MIRCEA BECHEANU
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ROMANIAN MATHEMATICAL COMPETITIONS

SOCIETATEA DE ŞTIINŢE MATEMATICE DIN ROMÂNIA

ROMANIAN MATHEMATICAL COMPETITION

MIRCEA BECHEANU, MIHAI BĂLUNĂ, BOGDAN ENESCU

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FOREWORD

Mathematical competitions have a long tradition in Romania. The first mathematical contest was held in 1898, when the Ministry of Public Education organized a national contest for the secondary schools, a part of which was an examination in mathematics.

In 1902 was held the first competition in mathematics by the journal " Gazeta Matematica" (founded in 1895). Since 1904, this competition was organized annualy, except for the years of the first world war and the years 1930-1932, because of an unsuccessful reform of the educational system. The name of the competition was "The Annual Contest Gazeta Matematica", and the competitors were selected from the correspondents of the journal. "The Annual Contest Gazeta Matematica" was scientifically supported by well known mathematicians like Traian Lalescu, Gheorghe Țițeica, Dan Barbilian, Octav Onicescu, etc.

In 1950 the first National Mathematical Olympiad was organized by the Ministry of Education and the Romanian Society for Mathematical Sciences. The olympiads were held each year, becoming a very popular competition in all schools in the country. These olympiads are organized for each grade, in four rounds: school level, city level, region level and national level. This year was held the 47-th olympiad in which approximately 100000 students participated. The final round was organized in Buzău with the 650 students that passed the regional round. One thing that distinguish the Romanian Olympiad is the fact that the problems are proposed acccordingly to the grade of the students and school curriculum. The contest problems are selected by a committee from a set of problems proposed by teachers all around the country. The problems must be original ones and have to respect the curriculum. The olympiad rules are similar to those of the IMO. The problems proposed in the regional and country levels between 1950-1990 were published under the coordination of prof. Ion Tomescu. The problems of each olympiad were published in " Gazeta Matematică".

The Romanian Society for Mathematical Sciences initiated in 1959 the first International Mathematical Olympiad in which eight of the ex-communist countries were invited. The existence of the international olympiads led to the developement of national competitions, problem proposers and contestants being stimulated. Ever since the scientific level of the olympiads raised every year. Furthermore, it had a favorable influence in mathematical teaching.

In this small book we present the problems proposed in the final round of the 47-th Olympiad, problems used in the four selection tests as well as the problems proposed in the 13-th Balkan Mathematic Olympiad, which was held this year in Romania.

Special thanks to GIL Publishing House for helping us to offer you this book.

Mircea Becheanu

THE 47-th NATIONAL MATHEMATICAL OLYMPIAD
BUZĂU 23-28 MARCH 1996

7-th GRADE

Problem I

Find all pairs of real numbers (x, y) such that :

$a) x \geq y \geq 1$

$b) 2x^2 - xy - 5x + y + 4 = 0$

Ştefan Smarandache

Problem II

Find all real numbers x for which the following equality holds :

$$\sqrt{\frac{x-7}{1989}} + \sqrt{\frac{x-6}{1990}} + \sqrt{\frac{x-5}{1991}} = \sqrt{\frac{x-1989}{7}} + \sqrt{\frac{x-1990}{6}} + \sqrt{\frac{x-1991}{5}}$$

Călin Burduşel

Problem III

Let ABCD be a rectangle with AB=1. If $m(B\hat{D}C) = 82°30'$, compute the length of BD and the cosine of $82°30'$.

Constantin Apostol

Problem IV

In the right triangle ABC ($m(\hat{A}) = 90°$) D is the foot of the altitude from A. The bisectors of the angles ABD and ADB intersect in I_1 and the bisectors of the angles ACD and ADC in I_2. Find the angles of the triangle if the sum of distances from I_1 and I_2 to AD is equal with $1/4$ of the length of BC.

Adrian Ghioca

8-th GRADE

Problem I

Let a and b be real numbers such that $a + b = 2$. Show that :

$$min\{|a|,|b|\} < 1 < max\{|a|,|b|\} \Leftrightarrow a,b \in (-3,1).$$

Dan Zaharia

Problem II

Find all the polynomials

$$P(X) = a_n X^n + a_{n-1} X^{n-1} + \ldots + a_1 X + a_0, \quad n \geq 2,$$

with real, non-zero coefficients such that

$P(X) - P_1(X) P_2(X) \ldots P_{n-1}(X)$ is a constant polynomial,

where $P_1(X) = a_1 X + a_0$, $P_2(X) = a_2 X^2 + a_1 X + a_0$,

$\ldots, P_{n-1}(X) = a_{n-1} X^{n-1} + \ldots + a_1 X + a_0$.

Adrian Ghioca, Eugen Păltănea

Problem III

Let N, P be the centers of the faces ABB'A' and ADD'A', respectively, of a right parallelepiped ABCDA'B'C'D' and M∈(A'C) such that $A'M = \frac{1}{3} A'C$. Prove that MN⊥AB' and MP⊥AD' if and only if the parallelepiped is a cube.

Petre Bătrânețu

Problem IV

a) Let ABCD be a regular tetrahedron. On the sides AB, AC and AD, the points M, N and P are considered. Determine the volume of the tetrahedron AMNP in terms of x, y, z, where x=AM, y=AN, z=AP.

b) Show that for any real numbers $x, y, z, t, u, v \in (0,1)$:

$$xyz + uv(1-x) + (1-y)(1-v)t + (1-z)(1-u)(1-t) < 1.$$

Sfetoslav Cremarenco

9-th GRADE

Problem I

Let a, b, $c \in \mathbf{R}$, $a \neq 0$, such that a and $4a+3b+2c$ have the same sign. Show that the equation $ax^2+bx+c=0$ cannot have both roots in the interval (1,2).

Cristinel Mortici

Problem II

Let the real numbers a, b, c, $d \in [0,1]$ and x, y, z, $t \in \left[0, \frac{1}{2}\right]$ such that $a + b + c + d = x + y + z + t = 1$. Show that:

a) $ax + by + cz + dt \geq \min\left\{\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+d}{2}, \frac{d+a}{2}, \frac{a+c}{2}, \frac{b+d}{2}\right\}$

b) $ax + by + cz + dt \geq 54abcd$.

Octavian Purcaru

Problem III

Show that :

$$\cos^7 x + \cos^7\left(x + \frac{2\pi}{3}\right) + \cos^7\left(x + \frac{4\pi}{3}\right) = \frac{63}{64}\cos 3x \ , (\forall)\, x \in \mathbf{R}$$

Radu Dâmboianu, Viorel Drăghici

Problem IV

In the triangle ABC the incircle $\mathfrak{I}$ touches the sides BC, CA, AB in D, E, F, respectively. The segments (BE) and (CF) intersect $\mathfrak{I}$ in G, H. If B and C are fixed points, find the loci of points A, D, E, F, G, H if GH || BC and the loci of the same points if BCHG is an inscriptible quadrilateral.

Dan Brânzei

Problem I

For $n, p \in \mathbf{N}^*$, $1 \le p \le n$, we define

$$R_n^p = \sum_{k=0}^{p} (p-k)^n (-1)^k C_{n+1}^k .$$

Show that : $R_n^{n-p+1} = R_n^p$.

Viorel Drăghici, Radu Dămboianu

Problem II

Let ABCD a tetrahedron and M a variable point on the face BCD. The line perpendicular to (BCD) in M intersects the planes (ABC), (ACD), and (ADB) in M_1, M_2, and M_3. Show that the sum $MM_1 + MM_2 + MM_3$ is constant if and only if the perpendicular dropped from A to (BCD) passes through the centroid of triangle BCD.

Vasile Pop

Problem III

Let P a convex regular polygon with n sides, having the center O and $x\hat{O}y$ an angle of measure α, $\alpha \in (0,\pi)$. Let S be the area of the common part of the interiors of the polygon and the angle. Find, as a function of n, the values of α such that S remains constant when $x\hat{O}y$ is rotating around O.

Adrian Ghioca

Problem IV

Let a, b, c be integers, a even and b odd. Show that for any positive integer n, there exists a positive integer x such that :

$$2^n \mid ax^2 + bx + c .$$

Mircea Becheanu

11-th GRADE

Problem I

Let $I \subset \mathbf{R}$ a non-degenerate interval and $f : I \to \mathbf{R}$ a derivable function.

Let $J = \left\{ \dfrac{f(b) - f(a)}{b - a} \;\middle|\; a,\, b \in I,\, a < b \right\}$.

Show that:

a) J is an interval;

b) $J \subset f'(I)$ and $f'(I) - J$ contains at most two elements;

c) Using a), b) deduce that f' has the Darboux property.

Ioan Raşa

Problem II

a)Let $f_1,\ f_2, \dots,\ f_n : \mathbf{R} \to \mathbf{R}$ periodical functions such that the function $f : \mathbf{R} \to \mathbf{R}$, $f = f_1 + f_2 + \dots + f_n$ has a finite limit at $+\infty$. Show that f is constant;

b) Show that if a_1, a_2, a_3 are real numbers and $a_1 \cos a_1 x + a_2 \cos a_2 x + a_3 \cos a_3 x \geq 0$, $\forall x \in \mathbf{R}$, then $a_1 a_2 a_3 = 0$.

Sorin Rădulescu, Mihai Piticari

Problem III

Let A, B $\in M_2(\mathbf{R})$ such that $\det(AB + BA) \leq 0$. Show that $\det(A^2 + B^2) \geq 0$.

Cristinel Mortici

Problem IV

Let A, B, C, D $\in M_n(C)$, A and C inversible. If $A^k B = C^k D$, $\forall k \in N^*$ show that B = D.

Marius Cavachi

Problem I

Let G be a group in which exactly two elements (different from the unit element) are commuting. Show that G is isomorphic to either $\mathbf{Z}_3$ or S_3.

Marius Gârjoabă

Problem II

Let $f : [a, b] \to \mathbf{R}$ a monotone function such that for any x_1, $x_2 \in [a, b]$, $x_1 < x_2$, there exists $c \in (a, b)$ such that

$$\int_{x_1}^{x_2} f(x)dx = f(c)(x_1 - x_2).$$

a) Show that f is continuous on (a, b);

b) Does the conclusion of a) still hold if f is integrable on $[a, b]$ but is not monotone ?

Marcel Chiriţă, Mihai Piticari

Problem III

Let A be a commutative ring with $0 \neq 1$, having the property that for every $x \in A - \{0\}$ there exist $m, n \in \mathbf{N}^*$ such that $(x^m + 1)^n = x$. Show that every endomorphism of A is an automorphism.

Marian Andronache, Ion Savu

Problem IV

Let $f : [0,1) \to \mathbf{R}$ a monotone function. Prove that the limits $\lim\limits_{\substack{x \to 1 \\ x < 1}} \int_0^x f(t)dt$ and $\lim\limits_{n \to \infty} \frac{1}{n}\left[f(0) + f\left(\frac{1}{n}\right) + \ldots + f\left(\frac{n-1}{n}\right)\right]$ exist and are equal.

Mihai Bălună

FOR THE 37th IMO

BUZAU, March 28, 1996

Problem I

Let n, $n > 2$, be an integer number and $f: \mathbf{R}^2 \to \mathbf{R}$ be a function such that for any regular n-gon, $A_1A_2 \ldots A_n$,

$$f(A_1) + f(A_2) + \ldots + f(A_n) = 0.$$

Prove that f is the zero function.

Gefry Barad

Problem II

Find the greatest positive integer n such that the following proposition is true:

"There exist n non-negative integer numbers $x_1, x_2, \ldots, x_n$, at least one different from zero such that for any system of numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, $\varepsilon_i \in \{-1, 0, 1\}$, at least one different from zero, n^3 does not divide $\varepsilon_1 x_1 + \varepsilon_2 x_2 + \ldots + \varepsilon_n x_n$."

Dorel Miheţ

Problem III

Let x,y be real numbers. Show that if the set

$$A_{x,y} = \{\cos n\pi x + \cos n\pi y \mid n \in \mathbf{N}\}$$

is finite then $x \in \mathbf{Q}$ and $y \in \mathbf{Q}$.

