \sum_{A \in \mathcal{F}} t^{|A|} (1-t)^{|S \setminus A|}$$

is nondecreasing ($|A|$ denotes the number of elements of A). [10 points]

Category I

Problem 1. *Can the set of positive rationals be split into two nonempty disjoint subsets Q_1 and Q_2, such that both are closed under addition, i.e. $p + q \in Q_k$ for every $p, q \in Q_k$, $k = 1, 2$? Can it be done when addition is exchanged for multiplication, i.e. $p \cdot q \in Q_k$ for every $p, q \in Q_k$, $k = 1, 2$?*

Solution. (a) No. If $\frac{p}{q}, \frac{r}{s} \in Q_k$ then of course $\frac{ps+qr}{qs} \in Q_k$. Adding n times $\frac{p}{q}$ and m times $\frac{r}{s}$ gives $\frac{nps+mqr}{qs} \in Q_k$ for all positive integers n, m, hence $\tilde{n}p + \tilde{m}r \in Q_k$ for all positive integers $\tilde{n}, \tilde{m}$. So if $\frac{p_k}{q_k}, \frac{r_k}{s_k} \in Q_k$ we get that $p_1p_2 + r_1r_2 \in Q_1 \cap Q_2$.

(b) Yes, for instance

$$Q_1 = \left\{ \frac{m}{n} \in \mathbb{Q}^+ : (m, n) = 1 \text{ and } 2 \mid n \right\} \quad \text{and} \quad Q_2 = \mathbb{Q}^+ \setminus Q_1 \, .$$

$\square$

Problem 2. *Alice has got a circular key ring with n keys, $n \geq 3$. When she takes it out of her pocket, she does not know whether it got rotated and/or flipped. The only way she can distinguish the keys is by colouring them (a colour is assigned to each key). What is the minimum number of colors needed?*

Solution. Clearly at least two colors are needed in any case to distinguish between at least two keys. For three, four or five keys on the ring, we will show that three colors are necessary. For six or more keys on the ring, we will show that two colors suffice. Choose one key and denote it with k_1. Order all other keys in natural order as they follow each other going from k_1 around the ring in one direction. For $1 \leq i \leq n$ denote with $c(k_i)$ color of the key k_i. Without loss of generality let $c(k_1) = 1$.

Suppose that two colors suffice for $n = 3$. Then there are two similar possibilities for coloring the keys. Either $c(k_2) = c(k_3) = 2$ or $c(k_2) = 1$. In the first case one can not distinguish between keys k_2 and k_3. In the second case one can not distinguish between keys k_1 and k_2. Hence for $n = 3$ we need three colors.

Suppose that two colors suffice for $n = 4$. Then there are four possibilities for coloring the keys. If $c(k_2) = c(k_3) = c(k_4) = 2$, then k_2 and k_4 can not be distinguished (rotation of the key ring through the line across k_1 and k_3 interchanges k_2 and k_4). If $c(k_2) = 1$ and $c(k_3) = c(k_4) = 2$ then there is a rotation that interchanges k_1 and k_2 and also interchanges k_3 and k_4 (similar is the case when $c(k_4) = 1$ and $c(k_2) = c(k_3) = 2$). If $c(k_3) = 1$ and $c(k_2) = c(k_4) = 2$ then there is a rotation that interchanges k_1 and k_3 and there is also other rotation that interchanges k_2 and k_4. Hence for $n = 4$ at least three colors are needed. Consider the following coloring: $c(k_1) = 1$, $c(k_2) = 2$, $c(k_3) = 3$ and $c(k_4) = 1$ (one possibility). Keys k_1 and k_4 have the same color, but one can distinguish between them since k_1 has a neighbor colored with color 1 and a neighbor colored with color 2, while k_4 has also one neighbor colored with color 1, but the other neighbor is colored with color 3. Hence three colors suffice for $n = 4$.

Suppose that two colors suffice for $n = 5$. Then there are two possibilities for coloring the keys: all other keys than k_1 are colored with color 2 (the similar is the case when one key gets color 1, only the roles of the colors are interchanged) or one of them gets color 1 and other three get color 2 (the same is the case when two keys get color 2, only the roles of the colors are interchanged). In first case one can not distinguish between keys k_2 and k_5 and also between keys k_3 and k_4 (there is a rotation of the key ring where keys in both pairs interchange, while k_1 is fixed). When there is a key other than k_1 with color 1 we need to consider two subcases. If $c(k_2) = 1$ (similar is the case when $c(k_5) = 1$) we can not distinguish between k_1 and k_2 (also between k_3 and k_5). If $c(k_3) = 1$ (similar is the case when $c(k_4) = 1$) we can not distinguish between k_1 and k_3 (also between k_4 and k_5). Hence for $n = 5$ at least three colors are needed. Consider the following coloring: $c(k_1) = 1$, $c(k_2) = 2$, $c(k_3) = 3$ and $c(k_4) = c(k_5) = 2$ (one possibility). Keys k_2, k_4 and k_5 have the same color, but one can distinguish between them since k_2 is the only one between them that has a neighbor colored with color 1 and a neighbor colored with color 3, while only k_4 has a neighbor colored with color 3 and a neighbor colored with color 2. Hence three colors suffice for $n = 5$.

For $n \geq 6$ consider the following coloring: $c(k_1) = 1$, $c(k_n) = 2$, $c(k_{n-1}) = c(k_{n-2}) = 1$ and $c(k_i) = 2$ for $2 \leq i \leq n-3$. Then k_1 is the only key of color 1 with both neighbors colored with color 2. Keys k_{n-1} and k_{n-2} both have neighbors of two different colors, but the distance (the smallest of the two numbers: number of the keys lying between the two keys in one and other direction) between k_{n-1} and k_1 is one while the distance between k_{n-2} and k_1 is two. Hence one can distinguish between all three keys colored with color 1. Among keys colored with color 2 only k_n has both neighbors colored with color 1. All other keys: k_i for $2 \leq i \leq n-3$ have either one or two neighbors colored with color 2. But any k_i, where $2 \leq i \leq n-3$, has a pair of distances: distance between k_i and k_1 and distance between k_i and k_{n-2} that is different from any other pair of distances of some key $k_j \neq k_i$ for $2 \leq j \leq n-3$. Hence we can distinguish also between keys colored with color 2. $\square$

Problem 3. *A function* $f\colon [0,\infty) \to \mathbb{R} \setminus \{0\}$ *is called* slowly changing *if for any* $t > 1$ *the limit* $\lim_{x\to\infty} \frac{f(tx)}{f(x)}$ *exists and is equal to 1. Is it true that every slowly changing function has for sufficiently large* x *a constant sign (that is — it is true that for every slowly changing* f *there exists* N *such that for every* $x, y > N$ *we have* $f(x)f(y) > 0$*?)*

Remark. The assumption $f(x) \neq 0$ is only technical, to avoid explaining what does the limit mean in the other case, and in reality changes nothing.

Remark. The reader is encouraged to try and solve the problem himself before reading the solution. The author's and the proposer's opinion is that although the solution is simple, it is not so easy to find it (both tried, both succeeded, but both spent some time on it before getting the correct idea).

Solution. Take $t = 2$. Take such a $N > 0$ that for $x > N$ we have $\frac{f(2x)}{f(x)} > 0$. This means $f(2x)$ and $f(x)$ are of the same sign for $x > N$. Suppose that for any $x > N$ we have that $f(x)$ and $f(N)$ are of a different sign. Let $t = \frac{x}{N}$. Then $\frac{f(tN)}{f(N)} < 0$, and by easy induction $\frac{f(t2^kN)}{f(2^kN)} < 0$ for any $k \in \mathbb{N}$, which contradicts the assumption $\frac{f(tx)}{f(x)} \to 1$ when x tends to ∞. The contradiction proves the thesis. $\square$

Problem 4. *Let $f\colon [0,1]\to[0,\infty)$ be an arbitrary function satisfying*

$$\frac{f(x)+f(y)}{2} \le f\Big(\frac{x+y}{2}\Big)+1 \tag{1}$$

for all pairs $x,y\in[0,1]$. Prove that for all $1\le u<v<w\le 1$,

$$\frac{w-v}{w-u}f(u)+\frac{v-u}{w-u}f(w)\le f(v)+2\,.$$

Solution. Let

$$M(u,w)=\sup_{v\in(u,w)}\Big(\frac{w-v}{w-u}f(u)+\frac{v-u}{w-u}f(w)-f(v)\Big)\,;$$

we have to prove $M(u,w)\le 2$. Note that $M(u,w)$ is finite, because

$$\frac{w-v}{w-u}f(u)+\frac{v-u}{w-u}f(w)-f(v)\le 1\cdot f(u)+1\cdot f(w)-0=f(u)+f(w)\,.$$

Let $\varepsilon>0$ be an arbitrary positive real number. Choose v such that

$$\frac{w-v}{w-u}f(u)+\frac{v-u}{w-u}f(w)-f(v)>M(u,w)-\varepsilon\,.$$

If $v\le\frac{u+w}{2}$, then apply (1) for $x=u$ and $y=u+2(v-u)=2v-u$:

$$\frac{f(u)+f(2v-u)}{2}\le f(v)+1\,;$$

$$\begin{aligned}
M(u,w)-\varepsilon &< \frac{w-v}{w-u}f(u)+\frac{v-u}{w-u}f(w)-f(v)\\
&\le \frac{w-v}{w-u}f(u)+\frac{v-u}{w-u}f(w)-\frac{f(u)+f(2v-u)}{2}+1\\
&= \frac{1}{2}\Big(\frac{w-(2v-u)}{w-u}f(u)+\frac{(2v-u)-u}{w-u}f(w)-f(2v-u)\Big)+1\\
&\le \frac{1}{2}M(u,w)+1\,;
\end{aligned}$$

$$M(u,w)\le 2+2\varepsilon\,.$$

Otherwise, if $\frac{u+w}{2}<v$, apply $x=w-2(w-v)=2v-w$ and $y=v$ in (1):

$$\frac{f(2v-w)+f(w)}{2}\le f(v)+1\,;$$

$$\begin{aligned}
M(u,w)-\varepsilon &< \frac{w-v}{w-u}f(u)+\frac{v-u}{w-u}f(w)-f(v)\\
&\le \frac{w-v}{w-u}f(u)+\frac{v-u}{w-u}f(w)-\frac{f(2v-w)+f(w)}{2}+1\\
&= \frac{1}{2}\Big(\frac{w-(2v-w)}{w-u}f(u)+\frac{(2v-w)-u}{w-u}f(w)-f(2v-w)\Big)+1\\
&\le \frac{1}{2}M(u,w)+1\,;
\end{aligned}$$

$$M(u,w)\le 2+2\varepsilon\,.$$

In both cases we obtained $M(u,w)\le 2+2\varepsilon$. This holds for all ε, therefore $M(u,w)\le 2$. □

Category II

Problem 1. *Construct a set* $A \subset [0,1] \times [0,1]$ *such that* A *is dense in* $[0,1] \times [0,1]$ *and every vertical and every horizontal line intersects* A *in at most one point.*

Solution. Take $\alpha, \beta \notin \mathbb{Q}$ such that $\frac{\alpha}{\beta} \notin \mathbb{Q}$. Then

$$A := \left\{ (\{n\alpha\}, \{n\beta\}) : n \in \mathbb{N} \right\},$$

where $\{x\}$ denotes the fractional part of x, fulfills the assumptions. □

Problem 2. *Let A be a real $n \times n$ matrix satisfying*

$$A + A^t = I\,,$$

where A^t denotes the transpose of A and I the $n \times n$ identity matrix. Show that $\det A > 0$.

Solution. The assumption $A + A^t = I$ is equivalent to saying $A = S + \frac{1}{2}I$ where S denotes an arbitrary real skew symmetric matrix. In particular, there exists some orthogonal matrix T that diagonalizes S and for which $D := T^t S T$ contains the eigenvalues of S. They are either zero or purely imaginary and pairwise conjugated, i.e. of the form

$$r_1\mathrm{i}, -r_1\mathrm{i}, \ldots, r_s\mathrm{i}, -r_s\mathrm{i}, 0, \ldots, 0$$

with $r_k \in \mathbb{R}$ for all $k = 1, \ldots, s$. The determinant of A is evaluated as follows:

$$\det A = \det\Big(S + \frac{1}{2}I\Big) = \det\Big(D + \frac{1}{2}I\Big)$$

since $\det(T^t T) = 1$ and with the notations from above this expression is

$$\Big(\frac{1}{2}\Big)^{n-2s} \prod_{i=1}^{s} \Big(\frac{1}{2} + r_k\mathrm{i}\Big)\Big(\frac{1}{2} - r_k\mathrm{i}\Big) = \Big(\frac{1}{2}\Big)^{n-2s} \prod_{i=1}^{s} \Big(\frac{1}{4} + r_k^2\Big)\,.$$

As all factors are strictly positive the result follows. □

Problem 3. *Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1) = 0$. Prove that the set*

$$A := \big\{h \in [0,1] : f(x+h) = f(x) \text{ for some } x \in [0,1]\big\}$$

has Lebesgue measure at least $\frac{1}{2}$.

Solution. Let us observe, that if f is continuous then A is closed, thus A is Lebesgue measurable. Moreover the set

$$B := \big\{h \in [0,1] : 1-h \in A\big\}$$

has the same Lebesgue measure as the set A. We show that $A \cup B = [0,1]$.

For any $h \in [0,1]$ we define a function $g\colon [0,1] \to \mathbb{R}$ by

$$g(x) = f(x+h) - f(x) \quad \text{if } x+h \leq 1$$

and

$$g(x) = f(x+h-1) - f(x) \quad \text{if } x+h > 1\,.$$

From the assumption we have that g is continuous. If f has its minimum and maximum, respectively, in x_0 and x_1, then $g(x_0) \geq 0$ and $g(x_1) \leq 0$. From Darboux property we have that, there exists x_2 such that $g(x_2) = 0$, therefore $h \in A$ or $h \in B$. This completes the proof. □

Problem 4. *Let S be a finite set with n elements and $\mathcal{F}$ a family of subsets of S with the following property:*

$$A \in \mathcal{F},\, A \subseteq B \subseteq S \Longrightarrow B \in \mathcal{F}$$

Prove that the function $f\colon [0,1] \to \mathbb{R}$ given by

$$f(t) := \sum_{A \in \mathcal{F}} t^{|A|}(1-t)^{|S \setminus A|}$$

is nondecreasing ($|A|$ denotes the number of elements of A).

Solution. Without loss of generality assume $S = \{1, 2, \dots, n\}$. For each subset A and every $t \in [0,1]$ construct a set $I_{t,A} := \prod_{j=1}^{n} I_{t,A}^{(j)}$ in $\mathbb{R}^n$, where

$$I_{t,A}^{(j)} := \begin{cases} [0,t) & \text{if } j \in A \\ [t,1] & \text{if } j \notin A\,. \end{cases}$$

It's clear that for any two different subsets A and B the sets $I_{t,A}$ and $I_{t,B}$ are disjoint. Since the volume of $I_{t,A}$ is equal to $t^{|A|}(1-t)^{|A^c|}$ we have that $f(t)$ is equal to the volume of $\bigcup_{A \in \mathcal{F}} I_{t,A}$. So the claim will be proved if we prove that

$$\bigcup_{A \in \mathcal{F}} I_{t_1,A} \subseteq \bigcup_{A \in \mathcal{F}} I_{t_2,A} \quad \text{for all } 0 < t_1 < t_2 < 1\,. \tag{1}$$

Take an arbitrary $x = (x_1, x_2, \dots, x_n) \in I_{t_1,A}$ for some $A \in \mathcal{F}$. Construct a set $B \subseteq S$ such that $j \in B$ if and only if $x_j \le t_2$. If $j \notin B$ then $x_j > t_2 > t_1$ which implies $j \notin A$. So $A \subseteq B$ and thus $B \in \mathcal{F}$. Moreover, from the definition of B, we have $x \in I_{t_2,B}$. This proves (1) and the problem is solved. $\square$

The 18th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 2nd April 2008
Category I

Problem 1. *Find all complex roots (with multiplicities) of the polynomial*

$$p(x) = \sum_{n=1}^{2008} \bigl(1004 - |1004 - n|\bigr) x^n \,.$$

[10 points]

Problem 2. *Find all functions* $f\colon (0,\infty) \to (0,\infty)$ *such that*

$$f(f(f(x))) + 4f(f(x)) + f(x) = 6x \,.$$

[10 points]

Problem 3. *Find all* $c \in \mathbb{R}$ *for which there exists an infinitely differentiable function* $f\colon \mathbb{R} \to \mathbb{R}$ *such that for all* $n \in \mathbb{N}$ *and* $x \in \mathbb{R}$ *we have*

$$f^{(n+1)}(x) > f^{(n)}(x) + c \,.$$

[10 points]

Problem 4. *The numbers of the set* $\{1, 2, \ldots, n\}$ *are colored with* 6 *colors. Let*

$$S := \bigl\{(x,y,z) \in \{1,2,\ldots,n\}^3 : x+y+z \equiv 0 \pmod{n} \\ \text{ and } x,y,z \text{ have the same color}\bigr\}$$

and

$$D := \bigl\{(x,y,z) \in \{1,2,\ldots,n\}^3 : x+y+z \equiv 0 \pmod{n} \\ \text{ and } x,y,z \text{ have three different colors}\bigr\} \,.$$

Prove that

$$|D| \le 2|S| + \frac{n^2}{2} \,.$$

(For a set A, $|A|$ denotes the number of elements in A.) [10 points]

The 18th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 2nd April 2008
Category II

Problem 1. *Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that*

$$19f(x) - 17f(f(x)) = 2x$$

for all $x \in \mathbb{Z}$. [10 points]

Problem 2. *Find all continuously differentiable functions $f\colon [0,1] \to (0,\infty)$ such that $\frac{f(1)}{f(0)} = \mathrm{e}$ and*

$$\int_0^1 \frac{\mathrm{d}x}{f(x)^2} + \int_0^1 f'(x)^2 \,\mathrm{d}x \leq 2\,.$$

[10 points]

Problem 3. *Find all pairs of natural numbers (n,m) with $1 < n < m$ such that the numbers 1, $\sqrt[n]{n}$ and $\sqrt[m]{m}$* are linearly dependent *over the field of rational numbers $\mathbb{Q}$.*
[10 points]

Problem 4. *We consider the following game for one person. The aim of the player is to reach a fixed capital $C > 2$. The player begins with capital $0 < x_0 < C$. In each turn let x be the player's current capital. Define $s(x)$ as follows:*

$$s(x) := \begin{cases} x & \text{if } x < 1 \\ C - x & \text{if } C - x < 1 \\ 1 & \text{otherwise.} \end{cases}$$

Then a fair coin is tossed and the player's capital either increases or decreases by $s(x)$, each with probability $\frac{1}{2}$. Find the probability that in a finite number of turns the player wins by reaching the capital C. [10 points]

Problem j18-I-1. *Find all complex roots (with multiplicities) of the polynomial*

$$p(x) = \sum_{n=1}^{2008} \bigl(1004 - |1004 - n|\bigr) x^n \,.$$

Solution. Observe, by comparison of coefficients, that

$$p(x) = x \Bigl(\sum_{n=0}^{1003} x^n \Bigr)^2 \,.$$

Since $\sum_{n=0}^{1003} x^n = \frac{x^{1004}-1}{x-1}$, we conclude that p has the simple root 0 and the roots $\exp \frac{\pi i n}{502}$, $n = 1, 2, \ldots, 1003$, with multiplicity 2. $\square$

Problem j18-I-2. *Find all functions* $f\colon (0,\infty) \to (0,\infty)$ *such that*

$$f(f(f(x))) + 4f(f(x)) + f(x) = 6x\,.$$

Solution. Let $a \in \mathbb{R}^+$ be arbitrary. Set $a_0 = a$, $a_n = f(a_{n-1})$ for $n > 0$. Then we obtain recurrence relation

$$a_{n+3} + 4a_{n+2} + a_{n+1} - 6a_n = 0\,.$$

Characteristic equation is

$$y^3 - 4y^2 + y - 6 = 0$$

with roots -2, -3 and 1. The general solution of recurrence relation is

$$a_n = A(-3)^n + B(-2)^n + C\,.$$

If A or B are not equal to 0, we have a contradiction because in range of f we could find negative values. So the only possible solution is $a_n = C$. Because of $a_0 = a$ we have $a_n = a$ for all $n \in \mathbb{N}_0$. Substituting $n = 1$ we obtain

$$f(a) = f(a_0) = a_1 = a\,,$$

so for all $a \in \mathbb{R}^+$ we have $f(a) = a$.

The only solution of the equation is $f(x) = x$, what can be easily checked. $\square$

Problem j18-I-3. *Find all $c \in \mathbb{R}$ for which there exists an infinitely differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ we have*

$$f^{(n+1)}(x) > f^{(n)}(x) + c\,. \tag{1}$$

Solution. For $c \le 0$ we can take $f(x) = \mathrm{e}^{2x}$. Then $f^{(n+1)}(x) = 2^{n+1}\mathrm{e}^{2x} > 2^n\mathrm{e}^{2x} = f^{(n)}(x)$.

For positive c no function satisfies (1). We begin with two simple lemmas.

Lemma 1. *If f satisfies* (1)*, then for any $x \in \mathbb{R}$ there exists an $y \le x$ such that $f(y) \le -\frac{c}{2}$.*

Proof. If $f(t) > -\frac{c}{2}$ on $(-\infty, x]$, then $f'(t) > \frac{c}{2}$ for any $t < x$, thus

$$f(y) = f(x) - \int_y^x f'(t)\,\mathrm{d}t \le f(x) - (x-y)\frac{c}{2}$$

for any $y < x$, thus for sufficiently small y we have $f(y) < 0$, a contradiction. ▫

Lemma 2. *If f satisfies* (1)*, then for any $x \in \mathbb{R}$ such that $f(x) < \frac{c}{2}$ we have $f(y) < \frac{c}{2}$ for any $y \le x$.*

Proof. Suppose that there exists a $y \le x$ such that $f(y) \ge -\frac{c}{2}$. Let $z := \sup\{t \le x : f(t) \ge -\frac{c}{2}\}$. By the continuity of f (f is differentiable, thus continuous) we have $f(z) \ge -\frac{c}{2}$. By the assumption upon x we have $z \ne x$. However by (1) we have $f'(z) \ge \frac{c}{2}$, thus f' is positive on $[z, z+\varepsilon]$ for some $\varepsilon > 0$, f is increasing, thus $f(t) \ge f(z) \ge -\frac{c}{2}$ for $t \in [z, z+\varepsilon]$, a contradiction with the definition of z. Thus by contradiction the thesis is proved. ▫

Now if f satisfies (1), then obviously f' also satisfies (1). Thus by Lemmas 1 and 2, there exists an x_0 such that $f'(t) < -\frac{c}{2}$ on $(-\infty, x_0]$. This, however, means $f(t) > f(x_0) + (x_0 - t)\frac{c}{2}$ for $t < x_0$, so for sufficiently small $t_0 < x_0$ we have $f(t_0) > -\frac{3c}{2} > f'(t_0) - c$, which is a contradiction with (1). Thus no such f exists. □

Problem j18-I-4. *The numbers of the set $\{1,2,\dots,n\}$ are colored with 6 colors. Let*

$$S := \big\{(x,y,z)\in\{1,2,\dots,n\}^3 : x+y+z\equiv 0 \pmod n \\ \text{and } x,y,z \text{ have the same color}\big\}$$

and

$$D := \big\{(x,y,z)\in\{1,2,\dots,n\}^3 : x+y+z\equiv 0 \pmod n \\ \text{and } x,y,z \text{ have three different colors}\big\}.$$

Prove that

$$|D| \le 2|S| + \frac{n^2}{2}.$$

(For a set A, $|A|$ denotes the number of elements in A.)

Solution. Denote by n_1,n_2,n_3,n_4,n_5,n_6 the number of occurences of the colors. Clearly $n_1+\ldots+n_6=n$. We prove that

$$|S| - \frac{1}{2}|D| = \sum_{u=1}^{6} n_u^2 - \sum_{1\le u<v\le 6} n_u n_v. \tag{1}$$

For arbitrary $u,v,w\in\{1,2,\dots,6\}$, denote by N_{uvw} the number of triples (x,y,z), satisfying $x+y+z\equiv 0 \pmod n$ and having colors u,v and w, respectively. For any u,v we obviously have $\sum\limits_{w=1}^{6} N_{uvw} = n_u n_v$ and therefore

$$\begin{aligned}
|S| - \frac{1}{2}|D| &= \sum_{u=1}^{6} N_{uuu} - \sum_{1\le u<v\le 6}\sum_{w\ne u,v} N_{uvw} \\
&= \sum_{u=1}^{6}\Big(n_u^2 - \sum_{v\ne u} N_{uuv}\Big) - \sum_{1\le u<v\le 6}\big(n_u n_v - N_{uuv} - N_{uvv}\big) \\
&= \sum_{u=1}^{6} n_u^2 - \sum_{1\le u<v\le 6} n_u n_v.
\end{aligned}$$

Now, applying the AM-QM inequality,

$$\begin{aligned}
|S| - \frac{1}{2}|D| &= \sum_{u=1}^{6} n_u^2 - \sum_{1\le u<v\le 6} n_u n_v = \frac{3}{2}\sum_{u=1}^{6} n_u^2 - \frac{1}{2}\Big(\sum_{u=1}^{6} n_u\Big)^2 \\
&\ge \Big(\frac{1}{4} - \frac{1}{2}\Big)\Big(\sum_{u=1}^{6} n_u\Big)^2 = -\frac{n^2}{4}.
\end{aligned}$$

□

Second solution. We present a different proof for the relation (1). We use the notation N_{uvw} as well.

For every $u=1,2,\dots,6$, let C_u be the set of those numbers from $\{1,2,\dots,n\}$ which have the uth color and let $f_u(t) := \sum\limits_{x\in C_u} t^x$.

Let $\varepsilon := \mathrm{e}^{2\pi \mathrm{i}/n}$. We will use that for every integer s,

$$\frac{1}{n}\sum_{j=0}^{n-1}\varepsilon^{js} = \begin{cases} 1 & \text{if } s\equiv 0 \pmod n \\ 0 & \text{if } s\not\equiv 0 \pmod n. \end{cases}$$

Then, for arbitrary colors u,v,w,

$$\begin{aligned}
N_{uvw} &= \sum_{x\in C_u}\sum_{y\in C_v}\sum_{z\in C_w}\frac{1}{n}\sum_{j=0}^{n-1}\varepsilon^{j(x+y+z)} \\
&= \frac{1}{n}\sum_{j=0}^{n-1}\Big(\sum_{x\in C_u}\varepsilon^{jx}\Big)\Big(\sum_{y\in C_v}\varepsilon^{jy}\Big)\Big(\sum_{z\in C_w}\varepsilon^{jz}\Big) = \frac{1}{n}\sum_{j=0}^{n-1} f_u(\varepsilon^j) f_v(\varepsilon^j) f_w(\varepsilon^j)
\end{aligned}$$

and

$$
\begin{aligned}
|S| - \frac{1}{2}|D| &= \frac{1}{n}\sum_{j=0}^{n-1}\Big(\sum_{u=1}^{6} f_u^3(\varepsilon^j) - 3\sum_{u<v<w} f_u(\varepsilon^j)f_v(\varepsilon^j)f_w(\varepsilon^j)\Big) \\
&= \frac{1}{n}\sum_{j=0}^{n-1}\Big(\sum_{u=1}^{6} f_u(\varepsilon^j)\Big)\Big(\sum_{u=1}^{6} f_u^2(\varepsilon^j) - \sum_{u<v} f_u(\varepsilon^j)f_v(\varepsilon^j)\Big) \\
&= \sum_{j=0}^{n-1}\Big(\frac{1}{n}\sum_{x=1}^{n}\varepsilon^{jx}\Big)\Big(\sum_{u=1}^{6} f_u^2(\varepsilon^j) - \sum_{u<v} f_u(\varepsilon^j)f_v(\varepsilon^j)\Big)\,.
\end{aligned}
$$

The first factor is 0 except if $j = 0$. Hence,

$$
|S| - \frac{1}{2}|D| = \sum_{u=1}^{6} f_u^2(1) - \sum_{u<v} f_u(1)f_v(1) = \sum_{u=1}^{6} n_u^2 - \sum_{u<v} n_u n_v\,.
$$

□

Problem j18-II-1. *Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that*

$$19f(x) - 17f(f(x)) = 2x \tag{1}$$

for all $x \in \mathbb{Z}$.

Solution. Suppose that there exists a function $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying the above equation. Then define a function $g\colon \mathbb{Z} \to \mathbb{Z}$ by

$$g(x) = x - f(x)\,. \tag{2}$$

Taking into account (1) and (2), we get

$$17g(f(x)) = 2g(x)\,. \tag{3}$$

Let us fix $y \in \mathbb{Z}$ and let $a := g(y)$. Define a sequence $(x_n)_{n\geq 0}$ as follows

$$x_0 := y\,, \quad x_1 := f(x_0)\,, \quad \ldots, \quad x_n := f(x_{n-1})\,, \quad \ldots$$

for any $n \in \mathbb{N}$. Now substituting x_n into (3) in turn, we get

$$a = g(x_0) = \frac{17}{2}g(x_1) = \ldots = \frac{17^n}{2^n}g(x_n)$$

for any $n > 0$. Consequently, we infer that

$$2^n a = 17^n g(x_n)$$

for any $n > 0$. Since 2 and 17 are relatively prime, we deduce that $17^n \mid a$ for any $n > 0$ and therefore $a = 0$. Moreover, since y was arbitrary, it follows that $g(y) = 0$ for any $y \in \mathbb{Z}$. Thus $y - f(y) = 0$ for any $y \in \mathbb{Z}$ and hence $f(y) = y$ for any $y \in \mathbb{Z}$. This implies that only one function satisfies the equation (1). So, this completes the solution. $\square$

Problem j18-II-2. *Find all continuously differentiable functions* $f\colon [0,1] \to (0,\infty)$ *such that* $\frac{f(1)}{f(0)} = \mathrm{e}$ *and*

$$\int_0^1 \frac{\mathrm{d}x}{f(x)^2} + \int_0^1 f'(x)^2\,\mathrm{d}x \le 2\,.$$

Solution. First, we note that if f is such function, then

$$\begin{aligned}
0 &\le \int_0^1 \Bigl(f'(x) - \frac{1}{f(x)}\Bigr)^2\,\mathrm{d}x = \int_0^1 f'(x)^2\,\mathrm{d}x - 2\int_0^1 \frac{f'(x)}{f(x)}\,\mathrm{d}x + \int_0^1 \frac{\mathrm{d}x}{f(x)^2}\\
&= \int_0^1 f'(x)^2\,\mathrm{d}x - 2\int_0^1 (\ln f(x))'\,\mathrm{d}x + \int_0^1 \frac{\mathrm{d}x}{f(x)^2}\\
&= \int_0^1 f'(x)^2\,\mathrm{d}x - 2\ln\frac{f(1)}{f(0)} + \int_0^1 \frac{\mathrm{d}x}{f(x)^2}\,\mathrm{d}x \le 0\,,
\end{aligned}$$

since $\frac{f(1)}{f(0)} = \mathrm{e}$ and $\int_0^1 \frac{\mathrm{d}x}{f(x)^2} + \int_0^1 f'(x)^2\,\mathrm{d}x \le 2$. Therefore

$$\int_0^1 \Bigl(f'(x) - \frac{1}{f(x)}\Bigr)^2\,\mathrm{d}x = 0\,. \tag{1}$$

Since f is continuously differentiable function on $[0,1]$, the equality (1) is equivalent to

$$f'(x)f(x) = 1 \quad \forall x \in [0,1]\,. \tag{2}$$

All positive solutions of the differential equation (2) are in the form $f(x) = \sqrt{2x + C}$ for some $C > 0$. Since $\frac{f(1)}{f(0)} = \mathrm{e}$, we have $C = \frac{2}{\mathrm{e}^2-1}$, and thus

$$f(x) = \sqrt{2x + \frac{2}{\mathrm{e}^2 - 1}}$$

is the unique function satisfying the conditions from the statement. $\square$

Problem j18-II-3. *Find all pairs of natural numbers (n, m) with $1 < n < m$ such that the numbers 1, $\sqrt[n]{n}$ and $\sqrt[m]{m}$ are* linearly dependent *over the field of rational numbers $\mathbb{Q}$.*

Solution. The answer is $n = 2, m = 4$.

We begin with the following

Lemma. *The minimal (over $\mathbb{Q}$) polynomial $f(X)$ for $\sqrt[n]{n}$ equals $X^k - (\sqrt[n]{n})^k$, where k is the minimal satisfying $(\sqrt[n]{n})^k \in \mathbb{N}$.*

Proof. $\sqrt[n]{n}$ is a root of $X^n - n = 0$. So there is some nonempty subset A of $\{0, 1, ..., n-1\}$ such that

$$f(X) = \prod_{l \in A} (X - \zeta^l)\,,$$

where $\zeta = \cos\left(\frac{2\pi}{n}\right) + \mathrm{i}\sin\left(\frac{2\pi}{n}\right)$.

The free term of $f(X)$ has an absolute value equal to $(\sqrt[n]{n})^{|A|}$. Hence $(\sqrt[n]{n})^{\deg f(X)}$ is integer, and $\deg f(X) \geq k$ follows (k is as in the lemma). But, clearly $\sqrt[n]{n}$ is a root of $X^k - (\sqrt[n]{n})^k$, which has integer coefficients. ▫

Let us assume that $1, \sqrt[n]{n}, \sqrt[m]{m}$ are linearly dependent over $\mathbb{Q}$, i.e. there are rational a, b, c not all equal 0 such that $a + b\sqrt[n]{n} + c\sqrt[m]{m} = 0$.

Case $a \neq 0$. Then, as $\sqrt[n]{n}$ is irrational, we have $b, c \neq 0$. But $a + b\sqrt[n]{n} = -c\sqrt[m]{m}$ has the same degree of a minimal polynomial as $\sqrt[n]{n}$, and as $\sqrt[m]{m}$. Let k be the degree of the minimal polynomial for $\sqrt[m]{m}$. Then $y = \sqrt[n]{n}$ satisfies

$$(a + by)^k = (\sqrt[m]{m})^k\,,$$

but y^k and $(\sqrt[m]{m})^k$ are rational, and as $a, b \neq 0$ we obtain that there is a nonzero polynomial with rational coefficients vanishing $\sqrt[n]{n}$ of degree smaller than k, a contradiction.

Case $a = 0$. Hence $\frac{\sqrt[n]{n}}{\sqrt[m]{m}}$ is rational, and this is equivalent to $\frac{n^m}{m^n}$ is a mn-th power of a rational. Let p be any prime, and $p^a \parallel n$, $p^b \parallel m$. So we must have $mn \mid am - bn$. But $am - bn \leq am < mn$, in view of $a \leq \log_2 n < n$. In a similar way one obtains $am - bn > -mn$. So we must have $am = bn$, the relation independent of the choice of prime p. Thus

$$n = m^{m/n},$$

and $\sqrt[n]{n} = \sqrt[m]{m}$ follows. As the function $\sqrt[x]{x}$ has maximum at $x = \mathrm{e}$, we see that $\sqrt[n]{n} = \sqrt[m]{m}$ holds only for $n = 2, m = 4$. □

Problem j18-II-4. *We consider the following game for one person. The aim of the player is to reach a fixed capital $C > 2$. The player begins with capital $0 < x_0 < C$. In each turn let x be the player's current capital. Define $s(x)$ as follows:*

$$s(x) := \begin{cases} x & \text{if } x < 1 \\ C - x & \text{if } C - x < 1 \\ 1 & \text{otherwise.} \end{cases}$$

Then a fair coin is tossed and the player's capital either increases or decreases by $s(x)$, each with probability $\frac{1}{2}$. Find the probability that in a finite number of turns the player wins by reaching the capital C.

Solution. Let us denote by $f(x)$ the probability that player wins with starting capital x.

If $x \leq 1$, then he loses if loses the first turn, and if he wins the first turn, he has capital $2x$. Thus $f(x) = \frac{1}{2} f(2x)$.

If $x \geq C - 1$ the player wins if he wins the first turn, and has $2x - C$ in other case, thus $f(x) = \frac{1}{2} + \frac{1}{2} f(2x - C)$.

In all other cases there is $f(x) = \frac{1}{2}\big(f(x-1) + f(x+1)\big)$.

We will prove that this implies $f(x) = \frac{x}{C}$.

Let us define $g(x) = f(x) - \frac{x}{C}$. It is bounded on $[0, C]$ (as $f(x) \in [0, 1]$), and we have

$$g(x) = \begin{cases} \frac{1}{2} f(2x) - \frac{x}{C} = \frac{1}{2}\big(f(2x) - \frac{2x}{C}\big) = \frac{1}{2} g(2x) & \text{for } x \leq 1, \\ \frac{1}{2}\big(f(x-1) + f(x+1)\big) - \frac{x}{C} & \\ \quad = \frac{1}{2}\big(f(x-1) - \frac{x-1}{C} + f(x+1) - \frac{x+1}{C}\big) & \\ \quad = \frac{1}{2}\big(g(x-1) + g(x+1)\big) & \text{for } x \in (1, C-1), \\ \frac{1}{2} + \frac{1}{2} f(2x - C) - \frac{x}{C} & \\ \quad = \frac{1}{2}\big(f(2x - C) - \frac{2x - C}{C}\big) = \frac{1}{2} g(2x - C) & \text{for } x \geq C - 1. \end{cases}$$

Obviously $g(0) = g(C) = 0$. Let $K = \sup\limits_{t \in [0,C]} f(t) \in [0, \infty)$. Denote $n_0 = [C] - 1 \geq 1$.

We will prove for any natural $0 < n \leq n_0$ and $x \in (n-1, n]$ there is $g(x) \leq \frac{2^n - 1}{2^n} K$.

If $x \in (0, 1]$ there is $g(x) = \frac{1}{2} g(2x) \leq \frac{K}{2}$.

Assume, that for $x \leq n - 1$ and take $\bar{x} \in (n-1, n]$. There is $g(\bar{x} - 1) \leq \frac{2^{n-1} - 1}{2^{n-1}} K$ as $\bar{x} - 1 \in (n-2, n-1]$, and $g(\bar{x} + 1) \leq K$. Thus

$$g(\bar{x}) = \frac{1}{2}\big(g(\bar{x} - 1) + g(\bar{x} + 1)\big) \leq \frac{1}{2}\Big(\frac{2^{n-1} - 1}{2^{n-1}} K + K\Big) = \frac{2^n - 1}{2^n} K$$

as required.