Vasile Pop

Problem IV

Let ABCD be an inscriptible quadrilateral and M be the set of the 4x4=16 centers of all incircles and excircles of the triangles BCD, ACD, ABD and ABC. Show that there exist two sets of parallel lines K and L, each set consisting of four lines, such that any line of $K \cup L$ contains exactly four points of M.

Dan Brânzei

THE SECOND SELECTION EXAMINATION
BUCHAREST, APRIL 23, 1996

Problem I

On a circle ζ with center O two points A,B are given such that OA and OB are perpendicular. The circles ζ_1 and ζ_2 are tangent from the inside to ζ at the points A and B, respectively and also are tangent to each other from the outside. The circle ζ_3 lies in the interior of the angle AOB and is tangent from the inside to ζ in the point C and tangent from the outside to ζ_1 and ζ_2 in the points S and T respectively. Find the angular measure of $\sphericalangle$ SCT.

Czech and Slovak Math. Olympiad

Problem II

A semicircle with center O and diameter AB is given. The line d intersects AB in M and the semicircle in C and D such that MB<MA and MD<MC. The circumcircles of the triangles AOC and DOB intersect second time in the point K. Show that the lines MK and KO are perpendicular.

Russian Olympiad

Problem III

Let $a \in \mathbf{R}$ and $f_1, f_2, \ldots, f_n : \mathbf{R} \to \mathbf{R}$ additive functions such that $f_1(x)f_2(x) \ldots f_n(x) = ax^n$, for all $x \in \mathbf{R}$. Prove that there exist $b \in \mathbf{R}$ and $i \in \{1,2,\ldots,n\}$ such that $f_i(x) = bx$, for all $x \in \mathbf{R}$.

Mihai Piticari and Sorin Radulescu

Problem IV

The sequence $(a_n)_{n\geq 2}$ is defined as follows: if the distinct prime divisors of n are $p_1, p_2, \ldots, p_k$ then $a_n = \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k}$.

Show that for any positive integer N, $N \geq 2$,

$$\sum_{n=2}^{N} a_2 a_3 \ldots a_n < 1.$$

Laurenţiu Panaitopol

THE THIRD SELECTION EXAMINATION FOR THE 37th IMO BUCHAREST, March 28, 1996

Problem I

Let $n\geq 3$ be an integer number and $x_1,x_2,\dots,x_{n-1}$ be positive integers such that

(i) $x_1+\dots+x_{n-1}=n$

(ii) $x_1+2x_2+\dots+(n-1)x_{n-1}=2n-2$.

Find the minimum of the sum:

$$F(x_1,\dots,x_{n-1})=\sum_{k=1}^{n-1} kx_k(2n-k)$$

Ioan Tomescu

Problem II

Let n,r be positive integers and A be a set of laticial points in the plane, such that in any open disc of radius r there exists a point from A. Show that for any coloring of the points from A using n colours, there exist four points which have the same colour and are the vertices of a rectangle.

Vasile Pop

Problem III

Find all prime numbers for which the congruence $\alpha^{3pq}\equiv\alpha \bmod 3pq$ holds for all integers α.

Proposed by Turkey for B.M.O

Problem IV

Let $n\geq 3$ be an integer and $p\geq 2n-3$ be a prime number. let M be a set of n points in the plane such that no three points are colinear and $f:M\to\{0,1,\dots,p-1\}$ be a function such that:

(i) only one point of M has the value 0.

(ii) if the points A,B,C are distinct points of M and C(ABC) is the circumscribed circle of the triangle ABC then

$$\sum_{P\in M\cap C(ABC)} f(P)\equiv 0 \pmod p.$$

Show that all the points of M are on a circle.

Marian Andronache, Ion Savu

THE FOURTH SELECTION EXAMINATION
FOR THE 37th IMO
BUCHAREST, March 28, 1996

Problem I.

Let $x_1, x_2, \ldots, x_n, x_{n+1}$ be positive reals such that $x_1+x_2+\ldots+x_n=x_{n+1}$.

Prove that $\sum_{i=1}^{n}\sqrt{x_i(x_{n+1}-x_i)} \le \sqrt{\sum_{i=1}^{n} x_{n+1}(x_{n+1}-x_i)}$.

Mircea Becheanu

Problem II

Let x,y,z be real numbers. Prove that the following conditions are equivalent:

i) $x>0, y>0, z>0$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\le 1$.

ii) for every quadrilateral with sides a,b,c,d, $a^2x+b^2y+c^2z>d^2$.

Laurenţiu Panaitopol

Problem III

Let $n\in\mathbf{N}^*$ and D be a set of n concentric circles of a plane. Prove that if the function $f: D\to D$ satisfies: $d(f(A),f(B))\ge d(A,B)$ for every $A,B\in D$ then $d(f(A),f(B))=d(A,B)$ for every $A,B\in D$.

Dinu Şerbănescu

Problem IV

Let $n\ge 3$ be an integer and $X\subset\{1,2,3,\ldots,n^3\}$ be a set with $3n^2$ elements. Prove that one can find nine pairwisely distinct numbers a_1,a_2,a_3, b_1,b_2,b_3, c_1,c_2,c_3 from X such that the system

$$a_1x+a_2y+a_3z=0$$
$$b_1x+b_2y+b_3z=0$$
$$c_1x+c_2y+c_3z=0$$

has a solution (x_0,y_0,z_0) with x_0,y_0,z_0 integers and $x_0y_0z_0\ne 0$.

Marius Cavachi

THE 13-th BALKAN MATHEMATICAL OLYMPIAD

BACAU, ROMANIA, APRIL 30, 1996

Problem I

Let O, G be the circumcentre and the barycentre of a triangle ABC, respectively. If R is the circumradius and r is the inradius of ABC, show that

$$OG \leq \sqrt{R(R-2r)}$$

proposed by Greece

Problem II

Let $p>5$ be a prime number and $X = \{ p-n^2 \mid n \in N^* \text{ and } n^2 < p \}$. Prove that X contains two different elements x, y such that $x \neq 1$ and x divides y.

proposed by Albania

Problem III

Let ABCDE be a convex pentagon. Denote by M, N, P, Q, R the midpoints of the segments AB, BC, CD, DE, EA, respectively. If the segments AP, BQ, CR, DM have a common point, prove that this point also belongs to the segment EN.

proposed by Yugoslavia

Problem IV

Show that there exists a subset A of the set $\{ 1, 2, \ldots, 2^{1996}-1 \}$ having the following properties: a) $1 \in A$ and $2^{1996}-1 \in A$; b) every element of A except 1 is the sum of two (not necessarily distinct) elements of A; c) the number of elements of A does not exceed 2012.

proposed by Romania

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THE 47-th NATIONAL MATHEMATICAL OLYMPIAD

BUZĂU 23-28 MARCH 1996

7-th GRADE

Problem I

Find all pairs of real numbers (x, y) such that :

$a) x \geq y \geq 1$

$b) 2x^2 - xy - 5x + y + 4 = 0$

Ştefan Smarandache

Solution

Condition b) is equivalent to: $2x^2 - 5x + 4 + y(1-x) = 0$

From a) we get : $\left.\begin{matrix} x \geq 1 \Leftrightarrow 1 - x \leq 0 \\ x \geq y \end{matrix}\right\} \Rightarrow x(1-x) \leq y(1-x)$

Therefore

$0 = 2x^2 - 5x + 4 + y(1-x) \geq 2x^2 - 5x + 4 + x(1-x) = (x-2)^2$,

it follows : $(x-2)^2 \leq 0 \Leftrightarrow x = 2$; Finally : $x = y = 2$.

Problem II

Find all real numbers x for which the following equality holds :

$$\sqrt{\frac{x-7}{1989}} + \sqrt{\frac{x-6}{1990}} + \sqrt{\frac{x-5}{1991}} = \sqrt{\frac{x-1989}{7}} + \sqrt{\frac{x-1990}{6}} + \sqrt{\frac{x-1991}{5}}$$

Călin Burduşel

Solution

For the existence of the radicals it is necessary that : $x \geq 1991$.
The equality is equivalent to :

$$\left(\sqrt{\frac{x-7}{1989}} - \sqrt{\frac{x-1989}{7}}\right) + \left(\sqrt{\frac{x-6}{1990}} - \sqrt{\frac{x-1990}{6}}\right) + \left(\sqrt{\frac{x-5}{1991}} - \sqrt{\frac{x-1991}{5}}\right) = 0 (*)$$

Let us show that the three numbers that are added have the same sign or are zero. Let $b > a > 0$, with $a+b=1996$. The sign of $\sqrt{\frac{x-a}{b}}-\sqrt{\frac{x-b}{a}}$ (where $x \geq b$) is the same with the sign of : $\frac{x-a}{b}-\frac{x-b}{a}=\frac{(a-b)(x-a-b)}{ab}$, so is the sign of the number $a+b-x=1996-x$.

Therefore :

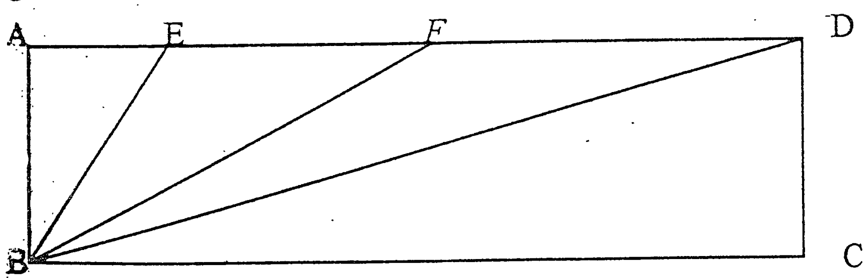
- if $x < 1996$, all numbers are strictly positive and the equality (*) is impossible;
- if $x > 1996$, all numbers are strictly negative and the equality (*) is impossible;
- for $x = 1996$, all the numbers are zero and the equality is true.

Problem III

Let ABCD be a rectangle with AB=1. If $m(B\hat{D}C)=82^\circ 30'$, compute the length of BD and the cosine of $82^\circ 30'$.

Constantin Apostol

Solution



Let $m(B\hat{D}C)=82^\circ 30' \Rightarrow m(A\hat{D}B)=7^\circ 30'$.

We take on (AD) the point F, such that : $m(F\hat{B}D)=m(F\hat{D}B)=7^\circ 30'$.

It results : $m(A\hat{F}B)=2m(A\hat{D}B)=15^\circ$.

We take then on (AD) the point E, such that : $m(E\hat{B}F)=m(E\hat{F}B)=15^\circ$. It results : $m(A\hat{E}B)=2(A\hat{F}B)=30^\circ$.

In right triangle AEB($m(\hat{A})=90^\circ$) we have the theorem of the angle of 30°:

$$\left.\begin{array}{l} BE=2 \\ AE=\sqrt{3} \end{array}\right\} \Rightarrow AF=2+\sqrt{3} \Rightarrow BF=\sqrt{8+4\sqrt{3}}$$

It results : $BD=\sqrt{1+(2+\sqrt{3}+\sqrt{8+4\sqrt{3}})^2}$

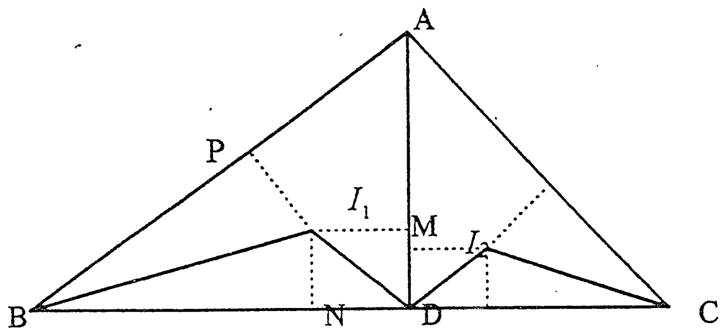
$$\cos 82^\circ 30' = \frac{DC}{BD} \Rightarrow \cos 82^\circ 30' = \frac{1}{\sqrt{1+(2+\sqrt{3}+\sqrt{8+4\sqrt{3}})^2}}.$$

Problem IV

In the right triangle ABC $(m(\hat{A})=90^\circ)$ D is the foot of the altitude from A. The bissectors of the angles ABD and ADB intersect in I_1, and the bissectors of the angles ACD and ADC in I_2. Find the angles of the triangle if the sum of distances from I_1 and I_2 to AD is equal with $\frac{1}{4}$ of the length of BC.

Adrian Ghioca

Solution



Denote by m=BD, n=CD, d_1 the distance from I_1 to AD, d_2 the distance from I_2 to AD, h=AD, the projection of I_1 on AD with M, on BC with N and on AB with P . Clearly BN ≡ BP , AP ≡ AM. It follows that AB=AM+BN therefore $c=h-d_1+m-d_1$. Analogously $b=h-d_2+n-d_2$. Adding then equalities, we get : $2h+(m+n)-2(d_1+d_2)=b+c$. But $h=\frac{bc}{a}$, $m+n=a$, $d_1+d_2=\frac{a}{4}$ therefore the relation above becomes:

$$2\frac{bc}{a}+a-2\frac{a}{4}=b+c \Rightarrow 4bc+a^2=2ab+2ac \Rightarrow 2c(2b-a)-a(2b-a)=0 \Rightarrow$$

$$\Rightarrow (2b-a)(2c-a)=0 \Rightarrow a=2b \text{ or } a=2c \Rightarrow m(\hat{B})=30^0 \text{ and } m(\hat{C})=60^0$$

or $m(\hat{C})=30^0$ and $m(\hat{B})=60^0$

8-th GRADE

Problem I

Let a and b be real numbers such that $a + b = 2$. Show that :

$$min\{|a|,|b|\} < 1 < max\{|a|,|b|\} \Leftrightarrow a,b \in (-3,1).$$

Dan Zaharia

Solution

Let a, $b\in\mathbf{R}$ such that $a+b=2$. We have :

$$\min\{|a|,|b|\} < 1 < \max\{|a|,|b|\} \Leftrightarrow |a| < 1 < |b| \text{ or } |b| < 1 < |a| \Leftrightarrow a^2 < 1 < b^2 \text{ or } b^2 < 1 < a^2 \Leftrightarrow$$

$$\Leftrightarrow (a^2-1)(b^2-1) < 0 \Leftrightarrow a^2b^2 - (a^2+b^2) + 1 < 0 \Leftrightarrow (ab)^2 - (4-2ab) + 1 < 0 \Leftrightarrow$$

$$\Leftrightarrow (ab+1)^2 < 4 \Leftrightarrow |ab+1| < 2 \Leftrightarrow -2 < ab+1 \Leftrightarrow ab \in (-3,1)$$

Problem II

Find all the polynomials

$$P(X) = a_nX^n + a_{n-1}X^{n-1} + ...+a_1X + a_0 \ , \quad n \ge 2 \ ,$$

with real, non-zero coefficients such that

$P(X) - P_1(X)P_2(X).......P_{n-1}(X)$ is a constant polynomial,

where $P_1(X) = a_1X + a_0$, $P_2(X) = a_2X^2 + a_1X + a_0$,

...., $P_{n-1}(X) = a_{n-1}X^{n-1} + ...+a_1X + a_0$.

Adrian Ghioca, Eugen Păltănea

Solution

Since the polynomial $P(X) - P_1(X)P_2(X).......P_{n-1}(X)$ is constant, we have deg P=deg $(P_1P_2.....P_n)$ .But $a_k \neq 0$, $k=\overline{1,n-1}$, therefore $\deg(P_1P_2.....P_n) =$

$$= \deg P_1 + \text{grad } P_2 + ... + \deg P_{n-1} = 1+2+...+(n-1) = \frac{n(n-1)}{2}.$$

We obtain $n = \dfrac{n(n-1)}{2}$, so $n\in\{0,3\}$. Since $n \ge 2$, we conclude that $n = 3$. In this case, $P(X) = a_3X^3 + a_2X^2 + a_1X + a_0$ and $P(X) - P_1(X)P_2(X) = k$, $k\in\mathbf{R}$, if and only if :

(1) $a_3 = a_1a_2$

(2) $a_2 = a_1^2 + a_0a_2$

(3) $a_1 = 2a_0a_1$

(4) $a_0 = a_0^2 + k$

From (3) we get $a_1(1-2a_0)=0$ and since $a_1 \neq 0$, $a_0 = \frac{1}{2}$.

From (4) we get $k = \frac{1}{4}$.

Let $a_1 = a \in \mathbf{R}^*$, from (2) it results $a_2 = a^2 + \frac{1}{2}a_2$, therefore $a_2 = 2a^2$ and then $a_3 = 2a^3$. Finally, the answer is :

$$P(X) = 2a^3X^3 + 2a^2X^2 + aX + \frac{1}{2},\ a \in \mathbf{R}^*$$

Problem III

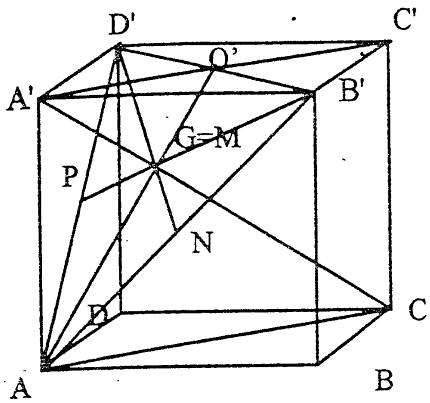
Let N, P be the centers of the faces ABB'A' and ADD'A', respectively, of a right parallelepiped ABCDA'B'C'D' and $M \in (A'C)$ such that $A'M = \frac{1}{3}A'C$. Prove that $MN \perp AB'$ and $MP \perp AD'$ if and only if the parallelepiped is a cube.

Petre Bătrânețu

Solution

Let $AO' \cap A'C = \{G\}$, where O' is the center of A'B'C'D'. In the rectangle ACC'A', since the triangle A'GO' and CGA are similar, one obtains $A'G = \frac{1}{2}GC$ and $A'G = \frac{1}{3}A'C$, therefore G=M. It follows $M \in (AO')$.

Analogously, from the same similarity it follows that $MO' = \frac{1}{3}AO'$. Since [AO'] is a median in triangle AB'D', we get M is the centroid of the triangle AB'D'.



Now $MN \perp AB'$ and $MP \perp AD'$ so the medians [D'N] and [B'P] are also altitudes in the triangle AB'D', therefore is an equilateral triangle. It follows that : $\Delta AA'B \equiv \Delta D'A'B' \equiv \Delta D'A'A$ and $A'A = A'B' = A'D'$, therefore the parallelepiped is a cube. Conversely, if the parallelepiped is a cube then the triangle AB'D' is equilateral and its medians are also altitudes. It follows $MN \perp AB'$ and $MP \perp AD'$.

Problem IV

a) Let ABCD be a regular tetrahedron. On the sides AB, AC and AD, the points M, N and P, are considered. Determine the volume of the tetrahedron AMNP in terms of x, y, z, where x=AM, y=AN, z=AP.

b) Show that for any real numbers $x, y, z, t, u, v \in (0,1)$:
$xyz + uv(1-x) + (1-y)(1-v)t + (1-z)(1-u)(1-t) < 1$.

Sfetoslav Cremarenco

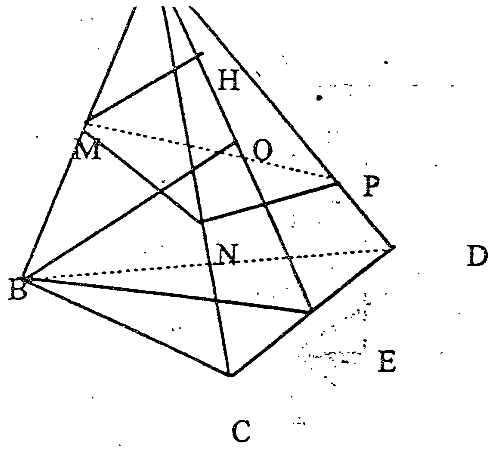
Solution

a)Let E be the midpoint of (CD). It follows $BE \perp CD$, $AE \perp CD$. Since ABCD is a regular tetrahedron, $H = pr_{(ACD)}M \in AE$. If we consider $O = pr_{(ACD)}B \in AE$, it follows MH||BO, then $\Delta AMH \sim \Delta ABO$, and we deduce $\frac{AM}{AB} = \frac{MH}{BO}$, that is $\frac{x}{a} = \frac{MH}{a\sqrt{\frac{2}{3}}}$ (because if AB=a, then

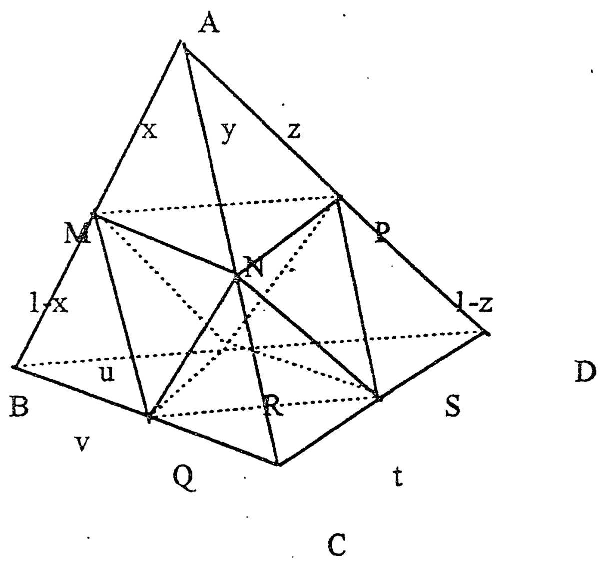
$BO = a\sqrt{\frac{2}{3}}$).

Since $MH = x\sqrt{\frac{2}{3}}$ it results:

$$Vol(AMNP) = \frac{S_{ANP} \cdot MH}{3} = \frac{xyz\sqrt{2}}{12}.$$



b)



b) Let us consider a regular tetrahedron ABCD with side 1 and the points M∈(AB), N∈(AC), P∈(AD), Q∈(BC), R∈(BD), S∈(CD) such that AM=x, AN=y, AP=z, BQ=v, BR=u, and, respectively CS=t where x, y, z, t, v, u∈(0,1). We have :
Vol(AMNP) + Vol(BMQR) + Vol(CNQS) + Vol(DPRS) < Vol(ABCD).
From a) we deduce :

$$\frac{xyz\sqrt{2}}{12}+\frac{(1-x)uv\sqrt{2}}{12}+\frac{(1-v)(1-u)z\sqrt{2}}{12}+\frac{(1-z)(1-t)(1-u)\sqrt{2}}{12}<\frac{\sqrt{2}}{12}$$

which proves the given inequality.

9-th GRADE

Problem I

Let a, b, $c \in \mathbb{R}$, $a \neq 0$, such that a and $4a+3b+2c$ have the same sign. Show that the equation $ax^2+bx+c=0$ cannot have both roots in the interval (1,2).

Cristinel Mortici

Solution

We have

$$0 \leq \frac{4a+3b+2c}{a} = 4+3\frac{b}{a}+2\frac{c}{a} = 2x_1x_2-3(x_1+x_2)+4 =$$

$= (x_1-1)(x_2-2)+(x_1-2)(x_2-1)$. If x_1 and x_2 belong to (1, 2) then each term of the sum would be strictly negative, which is a contradiction.

Problem II

Let the real numbers $a, b, c, d \in [0,1]$ and $x, y, z, t \in \left[0, \frac{1}{2}\right]$ such that $a+b+c+d = x+y+z+t = 1$. Show that:

a) $ax+by+cz+dt \geq \min\left\{\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+d}{2}, \frac{d+a}{2}, \frac{a+c}{2}, \frac{b+d}{2}\right\}$

b) $ax+by+cz+dt \geq 54abcd$.