$g(x) \leq= \frac{1}{2} g(2x - C) \leq \frac{K}{2}$ for $x \geq C - 1$.

Now take $x \in (n_0, C - 1)$ (it is empty set for integer C). We have proved that $g(x-1) \leq \frac{2^{n_0} - 1}{2^{n_0}} K$ (as $x - 1 \in (n_0 - 1, n_0)$) and $g(x+1) \leq \frac{K}{2}$ $(x + 1 > C - 1)$. Thus $g(x) \leq \frac{2^{n_0} - 1}{2^{n_0}} K$.

Thus we have proved, that $g(x) \leq \frac{2^{n_0} - 1}{2^{n_0}} K$ for every $x \in [0, C]$, which means that $K = 0$. Similarly one can prove, that $\inf\limits_{t \in [0,C]} f(t) = 0$. Thus $g(x) \equiv 0$, so $f(x) = \frac{x}{C}$. $\square$

The 19$^{\text{th}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 1$^{\text{st}}$ April 2009
Category I

Problem 1 *Let ABC be a non-degenerate triangle in the euclidean plane. Define a sequence $(C_n)_{n=0}^{\infty}$ of points as follows: $C_0 := C$, and C_{n+1} is the center of the incircle of the triangle ABC_n. Find $\lim_{n\to\infty} C_n$.*

[10 points]

Problem 2 *Prove that the number*

$$2^{2^k-1} - 2^k - 1$$

is composite (not prime) for all positive integers $k > 2$. [10 points]

Problem 3 *Let k and n be positive integers such that $k \leq n-1$. Let $S := \{1, 2, \ldots, n\}$ and let $A_1, A_2, \ldots, A_k$ be nonempty subsets of S. Prove that it is possible to color some elements of S using two colors, red and blue, such that the following conditions are satisfied:*

(i) *Each element of S is either left uncolored or is colored red or blue.*

(ii) *At least one element of S is colored.*

(iii) *Each set A_i $(i = 1, 2, \ldots, k)$ is either completely uncolored or it contains at least one red and at least one blue element.*

[10 points]

Problem 4 *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. We say that the sequence $(a_n)_{n=1}^{\infty}$* covers *the set of positive integers if for any positive integer m there exists a positive integer k such that $\sum_{n=1}^{\infty} a_n^k = m$.*

a) *Does there exist a sequence of real positive numbers which covers the set of positive integers?*

b) *Does there exist a sequence of real numbers which covers the set of positive integers?*

[10 points]

The 19th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 1st April 2009
Category II

Problem 1 *A positive integer m is called* self-descriptive in base b, *where $b \geq 2$ is an integer, if:*

i) The representation of m in base b is of the form $(a_0 a_1 \ldots a_{b-1})_b$
(that is $m = a_0 b^{b-1} + a_1 b^{b-2} + \cdots + a_{b-2} b + a_{b-1}$, where $0 \leq a_i \leq b-1$ are integers).

ii) a_i is equal to the number of occurences of the number i in the sequence $(a_0 a_1 \ldots a_{b-1})$.

For example, $(1210)_4$ is self-descriptive in base 4, because it has four digits and contains one 0, two 1s, one 2 and no 3s.

a) Find all bases $b \geq 2$ such that no number is self-descriptive in base b.

b) Prove that if x is a self-descriptive number in base b then the last (least significant) digit of x is 0.

[10 points]

Problem 2 *Let E be the set of all continuously differentiable real valued functions f on $[0,1]$ such that $f(0) = 0$ and $f(1) = 1$. Define*

$$J(f) = \int_0^1 (1 + x^2)(f'(x))^2 \, \mathrm{d}x \, .$$

a) Show that J achieves its minimum value at some element of E.

b) Calculate $\min\limits_{f \in E} J(f)$.

[10 points]

Problem 3 *Let A be an $n \times n$ square matrix with integer entries. Suppose that $p^2 A^{p^2} = q^2 A^{q^2} + r^2 I_n$ for some positive integers p, q, r where r is odd and $p^2 = q^2 + r^2$. Prove that $|\det A| = 1$.*
(Here I_n means the $n \times n$ identity matrix.)

[10 points]

Problem 4 *Let k, m, n be positive integers such that $1 \leq m \leq n$ and denote $S = \{1, 2, \ldots, n\}$. Suppose that $A_1, A_2, \ldots, A_k$ are m-element subsets of S with the following property: for every $i = 1, 2, \ldots, k$ there exists a partition $S = S_{1,i} \cup S_{2,i} \cup \cdots \cup S_{m,i}$ (into pairwise disjoint subsets) such that*

(i) A_i has precisely one element in common with each member of the above partition.

(ii) Every A_j, $j \neq i$ is disjoint from at least one member of the above partition.

Show that $k \leq \binom{n-1}{m-1}$.

[10 points]

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Category I

Problem 1 *Let ABC be a non-degenerate triangle in the euclidean plane. Define a sequence $(C_n)_{n=0}^{\infty}$ of points as follows: $C_0 := C$, and C_{n+1} is the center of the incircle of the triangle ABC_n. Find $\lim_{n\to\infty} C_n$.*

[10 points]

Solution If α is the angle at A, β the angle at B, then the limit is the point on the side $\overline{AB}$ dividing it in the ratio $\alpha : \beta$. Let α_i and β_i be the angles at A and B in ABC_i, respectively. Since the center of the incircle is the intersection of the angle bisectors, we have $\alpha_{i+1} = \frac{\alpha_i}{2}$ and $\beta_{i+1} = \frac{\beta_i}{2}$; so the limit point will obviously lie on $\overline{AB}$; furthermore, $\frac{\alpha_i}{\beta_i} = \frac{\alpha}{\beta} =: q$ for all i. Thus, if K_i is the circumcircle of ABC_i, $S_{1,i}$ and $S_{2,i}$ the arcs over $\overline{AC_i}$ and $\overline{BC_i}$, respectively, then $\frac{|S_{1,i}|}{|S_{2,i}|} = q$ for all i. Now, as the C_i approache $\overline{AB}$, the arcs converge to the corresponding sides of the triangle. Hence, the result follows. $\square$

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Category I

Problem 2 *Prove that the number*

$$2^{2^k - 1} - 2^k - 1$$

is composite (not prime) for all positive integers $k > 2$. [10 points]

Solution Denote

$$M = 2^{2^k - 1} - 2^k - 1\,.$$

If k is even then $3 \mid M$, and M is composite, since $M > 3$ for $k > 2$.

Suppose k is odd. Then

$$2M = 2^{2^k} - 1 - \left(2^{k+1} + 1\right) = \left(2^{2^{k-1}} + 1\right)\left(2^{2^{k-2}} + 1\right) \dots \left(2^{2^1} + 1\right) - \left(2^{k+1} + 1\right)\,.$$

Let $k + 1 = 2^a q$ with positive odd integer q and $a \geq 1$. Then $\left(2^{2^a} + 1\right) \mid 2M$. Indeed, $\left(2^{2^a} + 1\right) \mid \left(2^{k+1} + 1\right)$ and

$$\left(2^{2^a} + 1\right) \mid \left(2^{2^{k-1}} + 1\right)\left(2^{2^{k-2}} + 1\right) \dots \left(2^{2^1} + 1\right)\,,$$

since $a \leq k - 1$ for $k > 2$. $\square$

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Problem 3 *Let k and n be positive integers such that $k \leq n-1$. Let $S := \{1, 2, \ldots, n\}$ and let $A_1, A_2, \ldots, A_k$ be nonempty subsets of S. Prove that it is possible to color some elements of S using two colors, red and blue, such that the following conditions are satisfied:*

(i) Each element of S is either left uncolored or is colored red or blue.

(ii) At least one element of S is colored.

(iii) Each set A_i ($i = 1, 2, \ldots, k$) is either completely uncolored or it contains at least one red and at least one blue element.

[10 points]

Solution Consider the following system of k linear equations in n real variables $x_1, x_2, \ldots, x_n$:

$$\sum_{j \in A_i} x_j = 0\,, \quad i = 1, 2, \ldots, k\,.$$

Since $k < n$, this system has a nontrivial solution $(x_1, x_2, \ldots, x_n)$, i.e. a solution with at least one nonzero x_j. Now color red all elements of the set $\{j \in S : x_j > 0\}$, color blue all elements of the set $\{j \in S : x_j < 0\}$, and leave uncolored all elements of $\{j \in S : x_j = 0\}$.

Since the solution is nontrivial, at least one element is colored. If A_i contains some red element $j \in S$ then $x_j > 0$, and from $\sum_{j \in A_i} x_j = 0$ we see that there exists some $j' \in A_i$ such that $x_{j'} < 0$, i.e. j' is colored blue. Thus A_i must have elements of both colors. Analogously we argue when A_i contains a blue element. Therefore we see that the above coloring satisfies all requirements. $\square$

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Problem 4 *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. We say that the sequence $(a_n)_{n=1}^{\infty}$* covers *the set of positive integers if for any positive integer m there exists a positive integer k such that $\sum\limits_{n=1}^{\infty} a_n^k = m$.*

a) Does there exist a sequence of real positive numbers which covers the set of positive integers?

b) Does there exist a sequence of real numbers which covers the set of positive integers?

[10 points]

Solution The answer to the second question is positive.

First we shall prove that for any n there exists a finite sequence $(x_i)_{i=1}^{k_n}$ of real numbers such that

$$\sum_{i=1}^{k_n} x_i^{2m+1} = 0 \quad \text{for } 0 \le m < n$$

and

$$\sum_{i=1}^{k_n} x_i^{2m+1} \neq 0 \quad \text{for } m \ge n\,.$$

For the simplicity of notation we shall write $S_m(x_i)$ for $\sum_{i=1}^{k_n} x_i^{2m+1}$. We shall prove the thesis by induction upon n. For $n = 0$ the appropriate sequence is $x_1 = 1$.

Assume the thesis for n. For $n+1$ consider the sequence

$$(y_i)_{i=1}^{3k_n} = (-x_1, -x_2, \dots, -x_{k_n}, \alpha x_1, \alpha x_2, \dots, \alpha x_{k_n}, \alpha x_1, \alpha x_2, \dots, \alpha x_{k_n})\,,$$

where $\alpha = 2^{-1/(2n+1)}$. As $S_m(x_i) = 0$ for $m < n$, we also have $S_m(y_i) = 0$. We also have

$$S_n(y_i) = -S_n(x_i) + 2^{-1}S_n(x_i) + 2^{-1}S_n(x_i) = 0\,.$$

For $m > n$ we have

$$S_m(y_i) = (1 - 2\cdot 2^{-(2m+1)/(2n+1)})S_m(x_i) \neq 0\,.$$

Thus the induction step is finished, and the thesis is proved. Moreover it is easy to notice that $|x_i| \le 1$ and the length of the sequence is 3^n. Denote by $x(n)$ the sequence of length 3^n with $S_m(x(n)) = 0$ for $m < n$.

Now to give the required sequence (a_i). Our sequence will be a concatenation of multiples of the finite sequences $x(n)$ given above. We begin with $a_1 = 1$ (that is we begin by taking $x(0)$). In the n-th step we assume that we have some finite sequence a_i, with $S_m(a_i) = m+1$ for $m \le n$. We also assume that the elements added in the n-th step will be no larger than $\frac{1}{n}$.

To pass to the $(n+1)$-st step let $c = n+2-S_{n+1}(a_i)$, and let $d = S_{n+1}(x(n+1))$. Take an integer $N > \left|\frac{(n+1)c}{d}\right|$, let $\alpha = \frac{c}{dN}$, $|\alpha| < \frac{1}{n+1}$. We add N copies of the sequence $\alpha x(n+1)$ to the end of a_i. This does not change $S_m(a_i)$ for $m < n+1$ (as $S_m(x(n+1)) = 0$, and after the addition we have $S_{n+1}(a_i) = n+2$. Also all the added elements are of absolute value no larger than $\frac{1}{n+1}$.

Now to prove that for this series we have $G_{2k+1} = k+1$. As $S_m(a_i) = m+1$ after every step, no other limit is possible, we only have to check convergence. Note, however, that after the n–th step we only add sequences $x(m)$ for $m > n$, which in turn are concatenations of sequences $x(n)$, with some coefficients. Thus every 3^n–th partial sum in the series $\sum a_i^{2n+1}$ is going to be exactly equal to $n+1$. The partial sums "in the middle" cannot differ from this value by more than $\frac{3^n}{2}$ times the value of the maximal element $|a_i|$ in the appropriate interval, and this converges to zero. Thus for any n we do, in fact, have convergence.

For the first question, obviously the same series suffices.

For the last question, the answer is negative. As a_i are positive, we may rearrange them in decreasing order. Take k_0 to be the first k for which G_k is finite. For G_{k_0} to be finite, we have to have a_i convergent to zero, thus only a finite number of terms is larger than 1, assume these are the first n terms. Note that as for $i > n$ we have $a_i \le 1$, we also have that a_i^k decreases with k, and thus $\sum_{i=n+1}^{\infty} a_i^k$ decreases with k, and thus is bounded by $C := \sum_{i=n+1}^{\infty} a_i^{k_0}$. As G_k are assumed to attain unbounded values, we have to have terms larger than 1, thus $n > 0$.

Assume the first m terms of a_i are equal, $1 \le m \le n$. Then for $k \ge k_0$ and $l \le k$ we have

$$G_l \le m a_1^k + n a_{m+1}^k + C\,.$$

On the other hand $G_l \geq ma_1^{k+1}$ for $l \geq k$. For sufficiently large k, however, we have

$$ma_1^{k+1} > ma_1^k + na_{m+1}^k + C + 2\,,$$

which means that there is an integer number between ma_1^{k+1} and $ma_1^k + na_{m+1}^k + C$ which is not the value of G_k for any k. $\square$

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Category II

Problem 1 *A positive integer m is called* self-descriptive in base b, *where $b \geq 2$ is an integer, if:*

i) The representation of m in base b is of the form $(a_0 a_1 \dots a_{b-1})_b$
(that is $m = a_0 b^{b-1} + a_1 b^{b-2} + \dots + a_{b-2} b + a_{b-1}$, where $0 \leq a_i \leq b-1$ are integers).

ii) a_i is equal to the number of occurences of the number i in the sequence $(a_0 a_1 \dots a_{b-1})$.

For example, $(1210)_4$ is self-descriptive in base 4, because it has four digits and contains one 0, two 1s, one 2 and no 3s.

a) Find all bases $b \geq 2$ such that no number is self-descriptive in base b.

b) Prove that if x is a self-descriptive number in base b then the last (least significant) digit of x is 0.

[10 points]

Solution

1. For $b \geq 7$ it is easy to verify that the number of the form $(b-4)b^{b-1} + 2b^{b-2} + b^{b-3} + b^4$ is a self descriptive number (it contains $b-4$ instances of digit 0, two instances of digit 1, one instance of digit 2 and one instance of digit $b-4$), and numbers $21200_{(5)}$ and $2020_{(4)}$ are self-descriptive numbers in bases 5 and 4, respectively.

   It remains to show that for bases $2, 3$ and 6 no self descriptive numbers exist. First note, that a self-descriptive number (in any admissible base) contains at least one instance of the digit 0. If it does not, then the first digit is 0, which is a contradiction.

   It is easy to prove the claim for $b = 2, 3$.

   Let us prove it for $b = 6$. Assume there exists $x = (b_0 b_1 b_2 b_3 b_4 b_5)_{(6)}$, where x is a self-descriptive number.

   We observe the following about x:

   (a) $\sum_{i=0}^{5} b_i = 6$

   (b) $b_0 \neq 0$

   (c) $\sum_{i=1}^{5} b_i = |\{b_i, b_i \neq 0, i \geq 1\}| + 1$

   (d) Other than the first digit, the set of all other non-zero digits consists of several 1's and one 2.

   Observation 1d implies that all but one of the digits b_3, b_4 and b_5 are 0, now it is easy to check, that no such number is self-descriptive, which is a contradiction. Therefore base $b = 6$ contains no self-descriptive numbers.

2. Assume that there is in fact a self-descriptive number x in base b that it is b-digits long but not a multiple of b. The digit at position $b-1$ must be at least 1, meaning that there is at least one instance of the digit $b-1$ in x. At whatever position a that digit $b-1$ falls, there must be at least $b-1$ instances of digit a in x. Therefore, we have at least one instance of the digit 1, and $b-1$ instances of a. If $a > 1$, then x has more than b digits, leading to a contradiction of our initial statement. And if $a = 0$ or $a = 1$, that also leads to a contradiction.

3. These numbers are: 1210, 2020, 21200, 3211000, 42101000, 521001000, 6210001000. That these are the only such numbers, follows from previous observations.

□

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Problem 2 *Let E be the set of all continuously differentiable real valued functions f on $[0,1]$ such that $f(0)=0$ and $f(1)=1$. Define*

$$J(f)=\int_0^1 (1+x^2)(f'(x))^2\,\mathrm{d}x\,.$$

a) Show that J achieves its minimum value at some element of E.

b) Calculate $\min\limits_{f\in E} J(f)$.

[10 points]

Solution By the fundamental theorem of Calculus, we have

$$1=|f'(1)-f'(0)|=\Big|\int_0^1 f''(x)\,\mathrm{d}x\Big|\,.$$

Next, by using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\Big|\int_0^1 f''(x)\,\mathrm{d}x\Big| &= \Big|\int_0^1 \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} f''(x)\,\mathrm{d}x\Big| \\ &\le \Big(\int_0^1 (1+x^2)(f''(x))^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x\Big)^{1/2} \\ &= \Big(\int_0^1 (1+x^2)(f''(x))^2\,\mathrm{d}x\Big)^{1/2}\Big(\arctan x\Big|_0^1\Big)^{1/2} \\ &= \Big(\int_0^1 (1+x^2)(f''(x))^2\,\mathrm{d}x\Big)^{1/2}\frac{\sqrt{\pi}}{2}\,.\end{aligned}$$

Hence

$$\inf_{f\in E}\int_0^1 (1+x^2)(f''(x))^2\,\mathrm{d}x \ge \frac{4}{\pi}\,.$$

Finally, let

$$f(x):=\frac{4}{\pi}\int_0^x \arctan t\,\mathrm{d}t$$

for $x\in[0,1]$. Then $f'(x)=\frac{4}{\pi}\arctan x$ (by the fundamental theorem of Calculus) and $f''(x)=\frac{4}{\pi}\frac{1}{1+x^2}$, for $x\in[0,1]$. Consequently, we deduce that $f\in E$ and

$$J(f)=\int_0^1 (1+x^2)\Big(\frac{4}{\pi}\frac{1}{1+x^2}\Big)^2\,\mathrm{d}x=\frac{16}{\pi^2}\int_0^1\frac{1}{1+x^2}\,\mathrm{d}x=\frac{16}{\pi^2}\cdot\frac{\pi}{4}=\frac{4}{\pi}\,,$$

which proves that J attains its minimum on E. This completes the solution. $\square$

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Category II

Problem 3 *Let A be an $n \times n$ square matrix with integer entries. Suppose that $p^2A^{p^2} = q^2A^{q^2} + r^2I_n$ for some positive integers p, q, r where r is odd and $p^2 = q^2 + r^2$. Prove that $|\det A| = 1$.*
(Here I_n means the $n \times n$ identity matrix.) [10 points]

Solution Consider the function $f\colon \mathbb{R} \to \mathbb{R}$.

$$f(x) = p^2x^{p^2} - q^2x^{q^2} - r^2\,. \tag{1}$$

Observe that

$$f'(x) = p^4x^{q^2-1}\left(x^{r^2} - \left(\frac{q}{p}\right)^4\right).$$

The roots of equation $f'(x) = 0$ are $x_1 = 0$ and $x_2 = \left(\frac{q}{p}\right)^{\frac{4}{r^2}}$ ($r \neq 0$ and $q \neq 1$). From $f(0) = -r^2 < 0$ and $f\left(\left(\frac{q}{p}\right)^{\frac{4}{r^2}}\right) < 0$ we obtain

$$\operatorname{sgn} f(x) = \begin{cases} -1 & \text{if } x < 1\,, \\ 0 & \text{if } x = 1\,, \\ 1 & \text{if } x > 1\,. \end{cases} \tag{2}$$

So $x = 1$ is the only real root of equation $f(x) = 0$.

Since the matrix A verifies $f(A) = O_n$, some eigenvalue $\lambda \in \sigma_{\mathrm{P}}(A)$ satisfies the equation $f(\lambda) = 0$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of the matrix A. We show that $|\lambda_k| \leq 1$ for all k. The fact $f(\lambda) = 0$ can be written as

$$p^2\lambda^{p^2} = q^2\lambda^{q^2} + r^2. \tag{3}$$

Passing the relation (3) at modulus we obtain $p^2|\lambda|^{p^2} \leq q^2|\lambda|^{q^2} + r^2$ or

$$f(|\lambda|) \leq 0\,. \tag{4}$$

From (2) and (4) we obtain $0 \leq |\lambda| \leq 1$ or $0 \leq |\lambda_k| \leq 1$ for all $k = 1, \ldots, n$. Because $f(0) = -r^2 \neq 0$, it results that $\lambda_k \neq 0$ for all k.

Hence

$$0 < |\lambda_k| \leq 1 \quad \text{for all } k = 1, \ldots, n\,. \tag{5}$$

From $\det A = \lambda_1\lambda_2\cdots\lambda_n$ we obtain

$$|\det A| = |\lambda_1\lambda_2\cdots\lambda_n| = |\lambda_1||\lambda_2|\cdots|\lambda_n| \leq 1\,. \tag{6}$$

From (5) and (6) we obtain

$$0 < |\det A| \leq 1\,. \tag{7}$$

Since $A \in M_n(\mathbb{Z})$, it follows that $|\det A| \in \mathbb{N}$. From (7) we obtain the conclusion that $|\det A| = 1$. $\square$

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Problem 4 *Let k, m, n be positive integers such that $1 \le m \le n$ and denote $S = \{1, 2, \ldots, n\}$. Suppose that $A_1, A_2, \ldots, A_k$ are m-element subsets of S with the following property: for every $i = 1, 2, \ldots, k$ there exists a partition $S = S_{1,i} \cup S_{2,i} \cup \cdots \cup S_{m,i}$ (into pairwise disjoint subsets) such that*

(i) A_i has precisely one element in common with each member of the above partition.

(ii) Every A_j, $j \neq i$ is disjoint from at least one member of the above partition.

Show that $k \le \binom{n-1}{m-1}$. [10 points]

Solution Without loss of generality assume that $1 \in S_1^{(i)}$ for all $i = 1, 2, \ldots, k$, because otherwise we simply rename members of each partition.

For every $i = 1, 2, \ldots, k$ define the polynomial

$$P_i(x_2, x_3, \ldots, x_n) = \prod_{l=2}^{m} \Big(\sum_{s \in S_l^{(i)}} x_s \Big)$$

and regard it as a polynomial over $\mathbb{R}$ in variables $x_2, x_3, \ldots, x_n$.

Observe that P_i is a homogenous polynomial of degree $m-1$ in $n-1$ variables. Also observe that all monomials in P_i are products of different x's, i.e. there are no monomials with squares or higher powers. The last statement follows simply from the fact that $S_2^{(i)}, \ldots, S_m^{(i)}$ are mutually disjoint. Such polynomials form a linear space over $\mathbb{R}$ of dimension $\binom{n-1}{m-1}$ and polynomials P_i belong to that space. If we prove that polynomials P_i, $i = 1, 2, \ldots, k$ are linearly independent, the inequality $k \le \binom{n-1}{m-1}$ will follow from the dimension argument.

For any $i = 1, 2, \ldots, k$ let χ_i be the characteristic vector of $A \cap \{2, 3, \ldots, n\}$. In other words, $\chi_i \in \{0, 1\}^{n-1}$ where the j-th coordinate of χ_i equals 1 if $j + 1 \in A$, and 0 otherwise.

For every i we know that each $A_i \cap S_l^{(i)}$ has exactly one element and therefore

$$P_i(\chi_i) = \prod_{l=2}^{m} |A_i \cap S_l^{(i)}| = \prod_{l=2}^{m} 1 = 1\,.$$

On the other hand, if $j \neq i$ then either some $A_j \cap S_l^{(i)}$, $l \ge 2$ is empty, or all $A_j \cap S_l^{(i)}$, $l \ge 2$ are nonempty but $A_j \cap S_1^{(i)} = \emptyset$. In the latter case we must have $|A_j \cap S_l^{(i)}| = 2$ for some $l \ge 2$. In any case we have at least one even factor in the following product, and so

$$P_i(\chi_j) = \prod_{l=2}^{m} |A_j \cap S_l^{(i)}| \equiv 0 \pmod 2\,.$$

Therefore all diagonal entries in the matrix $[P_i(\chi_j)]_{i,j=1,2,\ldots,k}$ are odd, while all non-diagonal entries are even. Consequently, its determinant is an odd integer, in particular it is not 0, and thus the matrix is regular. If polynomials P_i were linearly dependent, we would conclude that rows of $[P_i(\chi_j)]_{i,j=1,2,\ldots,k}$ are also linearly dependent, but this is not the case. Therefore P_i, $i = 1, 2, \ldots, k$ must be linearly independent and this completes the proof. □

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Category I

Problem 1

a) *Is it true that for every bijection $f\colon \mathbb{N} \to \mathbb{N}$ the series*

$$\sum_{n=1}^{\infty} \frac{1}{nf(n)}$$

is convergent?

b) *Prove that there exists a bijection $f\colon \mathbb{N} \to \mathbb{N}$ such that the series*

$$\sum_{n=1}^{\infty} \frac{1}{n+f(n)}$$

is convergent.

($\mathbb{N}$ is the set of all positive integers.) [10 points]

Problem 2 *Let A and B be two complex 2×2 matrices such that $AB - BA = B^2$. Prove that $AB = BA$.* [10 points]

Problem 3 *Prove that there exist positive constants c_1 and c_2 with the following properties:*

a) *For all real $k > 1$,*

$$\left|\int_0^1 \sqrt{1-x^2}\,\cos(kx)\,\mathrm{d}x\right| < \frac{c_1}{k^{3/2}}\,.$$

b) *For all real $k > 1$,*

$$\left|\int_0^1 \sqrt{1-x^2}\,\sin(kx)\,\mathrm{d}x\right| > \frac{c_2}{k}\,.$$

[10 points]

Problem 4 *For every positive integer n let $\sigma(n)$ denote the sum of all its positive divisors. A number n is called* weird *if $\sigma(n) \geq 2n$ and there exists no representation*

$$n = d_1 + d_2 + \cdots + d_r\,,$$

where $r > 1$ and $d_1, \ldots, d_r$ are pairwise distinct positive divisors of n.
Prove that there are infinitely many weird numbers. [10 points]

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Category II

Problem 1 *Let a and b be given positive coprime integers. Then for every integer n there exist integers x, y such that*

$$n = ax + by\,.$$

Prove that $n = ab$ is the greatest integer for which $xy \le 0$ in all such representations of n. [10 points]

Problem 2 *Prove or disprove that if a real sequence (a_n) satisfies $a_{n+1} - a_n \to 0$ and $a_{2n} - 2a_n \to 0$ as $n \to \infty$, then $a_n \to 0$.* [10 points]

Problem 3 *Let A and B be two $n \times n$ matrices with integer entries such that all of the matrices*

$$A\,, \quad A+B\,, \quad A+2B\,, \quad A+3B\,, \quad \ldots\,, \quad A+(2n)B$$

are invertible and their inverses have integer entries, too. Show that $A + (2n+1)B$ is also invertible and that its inverse has integer entries. [10 points]

Problem 4 *Let $f\colon [0,1] \to \mathbb{R}$ be a function satisfying*

$$|f(x) - f(y)| \le |x - y|$$

for every $x, y \in [0,1]$. Show that for every $\varepsilon > 0$ there exists a countable family of rectangles (R_i) of dimensions $a_i \times b_i$, $a_i \le b_i$, in the plane such that

$$\big\{(x, f(x)) : x \in [0,1]\big\} \subset \bigcup_i R_i \quad \text{and} \quad \sum_i a_i < \varepsilon\,.$$

(The edges of the rectangles are not necessarily parallel to the coordinate axes.) [10 points]

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Category I

Problem 1

a) *Is it true that for every bijection $f\colon \mathbb{N} \to \mathbb{N}$ the series*

$$\sum_{n=1}^{\infty} \frac{1}{nf(n)}$$

is convergent?

b) *Prove that there exists a bijection $f\colon \mathbb{N} \to \mathbb{N}$ such that the series*

$$\sum_{n=1}^{\infty} \frac{1}{n+f(n)}$$

is convergent.

($\mathbb{N}$ is the set of all positive integers.) [10 points]

Solution a) Yes. Applying the inequality, if $0 \le a_1 \le \cdots \le a_n$ and $0 \le b_1 \le \cdots \le b_n$ and $\sigma\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$ is a permutation, then

$$\sum_{j=1}^{n} a_j b_{\sigma(j)} \le \sum_{j=1}^{n} a_j b_j,$$

for every n we get

$$\sum_{j=1}^{n} \frac{1}{jf(j)} \le \sum_{j=1}^{n} \frac{1}{j^2} \le \sum_{j=1}^{\infty} \frac{1}{j^2}.$$

Since the sequence $\left(\sum_{j=1}^{n} \frac{1}{jf(j)}\right)$ is increasing and bounded, it converges.

b) No. We will construct a permutation $f\colon \mathbb{N} \to \mathbb{N}$ such that the series

$$\sum_{n=1}^{\infty} \frac{1}{n+f(n)}$$

is convergent. Let $f\colon \mathbb{N} \to \mathbb{N}$ be given in the following way: $f(1) = 4$ and for $[(n!)^2+1, ((n+1)!)^2] \cap \mathbb{N}$ we put

$$f((n!)^2+k) = [(n+2)!]^2 - (k-1) \quad \text{if} \quad 1 \le k < [(n+1)!]^2 - 1 - \sum_{j=0}^{n-1} (-1)^j [(n-j)!]^2.$$

and

$$f([(n+1)!]^2 - k) = [(n-1)!]^2 + k + 1 \quad \text{if} \quad 0 \le k \le 1 + \sum_{j=0}^{n-1} (-1)^j [(n-j)!]^2.$$

Then

$$\begin{aligned}\sum_{j=(n!)^2+1}^{[(n+1)!]^2} \frac{1}{n+f(n)} &\le \frac{((n+1)!)^2 - (n!)^2}{(n!)^2 + [(n+2)!]^2 + 1} + \frac{(n!)^2 - [(n-1)!]^2}{[(n+1)!]^2 + [(n-1)!]^2 + 1} \\ &< \frac{1}{(n+2)^2} + \frac{1}{(n+1)^2}.\end{aligned}$$

Thus we show that the sequence $\left(\sum_{j=1}^{n} \frac{1}{j+f(j)}\right)$ is bounded. Since it is increasing, it converges. $\square$

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Category I

Problem 2 *Let A and B be two complex 2×2 matrices such that $AB - BA = B^2$. Prove that $AB = BA$.*

[10 points]

Solution We may conclude that $AB = BA$ if and only if $2 \neq 0$ in F (that is, $\operatorname{char} F \neq 2$).

If $\operatorname{char} F = 2$, take $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Assume that $\operatorname{char} F \neq 2$. Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $B^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$. We have $a^2 + d^2 + 2bc = \operatorname{trace} B^2 = \operatorname{trace} AB - \operatorname{trace} BA = 0$. If B is invertible, then $A = B(A+B)B^{-1}$, hence

$$\operatorname{trace} A = \operatorname{trace}(B(A+B)B^{-1}) = \operatorname{trace}(A+B) = \operatorname{trace} A + \operatorname{trace} B,$$

so $\operatorname{trace} B = 0$, $d = -a$, $\operatorname{trace} B^2 = 2(a^2 + bc) = 0$. Since $\operatorname{char} F \neq 2$, it implies $a^2 + bc = 0$, hence $B^2 = 0$ and $AB = BA$. If B is not invertible, then $\det B = ad - bc = 0$, so $(a+d)^2 = a^2 + d^2 + 2bc = 0$, $a + d = 0$, $a = -d$, $a^2 + bc = -ad + bc = 0$, so $B^2 = 0$. $\square$

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Problem 3 *Prove that there exist positive constants c_1 and c_2 with the following properties:*

a) For all real $k > 1$,

$$\left|\int_0^1 \sqrt{1-x^2}\,\cos(kx)\,\mathrm{d}x\right| < \frac{c_1}{k^{3/2}}\,.$$

b) For all real $k > 1$,

$$\left|\int_0^1 \sqrt{1-x^2}\,\sin(kx)\,\mathrm{d}x\right| > \frac{c_2}{k}\,.$$

[10 points]

Solution Put $f(x) = \sqrt{1-x^2}$.

1. Integrating by parts, we have

$$\int_0^1 f(x)\cdot\cos kx\,\mathrm{d}x = \Big[f(x)\cdot\frac{1}{k}\sin kx\Big]_0^1 - \int_0^1 f'(x)\cdot\frac{1}{k}\sin kx\,\mathrm{d}x\,.$$

The first term is $0 - 0 = 0$. The second term is $(-1/k)$ times

$$\int_0^{\sqrt{1-1/k}} f'(x)\cdot\sin kx\,\mathrm{d}x + \int_{\sqrt{1-1/k}}^1 f'(x)\cdot\sin kx\,\mathrm{d}x\,. \tag{1}$$

Here the first term equals

$$\Big[-f'(x)\cdot\frac{1}{k}\cos kx\Big]_0^{\sqrt{1-1/k}} + \int_0^{\sqrt{1-1/k}} f''(x)\cdot\frac{1}{k}\cos kx\,\mathrm{d}x\,,$$

whose absolute value is

$$\le -\frac{2}{k}f'\big(\sqrt{1-1/k}\big) = \frac{2}{k}\frac{\sqrt{1-1/k}}{\sqrt{1/k}} < \frac{2}{\sqrt{k}}\,.$$

The absolute value of the second term in (1) is

$$\le \int_{\sqrt{1-1/k}}^1 |f'(x)|\,\mathrm{d}x = -[f(x)]_{\sqrt{1-1/k}}^1 = \frac{1}{\sqrt{k}}\,.$$

Thus, we may choose $c_1 = 2 + 1 = 3$.

2. Integrating by parts, we have

$$\int_0^1 f(x)\cdot\sin kx\,\mathrm{d}x = -\Big[f(x)\cdot\frac{1}{k}\cos kx\Big]_0^1 + \int_0^1 f'(x)\cdot\frac{1}{k}\cos kx\,\mathrm{d}x\,.$$

The first term is $1/k$. The second term is $(1/k)$ times

$$\int_0^{\sqrt{1-1/k}} f'(x)\cdot\cos kx\,\mathrm{d}x + \int_{\sqrt{1-1/k}}^1 f'(x)\cdot\cos kx\,\mathrm{d}x\,. \tag{2}$$

Here the first term equals

$$\Big[f'(x)\cdot\frac{1}{k}\sin kx\Big]_0^{\sqrt{1-1/k}} - \int_0^{\sqrt{1-1/k}} f''(x)\cdot\frac{1}{k}\sin kx\,\mathrm{d}x\,,$$

whose absolute value is

$$\le -\frac{2}{k}f'\big(\sqrt{1-1/k}\big) = \frac{2}{k}\frac{\sqrt{1-1/k}}{\sqrt{1/k}} < \frac{2}{\sqrt{k}}\,.$$

The absolute value of the second term in (2) is

$$\le \int_{\sqrt{1-1/k}}^1 |f'(x)|\,\mathrm{d}x = -[f(x)]_{\sqrt{1-1/k}}^1 = \frac{1}{\sqrt{k}}\,.$$

Thus,

$$\int_0^1 f(x) \cdot \sin kx \, \mathrm{d}x > \frac{1}{k}\left(1 - \frac{3}{\sqrt{k}}\right).$$

This proves the desired claim for $k \geq 3\pi$.

The integral has a positive lower bound for $k < 3\pi$ as well, since

$$\int_0^1 f(x) \cdot \sin kx \, \mathrm{d}x = \int_0^1 \big(-f'(x)\big) \cdot \frac{1 - \cos kx}{k} \, \mathrm{d}x > 0\,.$$

$\square$

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Category I

Problem 4 *For every positive integer n let $\sigma(n)$ denote the sum of all its positive divisors. A number n is called* weird *if $\sigma(n) \geq 2n$ and there exists no representation*

$$n = d_1 + d_2 + \cdots + d_r\,,$$

where $r > 1$ and $d_1, \ldots, d_r$ are pairwise distinct positive divisors of n.
Prove that there are infinitely many weird numbers. [10 points]

Solution The idea is to show that given a weird number, one can construct a sequence of weird numbers tending to infinity.

We claim that for weird n and p a prime greater than $\sigma(n)$ and coprime to n, the number pn is also weird. In fact, if $1 = d_1, d_2, \ldots, d_k = n$ are the positive divisors of n, the ones of pn are $d_1, d_2, \ldots, d_k, pd_1, \ldots, pd_k$ and they are pairwise distinct as $(p, n) = 1$. Suppose now that we have

$$pn = d_{i_1} + \cdots + d_{i_r} + p(d_{j_1} + \cdots + d_{j_s})$$

with $i_k, j_l \in \{1, \ldots, k\}$. Then we have

$$d_{i_1} + \cdots + d_{i_r} = p(n - d_{j_1} - \cdots - d_{j_s})\,.$$

Note that $n \notin \{d_{j_1}, \ldots, d_{j_s}\}$ as the representation must have more than only one summand and the assumption that n is weird implies $n - d_{j_1} - \ldots - d_{j_s} \neq 0$. Hence as the right hand expression is divisible by p and non zero, so must be $d_{i_1} + \cdots + d_{i_r}$ which is impossible as $p > \sigma(n)$.

It remains to find a weird number. A possible reasoning could be: look for a number n with $\sigma(n) = 2n + 4$ that is not divisible by 3 and 4. Then the smallest possible divisors are $1, 2, 5$ so that it will be impossible to represent 4, and hence n, as a sum of pairwise distinct divisors of n. Checking for numbers with three distinct prime factors $2, p, q$ yields

$$\sigma(2pq) = 3(p+1)(q+1) = 3pq + 3p + 3q + 3$$

and hence we need

$$3pq + 3p + 3q + 3 = 4pq + 4 \iff (p-3)(q-3) = 8\,.$$

This equality is solved by $p = 5$ and $q = 7$ which yields the weird number $n = 70$. $\square$

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Category II

Problem 1 *Let a and b be given positive coprime integers. Then for every integer n there exist integers x, y such that*

$$n = ax + by \, .$$

Prove that $n = ab$ is the greatest integer for which $xy \leq 0$ in all such representations of n. [10 points]

Solution The greatest such integer is $a \cdot b$.