Octavian Purcaru

Solution

a) Without loss of generality, we can suppose that $a \leq b \leq c \leq d \Rightarrow$

$$\min\left\{\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}, \frac{d+a}{2}, \frac{a+c}{2}, \frac{d+a}{2}\right\} = \frac{a+b}{2}$$

The inequality becomes $E = 2ax+2by+2cz+2dt-a-b \geq 0$.

Since $x = 1-y-z-t$ and $c-a \geq b-a$, $d-a \geq b-a$ and $x, y, z, t \geq 0 \Rightarrow$

$$\Rightarrow E = 2y(b-a)+2z(c-a)+2t(d-a)-(b-a) \geq 2(b-a)\left(y+z+t-\frac{1}{2}\right) \geq 0$$

because $y+z+t-\frac{1}{2} \geq 0$.

b) We have to prove that $\frac{a+b}{2} \geq 54abcd$.

If $a = 0$ the assertion is obvious. Let $0 < a \leq b \leq c \leq d$.

$(1) \Leftrightarrow a+b \geq 108abc(1-a-b-c) \Leftrightarrow (a+b)(1+108abc)+108abc^2-108abc \geq 0 (2)$.

But

$a+b \geq 2\sqrt{ab}$ and $1+108abc \geq 2\sqrt{108abc} \Rightarrow (a+b)(1+108abc) \geq 24ab\sqrt{3c}$

Therefore $(2) \Leftrightarrow 2\sqrt{3c}+9c^2-9c \geq 0$ (3).

Let $\sqrt{3c} = u > 0$. Then (3) becomes

$u^3-3u+2 \geq 0 \Leftrightarrow (u-1)^2(u+2) \geq 0$, obviously.

Problem III

Show that :

$$\cos^7 x+\cos^7\left(x+\frac{2\pi}{3}\right)+\cos^7\left(x+\frac{4\pi}{3}\right)=\frac{63}{64}\cos 3x \ , (\forall)\ x \in \mathbf{R}$$

Radu Dâmboianu, Viorel Drăghici

Solution

First solution:

Let $E_n(x)=\cos^n x+\cos^n\left(x+\frac{2\pi}{3}\right)+\cos^n\left(x+\frac{4\pi}{3}\right)$.

Using $4\cos^3 x = \cos 3x + 3\cos x$ we obtain

$E_n(x)=\frac{\cos 3x}{4}E_{n-3}(x)+\frac{3}{4}E_{n-2}(x)$.

We have $E_0(x)=3$, $E_1(x)=0$, $E_2(x)=\frac{3}{2}$, $E_3(x)=\frac{3}{4}\cos 3x$,

$E_4(x)=\frac{9}{8}$, $E_5(x)=\frac{15}{16}\cos 3x$, $E_7(x)=\frac{63}{64}\cos 3x$.

Second solution:

We have $64\cos^7 x = \cos 7x + 7\cos 5x + 21\cos 3x + 35\cos x$. We replace x with $x+\frac{2\pi}{3}$ and then with $x+\frac{4\pi}{3}$. The equality then follows by adding the relations.

Problem IV

In the triangle ABC the incircle $\Im$ touches the sides BC, CA, AB in D, E, F, respectively. The segments (BE) and (CF) intersect $\Im$ in G, H. If B and C are fixed points, find the loci of points A, D, E, F, G, H if GH || BC and the loci of the same points if BCHG is an inscriptible quadrilateral.

Dan Brânzei

Solution

Let A the point of the locus (see figure 1). From BCHG it follows that ∠FHG≡∠GBC.

Since GH|| BC it follows that ∠GBC≡∠FCB. On the other hand, ∠FHG≡∠FEG. From these equalities we get that ∠FEB≡∠FCB. In conclusion, BCEF is an inscriptible trapezoid, therefore an isosceles one. It results that the triangle ABC is isosceles, with AB=AC, therefore A belongs to the straight line d perpendicular to BC that passes through its midpoint, O. Conversely, for an arbitrary point A∈d-{O}, one obtains GH || BC and the fact that BCHG is inscriptible. Therefore, the locus of A is d-{O}.Now, clearly, D is the midpoint of BC, so D=O, and the locus of D is the point O. Furthermore, BF=BD=$\frac{1}{2}a$, where a=BC, so F belong to a circle centered in B. As $m(\angle ABC)<90^0$, we get that the locus of F is a half-circle located in the halfplane bordered by the perpendicular to BC dropped in B that contains C, excepting the point O. Analogously, the locus of E is the circle $\mathcal{E}$ (see fig. 2).

Let T∈(BC) such that BT=$\frac{1}{3}a$. We have

$$BE^2 = \frac{5a^2}{4} - a^2\cos C$$ (by the cosine law in BEC).

$$BG \cdot BE = \frac{a^2}{4}$$ (the power of B with respect to I).

Therefore $BG = \frac{a^2}{4 \cdot BE}$.

Let α=$m(\angle EBC)$. It results $\cos\alpha = \dfrac{BE^2 + \frac{3}{4}a^2}{2a \cdot BE}$.

Then

$$GT^2 = BG^2 + \frac{a^2}{9} - 2 \cdot BC \cdot \frac{a}{3} \cdot \cos\alpha = \frac{9a^2 + 16a^2 \cdot BE^2 - 12a^2 \cdot BE^2 - 9a^2}{144 \cdot BE^2} = \left(\frac{a}{6}\right)^2$$

so GT= $\frac{a}{6}$=constant. Since T is a fixed point, we deduce that the locus of G is included in the circle $\mathcal{G}\left(T,\frac{1}{6}a\right)$. Since I is in the interior of the circle $\mathcal{L}$ with radius $\frac{1}{2}a$ with the center on d, the locus will be the open arc (QOQ'), except the point O (see the figure). In the same manner, we get that the locus of H is the arc POP', except the point O.

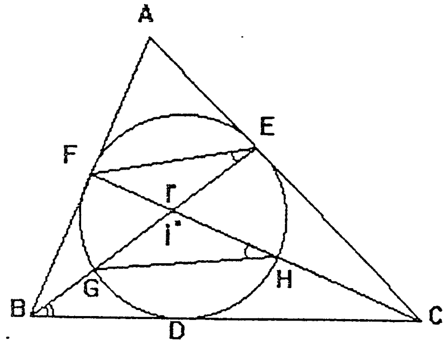


Figura 1

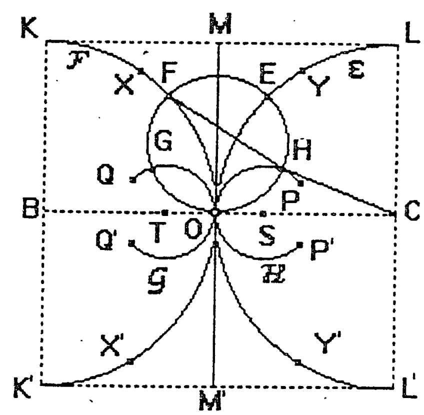


Figura 2

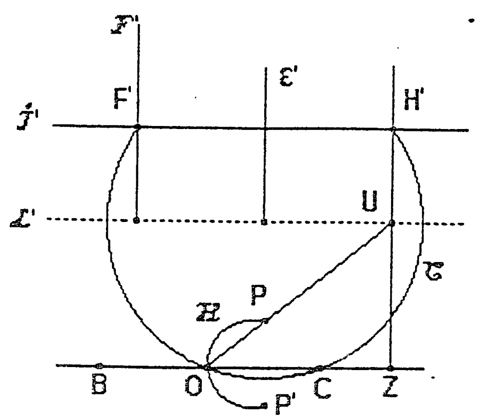


Figura 3

10-th GRADE

Problem I

For $n, p \in \mathbb{N}^*$, $1 \le p \le n$, we define

$$\mathrm{R}_n^p = \sum_{k=0}^{p} (p-k)^n (-1)^k C_{n+1}^k .$$

Show that : $\mathrm{R}_n^{n-p+1} = \mathrm{R}_n^p$.

Viorel Drăghici, Radu Dămboianu

Solution

$$R_n^{n-p+1} = \sum_{k=0}^{n-p+1} (n-p+1-k)^n (-1)^k C_{n+1}^k \overset{n-k+1=i}{=\!=\!=}$$

$$= \sum_{i=p}^{n+1} (i-p)^n (-1)^{n+1-i} C_{n+1}^{n+1-i} = \sum_{i=p}^{n+1} (i-p)^n (-1)^{n+1-i} C_{n+1}^i =$$

$$= -\sum_{i=p}^{n+1} (p-i)^n (-1)^i C_{n+1}^i = -\sum_{k=p}^{n+1} (p-k)^n (-1)^k C_{n+1}^k$$

$$R_n^{n-p+1} = R_n^p \Leftrightarrow$$

$$\Leftrightarrow \sum_{k=0}^{p} (p-k)^n (-1)^k C_{n+1}^k = -\sum_{k=p+1}^{n+1} (p-k)^n (-1)^k C_{n+1}^k \Leftrightarrow$$

$$\Leftrightarrow \sum_{t=0}^{n+1} (p-k)^n (-1)^k C_{n+1}^k = 0 \Leftrightarrow$$

$$\Leftrightarrow C_n^0 p^n \left(\sum_{k=0}^{n+1} (-1)^k C_{n+1}^k \right) - C_n^1 p^{n-1} \left(\sum_{k=0}^{n+1} (-1)^k k \cdot C_{n+1}^k \right) +$$

$$+ C_n^2 p^{n-2} \left(\sum_{k=0}^{n+1} (-1)^k k^2 C_{n+1}^k \right) - \ldots + C_n^n (-1)^n \left(\sum_{k=0}^{n+1} (-1)^k k^n C_{n+1}^k \right) = 0$$

We will show that the sums $\sum_{k=0}^{n+1} (-1)^k k^s C_{n+1}^k = 0, s = 0.\ldots.n$

We will prove that inductively on n :

For n=0 we have $\sum_{k=0}^{1} (-1)^k C_1^k = 0$ (where s=0)

Suppose now $\sum_{k=0}^{n+1}(-1)^k k^s C_{n+1}^k = 0,\ (\forall)\, s = 0\ldots n,$ (1)

We will show that $\sum_{k=0}^{n+2}(-1)^k k^s C_{n+2}^k = 0,\ (\forall)\, s = 0\ldots n+1.$

We have $\sum_{k=0}^{n+2}(-1)^k k^s C_{n+2}^k = \sum_{k=1}^{n+2}(-1)^k k^s \frac{n+2}{k}\cdot C_{n+1}^{k-1} =$

$= (n+2)\sum_{k=0}^{n+1}(-1)^{k+1}(k+1)^{s-1} C_{n+1}^k =$

$= -(n+2)\sum_{k=0}^{n+1}(-1)^k C_{n+1}^k\left(\sum_{i=0}^{s-1} C_{s-1}^i k^i\right) =$

$= -(n+2)\sum_{i=0}^{s-1} C_{s-1}^i\left(\sum_{k=0}^{n+1}(-1)^k k^i C_{n+1}^k\right) \overset{(1)}{=} 0,\ (\forall)\, s = 1\ldots n+1$

For s=0 obviously $\sum_{k=0}^{n+1}(-1)^k C_{n+1}^k = 0$, which ends the proof.

Problem II

Let ABCD a tetrahedron and M a variable point on the face BCD. The line perpendicular to (BCD) in M intersects the planes (ABC), (ACD), and (ADB) in M_1, M_2, and M_3. Show that the sum $MM_1 + MM_2 + MM_3$ is constant if and only if the perpendicular dropped from A to (BCD) passes through the centroid of triangle BCD.

Vasile Pop

Solution

Let α_1, α_2, α_3 the measures of the angles between the lateral faces and the base (BCD); O the projection of A on the base; h the length of the altitude from A; y_1, y_2, y_3 the distances from O to the sides of the base and x_1, x_2, x_3 the distances from M to the same sides.