If $ab = ax + by$, then $a \mid y$ and $b \mid x$. Thus if $x > 0$, then $x \geq b$ and $by = ab - ax \leq ab - ab = 0$, so $y \leq 0$.

Now let $n > ab$. Let $n = ax + by$ be the representation such that x is positive and as small as possible. Then since $n = a(x - b) + b(y + a)$ is another representation of n, $x - b$ must not be positive and therefore $x \leq b$. Hence $by = n - ax \geq n - ab > 0$, so $y > 0$. $\square$

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Category II

Problem 2 *Prove or disprove that if a real sequence (a_n) satisfies $a_{n+1} - a_n \to 0$ and $a_{2n} - 2a_n \to 0$ as $n \to \infty$, then $a_n \to 0$.* [10 points]

Solution The proposition is true.

From the condition $a_{n+1} - a_n \to 0$ we conclude by Cesaro's lemma that $\frac{a_n}{n} \to 0$. Since the sequence $a_{2n} - 2a_n$ must be bounded, we know that

$$C := \sup\{|a_{2n} - 2a_n| : n \in \mathbb{N}\} < \infty .$$

Considering the identity

$$\frac{a_n}{n} - \frac{a_{n\cdot 2^{m+1}}}{n \cdot 2^{m+1}} = \sum_{k=0}^{m} \left(\frac{a_{n\cdot 2^k}}{n \cdot 2^k} - \frac{a_{n\cdot 2^{k+1}}}{n \cdot 2^{k+1}} \right)$$

we conclude by letting $m \to \infty$ and n fixed that

$$\frac{a_n}{n} = \sum_{k=0}^{\infty} \left(\frac{a_{n\cdot 2^k}}{n \cdot 2^k} - \frac{a_{n\cdot 2^{k+1}}}{n \cdot 2^{k+1}} \right) .$$

Now from

$$\left| \frac{a_n}{n} \right| \le \sum_{k=0}^{\infty} \left| \frac{a_{n\cdot 2^k}}{n \cdot 2^k} - \frac{a_{n\cdot 2^{k+1}}}{n \cdot 2^{k+1}} \right| \le \sum_{k=0}^{\infty} \frac{C}{n \cdot 2^{k+1}} = \frac{C}{n}$$

we infer that $|a_n| \le C$, i. e. the sequence (a_n) must be bounded.

Now suppose that (a_n) does not converge to 0. Then, by Bolzano's theorem, there must exist a subsequence (a_{n_k}) converging to some number $a \neq 0$. From the hypothesis we conclude in turn that

$$\begin{aligned} a_{2n_k} &\to 2a \,, \\ a_{4n_k} &\to 4a \,, \\ &\vdots \end{aligned}$$

which would result in an unbounded set of accumulation points $a, 2a, 4a, \ldots$ of (a_n) in contradiction to (a_n) being bounded. $\square$

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Problem 3 *Let A and B be two $n \times n$ matrices with integer entries such that all of the matrices*

$$A\,, \quad A+B\,, \quad A+2B\,, \quad A+3B\,, \quad \dots\,, \quad A+(2n)B$$

are invertible and their inverses have integer entries, too. Show that $A+(2n+1)B$ is also invertible and that its inverse has integer entries. [10 points]

Solution Suppose that the $n \times n$ matrix M has integer entries and M has inverse matrix M^{-1} with integer entries. Then $M \cdot M^{-1} = I$ implies $\det M \cdot \det M^{-1} = 1$. Thus $\det M = 1$ or $\det M = -1$. Set $M(t) = A + tB$. The determinant of the matrix $M(t)$

$$\det M(t) = \det (A + tB) = \det A + \cdots + t^n \det B$$

is the polynomial of degree n in t. The polynomial $\det M(t)$ takes values 1 or -1 at points $t = 0, 1, 2, \dots, 2n$. Hence $\det M(t)$ takes the value 1 or the value -1 at least $n+1$ times. This implies that $\det M(t)$ is a constant polynomial: $M(t) = 1$ or $M(t) = -1$ for all t. Consequently, $\det M(2n+1) = \pm 1$. Hence the matrix $A + (2n+1)B$ is invertible. By Cramer's formula, the inverse matrix has integer entries, since the determinant is equal to 1 or -1. $\square$

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Problem 4 *Let $f\colon [0,1] \to \mathbb{R}$ be a function satisfying*

$$|f(x) - f(y)| \le |x - y|$$

for every $x, y \in [0,1]$. Show that for every $\varepsilon > 0$ there exists a countable family of rectangles (R_i) of dimensions $a_i \times b_i$, $a_i \le b_i$, in the plane such that

$$\big\{(x, f(x)) : x \in [0,1]\big\} \subset \bigcup_i R_i \quad \textit{and} \quad \sum_i a_i < \varepsilon\,.$$

(The edges of the rectangles are not necessarily parallel to the coordinate axes.) [10 points]

Solution Assume without loss of generality that $f(0) = 0$, thus $|f(x)| \le 1$ for $x \in [0,1]$.

First notice that if $C \subset [0,1]$ is a set of Lebesgue measure no larger than $\varepsilon/3$, then it can be covered by a countable family of intervals I_i of total measure at most $\varepsilon/2$, and thus $\{(x, f(x) : x \in C\}$ is covered by rectangles $I_i \times [-1,1]$, and their total width is at most $\varepsilon/2$.

Notice that as we are interested in only one dimension of the rectangle, and the graph we are to covered is bounded, we may as well think in terms of covering with strips instead of rectangles.

For now on fix $\varepsilon > 0$. We shall introduce a few definitions. Let $x, y \in [0,1]$. We say that the interval $[x,y]$ is covered, if $|f(z) - \alpha(z)| < \varepsilon|x - y|$ for all $z \in [x,y]$, where α is the linear function meeting f at x and y. The inclination of an interval $[x,y]$, denoted $i(x,y)$, is the number $|f(x) - f(y)|/|x - y|$. Notice the inclination of any interval cannot be larger than 1 as f is 1-Lipschitz.

Now we prove the following lemma.

Lemma *There exists a constant $\delta > 0$ such that the following holds. Consider any interval $[x,y] \subset [0,1]$. Then either $[x,y]$ is covered, or there exists a subinterval $[x',y'] \subset [x,y]$ of length $|y' - x'| > \delta|x - y|$ and inclination at least $i(x,y) + \varepsilon$.*

Proof The proof is pretty simple. If $[x,y]$ is not covered, then there exists a point $z \in [x,y]$ with $|f(z) - \alpha(z)| > \varepsilon|x - y|$. Without loss of generality assume $f(x) < f(y)$ and $f(z) - \alpha(z) > \varepsilon|x - y|$. The interval $[x,z]$ in this case has inclination

$$\begin{aligned} i(x,z) = |f(x) - f(z)|/|x - z| = \frac{f(z) - f(x)}{z - x} &\ge \frac{\alpha(z) + \varepsilon(y - x) - f(x)}{z - x} = \frac{\frac{f(y)-f(x)}{y-x}(z - x) + \varepsilon(y - x)}{z - x} \\ &= \frac{f(y) - f(x)}{x - y} + \varepsilon\frac{y - x}{z - x} \ge i(x,y) + \varepsilon. \end{aligned}$$

The cases of $f(x) > f(y)$ and (or) $f(z) - \alpha(z) < -\varepsilon|x - y|$ are similar. Moreover we have

$$f(z) > \alpha(z) + \varepsilon|x - y| = f(x) \pm i(x,y)(z - x) + \varepsilon(y - x)\,.$$

Thus

$$2|z - x| \ge |f(z) - f(x)| + i(x,y)|z - x| \ge f(z) - f(x) \pm i(x,y)(z - x) \ge \varepsilon|x - y|\,,$$

thus $|z - x| \ge \frac{\varepsilon|x-y|}{2}$, which finishes the proof of the lemma with $\delta = \varepsilon/2$. □

Take a constant $n > 1/\varepsilon$. If begin with an interval $[x,y]$ and apply the lemma n times, we end up with an interval of length at least $|x - y|\delta^n$, which is either covered, or has inclination at least $n\varepsilon$ — the second is impossible, however, as the inclination of any interval is at most 1. Thus for any interval we can find its subinterval of length at least δ^n times the length of the original, which is covered. Thus we have the following corollary: for any interval $[x,y] \subset [0,1]$ there exists a covered subinterval $[x',y']$ of $[x,y]$ of length at least $c|x - y|$ for some fixed constant c.

Now we are ready to solve the problem. We shall construct a family of disjoint intervals $C_i \subset [0,1]$, with the Lebesgue measure of $[0,1] \setminus \bigcup C_i$ no larger than ε. Each of these intervals will be covered, and thus we shall be able to cover the whole graph of f by rectangles — each interval is covered, and thus the appropriate piece of the graph is contained in a rectangle of width at most 2ε, while the remaining part can be covered by a countable family of vertical rectangles of total width at most 2ε. As ε was arbitrary, this will end the proof.

The construction of C_is follows directly from the corollary — we choose $C_0 = [x_0, y_0]$ to be the interval given by the corollary for $[0,1]$, then C_1 and C_2 the intervals for $[0, x_0]$ and $[y_0, 1]$ respectively, then (in the third step), C_3, C_4, C_5 and C_6 are given for $[0, x_1]$, $[y_1, x_0]$, $[y_0, x_2]$ and $[y_2, 1]$ respectively, and so on. In each step a constant fraction of measure is removed, thus after sufficiently many steps no more than ε measure remains. □

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Category I

Problem 1

(a) *Is there a polynomial $P(x)$ with real coefficients such that*

$$P\left(\frac{1}{k}\right) = \frac{k+2}{k}$$

for all positive integers k?

(b) *Is there a polynomial $P(x)$ with real coefficients such that*

$$P\left(\frac{1}{k}\right) = \frac{1}{2k+1}$$

for all positive integers k?

Problem 2 *Let $(a_n)_{n=1}^{\infty}$ be an unbounded and strictly increasing sequence of positive reals such that the arithmetic mean of any four consecutive terms a_n, a_{n+1}, a_{n+2}, a_{n+3} belongs to the same sequence. Prove that the sequence a_{n+1}/a_n converges and find all possible values of its limit.*

Problem 3 *Prove that*

$$\sum_{k=0}^{\infty} x^k \frac{1+x^{2k+2}}{(1-x^{2k+2})^2} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{(1-x^{k+1})^2}$$

for all $x \in (-1,1)$.

Problem 4 *Let a, b, c be elements of finite order in some group. Prove that if $a^{-1}ba = b^2$, $b^{-2}cb^2 = c^2$ and $c^{-3}ac^3 = a^2$ then $a = b = c = e$, where e is the unit element.*

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Category II

Problem 1 *Let $n > k$ and let $A_1, \ldots, A_k$ be real $n \times n$ matrices of rank $n-1$. Prove that*

$$A_1 \cdot \ldots \cdot A_k \neq 0\,.$$

Problem 2 *Let k be a positive integer. Compute*

$$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k (n_1 + \cdots + n_k + 1)}\,.$$

Problem 3 *Let p and q be complex polynomials with $\deg p > \deg q$ and let $f(z) = \dfrac{p(z)}{q(z)}$. Suppose that all roots of p lie inside the unit circle $|z| = 1$ and that all roots of q lie outside the unit circle. Prove that*

$$\max_{|z|=1} |f'(z)| > \frac{\deg p - \deg q}{2} \max_{|z|=1} |f(z)|\,.$$

Problem 4 *Let $\mathbb{Q}[x]$ denote the vector space over $\mathbb{Q}$ of polynomials with rational coefficients in one variable x. Find all $\mathbb{Q}$-linear maps $\Phi\colon \mathbb{Q}[x] \to \mathbb{Q}[x]$ such that for any irreducible polynomial $p \in \mathbb{Q}[x]$ the polynomial $\Phi(p)$ is also irreducible.*

(A polynomial $p \in \mathbb{Q}[x]$ is called irreducible if it is non-constant and the equality $p = q_1 q_2$ is impossible for non-constant polynomials $q_1, q_2 \in \mathbb{Q}[x]$.)

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Category I

Problem 1

(a) Is there a polynomial $P(x)$ with real coefficients such that

$$P\left(\frac{1}{k}\right) = \frac{k+2}{k},$$

for all positive integers k?

(b) Is there a polynomial $P(x)$ with real coefficients such that

$$P\left(\frac{1}{k}\right) = \frac{1}{2k+1},$$

for all positive integers k?

Solution (a) YES. It suffices to define a polynomial $W(x)$ as follows

$$W(x) = 2x + 1.$$

(b) NO. Suppose that such a polynomial $W(x)$ exists. Define a polynomial $F(x)$ as follows

$$F(x) = (x+2)W(x) - x.$$

Then

$$F\left(\frac{1}{k}\right) = \left(\frac{1}{k} + 2\right)W\left(\frac{1}{k}\right) - \frac{1}{k} = 0,$$

for all $k \in \mathbb{N}$. Hence, the polynomial $F(x)$ admits infinitely many zeros. Consequently,

$$(x+2)W(x) - x = 0,$$

for all $x \in \mathbb{R}$. But this implies that

$$W(x) = \frac{x}{x+2},$$

for all $x \in \mathbb{R}$ – a contradiction. $\square$

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Problem 2 *Let $(a_n)_{n=1}^{\infty}$ be unbounded and strictly increasing sequence of positive reals such that the arithmetic mean of any four consecutive terms a_n, a_{n+1}, a_{n+2}, a_{n+3} belongs to the same sequence. Prove that the sequence a_{n+1}/a_n converges and find all possible values of its limit.*

Solution Since $a_n < a_{n+1} < a_{n+2} < a_{n+3}$, one has

$$a_n < \frac{1}{4}(a_n + a_{n+1} + a_{n+2} + a_{n+3}) < a_{n+3},$$

thus $(a_n + a_{n+1} + a_{n+2} + a_{n+3})/4 \in \{a_{n+1}, a_{n+2}\}$. Hence for any $n \in \mathbb{N}$ precisely one of the two identities

$$a_n + a_{n+1} + a_{n+2} + a_{n+3} = 4a_{n+1} \tag{1}$$

or

$$a_n + a_{n+1} + a_{n+2} + a_{n+3} = 4a_{n+2} \tag{2}$$

holds. Let A be the set of indices $n \in \mathbb{N}$ for which (1) holds and let B be the set of indices $n \in \mathbb{N}$ for which (2) holds. Clearly, $A \cup B = \mathbb{N}$, $A \cap B = \emptyset$. We shall prove that one of A or B is finite. Indeed, suppose the contrary, that both A and B are infinite. Since A and B partition $\mathbb{N}$, there exists a positive integer k, such that $k \in B$, $k+1 \in A$. From (1) and (2), it follows that

$$a_k + a_{k+1} + a_{k+2} + a_{k+3} = 4a_{k+2} \qquad \text{and} \qquad a_{k+1} + a_{k+2} + a_{k+3} + a_{k+4} = 4a_{k+2}.$$

Hence $a_k = a_{k+4}$, which contradicts the fact that a_n is strictly increasing. We now consider two cases.

Case 1) The set A is infinite, the set B is finite. By (1), the sequence a_n satisfies a linear recurrence $a_n - 3a_{n+1} + a_{n+2} + a_{n+3} = 0$ for all $n > n_0$. The characteristic polynomial of the linear recurrence

$$\phi(\lambda) = \lambda^3 + \lambda^2 - 3\lambda + 1 = (\lambda - 1)(\lambda^2 + 2\lambda - 1)$$

has roots $\lambda_1 = 1$, $\lambda_2 = -1 - \sqrt{2}$, $\lambda_3 = -1 + \sqrt{2}$. Hence

$$a_n = C_1 + C_2(-1-\sqrt{2})^n + C_3(-1+\sqrt{2})^n, \qquad C_1, C_2, C_3 \in \mathbb{R}, \qquad n > n_0.$$

Observe that $\lambda_2 < -1$, $0 < \lambda_3 < 1$. If $C_2 \neq 0$, then $\lim_{n\to\infty} |a_n| = \infty$ and a_n alternates in sign for n sufficiently large which contradicts the monotonicity property. If $C_2 = 0$, then the sequence a_n is bounded, which leads to the contradiction again. Thus we reject the case one.

Case 2) The set A is finite, the set B is infinite. By (1), the sequence a_n satisfies a linear recurrence $a_n + a_{n+1} - 3a_{n+2} + a_{n+3} = 0$ for all $n > n_0$. The characteristic polynomial of the linear recurrence

$$\phi(\lambda) = \lambda^3 - 3\lambda^2 + \lambda + 1 = (\lambda - 1)(\lambda^2 - 2\lambda - 1)$$

has roots $\lambda_1 = 1$, $\lambda_2 = 1 - \sqrt{2}$, $\lambda_3 = 1 + \sqrt{2}$. Hence

$$a_n = C_1 + C_2(1-\sqrt{2})^n + C_3(1+\sqrt{2})^n, \qquad C_1, C_2, C_3 \in \mathbb{R}, \qquad n > n_0.$$

Note that $-1 < \lambda_2 < 0$, $\lambda_3 > 1$. If $C_3 \leq 0$, then the sequence a_n is bounded from above. Hence $C_3 > 0$ so $a_n \sim C_3\lambda_3^n$ as $n \to \infty$. The standard limit calculation now shows that b_n converges and has limit value

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda_3 = 1 + \sqrt{2}.$$

□

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Category I

Problem 3 *Prove that*

$$\sum_{k=0}^{\infty} x^k \frac{1+x^{2k+2}}{(1-x^{2k+2})^2} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{(1-x^{k+1})^2}$$

for all $x \in (-1,1)$.

Solution We use the binomial series

$$\frac{1}{(1-u)^2} = \sum_{j=0}^{\infty} (j+1)u^j, \ |u| < 1$$

to get

$$\begin{aligned}\sum_{k=0}^{\infty} x^k \frac{1+x^{2k+2}}{(1-x^{2k+2})^2} &= \sum_{k=0}^{\infty} x^k(1+x^{2k+2}) \sum_{j=0}^{\infty}(j+1)x^{j(2k+2)} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x^k(1+x^{2k+2})(j+1)x^{j(2k+2)} = \\ &= \sum_{j=0}^{\infty}(j+1)x^{2j}\sum_{k=0}^{\infty} x^k(1+x^{2k+2})x^{j2k} = \sum_{j=0}^{\infty}(j+1)x^{2j}\left(\frac{1}{1-x^{2j+1}} + \frac{x^2}{1-x^{2j+3}}\right) = \\ &= \sum_{j=0}^{\infty}\frac{(j+1)x^{2j}}{1-x^{2j+1}} + \sum_{j=1}^{\infty}\frac{jx^{2j}}{1-x^{2j+1}} = \sum_{j=0}^{\infty}\frac{(2j+1)x^{2j}}{1-x^{2j+1}} = -\frac{\mathrm{d}}{\mathrm{d}x}\sum_{j=0}^{\infty}\log(1-x^{2j+1})\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-x)^k}{(1-x^{k+1})^2} &= \sum_{k=0}^{\infty}(-x)^k\sum_{j=0}^{\infty}(j+1)x^{(k+1)j} = \sum_{j=0}^{\infty}(j+1)x^j\sum_{k=0}^{\infty}(-x)^k x^{kj} = \sum_{j=0}^{\infty}\frac{(j+1)x^j}{1+x^{j+1}} = \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{j=0}^{\infty}\log(1+x^{j+1}).\end{aligned}$$

The proposition now follows by logarithmic differentiation of the classical identity

$$\prod_{n=0}^{\infty}\frac{1}{1-x^{2n+1}} = \prod_{n=1}^{\infty}(1+x^n),$$

which can be proved as follows:

$$\prod_{n=1}^{\infty}(1+x^n) = \prod_{n=1}^{\infty}\frac{1-x^{2n}}{1-x^n} = \frac{\prod_{n=1}^{\infty}(1-x^{2n})}{\prod_{n=1}^{\infty}(1-x^n)} = \frac{\prod_{n=1}^{\infty}(1-x^{2n})}{\prod_{n=1}^{\infty}(1-x^{2n})\prod_{n=1}^{\infty}(1-x^{2n-1})} = \prod_{n=1}^{\infty}\frac{1}{1-x^{2n-1}}.$$

$\square$

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Category I

Problem 4 *Let a, b, c be elements of finite order in some group. Prove that if $a^{-1}ba = b^2$, $b^{-2}cb^2 = c^2$ and $c^{-3}ac^3 = a^2$, then $a = b = c = e$, where e is the unit element.*

Solution Let $r(g)$ denote the rank of $g \in G$. Assume that the assertion does not hold. Let p be the smallest prime number dividing $r(a)r(b)r(c)$. Without loss of generality we can assume that $p \mid r(b)$ (if $p \mid r(a)$ or $p \mid r(c)$, then the reasoning is the same). Then there exists k such that $r(b) = pk$. Let $d := b^k$. Then $r(d) = p$.

Lemma *For any $m \in \mathbb{N}$, $a^{-m}da^m = d^{2^m}$.*

Proof First we prove that

$$a^{-1}da = d^2.$$

Indeed, multiplying the equation $a^{-1}ba = b^2$ k-times with itself we get

$$(a^{-1}ba)(a^{-1}ba)\cdots(a^{-1}ba) = b^2b^2\cdots b^2;$$

and hence

$$a^{-1}b^ka = (b^2)^k = (b^k)^2.$$

Now, the assertion of the above lemma follows from the following calculations:

$$d = ad^2a^{-1} = a(ad^2a^{-1})^2a^{-1} = a^2d^{2^2}a^{-2} = a^2(ad^2a^{-1})^{2^2}a^{-2} = a^3d^{2^3}a^{-3} = \cdots = a^md^{2^m}a^{-m}. \tag{1}$$

□

Observe that Fermat's little theorem implies that $2^p \equiv 2 \pmod p$. Consequently,

$$a^{-p}da^p = d^{2^p} = d^2 = a^{-1}da. \tag{2}$$

Since $\gcd(r(a), p-1) = 1$, there exist integers r and s such that

$$r \cdot r(a) + s \cdot (p-1) = 1. \tag{3}$$

From (2) we get

$$a^{-l(p-1)}da^{l(p-1)} = d,$$

for all $l \in \mathbb{Z}$ (see the calculations in (1)). Finally, putting $l := s$, we obtain

$$d = a^{-s(p-1)}da^{s(p-1)} \overset{(3)}{=} a^{rr(a)-1}da^{-rr(a)+1} = a^{-1}da = d^2,$$

which implies that $d = e$, a contradiction. □

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Category II

Problem 1 *Let $n > k$ and let $A_1, \dots, A_k$ be real $n \times n$ matrices of rank $n-1$. Prove that*

$$A_1 \cdot \ldots \cdot A_k \neq 0\,.$$

Solution Consider two linear operators $V \xrightarrow{g} V \xrightarrow{f} V$ of an n-dimensional vector space V. If $\operatorname{Ker}(f) \subset \operatorname{Im}(g)$, then $\dim(\operatorname{Im}(fg)) = \dim(\operatorname{Im}(g)) - \dim(\operatorname{Ker}(f))$. But we have the inequality

$$\dim(\operatorname{Im}(fg)) \geq \dim(\operatorname{Im}(g)) - \dim(\operatorname{Ker}(f))$$

in the general case. Applying the correspondence between linear operators and matrices, we obtain the inequality $\operatorname{rank}(AB) \geq \operatorname{rank} B - (n - \operatorname{rank} A)$ for every two matrices A and B. The inequality $\operatorname{rank}(A_1 \cdot \ldots \cdot A_k) \geq (\operatorname{rank}(A_1) + \ldots + \operatorname{rank}(A_k)) - (k-1)\,n$ can be deduced from the inequality $\operatorname{rank}(AB) \geq \operatorname{rank} A + \operatorname{rank} B - n$ by the simple induction. We obtain the inequality $\operatorname{rank}(A_1 \cdot \ldots \cdot A_k) \geq k\,(n-1) - (k-1)\,n = n-k$ in our case. Thus, if $k < n$ then $\operatorname{rank}(A_1 \cdot \ldots \cdot A_k) \geq 1$ and the product $A_1 \cdot \ldots \cdot A_k$ can not be equal to zero. $\square$

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Category II

Problem 2 *Let k be a positive integer. Compute*

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{1}{n_1 n_2\ldots n_k(n_1+\ldots+n_k+1)}.$$

Solution

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{1}{n_1 n_2\ldots n_k(n_1+\ldots+n_k+1)}=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{1}{n_1 n_2\ldots n_k}\int_0^1 x^{n_1+\ldots+n_k}\,\mathrm{dx}=$$

$$=\int_0^1\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{x^{n_1+\ldots+n_k}}{n_1 n_2\ldots n_k}\,\mathrm{dx}=\int_0^1(-\log(1-x))^k\,\mathrm{dx}=[1-x=\mathrm{e}^{-u}]=\int_0^{\infty}u^k\mathrm{e}^{-u}\,\mathrm{du}=\Gamma(k+1)=k!$$

$\square$

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Category II

Problem 3 *Let p and q be complex polynomials with $\deg p > \deg q$ and let $f(z) = \frac{p(z)}{q(z)}$. Suppose that all roots of p lie inside the unit circle $|z| = 1$ and that all roots of q lie outside the unit circle. Prove that*

$$\max_{|z|=1} |f'(z)| > \frac{\deg p - \deg q}{2} \max_{|z|=1} |f(z)|.$$

Solution Without loss of generality we can assume that the maximum of $|f|$ is attained at 1.

Let $p(z) = a \prod_{k=1}^{n_1} (z - c_k)$ and $q(z) = b \prod_{\ell=1}^{n_2} (z - d_\ell)$ where $n_1 = \deg p$ and $n_2 = \deg q$. Then

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{n_1} \frac{1}{z - c_k} - \sum_{\ell=1}^{n_2} \frac{1}{z - d_\ell}.$$

Since $|c_k| < 1$ and $|d_\ell| > 1$ for all k and ℓ, we have

$$Re \frac{1}{1 - c_k} > \frac{1}{2}$$

and

$$Re \frac{1}{1 - d_k} < \frac{1}{2}.$$

Therefore,

$$\frac{|f'(1)|}{|f(1)|} \geq Re \frac{f'(1)}{f(1)} > n_1 \cdot \frac{1}{2} - n_2 \cdot \frac{1}{2} = \frac{\deg p - \deg q}{2}$$

and

$$\max_{|z|=1} |f'(z)| \geq |f'(1)| = \frac{|f'(1)|}{|f(1)|} \cdot |f(1)| \geq \frac{\deg p - \deg q}{2} \max_{|z|=1} |f(z)|.$$

□

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Category II

Problem 4 *Let $\mathbb{Q}[x]$ denote the vector space over $\mathbb{Q}$ of polynomials with rational coefficients in one variable x. Find all $\mathbb{Q}$-linear maps $\Phi : \mathbb{Q}[x] \to \mathbb{Q}[x]$ such that for any irreducible polynomial $p \in \mathbb{Q}[x]$ the polynomial $\Phi(p)$ is also irreducible.*

(A polynomial $p \in \mathbb{Q}[x]$ is called irreducible if it is non-constant and the equality $p = q_1 q_2$ is impossible for non-constant polynomials $q_1, q_2 \in \mathbb{Q}[x]$.)

Solution

The answer is $\Phi(p(x)) = ap(bx + c)$ for some non-zero rationals a, b and some rational c. It is clear that such operators preserve irreducibility. Let's prove that any irreducibility-preserving operator is of such form. We start with the following

Lemma 1 *Assume that $f, g \in \Pi$ are two polynomials such that for all rational numbers c the polynomial $f + cg$ is irreducible. Then either $g \equiv 0$, or f is non-constant linear polynomial and g is non-zero constant.*

Proof Let $g(x_0) \neq 0$ for some rational x_0. Then for $c = -f(x_0)/g(x_0)$ we have $(f + cg)(x_0) = 0$, so the polynomial $f + cg$ is divisible by $x - x_0$. Hence $f + cg = C(x - x_0)$ for some non-zero rational C. Choose $x_1 \neq x_0$ such that $g(x_1) \neq 0$. Then for $c_1 = -f(x_1)/g(x_1) \neq c$ (since $f(x_1) + cg(x_1) = C(x_1 - x_0) \neq 0$) we have $f + c_1 g = C_1(x - x_1)$. Subtracting we get that $(c_1 - c)g$ is linear, hence g is linear, hence f too. If $f(x) = ax + b$, $g(x) = a_1 x + b_1$, then $a \neq 0$ (since f is irreducible) and if $a_1 \neq 0$, then for $c = -a/a_1$ the polynomial $f + cg$ is constant, hence not irreducible. So $a_1 = 0$ and we are done. □

Now denote $g_k = \Phi(x^k)$.

Lemma 2 *g_0 is non-zero constant and g_1 is non-constant linear function.*

Proof Since $x+c$ is irreducible for any rational c, we get that $g_1 + cg_0$ is irreducible for any rational c. By Lemma 1 we have that either $g_0 = 0$ or g_0 is constant and g_1 is linear non-constant. Assume that $g_0 = 0$. Note that for any rational α one may find rational β such that $x^2 + \alpha x + \beta$ is irreducible, hence $g_2 + \alpha g_1 = \Phi(x^2 + \alpha x + \beta)$ is irreducible for any rational α. It follows by Lemma 1 that g_1 is constant, hence not irreducible. A contradiction, hence $g_0 \neq 0$ and we are done. □

Denote $g_0 = C$, $g_1(x) = Ax + B$. Consider the new operator $p(x) \to C^{-1}\Phi(p(A^{-1}Cx - A^{-1}B))$. This operator of course preserves irreducibility, consider it instead Φ.

Now $g_0 = 1$, $g_1(x) = x$ and our goal is to prove that $g_n = x^n$ for all positive integers n. We use induction by n. Assume that $n \geq 2$ and $g_k(x) = x^k$ is already proved for $k = 0, 1, \ldots, n-1$. Denote $h(x) = g_n(x) - x^n$ and assume that h is not identical 0. For arbitrary monic irreducible polynomial f of degree n we have $\Phi(f) = f + h$, hence $f + h$ is irreducible aswell. Choose rational x_0 such that $h(x_0) \neq 0$, our goal is to find irreducible f such that $f(x_0) = -h(x_0)$ and hence $f + h$ has a root in x_0.

There are many ways to do it, consider one of them, via Eisenstein's criterion. Recall it.

Eisenstein's criterion *Assume that $f(x) = a_n x^n + \cdots + a_0$ is a polynomial with rational coefficients and p is a prime number so that $a_k = b_k/c_k$ with coprime integers b_k, c_k such that b_k is divisible by p for $k = 0, 1, \ldots, n-1$, both b_n and c_n are not divisible by p and b_0 is not divisible by p^2. Then f is irreducible.*

Without loss of generality, $x_0 = 0$ (else denote $x - x_0$ by new variable). Then we want to find an irreducible polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x - h(0)$. Denote $-h(0) = u/v$ for coprime positive integer v and non-zero integer u. Take $L = 6uv$ and consider the prime divisor p of the number $vL^n/u - 1$. Clearly, p does not divide $6uvL$. Then consider the polynomial $(x + L)^n - L^n + u/v$. If $vL^n/u - 1$ is not divisible by p^2, then we are done by Eisenstein's criterion (with new variable $y = x + L$). If $vL^n/u - 1$ is divisible by p^2, then add px to our polynomial and now Eisenstein's criterion works.

Unless $h(x) = -x^n + \ldots$, the polynomial $f + h$ is not linear and so is not irreducible. If $n \geq 3$, then we may add px^2 or $2px^2$ to our polynomial f and get non-linear $f + h$ (but still irreducible f). Finally, if $n = 2$, and $h(x) = -x^2 + ax + b$, then choose irreducible polynomial of the form $f(x) = x^2 - ax + c$ and get $f + h$ being constant (hence not irreducible).

The induction step and the whole proof are finished. □

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Category I

Problem 1 *Let $f\colon [0,1] \to [0,1]$ be a differentiable function such that $|f'(x)| \neq 1$ for all $x \in [0,1]$. Prove that there exist unique points $\alpha, \beta \in [0,1]$ such that $f(\alpha) = \alpha$ and $f(\beta) = 1 - \beta$.*

Problem 2 *Determine all 2×2 integer matrices A having the following properties:*

1. *the entries of A are (positive) prime numbers,*
2. *there exists a 2×2 integer matrix B such that $A = B^2$ and the determinant of B is the square of a prime number.*

Problem 3 *Determine the smallest real number C such that the inequality*

$$\frac{x}{\sqrt{yz}} \cdot \frac{1}{x+1} + \frac{y}{\sqrt{zx}} \cdot \frac{1}{y+1} + \frac{z}{\sqrt{xy}} \cdot \frac{1}{z+1} \leq C$$

holds for all positive real numbers x, y and z with

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1\,.$$

Problem 4 *Find all positive integers n for which there exists a positive integer k such that the decimal representation of n^k starts and ends with the same digit.*

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Category II

Problem 1 *Let $f\colon [1,\infty) \to (0,\infty)$ be a non-increasing function such that*

$$\limsup_{n\to\infty} \frac{f(2^{n+1})}{f(2^n)} < \frac{1}{2}\,.$$

Prove that

$$\int_1^\infty f(x)\,\mathrm{d}x < \infty\,.$$

Problem 2 *Let M be the (tridiagonal) 10×10 matrix*

$$M = \begin{pmatrix} -1 & 3 & 0 & \cdots & \cdots & \cdots & 0 \\ 3 & 2 & -1 & 0 & & & \vdots \\ 0 & -1 & 2 & -1 & \ddots & & \vdots \\ \vdots & 0 & -1 & 2 & \ddots & 0 & \vdots \\ \vdots & & \ddots & \ddots & \ddots & -1 & 0 \\ \vdots & & & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Show that M has exactly nine positive real eigenvalues (counted with multiplicities).

Problem 3 *Let $(A,+,\cdot)$ be a ring with unity, having the following property: for all $x \in A$ either $x^2 = 1$ or $x^n = 0$ for some $n \in \mathbb{N}$. Show that A is a commutative ring.*

Problem 4 *Let a,b,c,x,y,z,t be positive real numbers with $1 \le x,y,z \le 4$. Prove that*

$$\frac{x}{(2a)^t} + \frac{y}{(2b)^t} + \frac{z}{(2c)^t} \ge \frac{y+z-x}{(b+c)^t} + \frac{z+x-y}{(c+a)^t} + \frac{x+y-z}{(a+b)^t}\,.$$

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Category I

Problem 1 *Let $f\colon [0,1] \to [0,1]$ be a differentiable function such that $|f'(x)| \neq 1$ for all $x \in [0,1]$. Prove that there exist unique points $\alpha, \beta \in [0,1]$ such that $f(\alpha) = \alpha$ and $f(\beta) = 1 - \beta$.*

Solution *Existence:* Since f is derivable in $[0,1]$, then f is continuous in $[0,1]$. Considering the functions $g(x) = f(x) - x$ and $h(x) = f(x) - (1 - x)$ that are continuous in $[0,1]$ and applying Bolzano's theorem we get that exists $\alpha \in [0,1]$ such that $g(\alpha) = 0$ and $\beta \in [0,1]$ with $h(\beta) = 0$. That is, there exist $\alpha, \beta \in [0,1]$ for which $f(\alpha) = \alpha$ and $f(\beta) = 1 - \beta$.

Uniqueness: Suppose that there exist $\alpha, \alpha' \in [0,1]$, $\alpha < \alpha'$) such that $f(\alpha) = \alpha$ and $f(\alpha') = \alpha'$. On account of Lagrange's theorem, there exists $\theta \in (\alpha, \alpha') \subset [0,1]$ such that

$$f'(\theta) = \frac{f(\alpha') - f(\alpha)}{\alpha' - \alpha} = \frac{\alpha' - \alpha}{\alpha' - \alpha} = 1$$

contradiction. Likewise, if we assume that there exist $\beta, \beta' \in [0,1]$, ($\beta < \beta'$) such that $f(\beta) = 1 - \beta$ and $f(\beta') = 1 - \beta'$. On account of Lagrange's theorem, there exists $\theta' \in (\beta, \beta') \subset [0,1]$ such that

$$f'(\theta') = \frac{f(\beta') - f(\beta)}{\beta' - \beta} = \frac{(1 - \beta') - (1 - \beta)}{\beta' - \beta} = -1$$

contradiction. This completes the proof. $\square$

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Category I

Problem 2 *Determine all 2×2 integer matrices A having the following properties:*

1. *the entries of A are (positive) prime numbers,*
2. *there exists a 2×2 integer matrix B such that $A = B^2$ and the determinant of B is the square of a prime number.*

Solution Let

$$A = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} = B^2,$$

and $d = \det(B) = q^2$ with $p_1, p_2, p_3, p_4, q \in \mathbb{P}$; here $\mathbb{P}$ denotes the set of positive prime numbers.

By Cayley-Hamilton Theorem,

$$B^2 = \operatorname{tr}(B)B - \det(B)E,$$

where E is the 2×2 identity matrix. Without loss of generality, we assume that $\operatorname{tr}(B) \geq 0$, otherwise, replace B by $-B$. The equality

$$\operatorname{tr}(B)B = B^2 + dE = A + dE = \begin{pmatrix} p_1 + d & p_2 \\ p_3 & p_4 + d \end{pmatrix}$$

implies that $\operatorname{tr}(B)$ divides the numbers p_2 and p_3. Moreover,

$$(\operatorname{tr}(B))^2 = \operatorname{tr}(\operatorname{tr}(B)B) = p_1 + p_4 + 2d \geq 2 + 2 + 8 = 12 \implies \operatorname{tr}(B) > 3.$$

It follows that

$$\operatorname{tr}(B) = p_2 = p_3, \quad \text{and} \quad B = \frac{1}{\operatorname{tr}(B)} \begin{pmatrix} p_1 + d & p_2 \\ p_3 & p_4 + d \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$$

for some positive integers a and b. Hence,

$$A = B^2 = \begin{pmatrix} a^2 + 1 & a + b \\ a + b & b^2 + 1 \end{pmatrix}.$$

The numbers $a^2 + 1, b^2 + 1, a + b$ cannot all be odd, thus, one of them equals 2. Since $ab = d + 1 = q^2 + 1 \geq 5$ we have $\max(a, b) \geq 3$. Hence, $a + b \geq 3 + 1 > 2$.