We have $MM_i = x_i \cdot tg\alpha_i$, i=1, 2, 3 and $tg\alpha_i = \frac{h}{y_i}$, i=1, 2, 3

$\Rightarrow MM_1 + MM_2 + MM_3 = k \Leftrightarrow h\left(\frac{x_1}{y_1} + \frac{x_2}{y_2} + \frac{x_3}{y_3}\right) = k.$ (1)

For M=D we have $x_2 = x_3 = 0$ and $x_1 = \frac{2\sigma(\text{BCD})}{\text{BC}} = \frac{2\text{S}}{\text{BC}}$

and from (1) we obtain $k = h\frac{2\text{S}}{\text{BC}\cdot y_1} \Leftrightarrow \frac{\sigma(\text{OBC})}{\sigma(\text{BCD})} = \frac{h}{k}$.

Analogously for M=B and M=C we obtain $\sigma(\text{OCD}) = \sigma(\text{ODB}) = \sigma(\text{OBC}) = \frac{h}{k}\sigma(\text{BCD}) \Rightarrow$ O=G and k=3.

Conversely, if O=G then $y_1 = \frac{1}{3}\cdot\frac{2\text{S}}{\text{BC}}, y_2 = \frac{1}{3}\cdot\frac{2\text{S}}{\text{CD}}, y_3 = \frac{1}{3}\cdot\frac{2\text{S}}{\text{BD}}$ and

$$\text{M}_1\text{M} + \text{M}_2\text{M} + \text{M}_3\text{M} = \frac{3h}{2\text{S}}(x_1\cdot\text{BC} + x_2\cdot\text{CD} + x_3\cdot\text{DB}) =$$

$$= \frac{3h}{2\sigma}\cdot 2\sigma = 3h = \text{constant}$$

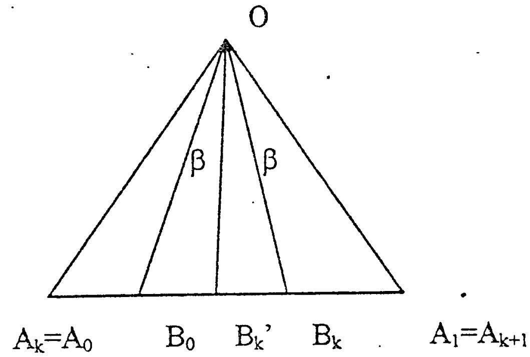
Problem III

Let P a convex regular polygon with n sides, having the center O and $x\hat{O}y$ an angle of measure α, $\alpha \in (0,\pi)$. Let S be the area of the common part of the interiors of the polygon and the angle. Find, as a function of n, the values of α such that S remains constant when $x\hat{O}y$ is rotating around O.

Adrian Ghioca

Solution

It is easy to check $\alpha = \frac{2k\pi}{n}$, $k = 1,\ldots,\left[\frac{n}{2}\right]$ satisfy the condition of the problem. We will prove that these are the only ones. Let $\alpha \notin \left\{\frac{2k\pi}{n} | k = 1,\ldots,\left[\frac{n}{2}\right]\right\}$. With the notations used in the figure and $\text{B}_k \neq \text{A}_{k+1}$ we have $x_0 \neq x_k$ (otherwise it results $\alpha = \frac{2k\pi}{n}$, $k = 1,\ldots,\left[\frac{n}{2}\right]$). Let $\beta = \mu(A_0OB_0) < \mu(A_kOB_k)$. After a rotation of angle β, $(\text{OB}_0$ coincides with $(\text{OA}_0$; $(\text{OB}_k$ becomes $(\text{OB}_k'$, $\text{B}_k' \in [\text{A}_k\text{A}_{k+1}]$. Overlapping triangles OA_0A and $\text{OA}_k\text{A}_{k+1}$, we obtain the configuration bellow in which the condition of the problem leads to $\sigma\lfloor OA_0B_0\rfloor=\sigma\lfloor OB_k'B_k\rfloor$, impossible since triangle A_0OA_1 is isosceles.



Problem IV

Let *a, b, c* be integers, *a* even and *b* odd. Show that for any positive integer *n*, exists a positive integer *x* such that :

$$2^n \mid ax^2 + bx + c .$$

Mircea Becheanu

Solution

We will prove the assertion by induction on $n \in \mathbb{N}$.

For $n=0$, let $x_0 \in \mathbb{N}$. It is clear that $2^0 | ax_0^2+bx_0+c$.

For $n \geq 0$, let $x_n \in \mathbb{N}$ such that $2^n | ax_n^2 + bx_n + c = P(x_n)$. We will choose $x_{n+1} \in \mathbb{N}$ such that $2^{n+1} | P(x_{n+1})$. If $2^{n+1} | P(x_n)$, let $x_{n+1} = x_n$. Otherwise $P(x_n) = 2^n \cdot d$, with $d \in \mathbb{Z}, d$ odd. Now $P(x) - P(x_n) = (x - x_n)(a(x + x_n) + b)$, where $a(x + x_n) + b$ is odd for $x \in \mathbb{N}$.

Let $x_{n+1} = x_n + 2^n \cdot f$, with $f \in \mathbb{N}$, odd

$\Rightarrow P(x_{n+1}) = P(x_n) + 2^n \cdot f(a(x_{n+1} + x_n) + b) =$

$= 2^n(d + f(a(x_{n+1} + x_n) + b)) \vdots 2^{n+1}$

Since $d + f(a(x_{n+1} + x_n) + b)$ is even, which ends the proof.

11-th GRADE

Problem I

Let $I \subset \mathbf{R}$ a non-degenerate interval and $f: I \to \mathbf{R}$ a derivable function.

Let $J = \left\{ \frac{f(b)-f(a)}{b-a} \,\middle|\, a,\ b \in I,\ a < b \right\}$.

Show that:

a) J is an interval;

b) $J \subset f'(I)$ and $f'(I)$ - J contains at most two elements;

c) Using a), b) deduce that f' has the Darboux property.

Ioan Raşa

Solution

a) We will use that J is an interval $\Leftrightarrow (\forall a,b \in J, a < b \Rightarrow (a,b) \subset J)$.

Let us consider

$u, v \in J,\ u < v,\ u = \frac{f(b_1)-f(a_1)}{b_1 - a_1},\ v = \frac{f(b_2)-f(a_2)}{b_2 - a_2}$ and

$p \in (u, v)$. Let $\varphi(t) = \frac{f(tb_1 + (1-t)b_2) - f(ta_1 + (1-t)a_2)}{t(b_1 - a_1) + (1-t)(b_2 - a_2)}$. Since φ is continuos, $\varphi(1) = u$ and $\varphi(0) = v$ we deduce that there exists t_0 such

$\varphi(t_0) = p$, so $p = \frac{f(b_0)-f(a_0)}{b_0 - a_0} \in J$, where

$b_0 = t_0 b_1 + (1-t)b_2,\ \ a_0 = t_0 a_1 + (1-t_0)a_2$.

b) From Lagrange's theorem, $\frac{f(b)-f(a)}{b-a}$ can be written as $f'(c)$, for some $c \in (a,b)$, so $J \subset f'(I)$.

If $x_0 = f'(c) \in f'(I)$ then $x_0 = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$, therefore x_0 is the limit of a sequence of points from J, which shows that only the endpoints of J can be in f'(I)-J.

c) It easy to see, from the results above, that $f'(I_1)$ is an interval for any $I_1 \subset I$, therefore f' has the Darboux property.

Problem II

a)Let $f_1, f_2, ..., f_n$: $\mathbf{R}\to\mathbf{R}$ periodical functions such that th function $f:\mathbf{R}\to\mathbf{R}$, $f=f_1+f_2+...+f_n$ has a finite limit at $+\infty$. Show that f is constant;

b) Show that if a_1, a_2, a_3 are real numbers an $a_1\cos a_1x+a_2\cos a_2x+a_3\cos a_3x\geq 0$, $\forall x\in\mathbf{R}$, then $a_1a_2a_3=0$.

Sorin Rădulescu, Mihai Piticai

Solution

a) We will prove the assertion inductively. For n=1 l f:$\mathbf{R}\to\mathbf{R}$, having the period T>0, and $\lim_{x\to\infty}f(x)=L\in\mathbf{R}$. If f is non constant, we can find $\alpha,\beta\in\mathbf{R}$ such that $f(\alpha)\neq f(\beta)$. But the sequenc $x_n=\alpha+nT$ and $y_n=\beta+nT$ have the limit $+\infty$ therefo $\lim_{n\to\infty}f(x_n)=f(\alpha)=L$, $\lim_{n\to\infty}f(y_n)=f(\beta)=L$, which is a contradiction Suppose the assertion true for n functions. L $f_1, f_2, ..., f_{n+1}:\mathbf{R}\to\mathbf{R}$ periodical functions and $f=f_1+f_2+...+f_{n+}$ with $\lim_{x\to\infty}f(x)=L\in\mathbf{R}$. If $T_1>0$ is a period of f_1, then the function $\mathbf{R}\to\mathbf{R}$, $g(x)=f(x+T_1)-f(x)=$

$=(f_2(x+T_1)-f_2(x))+...+(f_{n+1}(x+T_1)-f_{n+1}(x))$ is the sum of n periodic functions and $\lim_{x\to\infty}g(x)=L-L=0$. It follows that g is constant, $f(x+T_1)=f(x)$, $\forall x\in\mathbf{R}$, that is f is periodical and since $\lim_{x\to\infty}f(x)=L$, f constant.

b) Let $f(x)=a_1\cos a_1x+a_2\cos a_2x+a_3\cos a_3x$, f: $\mathbf{R}\to\mathbf{R}$ and $g(x)=\sin a_1x+\sin a_2x+\sin a_3x$, g:$\mathbf{R}\to\mathbf{R}$. Observe th $g'(x)=f(x)\geq 0,\forall x\in\mathbf{R}$, so g is increasing, and since is obviou bounded, there exists $\lim_{x\to\infty}g(x)=L\in\mathbf{R}$. But g is the sum of periodi functions, therefore constant. Hence $f(x)=0$, $\forall x\in\mathbf{R}$.

From $f(0)=f''(0)=0$ we get $a_1+a_2+a_3=a_1^3+a_2^3+a_3^3=0$.

But

$a_1^3+a_2^3+a_3^3-3a_1a_2a_3=(a_1+a_2+a_3)(a_1^2+a_2^2+a_3^2-a_1a_2-a_2a_3-a_1a_3)$

hence $a_1a_2a_3=0$.

Problem III

Let A, B $\in M_2(\mathbf{R})$ such that det (AB + BA) ≤ 0. Show that $\det(A^2+B^2) \geq 0$.

Cristinel Mortici

First solution

For any two matrices $X, Y \in M_2(\mathbf{R})$ we have :

$$\det(X+Y)+\det(X-Y)=2\det(X)+2\det(Y), \quad (1).$$

Indeed, the function $f: \mathbf{R}\to\mathbf{R}$, $f(t)=\det(X+tY)$ is a polynomial function of degree at most two for which the coefficient of t^2 is $\det(Y)$ and the free term is $\det(X)$. Therefore : $f(t)=\det(Y)\ t^2+at+\det(X)$, for some $a\in\mathbf{R}$.

Then $f(1)+f(-1)=2\det((Y)+2\det(X)$, which is equivalent to

$$\det(X+Y)+\det(X-Y)=2\det(X)+2\det(Y).$$

The equality (1) can be obtained also by a straight forward computation.

Taking in (1) $X=A^2+B^2$, $Y=AB+BA$ using that $A^2+B^2+AB+BA=(A+B)^2$, $A^2+B^2-AB-BA = (A-B)^2$ we obtain that

$$\det((A+B)^2+\det(A-B)^2)=2\det(A^2+B^2)+2\det(AB+BA).$$

Hence

$$\det(A^2+B^2) = \frac{(\det(A+B))^2+(\det(A-B))^2-2\det(AB+BA)}{2} \geq 0$$

Second solution

Let $f:\mathbf{R}\to\mathbf{R}$ $f(t)=\det(A^2+B^2+t(AB+BA))$ be a polynomial function of degree at most two in which the coefficient of t^2 is $\det(AB+BA)\leq 0$.

We shall distinguish two cases.

I) $\det(AB+BA)=0$. Then f is a linear function.