Now we assume that $a^2 + 1 \leq b^2 + 1$. Then $a^2 + 1 = 2$ and $a = 1$. Note that $d = ab - 1 = b - 1$ and $0 < p_2 = a+b = b+1 = d+2 = q^2+2$. If $q \neq 3$ then $q^2 \equiv 1 \mod 3 \implies p_2 \equiv 0 \mod 3 \implies p_2 = 3 \implies q^2 = 1$, which is impossible. Hence, $q = 3, b = p_2 - a = 3^2 + 2 - 1 = 10$,

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 10 \end{pmatrix}, \quad \text{and} \quad A = B^2 = \begin{pmatrix} 2 & 11 \\ 11 & 101 \end{pmatrix}.$$

Similarly, if $a^2 + 1 > b^2 + 1$ we obtain the matrix

$$A = \begin{pmatrix} 101 & 11 \\ 11 & 2 \end{pmatrix}.$$

Answer:

$$A = \begin{pmatrix} 2 & 11 \\ 11 & 101 \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 101 & 11 \\ 11 & 2 \end{pmatrix}.$$

$\square$

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Category I

Problem 3 *Determine the smallest real number C such that the inequality*

$$\frac{x}{\sqrt{yz}} \cdot \frac{1}{x+1} + \frac{y}{\sqrt{zx}} \cdot \frac{1}{y+1} + \frac{z}{\sqrt{xy}} \cdot \frac{1}{z+1} \leq C$$

holds for all positive real numbers x, y and z with

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1\,.$$

Solution In what follows we shall deal with the harder version of the problem only.

1. We consider the case $x = y = t$. Then
$$\frac{2}{t+1} + \frac{1}{z+1} = 1$$
that is
$$z = \frac{2}{t-1}.$$
Thus the inequality under consideration becomes
$$\frac{2t}{\sqrt{t \cdot \dfrac{2}{t-1}}} \cdot \frac{1}{t+1} + \frac{\frac{2}{t-1}}{\sqrt{t \cdot t}} \cdot \frac{1}{\frac{2}{t-1}+1} \leq C$$
that is
$$\frac{\sqrt{2} \cdot \sqrt{t} \cdot \sqrt{t-1}}{t+1} + \frac{2}{t(t+1)} \leq C.$$
Letting here $t \to \infty$ leads to $C \geq \sqrt{2}$.

2. We are now going to prove that always
$$\frac{x}{\sqrt{yz}} \cdot \frac{1}{x+1} + \frac{y}{\sqrt{zx}} \cdot \frac{1}{y+1} + \frac{z}{\sqrt{xy}} \cdot \frac{1}{z+1} < \sqrt{2}.$$
In order to achieve this goal we make use of the following transformation
$$a = \frac{1}{x+1}, \quad b = \frac{1}{y+1}, \quad c = \frac{1}{z+1}.$$
Then the three new variables satisfy $a, b, c \in (0; 1)$ and are subject to the condition $a + b + c = 1$. Furthermore
$$x = \frac{1-a}{a}, \quad y = \frac{1-b}{b}, \quad z = \frac{1-c}{c}$$
that is (due to $1 - a = b + c$, etc.)
$$x = \frac{b+c}{a}, \quad y = \frac{c+a}{b}, \quad z = \frac{a+b}{c}$$
yield for the claimed inequality
$$\frac{(a+b)\sqrt{ab}}{\sqrt{(b+c)(c+a)}} + \frac{(b+c)\sqrt{bc}}{\sqrt{(c+a)(a+b)}} + \frac{(c+a)\sqrt{ca}}{\sqrt{(a+b)(b+c)}} < \sqrt{2}.$$
Upon clearing fractions this inequality becomes
$$(a+b)\sqrt{ab(a+b)} + (b+c)\sqrt{bc(b+c)} + (c+a)\sqrt{ca(a+c)} < \sqrt{2(a+b)(b+c)(c+a)}.$$

We smuggle the condition $1 = a + b + c$ into the inequality and get

$$(a+b)\sqrt{ab(a+b)} + (b+c)\sqrt{bc(b+c)} + (c+a)\sqrt{ca(a+c)} < \sqrt{2(a+b)(b+c)(c+a)}(a+b+c).$$

Next, we deal with the right-hand expressions. For them we have

$$\sqrt{(a+b)(b+c)(c+a)} = \sqrt{ab(a+b) + bc(b+c) + ca(c+a) + 2abc}$$

and

$$\sqrt{2}(a+b+c) = \sqrt{2(a+b+c)^2} = \sqrt{(a+b)^2 + (a+c)^2 + (b+c)^2 + 2(ab+bc+ca)}$$

But, employing the Cauchy-Schwarz inequality yields for our inequality

$$(a+b)\sqrt{ab(a+b)} + (b+c)\sqrt{bc(b+c)} + (c+a)\sqrt{ca(a+c)} \leq \sqrt{(a+b)^2 + (b+c)^2 + (c+a)^2} \cdot \sqrt{ab(a+b) + ac(a+c) + bc(b+c)}.$$

This together with the two previously stated equations completes the proof. It is also evident that there cannot exist any triples (a, b, c), and thus also (x, y, z), yielding equality.

$\square$

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Category I

Problem 4 *Find all positive integers n for which there exists a positive integer k such that the decimal representation of n^k starts and ends with the same digit.*

Solution The number n^k ends with zero whenever n is divisible by 10 and starts with nonzero digit. We show that the claim is true for all other n's.

It can be easily shown that all the numbers

$$n, n^5, n^9, \dots, n^{4m+1}, \dots \tag{1}$$

ends with the same digit. In fact, $n^5 - n = n(n-1)(n+1)(n^2+1)$ is even and for each possible reminder of n modulo 5 there is a factor divisible by 5 in this product. Thus $n^5 - n$ is divisible by 10 and in the same fashion we can show this for $n^9 - n^5, n^{13} - n^9, \dots$

Now it suffices to show that for any nonzero digit c there is a number in the sequence (1) which starts with c. For any nonnegative integer m put $d_m = n^{4m+1}/10^l$, where l is the greatest integer for which $10^l \le n^{4m+1}$. Thus $1 \le d_m < 10$ and $\lfloor d_m \rfloor$ is the first digit of n^{4m+1}. Clearly all the d_m's are different, since for $m' > m$ we have

$$\frac{d_{m'}}{d_m} = \frac{n^{4m'+1}/10^{l'}}{n^{4m+1}/10^l} = \frac{n^{4(m'-m)}}{10^{l'-l}} \neq 1$$

(the numerator is not a power of 10 for n not divisible by 10).

The sequence $(d_m)_{m=1}^{\infty}$ has the following property: If $d_{m+i} = d_m \cdot q$, then $d_{m+2i} = d_m \cdot q^2 \cdot 10^{\varepsilon}$, where $\varepsilon \in \{-1, 0, 1\}$. This is true since when

$$d_m = n^{4m+1}/10^l, \quad d_{m+i} = n^{4(m+i)+1}/10^{l'} \quad \text{and} \quad d_{m+2i} = n^{4(m+2i)+1}/10^{l''},$$

we have $q = d_{m+i}/d_m = n^{4i}/10^{l'-l}$ and so

$$d_{m+2i}/d_m = n^{8i}/10^{l''-l} = q^2 \cdot 10^{2l'-l-l''} = q^2 \cdot 10^{\varepsilon}$$

for some integer ε. But $d_m, d_m \cdot q, d_m \cdot q^2 \cdot 10^{\varepsilon} \in [1, 10)$, i. e. $\varepsilon \in \{-1, 0, 1\}$.

Since all the terms of the sequence $(d_m)_{m=1}^{\infty}$ are different and all lie in the interval $[1, 10)$, there have to be two terms d_m and $d_{m'}$ such that $|d_{m'} - d_m| < \frac{1}{10}$. Without loss of generality let $m' > m$. There are two possibilities.

Let $d_{m'} > d_m$. Then we have $d_{m'} < d_m + \frac{1}{10}$. Thus

$$1 < q = d_{m'}/d_m < \frac{d_m + \frac{1}{10}}{d_m} = 1 + \frac{1}{10 d_m} \le 1 + \tfrac{1}{10}.$$

By previous remark $d_m \cdot q^2$ lies in the studied sequence, whenever it lies in the interval $[1, 10)$. Repeating this idea we have the numbers $d_m, d_m \cdot q, d_m \cdot q^2, d_m \cdot q^3, \dots, d_m \cdot q^i$ all lying in the studied sequence and after overrunning the value 10 we have the numbers $d_m \cdot q^{i+1}/10, d_m \cdot q^{i+2}/10, \dots$ in the sequence, and so on. Computing the difference of two consecutive terms in this recurrence we get

$$d_m \cdot q^{j+1} - d_m \cdot q^j = d_m \cdot q^j (q-1) < d_m \cdot q^j \cdot \tfrac{1}{10} < 1 \qquad \text{for } j < i,$$
$$d_m \cdot q^{j+1}/10 - d_m \cdot q^j/10 = d_m \cdot q^j (q-1)/10 < d_m \cdot q^j/10 \cdot \tfrac{1}{10} < 1 \qquad \text{for } j > i$$

and for the first term after overrunning 10 we obtain

$$d_m \cdot q^{i+1}/10 = d_m \cdot q^i/10 \cdot q < 10/10 \cdot (1 + \tfrac{1}{10}) = \tfrac{11}{10} < 2.$$

Since the difference is less then 1 and after overrunning we jump into the interval $[1, 2)$, we must get at least one $d_{m+j(m'-m)}$ in the interval $[c, c+1)$ for every nonzero digit c.

Let $d_{m'} < d_m$. Then we have $d_m < d_{m'} + \frac{1}{10}$. Thus

$$1 < q = d_m/d_{m'} < \frac{d_{m'} + \frac{1}{10}}{d_{m'}} = 1 + \frac{1}{10 d_{m'}} \le 1 + \tfrac{1}{10}.$$

In the very similar way as in the previous case (new terms are generated by dividing instead of multiplying by q) we obtain the new sequence of terms with consecutive differences less then 1 and after underrunning 1 we jump to

$$d_m/q^{i+1} \cdot 10 = d_m/q^i \cdot 10/q > 1 \cdot \frac{10}{1+\frac{1}{10}} = \tfrac{100}{11} > 9.$$

Thus also in this case we must obtain some $d_{m+j(m'-m)}$ in the interval $[c, c+1)$ for every nonzero digit c.

This ends the proof.

Answer. Integers satisfying the given conditions are all integers not divisible by 10. $\square$

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Category II

Problem 1 *Let $f\colon [1,\infty) \to (0,\infty)$ be a non-increasing function such that*

$$\limsup_{n\to\infty} \frac{f(2^{n+1})}{f(2^n)} < \frac{1}{2}\,.$$

Prove that

$$\int_1^\infty f(x)\,\mathrm{d}x < \infty\,.$$

Solution Since

$$\limsup_{n\to\infty} \frac{2^{n+1} f(2^{n+1})}{2^n f(2^n)} < 1,$$

then by ratio test we obtain that the series

$$\sum_{n=1}^{\infty} 2^n f(2^n)$$

converges. Using Cauchy condensation test we obtain that

$$\sum_{n=1}^{\infty} f(n)$$

converges. Now, by integral test for convergence we have

$$\int_1^\infty f(x)\,\mathrm{d}x < \infty.$$

$\square$

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Category II

Problem 2 *Let M be the (tridiagonal) 10×10 matrix*

$$M = \begin{pmatrix} -1 & 3 & 0 & \cdots & \cdots & \cdots & 0 \\ 3 & 2 & -1 & 0 & & & \vdots \\ 0 & -1 & 2 & -1 & \ddots & & \vdots \\ \vdots & 0 & -1 & 2 & \ddots & 0 & \vdots \\ \vdots & & \ddots & \ddots & \ddots & -1 & 0 \\ \vdots & & & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Show that M has exactly nine positive real eigenvalues (counted with multiplicities).

Solution Let $x^T = (0, x_1, ..., x_9)$. Then the direct calculation shows that

$$x^T M x = x_1^2 + (x_2 - x_1)^2 + \cdots + (x_9 - x_8)^2 + x_9^2. \tag{1}$$

Let $\lambda_{\min} := \min\{\lambda \mid \lambda \in \sigma(M)\}$ (recall that if a matrix M is symmetric then $\sigma(M) \subset \mathbb{R}$). Moreover, since M is symmetric, there exists an orthogonal matrix C such that $C^T M C = \operatorname{diag}\{\lambda_{\min}, \lambda_1 ..., \lambda_9\}$. Hence we infer that $y^T(\lambda_{\min} I - M)y \leq 0$ for $y \in \mathbb{R}^{10}$. Let $y^T = (1, -1, 0, ..., 0)$. Then $2\lambda_{\min} \leq y^T M y = -5$. Thus $\lambda_{\min} < 0$.
Let $V_1 = \{(0, x_1, ..., x_9) \mid x_i \in \mathbb{R}\} \subset \mathbb{R}^{10}$. Then, in view of (1), we have

$$y^T M y \geq 0 \tag{2}$$

for any $y \in V_1$ and $y^T M y = 0$ if and only if $y = 0$.

Suppose on the contrary that M admits at least two nonpositive eigenvalues $\lambda_1, \lambda_2 \in \sigma(M)$. Consequently, there exist $y_1, y_2 \in \mathbb{R}^{10}$ such that $y_1 \perp y_2$, $y_1^T y_1 = y_2^T y_2 = 1$ and $My_i = \lambda_i y_i$ $(i = 1, 2)$. Put $V_2 := \operatorname{span}\{y_1, y_2\}$. Then for any $y = \alpha_1 y_1 + \alpha_2 y_2 \in V_2$ one has

$$y^T M y = \alpha_1^2 \cdot \lambda_1 + \alpha_2^2 \cdot \lambda_2 \leq 0. \tag{3}$$

Finally, we obtain that

$$\dim V_1 + \dim V_2 = 9 + 2 = 11 > 10.$$

Therefore $V_1 \cap V_2 \neq \{0\}$. Take $0 \neq y \in V_1 \cap V_2$. Then, in view of (2), $y^T M y > 0$. But (3) implies that $y^T M y \leq 0$ – a contradiction. $\square$

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Problem 3 *Let $(A, +, \cdot)$ be a ring with unity, having the following property: for all $x \in A$ either $x^2 = 1$ or $x^n = 0$ for some $n \in \mathbb{N}$. Show that A is a commutative ring.*

Solution Denote by $U(A)$ the multiplicative group of units of the ring A ($U(A) = \{x \mid x \text{ is invertible}\}$). Note first that $(U(A), \cdot)$ is commutative, because if $x, y \in U(A)$, $(xy)^2 = 1 \Rightarrow xy \cdot xy = 1$, and multiplying by x to the left and by y to the right and using also the fact that $x^2 = 1 = y^2$, we get that

$$xy = yx. \tag{1}$$

We now show that if

$$x \notin U(A) \text{ then } 1 - x \in U(A).$$

Assume, by contradiction, that

$$\exists x \notin U(A) \text{ so } y = 1 - x \notin U(A). \tag{2}$$

By hypothesis,

$$\exists n \text{ and } m \in \mathbb{N} \text{ so } x^n = 0;\ y^m = 0$$

and as

$$xy = x(1-x) = x - x^2 = (1-x)x = yx$$

we get that

$$(x+y)^{n+m} = \sum_{i+j=n+m} C_{n+m}^i x^i y^j = 0,$$

Note that whenever $i + j = n + m$ we have

$$i \geq n \textit{ or } j \geq m \text{ and so } x^i = 0 \text{ or } y^j = 0;$$

So

$$1 = x + y \notin U(A),$$

which is a contradiction; thus (2) is proved.

Commutativity in A follows now from (1) and (2) with a case by case analysis: $x, y \in A$,

1. if $x \in U(A)$, $y \in U(A)$ then $(1) \Rightarrow xy = yx$;
2. if $x \in U(A)$, $y \notin U(A)$ then $(2) \Rightarrow 1 - y \in U(A)$ and from (1) we have $x(1-y) = (1-y)x \Rightarrow xy = yx$;
3. if $x \notin U(A)$, $y \in U(A)$ analogous to the case 2 and
4. if $x \notin U(A)$, $y \notin U(A)$ then $(2) \Rightarrow 1 - x,\ 1 - y \in U(A)$

and using

$$(1-x)(1-y) = (1-y)(1-x) \Leftrightarrow 1 - x - y + xy = 1 - y - x + xy \Leftrightarrow xy = yx.$$

Now cases $1 \to 4$ above show that A is a commutative ring. $\square$

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Problem 4 *Let a, b, c, x, y, z, t be positive real numbers with $1 \le x, y, z \le 4$. Prove that*

$$\frac{x}{(2a)^t} + \frac{y}{(2b)^t} + \frac{z}{(2c)^t} \ge \frac{y+z-x}{(b+c)^t} + \frac{z+x-y}{(c+a)^t} + \frac{x+y-z}{(a+b)^t}.$$

Solution We will use the following variant of Schur's inequality.

Lemma 1 *For arbitrary $A, B, C > 0$,*

$$x(A-B)(A-C) + y(B-A)(B-C) + z(C-A)(C-B) \ge 0.$$

Proof Without loss of generality we can assume $A \le B \le C$. Let $U = B - A$ and $V = C - B$. Then

$$LHS = xU(U+V) - yUV + z(U+V)V \ge U(U+V) - 4UV + (U+V)V = (U-V)^2 \ge 0.$$

□

Lemma 2 *For every $p > 0$,*

$$\frac{1}{p^k} = \frac{1}{\Gamma(k)} \int_0^\infty t^{k-1} e^{-pt} \mathrm{d}t.$$

Proof Substituting $u = pt$,

$$\int_0^\infty t^{k-1} e^{-pt}\, \mathrm{d}t = \frac{1}{p^k} \int_0^\infty u^{k-1} e^{-u}\, \mathrm{d}u = \frac{\Gamma(k)}{p^k}.$$

□

Now, applying Lemma 1 to $A = e^{-at}$, $B = e^{-bt}$, and $C = e^{-ct}$, the statement can be proved as

$$0 \le \int_0^\infty t^{k-1} \Big(x(e^{-at} - e^{-bt})(e^{-at} - e^{-ct}) + y(e^{-bt} - e^{-at})(e^{-bt} - e^{-ct}) + z(e^{-ct} - e^{-at})(e^{-ct} - e^{-bt}) \Big)\, \mathrm{d}t$$
$$= \int_0^\infty t^{k-1} \Big(xe^{-2at} + ye^{-2bt} + ze^{-2ct} - (y+z-x)e^{-(b+c)t} - (z+x-y)e^{-(c+a)t} - (x+y-x)e^{-(a+b)t} \Big)\, \mathrm{d}t$$
$$= \Gamma(k) \left(\frac{x}{(2a)^k} + \frac{y}{(2b)^k} + \frac{z}{(2c)^k} - \frac{y+z-x}{(b+c)^k} - \frac{z+x-y}{(c+a)^k} - \frac{x+y-z}{(a+b)^k} \right).$$

□

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Category I

Problem 1 *Let* $f\colon [0,\infty) \to \mathbb{R}$ *be a differentiable function with* $|f(x)| \leq M$ *and* $f(x)f'(x) \geq \cos x$ *for* $x \in [0,\infty)$, *where* $M > 0$. *Prove that* $f(x)$ *does not have a limit as* $x \to \infty$.