Since $f(-1)=\det(A^2+B^2-AB-BA)=\det((A-B)^2)=(\det(A-B))^2 \geq 0$

$f(1)=\det(A^2+B^2+AB+BA)=\det((A+B)^2)=(\det(A+B))^2 \geq 0$

It follows that $f(0)\geq 0$, hence $\det(A^2+B^2) \geq 0$.

II) $\det(AB+BA)<0$. Then f is a second degree function which is concave. As in the previous case we have $f(\pm 1)\geq 0$ and the concavity of f implies $f(0)\geq 0$, hence $\det(A^2+B^2)\geq 0$.

Problem IV

Let A, B, C, D $\in \mathcal{M}_n(\mathbb{C})$, A and C inversable. If $A^kB = C^kD$, $\forall\, k \in \mathbb{N}^*$ show that B = D.

Marius Cavachi

Solution

Let us consider the matrices I_n , A, A^2 , ..., A^{n^2}. The homogenous system having n^2+1 variables and n^2 equations $x_0I_n+x_1A+....+x_{n^2}A^{n^2}=0$ must have non-zero solutions hence there exists a non-zero polynomial function $f\in\mathbb{C}[X]$ such that $f(A)=0_n$.

Let f, g be the polynomials of minimum degree for which $f(A)= 0_n$ and $g(C)= 0_n$. Then $f(0),g(0)\neq0$. Indeed, if, for instance, $f(0)=0$ then, by multiplying with A^{-1} the equality $f(A)=0_n$ we obtain proof of the existence of a polynomial $f_1\in\mathbb{C}[X]$, $f_1\neq0$, $f_1(A)=0_n$ and $\deg f_1<\deg f$.

Let $h=f\cdot g\in\mathbb{C}[X]$. Then $h(A)=h(C)=0_n$, so $h(A)\cdot B=h(C)\cdot D$. If $h=h_0+h_1X+...+h_SX^S$,it results

$h(A)\cdot B=(h_0I_n+h_1A+...+h_SA^S)B=h_0B+h_1AB+...+h_SA^SB$ and $h(C)\cdot D=h_0D+h_1CD+...+h_SC^SD$. Since $A^kB=C^kD$ for any $k\geq1$ we obtain that $h_0B=h_0D$. But $h_0=h(0)=f(0)\cdot g(0)$, hence $h_0\neq0$, which leads to B=D.

12-th GRADE

Problem I

Let G a group in which exactly two elements (different from the unit element) are commuting. Show that G is isomorphic to either Z_3 or S_3.

Marius Gârjoabă

Solution

From the hypothesis, G contains at least 3 elements.

If all the elements of G have the order at most 2, then G is commutative and contains at least 4 elements, hence it does not verify the hypothesis of the problem.

Therefore G contains elements of order at least 3. Let $a\in G$, with $\mathrm{ord}(a)\geq 3$. If $\mathrm{ord}(a)>3$ then a, a^2, a^3 are distinct and commuting, which contradicts the hypothesis. Hence $\mathrm{ord}(a)=3$. It follows that $a\neq a^2$ and $a\cdot a^2 = a^2\cdot a$, so a and a^2 are the two elements that commute.

Let $H=\{e,a,a^2\}$. If $G=H$, then obviously $G\cong Z_3$. If $G\neq H$, every element of $G\setminus H$ must have the order 2. Let $b\in G\setminus H$.

If $x\in G\setminus H$, then $x\in bH=\{b,ba,ba^2\}$. Indeed, if $bx\in G\setminus H$ then $(bx)^2=e$, and since $b^2=x^2=e$, we obtain $bx=xb$, which is a contradiction. Then $bx\in H$, hence $x\in b^{-1}H= bH$ ($b= b^{-1}$). Therefore $G\setminus H \subset bH$ and, since $G\setminus H \supset bH$, we obtain $G\setminus H=bH$ hence G contains 6 element.

Now, since G is non-commutative, it follows that $G\cong S_3$. (Indeed, let $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \in S_3$ and $\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3$. It is easy to check that $f:G\to S_3$, $f(b^i a^j)=\tau^i\sigma^j$, $(\forall)$ $i\in\overline{0,1}$ şi $j\in\overline{0,2}$ is an isomorphism.)

Problem II

Let $f:[a,b]\to \mathbf{R}$ a monotone function such that for any $x_1, x_2\in[a,b]$, $x_1 < x_2$, there exists $c\in(a,b)$ such that $\int_{x_1}^{x_2} f(x)dx = f(c)(x_1-x_2)$.

a) Show that f is continuous on (a,b);

b) Does the conclusion of a) still hold if f is integrable on $[a, b]$ but is not monotone ?

Marcel Chiriţă, Mihai Piticari

Solution

a)Since the function is monotone on a closed interval, it is integrable hence bounded.

It follows that for every $x_0 \in (a, b)$ the left and right limits of f in x_0 denoted by l_s and l_d exist and are finite.(1)

Let us take $x_1 = x_0 - \frac{1}{n}; x_2 = x_0$.

It results

$$(\exists)\alpha_n \in \left(x_0 - \frac{1}{n}, x_0\right) \text{ a.î. } \int_{x_0-\frac{1}{n}}^{x_0} f(x)dx = f(\alpha_n)\cdot\frac{1}{n} \quad (2)$$

. Analogously, $(\exists)\beta_n \in \left(x_0, x_0 + \frac{1}{n}\right)$ a.î. $\int_{x_0}^{x_0+\frac{1}{n}} f(x)dx = f(\beta_n)\cdot\frac{1}{n}$ (3)

and $(\exists)\delta_n \in \left(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right)$ a.î. $\int_{x_0-\frac{1}{n}}^{x_0+\frac{1}{n}} f(x)dx = f(\delta_n)\cdot\frac{2}{n}$ (4)

Relations (2),(3),(4) lead to:

$f(\alpha_n)+f(\beta_n)=f(\delta_n)$ (5)

If $\delta_n \neq x_0$ holds for an infinite number of values of n, then at least one of the sets $A=\{n\in \mathbf{N} \mid \delta_n<x_0\}$, $B=\{n\in \mathbf{N} \mid \delta_n>x_0\}$ must be infinite.

If, for instance, A is infinite making $n\to\infty$ in (5) we obtain: $l_s+l_d=2l_s \Rightarrow l_s=l_d=f(x_0)$ (the last equality derives from the monotony of f). Therefore the function is continuos in x_0.

If A and B are finite, then $(\exists)\ n_0\in\mathbf{N}$ such that $\delta_n=x_0\ (\forall)n\geq n_0$ and letting $n\to\infty$ in (5) we get the conclusion.

b)We have the following counterexample: $f(x)=\begin{cases}\sin\frac{1}{x}, x\neq 0\\ 0, x=0\end{cases}$

Problem III

Let A be a commutative ring with $0\neq 1$, having the property that for every $x\in A-\{0\}$ there exist $m, n\in \mathbf{N}^*$ such that $(x^m+1)^n = x$. Show that every endomorphism of A is an automorphism.

Marian Andronache, Ion Savu

Let $x\in$A-{0} and $m,n\in\mathbf{N}^*$ such that $(x^m+1)^n=x$. Then $x\left(x^{mn-1}+C_n^1x^{m(n-1)-1}+C_n^{n-1}x^{m-1}-1\right)=-1$, hence x is inversible. Since every element of A-{0} is inversible, it follows that A is a field. Let $u\in$End(A) and a, $b\in$A such that $u(a)=u(b)$. Then $u(a)$-u$(b)=0$, hence $u(a-b)=0$. If $a-b\neq 0$, then $1=u(1)=u[(a-b)(a-b)^{-1}]=u(a-b)u[(a-b)^{-1}]=0$, false. It follows $a=b$ so that u is injective.

For the surjectivity, let us consider $b\in$A-{0} and $m,n\in\mathbf{N}^*$ such that $(b^m+1)^n=b$. Let $f=(X^m+1)^n-X\in A[X]$ and U_f the set of roots of f in A. If $\alpha\in U_f$ then

$$f(u(\alpha))=(u^m(\alpha)+1)^n-u(\alpha)=(u(\alpha^m)+1)^n-u(\alpha)=$$
$$=u^n(\alpha^m+1)-u(\alpha)=u((\alpha^m+1)^n)-u(\alpha)=u((\alpha^m+1)^n-\alpha)=$$
$$=u(0)=0,\text{ hence } u(\alpha)\in U_f$$

As A is a field, U_f is finite. As u is injective and $u(U_f)\subset U_f$, then $u(U_f)\subset U_f$, so there is $a\in U_f$ such that $u(a)=b$.

Problem IV

Let f : $[0,1)\to\mathbf{R}$ a monotone function. Prove that the limits $\lim\limits_{\substack{x\to 1\\x<1}}\int_0^x f(t)dt$ and $\lim\limits_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{1}{n}\right)+\ldots+f\left(\frac{n-1}{n}\right)\right]$ exist and are equal.

Mihai Bălună

Solution

We may assume that f is strictly increasing (otherwise, take $-f$ instead). Also we may assume $f(x)\geq 0$ (if not we replace f by $g(x)=f(x)-f(0)$).

Let F:$[0,1)\to\mathbf{R}$, $F(x)=\int_0^x f(t)dt$

Since $F'(x)=f(x)$ and $f(x)\geq 0$ it follows that F is increasing, hence there exists $\lim\limits_{\substack{x\to 1\\x<1}}F(x)=l$

We distinguish two cases :

Case 1: $l=\infty$. Then :

$$\frac{1}{n}\sum_{k=0}^{n-1}f\left(\frac{k}{n}\right)=\frac{1}{n}f(0)+\frac{1}{n}\sum_{k=1}^{n-1}f\left(\frac{k}{n}\right)\geq\frac{1}{n}f(0)+\int_0^{\frac{n-1}{n}}f(t)dt$$

It follows $\frac{1}{n}\sum_{k=0}^{n-1}f\left(\frac{k}{n}\right)\geq\frac{1}{n}f(0)+F\left(\frac{n-1}{n}\right)$

but $\lim_{n\to\infty} F\left(\frac{n-1}{n}\right) = \infty \Rightarrow \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) = \infty$.

Case 2: $l \in \mathbf{R}$. Then :

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) = \frac{1}{n}\sum_{k=0}^{n-2} f\left(\frac{k}{n}\right) + \frac{1}{n} f\left(\frac{n-1}{n}\right) \le \frac{1}{n} f\left(\frac{n-1}{n}\right) + \int_0^{\frac{n-1}{n}} f(t)dt =$$

$$= F\left(\frac{n-1}{n}\right) + 2\cdot\frac{1}{2n} f\left(\frac{n-1}{n}\right) \le F\left(\frac{n-1}{n}\right) + 2\int_{\frac{n-1}{n}}^{\frac{2n-1}{2n}} f(t)dt$$

It follows $\frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) = F\left(1-\frac{1}{n}\right) + 2\left[F\left(1-\frac{1}{2n}\right) - F\left(1-\frac{1}{n}\right)\right]$

Hence $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) = l + 2l - 2l = l$

THE FIRST SELECTION EXAMINATION
FOR THE 37th IMO
BUZAU, March 28, 1996

Problem I

Let n, $n > 2$, be an integer number and $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that for any regular n-gon, $A_1A_2...A_n$,

$$f(A_1) + f(A_2) + ... + f(A_n) = 0.$$

Prove that f is the zero function.

Gefry Barad

Solution

It is obvious that in this problem we identify a pair (x,y) of real numbers with the corresponding point P(x,y) from the plane. Let P be a point in the plane and let us consider the regular n-gon $PA_1A_2...A_{n-1}$. After a rotation centered in P through the angle $\frac{2k\pi}{n}$, $k=0,1,2,...,n-1$, of the given n-gon, we obtain the regular n-gon $A_{k0}A_{k1}...A_{kn-1}$, where $A_{k0}=P$ and A_{ki} is the point obtained by rotating the point A_i, for all $i=1,2,...,n-1$. Taking into account the hypothesis for each regular n-gon before obtained, we obtain:

$$\sum_{k=0}^{n-1}\sum_{i=0}^{n-1} f(A_{ki}) = 0.$$

In this sum, the number $f(P)$ appears n times and then

$$(1)\ nf(P) + \sum_{k=0}^{n-1}\sum_{i=1}^{n-1} f(A_{ki}) = 0.$$

After a small analyze of the sum it is obvious that:

$$(2)\ \sum_{k=0}^{n-1}\sum_{i=1}^{n-1} f(A_{ki}) = \sum_{i=1}^{n-1}\sum_{k=0}^{n-1} f(A_{ki}) = 0$$

because $A_{0i}A_{1i}...A_{n-1i}$ are all regular n-gons.

From (1) and (2) one gets $f(P)=0$ and then $f=0$.

Problem II

Find the greatest positive integer n such that the following proposition is true:

"There exist n non-negative integer numbers $x_1, x_2, \ldots, x_n$, at least one different from zero such that for any system of numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, $\varepsilon_i \in \{-1,0,1\}$, at least one different from zero, n^3 does not divide $\varepsilon_1 x_1 + \varepsilon_2 x_2 + \ldots + \varepsilon_n x_n$."

Dorel Miheţ

Solution

For n=9, take the numbers $1, 2, 2^2, \ldots, 2^8$. Then for arbitrary $\varepsilon_i \in \{-1,0,1\}$.

$$|\varepsilon_1 + 2\varepsilon_2 + \ldots + 2^8\varepsilon_9| \leq 1 + 2 + \ldots + 2^8 = 2^9 - 1 < 9^3.$$

If $9^3 | (\varepsilon_1 + 2\varepsilon_2 + \ldots + 2^8\varepsilon_9)$, then $\varepsilon_1 + 2\varepsilon_2 + \ldots + 2^8\varepsilon_9 = 0$ and because the sum must be an even number it follows $\varepsilon_1 = 0$. Then we simplify by 2 and by the same argument it follows $\varepsilon_2 = 0$ etc. In this way it is clear that the number 9 satisfies the enounced condition.

Let us suppose $n \geq 10$. It is proved, by mathematical induction that $2^n > n^3$. Let $A = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct non-negative integers and $P(A)$ be the set of all subsets of A. Because $|P(A)| = 2^n$ and $2^n > n^3$, using the pigeonhole principle, it follows that there exist subsets $B \subset A$, $C \subset A$ and $B \neq C$ such that

$$\sum_{x \in B} x \equiv \sum_{x \in C} x \pmod{n^3}.$$

This can be written in the form:

$$\sum_{x \in B} x - \sum_{y \in C} y \equiv 0 \pmod{n^3}.$$

Such a congruence can be interpreted in the following way: there exist numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{-1,0,1\}$, but not all zeros, such that

$$n^3 \Big| \sum_{i=1}^{n} \varepsilon_i x_i.$$

It follows that the desired number is 9.

Problem III

Let x,y be real numbers. Show that if the set

$$A_{x,y} = \{\cos n\pi x + \cos n\pi y \mid n \in \mathbf{N}\}$$

is finite then $x \in \mathbf{Q}$ and $y \in \mathbf{Q}$.

Vasile Pop

Solution

Denote $\alpha_n = \cos n\pi x$ and $\beta_n = \cos n\pi y$. Then:

$$(\alpha_n + \beta_n)^2 + (\alpha_n - \beta_n)^2 = 2(\alpha_n^2 + \beta_n^2) = 2 + (\alpha_{2n} + \beta_{2n}).$$

Hence: $(\alpha_n-\beta_n)^2=2+(\alpha_{2n}+\beta_{2n})-(\alpha_n+\beta_n)^2$
If we suppose that the set $A_{x,y}$ is finite, then the set $B_{x,y}=\{\alpha_n-\beta_n \mid n\in\mathbf{N}\}$ is also finite. From the two equalities:

$$\alpha_n=[(\alpha_n+\beta_n)+(\alpha_n-\beta_n)]/2$$
$$\beta_n=[(\alpha_n+\beta_n)-(\alpha_n-\beta_n)]/2$$

it follows that the sets $A=\{\alpha_n \mid n\in\mathbf{N}\}$ and $B=\{\beta_n \mid n\in\mathbf{N}\}$ are finite sets. It follows that there exist positive integers m, n, $m\neq n$, such that $\alpha_n=\alpha_m$. From the equivalences $\alpha_n=\alpha_m \Leftrightarrow \cos n\pi x=\cos m\pi x \Leftrightarrow n\pi x\pm m\pi x=2k\pi$ where $k\in\mathbf{Z}$, one obtains $x\in\mathbf{Q}$, and similary, $y\in\mathbf{Q}$.

Problem IV

Let ABCD be an inscriptible quadrilateral and M be the set of the 4x4=16 centers of all incircles and excircles of the triangles BCD, ACD, ABD and ABC. Show that there exist two sets of parallel lines K and L, each set consisting of four lines, such that any line of $K\cup L$ contains exactly four points of M.

Dan Brânzei

Author's solution.

We shall use two lemmas:

Lemma A. If L is the midpoint of the arc AB of the circumcircle ABCD and I, I_C are the incenter, respectively the excenter of the triangle ABC, then $LI=LA=LB=LI_C$.(see fig.1)

Proof. From
$\angle LAI=\angle LAB+\angle BAI=(\angle C+\angle A)/2=\angle ICA+\angle IAC=$
$=\angle LIA$ follows LI=LA. In the right triangle AI_CI, $\angle AI_CL=90°-\angle AIL=$
$=90°-\angle LAI=\angle LAI_C$. Then $LA=LI_C$. And finally, of course LA=LB.

Lemma B. The midpoint U of the segment I_BI_C and the midpoint of the arc BAC of circumcircle ABCD coincide.(see fig.2)

Proof. The line I_BI_C bisects the exterior angles of the triangle ABC in the vertex A. Then the midpoint U' belongs to the segment I_BI_C. In the right triangles I_cBI_B and I_cCI_B, the following equalities are valid: $BU= I_BI_C/2=CU$. Hence, U belongs to the midperpendicular of the segment BC. Moreover, U and A are in the same halfplane defined by the line BC because I_C and I_B have this property.

To determine 12 points of the set M, let us consider the midpoints E, F, G, H of the arcs AB, BC, CD, DA respectively, all belonging to the circumcircle ABCD. We shall use the following notations: A', B', C', D' are the incenters of the triangles BCD, CDA, DAB and ABC respectively; A_B, A_C, A_D are the centers of the excircles of the triangle BCD, and so on.

Using the lemma A, it is easy to see that $C'D'C_DD_C$ is a rectangle with the center E, the diagonals of $C'D'C_DD_C$ contain the points C and D and have the length 2EA=2EB. In the same way are obtained the rectangles $D'A'D_AA_D$, $A'B'A_BB_A$ and $B'C'B_CC_B$ having their centers in the points F,G,H respectively.(fig. 3)

It is now necessary to consider the centers of the form X_Y, where X and Y are opposite vertices in the quadrilateral ABCD. We shall prove that $K=\{B_CC_B, C'B', D'A', A_DD_A\}$ and $L=\{C_DD_C, D'C', A'B', B_AA_B\}$. Consider the rectangle $B_CD'B_AM$. From the lemma B, the midpoint K of the diagonal B_CB_A is the midpoint of he arc CDA, hence it belongs to the interior bisector line BK of the triangle ABC. Using once again the lemma A, it follows that the center D_A of the exscribed circle of ABC which is tangent to AB and the point M coincide. Hence, $D_A \in B_CC_B$ ($B_CC_B \in K$) and $D_A \in B_AA_B$ ($B_AA_B \in L$). In the same way can be proved the corresponding properties for the points C_A, B_D and A_C.

Remark 1. If a line l of $K \cup L$ intersects AL and BD in X and Y respectively, then there exists a "Thébault circle" tangent to the circle ABCD and tangent to the lines AC and BD in X and Y respectively. The Thébault's problem was proposed in American Mathematical Monthly, no 9(1938) and was solved in 1983. The given solution was too long to be published. A nice generalization was given by John F. Rigby in Journal of Geometry, 54(1995), pp 134-147.

Remark 2. In the examination, four students have presented complete solutions of this problem. We thank professor Dan Branzei for his kindly permission to publish the solution and history of this problem.

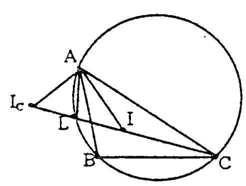


Fig 1

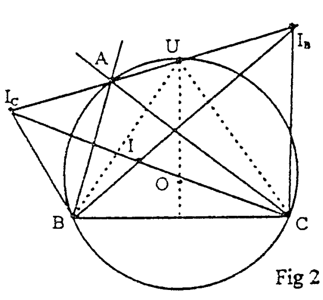


Fig 2

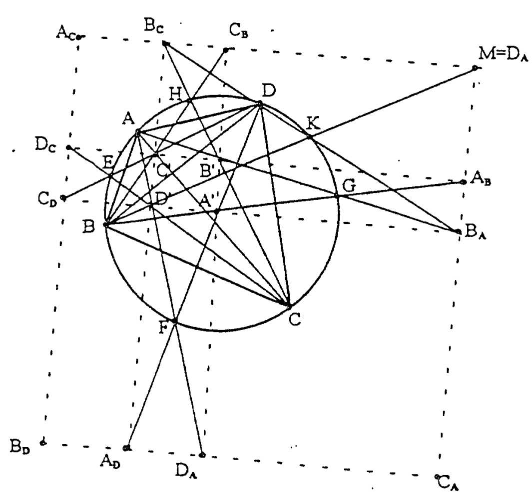


Fig. 3

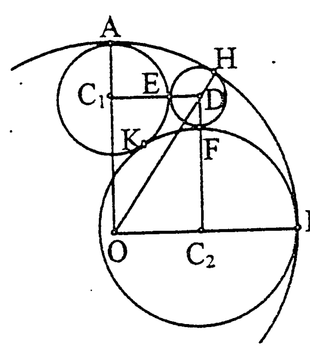
THE SECOND SELECTION EXAMINATION
BUCHAREST, APRIL 23, 1996

Problem 1

On a circle ζ with center O two points A,B are given such that OA and OB are perpendicular. The circles ζ_1 and ζ_2 are tangent from the inside to ζ at the points A and B, respectively and also are tangent to each other from the outside. The circle ζ_3 lies in the interior of the angle AOB and is tangent from the inside to ζ in the point C and tangent from the outside to ζ_1 and ζ_2 in the points S and T respectively. Find the angular measure of $\sphericalangle SCT$.

Czech and Slovak Math. Olympiad

Solution



Let us denote C_1 and C_2 the centers of the circles ζ_1 and ζ_2 respectively and D be the fourth vertex of the rectangle C_2OC_1D. If the circles ζ_1 and ζ_2 touch in K, then the points C_1, K, C_2 are collinear points. Hence $C_1C_2 = R_1 + R_2$ where R_i, $i = 1,2$ is the radius of the circle ζ_i. Therefore $OD = C_1C_2 = R_1 + R_2$. In the triangle C_1OC_2, $OC_1 = R - R_1$ and $OC_2 = R - R_2$, where R denotes the radius of the circle ζ.

Using the triangle's inequality: $C_1C_2 < OC_1 + OC_2$ it follows: $R_1 + R_2 < (R - R_1) + (R - R_2) \Rightarrow R > R_1 + R_2$. Hence, the point D is interior to the circle ζ. Let H be the intersection of the ray OD with ζ and E, F be the intersections of the circles ζ_1, resp. ζ_2 with the sides C_1D and C_2D respectively. By simple computations:

$$DE = DF = DH = R - (R_1 + R_2).$$

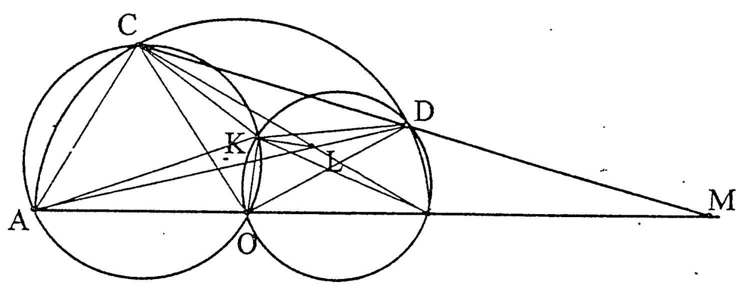
Hence, the point D is the center of the circle ζ_3 and $F \equiv S$, $G \equiv T$, $H \equiv C$. Therefore, $\sphericalangle SCT = 45°$.