Problem 2 *Let* $A = (a_{ij})$ *and* $B = (b_{ij})$ *be two real* 10×10 *matrices such that* $a_{ij} = b_{ij} + 1$ *for all* i, j *and* $A^3 = 0$. *Prove that* $\det B = 0$.

Problem 3 *Let* S *be a finite set of integers. Prove that there exists a number* c *depending on* S *such that for each non-constant polynomial* f *with integer coefficients the number of integers* k *satisfying* $f(k) \in S$ *does not exceed* $\max(\deg f, c)$.

Problem 4 *Let* n *and* k *be positive integers. Evaluate the following sum*

$$\sum_{j=0}^{k} \binom{k}{j}^2 \binom{n+2k-j}{2k}$$

where $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$.

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Category II

Problem 1 *Let S_n denote the sum of the first n prime numbers. Prove that for any n there exists the square of an integer between S_n and S_{n+1}.*

Problem 2 *An n-dimensional cube is given. Consider all the segments connecting any two different vertices of the cube. How many distinct intersection points do these segments have (excluding the vertices)?*

Problem 3 *Prove that there is no polynomial P with integer coefficients such that $P(\sqrt[3]{5} + \sqrt[3]{25}) = 5 + \sqrt[3]{5}$.*

Problem 4 *Let $\mathcal{F}$ be the set of all continuous functions $f\colon [0,1] \to \mathbb{R}$ with the property*

$$\left| \int_0^x \frac{f(t)}{\sqrt{x-t}} \, \mathrm{d}t \right| \le 1 \quad \textit{for all } x \in (0,1] \, .$$

Compute $\sup\limits_{f \in \mathcal{F}} \left| \int_0^1 f(x) \, \mathrm{d}x \right|$.

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Category I

Problem 1 *Let $f\colon [0,\infty) \to \mathbb{R}$ be a differentiable function with $|f(x)| \leq M$ and $f(x)f'(x) \geq \cos x$ for $x \in [0,\infty)$, where $M > 0$. Prove that $f(x)$ does not have a limit as $x \to \infty$.*

Solution Consider a function $F\colon [0,\infty) \to \mathbb{R}$ given by

$$F(x) := f^2(x) - 2\sin x.$$

Then:

- $|F(x)| \leqslant f^2(x) + 2|\sin x| \leqslant M + 2.$
- $F'(x) = 2f(x)f'(x) - 2\cos x \geqslant 0.$

Hence we infer that F is increasing and bounded. Let

$$x_n = \begin{cases} n\pi & \text{if} \quad n = 2k-1, \\ n\pi + \frac{\pi}{2} & \text{if} \quad n = 2k. \end{cases}$$

Then $(F(x_n))$ is increasing and bounded and hence convergent. Assume on the contrary that $\lim\limits_{x\to\infty} f(x)$ exists. In turn, this implies that $\lim\limits_{n\to\infty} f^2(x_n)$ exists. Thus the sequence $F(x_n) - f^2(x_n)$ is convergent. But

$$F(x_n) - f^2(x_n) = -2\sin(x_n)\,.$$

Consequently we get that the sequence $(\sin(x_n))$ is convergent. This contradicts the fact that $(\sin(x_n))$ is not convergent since

$$\sin(x_n) = \begin{cases} 0 & \text{if} \quad n = 2k-1, \\ 1 & \text{if} \quad n = 2k. \end{cases}$$

$\square$

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Category I

Problem 2 *Let* $A = (a_{ij})$ *and* $B = (b_{ij})$ *be two real* 10×10 *matrices such that* $a_{ij} = b_{ij} + 1$ *for all* i, j *and* $A^3 = 0$. *Prove that* $\det B = 0$.

Solution Let H be the matrix 10×10 consisting of units. Then $A = B + H$. As $A^3 = 0$ then

$$B^3 = (A - H)^3 = A^3 + \text{a sum of 7 matrices of the rank} \leq 1.$$

Therefore $\operatorname{rank} B^3 \leq 7$. Since B is of size 10×10, B is degenerate. $\square$

Problem 3 *Let S be a finite set of integers. Prove that there exists a number c depending on S such that for each non-constant polynomial f with integer coefficients the number of integers k satisfying $f(k) \in S$ does not exceed* $\max(\deg f, c)$.

Solution For each set $T \subseteq \mathbb{Z}$ let $N(f,T)$ denote the number of distinct integers k for which $f(k) \in T$. Suppose that the cardinality of S is at least 2 and suppose for some two elements $s_1 \neq s_2$ of S the equations $f(x) = s_1$ and $f(x) = s_2$ both have integer solutions, say, $x = k_1$ and $x = k_2$, respectively. (Otherwise, we immediately obtain $N(f,S) \leq \deg f$.) Put $d = d(S)$ for the difference between the largest and the smallest elements of S. We claim that then $N(f,S) \leq 4d(S)$.

Indeed, if for some $k \in \mathbb{Z}$ we have $f(k) = s \in S$, where $s \neq s_1$ (and so $k \neq k_1$), then $k - k_1$ divides the integer $f(k) - f(k_1) = s - s_1$. Thus $|k - k_1| \leq |s - s_1| \leq d$. Clearly, there are at most $2d$ of such integers k (since $k \neq k_1$), so $N(f, S \setminus \{s_1\}) \leq 2d$. By the same argument, we must have $N(f, S \setminus \{s_2\}) \leq 2d$. Since S is contained in the union of the sets $S \setminus \{s_1\}$ and $S \setminus \{s_2\}$, we deduce that

$$N(f,S) \leq N(f, S \setminus \{s_1\}) + N(f, S \setminus \{s_2\}) \leq 2d + 2d = 4d.$$

Therefore, $N(f,S) \leq \max(\deg f, 4d(S))$. $\square$

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Category I

Problem 4 *Let n and k be positive integers. Evaluate the following sum*

$$\sum_{j=0}^{k} \binom{k}{j}^2 \binom{n+2k-j}{2k}$$

where $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$.

Solution We show that

$$\sum_{j=0}^{k} \binom{k}{j}^2 \binom{n+2k-j}{2k} = \binom{n+k}{k}^2. \tag{1}$$

Multiplying equation (1) by $\dfrac{(2k)!n!}{(n+k)!k!}$ we get

$$\sum_{j=0}^{k} \binom{k}{j} \frac{k!}{j!(k-j)!} \frac{(n+2k-j)!}{(2k)!(n-j)!} \frac{(2k)!n!}{(n+k)!k!} = \sum_{j=0}^{k} \binom{k}{j} \frac{n!}{j!(n-j)!} \frac{(n+2k-j)!}{(n+k)!(k-j)!}$$
$$= \sum_{j=0}^{k} \binom{k}{j} \binom{n}{j} \binom{n+2k-j}{k-j}. \tag{2}$$

On the right side in the formula (1) after multiplying we obtain

$$\binom{n+k}{k} \frac{(n+k)!}{k!n!} \frac{(2k)!n!}{(n+k)!k!} = \binom{n+k}{k}\binom{2k}{k}.$$

Applying Cauchy identity

$$\binom{m+n}{k} = \sum_{r=0}^{k} \binom{n}{r}\binom{m}{k-r},$$

to formula (2) we have

$$\sum_{j=0}^{k} \binom{k}{j}\binom{n}{j} \sum_{r=0}^{k-j} \binom{n-j}{r}\binom{2k}{k-j-r}. \tag{3}$$

By changing the order of summation in formula (3) putting $s = r + j$ we get

$$\sum_{j=0}^{k} \binom{k}{j}\binom{n}{j} \sum_{s=j}^{k} \binom{n-j}{s-j}\binom{2k}{k-s} =$$
$$\sum_{j=0}^{k} \binom{k}{j}\binom{n}{j} \sum_{s=0}^{k} \binom{n-j}{s-j}\binom{2k}{k-s}. \tag{4}$$

Once again by changing the order of summation in formula (4) it follows

$$\sum_{s=0}^{k} \binom{2k}{k-s} \sum_{j=0}^{s} \binom{k}{j}\binom{n}{j}\binom{n-j}{s-j}.$$

On account of the Cauchy identity we have

$$\binom{2k}{k} \sum_{s=0}^{k} \binom{n}{s}\binom{k}{k-s}.$$

Finally we show that

$$\binom{2k}{k-s}\sum_{j=0}^{s}\binom{k}{j}\binom{n}{j}\binom{n-j}{s-j}=\binom{2k}{k}\binom{n}{s}\binom{k}{k-s}.$$

By applying well-known formula

$$\binom{n}{m}\binom{m}{k}=\binom{n}{k}\binom{n-k}{m-k}.$$

it follows

$$\begin{aligned}\binom{2k}{k-s}\sum_{j=0}^{s}\binom{k}{j}\binom{n}{j}\binom{n-j}{s-j}&=\binom{2k}{k+s}\sum_{j=0}^{s}\binom{k}{j}\binom{n}{s}\binom{s}{j}=\binom{2k}{k+s}\binom{n}{s}\sum_{j=0}^{s}\binom{k}{j}\binom{s}{s-j}\\&=\binom{2k}{k+s}\binom{n}{s}\binom{k+s}{s}=\binom{2k}{k+s}\binom{n}{s}\binom{k+s}{k}=\binom{n}{s}\binom{2k}{k}\binom{2k-k}{k+s-k}\\&=\binom{n}{s}\binom{2k}{k}\binom{k}{s}=\binom{n}{s}\binom{2k}{k}\binom{k}{k-s}.\end{aligned}$$

This completes the proof of Li-en-Szua formula. $\square$

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Category II

Problem 1 *Let S_n denote the sum of the first n prime numbers. Prove that for any n there exists the square of an integer between S_n and S_{n+1}.*

Solution We have
$$\sqrt{x} < m < \sqrt{y} \Rightarrow x < m^2 < y,$$
so if $\sqrt{y} - \sqrt{x} > 1$, there is certainly a square between x and y.
We have
$$\sqrt{y} - \sqrt{x} > 1 \Rightarrow y - x > 1 + 2\sqrt{x},$$
hence it suffices to prove
$$S_{n+1} - S_n > 1 + 2\sqrt{S_n}.$$

For $n = 1, 2, 3, 4$ the assertion can be seen directly. For $n \geq 5$, we use
$$S_n < 1 + 3 + 5 + \ldots + p_n,$$
where the sum contains all odd integers up to p_n. Their sum equals $1/4(1+p_n)^2$, so it follows that $2\sqrt{S_n} < 1+p_n$. As p_{n+2} is at least $p_n + 2$, we get $S_{n+1} - S_n > 1 + 2\sqrt{S_n}$ as desired. $\square$

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Category II

Problem 2 *An n-dimensional cube is given. Consider all the segments connecting any two different vertices of the cube. How many distinct intersection points do these segments have (excluding the vertices)?*

Solution We may think that every vertex of the cube has a view $(\varepsilon_1, \ldots, \varepsilon_n)$ where $\varepsilon_i \in \{0, 1\}$ for $i = 1, 2, \ldots, n$. A cross-point of two segments has a view $(\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in \{0, \frac{1}{2}, 1\}$. For example, if $A = (0, 0, 0, 1, 1)$, $B = (1, 0, 0, 0, 1)$, $C = (1, 0, 0, 1, 1)$, $D = (0, 0, 0, 0, 1)$ then $AB \cap CD = (\frac{1}{2}, 0, 0, \frac{1}{2}, 1)$. However a row containing less than 2 of $\frac{1}{2}$ may be not a cross-point. Therefore, there are exactly $3^n - 2^n - n2^{n-1}$ of cross-points. $\square$

Problem 3 *Prove that there is no polynomial P with integer coefficients such that $P(\sqrt[3]{5}+\sqrt[3]{25})=5+\sqrt[3]{5}$.*

Solution First we prove two lemmas.

Lemma 1. *There is no polynomial $w(x)=ax+b$ with integer coefficients such that $w(\sqrt[3]{5}+\sqrt[3]{25})=5+\sqrt[3]{5}$;*

Proof Assume on the contrary that such a polynomial $w(x)=ax+b$ exists. Since $\sqrt[3]{5}$ and $\sqrt[3]{25}$ are irrational, it follows that $a\neq 0$ and $a\neq 1$. Furthermore, one has

$$\begin{aligned}
&a(\sqrt[3]{5}+\sqrt[3]{25})+b=5+\sqrt[3]{5}\Longrightarrow (a-1)\sqrt[3]{5}+a\sqrt[3]{25}\in\mathbb{Q}\\
&\Longrightarrow \Big((a-1)\sqrt[3]{5}+a\sqrt[3]{25}\Big)^2\in\mathbb{Q}\Longrightarrow (a-1)^2\sqrt[3]{25}+5a^2\sqrt[3]{5}\in\mathbb{Q}\\
&\Longrightarrow \frac{5a^2}{(1-a)}\Big((a-1)\sqrt[3]{5}+a\sqrt[3]{25}\Big)+\Big((a-1)^2\sqrt[3]{25}+5a^2\sqrt[3]{5}\Big)\in\mathbb{Q}\\
&\Longrightarrow \left(\frac{(a-1)^3-5a^3}{(a-1)}\right)\sqrt[3]{25}\in\mathbb{Q}\Longrightarrow \sqrt[3]{25}\in\mathbb{Q},
\end{aligned}$$

which contradicts the fact that $\sqrt[3]{25}\in n\mathbb{Q}$, where $\mathbb{Q}$ and $n\mathbb{Q}$ denote the set of rational and irrational numbers, respectively. This completes the proof of the lemma. □

Lemma 2. *There exists exactly one polynomial $w(x)$ of degree two and rational coefficients such that $w(\sqrt[3]{5}+\sqrt[3]{25})=5+\sqrt[3]{5}$;*

Proof Consider a polynomial $w(x)=ax^2+bx+c$, where $a,b,c\in\mathbb{Q}$. Then

$$w(\sqrt[3]{5}+\sqrt[3]{25})=5+\sqrt[3]{5}\Longleftrightarrow a(\sqrt[3]{5}+\sqrt[3]{25})^2+b(\sqrt[3]{5}+\sqrt[3]{25})+c=5+\sqrt[3]{5}$$

$$\Longleftrightarrow\left\{\begin{array}{rcl} a+b&=&0\\ 5a+b&=&1\\ 10a+c&=&5\end{array}\right.\Longleftrightarrow\left\{\begin{array}{rcr} a&=&1/4\\ b&=&-1/4\\ c&=&10/4\end{array}\right.$$

This implies that there exists only one polynomial $w(x)$ with the required properties, i.e.,

$$w(x)=\frac{1}{4}x^2-\frac{1}{4}x+\frac{10}{4}\text{ and }w(\sqrt[3]{5}+\sqrt[3]{25})=5+\sqrt[3]{5},$$

which completes the proof of the second lemma.

□

Now we are ready to solve the problem. Let $x_0:=\sqrt[3]{5}+\sqrt[3]{25}$. Then

$$x_0^3=(\sqrt[3]{5}+\sqrt[3]{25})^3=5+3\sqrt[3]{5^4}+3\sqrt[3]{5^5}+25=30+15\sqrt[3]{5}+15\sqrt[3]{5}=15x_0+30.$$

We put $Q(x):=x^3-15x-30$. Then $Q(x_0)=0$. Assume on the contrary that such a polynomial $P(x)$ exists. Then there exist two polynomials $R(x)$ and $w(x)$ with integer coefficients such that

$$P(x)=Q(x)R(x)+w(x),$$

where the degree $\deg w(x)$ of $w(x)$ is less than or equal 2. Consequently we obtain

$$5+\sqrt[3]{5}=P(\sqrt[3]{5}+\sqrt[3]{25})=Q(\sqrt[3]{5}+\sqrt[3]{25})R(\sqrt[3]{5}+\sqrt[3]{25})+w(\sqrt[3]{5}+\sqrt[3]{25})=w(\sqrt[3]{5}+\sqrt[3]{25}).$$

From this it follows that there exists a polynomial $w(x)$ of degree less than or equal 2 with integer coefficients such that

$$w(\sqrt[3]{5}+\sqrt[3]{25})=5+\sqrt[3]{5},$$

a contradiction with Lemma 1 and Lemma 2. This completes the solution. □

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Category II

Problem 4 *Let $\mathcal{F}$ be the set of all continuous functions $f\colon [0,1] \to \mathbb{R}$ with the property*

$$\left|\int_0^x \frac{f(t)}{\sqrt{x-t}}\,\mathrm{d}t\right| \le 1 \quad \textit{for all } x \in (0,1]\,.$$

Compute $\sup\limits_{f\in\mathcal{F}} \left|\displaystyle\int_0^1 f(x)\,\mathrm{d}x\right|$.

Solution We will use the following lemma.

Lemma *For every functions $f \in L_1[0,1]$,*

$$\int_0^1 \left(\int_0^x \frac{f(t)\mathrm{d}t}{\sqrt{x-t}}\right)\frac{\mathrm{d}x}{\sqrt{1-x}} = \pi\int_0^1 f.$$

Proof Changing the order of integration then substituting $t = -1 + 2\frac{x-t}{1-t}$,

$$\begin{aligned}\int_0^1 \left(\int_0^x \frac{f(t)\mathrm{d}t}{\sqrt{x-t}}\right)\frac{\mathrm{d}x}{\sqrt{1-x}} &= \int_0^1 f(t)\left(\int_t^1 \frac{\mathrm{d}x}{\sqrt{(x-t)(1-x)}}\right)\mathrm{d}t\\ &= \int_0^1 f(t)\left(\int_{-1}^1 \frac{\mathrm{d}t}{\sqrt{(1+t)(1-t)}}\right)\mathrm{d}t = \pi\int_0^1 f.\end{aligned}$$

□

Now, by Lemma, for all $f \in \mathcal{F} \subset L_1[0,1]$ we have

$$\left|\int_0^1 f\right| \le \frac{1}{\pi}\int_0^1 \left|\int_0^x \frac{f(t)\mathrm{d}t}{\sqrt{x-t}}\right|\frac{\mathrm{d}x}{\sqrt{1-x}} \le \frac{1}{\pi}\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x}} = \frac{2}{\pi}$$

so $\sup_{f\in\mathbb{F}}\left|\int_0^1 f\right| \le \frac{2}{\pi}$.

For the function $g(x) = \dfrac{1}{\pi\sqrt{x}}$ we have

$$\int_0^x \frac{g(t)\mathrm{d}t}{\sqrt{x-t}} = \frac{1}{\pi}\int_0^x \frac{\mathrm{d}t}{\sqrt{t(x-t)}} = 1.$$

Define a sequence $f_1, f_2, \ldots$ of $[0,1] \to \mathbb{R}$ functions as $f_n(x) = \dfrac{1}{\pi\sqrt{x+\frac{1}{n}}}$. Then $f_n \in C[0,1]$ and $0 < f \le g$, so $f_n \in \mathcal{F}$. As $f_n(x) \to g(x)$ pointwise, we have $\int_0^1 f_n \to \int_0^1 g = \frac{2}{\pi}$.

Hence, $\sup_{f\in\mathcal{F}}\left|\int_0^1 f\right| = \frac{2}{\pi}$. □