Problem II

A semicircle with center O and diameter AB is given. The line d intersects AB in M and the semicircle in C and D such that MB<MA and MD<MC. The circumcircles of the triangles AOC and DOB intersect second time in the point K. Show that the lines MK and KO are perpendicular.

Russian Olympiad

Solution



Consider the inscribed quadrilaterals: AOKC, BOKD, ABCD and let L be the intersection of the diagonals AD and BC.

Because $\sphericalangle OKB = \sphericalangle ODB = \sphericalangle OBD$ and
$\sphericalangle AKO = \sphericalangle ACO = \sphericalangle CAO$, follows

$$\sphericalangle AKB = \sphericalangle OBD + \sphericalangle OAC = \frac{1}{2}(\overset{\frown}{ACD} + \overset{\frown}{CDB}) = 90° + \frac{\overset{\frown}{CD}}{2}$$

It is obvious that $\sphericalangle ALB = 90° + \frac{\overset{\frown}{CD}}{2}$. Hence the quadrilateral AKLB is inscribed. From this property one obtains: $\sphericalangle LKO = \sphericalangle LKA - \sphericalangle AKO =$

$$180° - \sphericalangle LBO - \sphericalangle AKO = 180° - \frac{\overset{\frown}{AC}}{2} - \frac{\overset{\frown}{BC}}{2} = 90°.$$

Hence $OK \perp KL$.

The quadrilateral LKCD is also inscribed because:

$\sphericalangle DKC = 360° - \sphericalangle DKO - \sphericalangle CKO = \sphericalangle DBO + \sphericalangle CAO = \frac{AC}{2} + \frac{BC}{2} =$

$\frac{180° + \overset{\frown}{CD}}{2} = \sphericalangle DLC$

If one considers the circumscribed circles ζ_1 to AKLB and ζ_2 to LKCD, the point M has the same powers to respect ζ_1 and ζ_2. Hence M belongs to radical axes of the two circles. This axis is the line KL.

Problem III

Let $a \in \mathbf{R}$ and $f_1, f_2, \ldots, f_n : \mathbf{R} \to \mathbf{R}$ additive functions such that $f_1(x)f_2(x) \ldots f_n(x) = ax^n$, for all $x \in \mathbf{R}$. Prove that there exist $b \in \mathbf{R}$ and $i \in \{1,2,\ldots,n\}$ such that $f_i(x) = bx$, for all $x \in \mathbf{R}$.

Mihai Piticari and Sorin Radulescu

Solution

An additive function $f: \mathbf{R} \to \mathbf{R}$ has the properties:
$f(0) = 0$ and $f(m) = mf(1)$ for any $m \in Z$.

In our problem, let us denote $f_i(1) = c_i$ and let x be an arbitrary real number. For any integer number one obtains:

(1) $$\prod_{i=1}^{n} f_i(1+mx) = \prod_{i=1}^{n} [c_i + mf_i(x)] = a(1+mx)^n$$

Let us consider the real polynomials:

$$P_x(T) = \prod_{i=1}^{n} [c_i + f_i(x)T] \text{ and}$$

$$Q_x(T) = a(1+xT)^n$$

We shall distinguish two cases:

<u>First case:</u> $a \neq 0$. Then

$$a = \prod_{i=1}^{n} f_i(1) = \prod_{i=1}^{n} c_i \neq 0,$$

and therefore $c_i \neq 0$ for any i, $1 \leq i \leq n$. In this case, the polynomials $P_x(T)$ and $Q_x(T)$ are different from zero and from (1) we conclude that $P_x(T) = Q_x(T)$. Consequently, using the unique decomposition in factors of polynomials, it follows that there exist real numbers b_i, i=1,2,...,n such that $c_i + f_i(x)T = b_i(1+xT)$. It follows, $c_i = b_i$ and $f_i(x) = xb_i = xf_i(1)$. The last equality is valid for arbitrary $x \in \mathbf{R}$ and for all i. The conclusion is: $f_i(x) = c_i x$, all $x \in \mathbf{R}$ and all i.

The second case: a = 0. In this case we have to prove the following: if $f_1,...,f_n : R \to R$ are additive functions such that

$$\prod_{i=1}^{n} f_i(x) = 0$$

then there exists i such that $f_i(x) = 0$, for all $x \in \mathbf{R}$. We shall prove by mathematical induction. For n = 1, the conclusion is obvious. Suppose the property valid for n functions and let $f_1,...,f_n,f_{n+1}$ be additive functions such that

$$\prod_{i=1}^{n+1} f_i(x) = 0$$

and that $f_{n+1}(x) \neq 0$. Then there exists $x_0 \in \mathbf{R}$ with $f_{n+1}(x_0) \neq 0$. Let y be an arbitrary real number and consider the product:

$$0 = \prod_{i=1}^{n+1} f_i(x_0 + my) = \prod_{i=1}^{n+1} [f_i(x_0) + mf(y)],$$

where $m \in Z$.

From (2) follows that the real polynomial

$$P_{x_0,y}(T) = \prod_{i=1}^{n+1} [f_i(x_0) + f_i(y)T]$$

is zero. Because $f_{n+1}(x_0) + f(y)T \neq 0$, it follows:

$$Q_{x_0,y}(T) = \prod_{i+1}^{n} [f_i(x_0) + f_i(y)T] = 0.$$

Hence: $\prod_{i=1}^{n} f_i(y) = 0$, for all $y \in \mathbf{R}$. The conclusion follows by induction.

Alternative solution for the second case: Suppose

$$\prod_{i=1}^{n} f_i(x) = 0, \forall x \in \mathbf{R}$$

and that for any i, there exists $a_i \in \mathbf{R}$ such that $f_i(a_i) \neq 0$. Consider the real number

$$x_m = a_1 + ma_2 + ... + m^{n-1}a_n$$

where $m \in Z$ is arbitrary. Then

$$0=\prod_{i=1}^{n} f_i(x_m)=\prod_{i=1}^{n}[f_i(a_1)+f_i(a_2)m+\ldots+f_i(a_n)m^{n-1}].$$

This shows that the polynomial

$$\prod_{i=1}^{n}[f_i(a_1)+f_i(a_2)T+\ldots+f_i(a_n)T^{n-1}]$$

is the zero polynomial. Hence

$$f_i(a_1)+f_i(a_2)T+\ldots+f_i(a_n)T^{n-1}=0, \text{ for all i}$$

and then $f_i(a_i)=0$. This is a contradiction.

Problem IV

The sequence $(a_n)_{n\geq 2}$ is defined as follows: if the distinct prime divisors of n are $p_1, p_2, \ldots, p_k$ then $a_n=\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_k}$.

Show that for any positive integer N, $N \geq 2$,

$$\sum_{n=2}^{N} a_2 a_3 \ldots a_n < 1.$$

Laurenţiu Panaitopol

Solution

It is easy to see the following equality holds:

$$\sum_{k=2}^{n} a_k=\sum_{k=2}^{n}\left(\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_k}\right)=\sum_{\substack{p\leq n\\ p\ prime}}\frac{1}{p}\left[\frac{n}{p}\right]$$

The folowing inequalities are obvious for n >> 0 :

$$\sum_{\substack{p\leq n\\ p\ prime}}\frac{1}{p}\left[\frac{n}{p}\right]\leq \sum_{\substack{p\leq n\\ p\ prime}}\frac{1}{p}\cdot\frac{n}{p}=n\cdot\sum_{\substack{p\leq n\\ p\ prime}}\frac{1}{p^2}\leq n\cdot\left(\frac{1}{4}+\sum_{k=1}^{\infty}\frac{1}{(2k+1)^2}\right)<$$

$$< n\left(\frac{1}{4}+\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\right)=\frac{n}{2}$$

Therefore, $\sum_{k=2}^{n} a_k<\frac{n}{2}$

By the geometric-arithmetic mean inequality

$$(a_2 a_3 \ldots a_n)^{\frac{1}{n-1}}<\frac{a_2+a_3+\ldots+a_n}{n-1}$$

Hence, the folowing inequalities hold:

$$a_2a_3\ldots a_n < \left(\frac{a_2+a_3+\ldots+a_n}{n-1}\right)^{n-1} < \frac{1}{2^{n-1}}\left(1+\frac{1}{n-1}\right)^{n-1} < \frac{e}{2^{n-1}} < \frac{3}{2^{n-1}}$$

Add these inequalities and obtain:

$$\sum_{n=2}^{\infty} a_2\ldots a_n = a_2 + a_2a_3 + a_2a_3a_4 + a_2a_3a_4a_5 + \ldots < \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{60} +$$

$$+3\left(\frac{1}{2^5} + \frac{1}{2^6} + \ldots\right) = \frac{30+10+5+1}{60} + \frac{3}{2^5}\left(1+\frac{1}{2}+\ldots\right) = \frac{46}{60} + \frac{6}{32} = \frac{229}{240} < 1$$

THE THIRD SELECTION EXAMINATION
FOR THE 37-th IMO
BUCHAREST, March 28, 1996

Problem I

Let $n \geq 3$ be an integer number and $x_1, x_2, \ldots, x_{n-1}$ be positive integers such that

(i) $x_1 + \ldots + x_{n-1} = n$

(ii) $x_1 + 2x_2 + \ldots + (n-1)x_{n-1} = 2n-2$.

Find the minimum of the sum:

$$F(x_1, \ldots, x_{n-1}) = \sum_{k=1}^{n-1} kx_k(2n-k)$$

Ioan Tomescu

Solution

We shall consider two cases : A) $x_{n-1}=0$ and B) $x_{n-1}=1$(From (i) and (ii) we deduce that $x_{n-1} \in \{0,1\}$).

A) In this case (i) and (ii) become :

$$\begin{cases} x_1 + \ldots + x_{n-2} = n \\ x_1 + 2x_2 + \ldots + (n-2)x_{n-2} = 2n-2 \end{cases} (\alpha)$$

If there exists an index m, $1 \leq m \leq n-2$ such that $x_m > 0$ and $x_i = 0$ for any $i \neq m$ then (α) is not compatible for $n \geq 3$.

If there exist two indices i, j such that $x_i, x_j > 0$, $1 \leq i < j \leq n-2$ and $j \geq i+2$ then we shall define: $x'_i = x_i + 1$, $x'_{i+1} = x_{i+1} - 1$, $x'_j = x_j - 1$, $x'_{j+1} = x_{j+1} + 1$ and $x'_k = x_k$ for every $k \neq i, i+1, j, j+1$. We deduce that $\sum_{i=1}^{n-1} x'_i = n$; $\sum_{i=1}^{n-1} ix'_i = 2n-2$ and $F(x_1, \ldots, x_{n-1}) - F(x'_1, \ldots, x'_{n-1}) = ix_i(2n-i) + (i+1)x_{i+1}(2n-i-1) + jx_j(2n-j) + (j+1)x_{j+1}(2n-j-1) - i(x_i+1)(2n-i) - (i+1)x_{i+1}(2n-i-1) - j(x_j-1)(2n-j) - (j+1)(x_{j+1}+1)(2n-j-1) = 2j-2i > 0$ and $F(x_1, \ldots, x_{n-1})$ cannot be minimum.

Otherwise there exist two indices i,j such that $1 \leq i < j \leq n-1$, x_i, $x_j \neq 0$ and $j = i+1$ and (α) becomes:

$$\begin{cases} x_i + x_{i+1} = n \\ ix_i + (i+1)x_{i+1} = 2n-2 \end{cases}$$

which implies that $x_{i+1}=(2-i)n-2$. But $x_{i+1}\geq 0$, hence $i=1$, $x_1=2$, $x_2=n-2$, $x_3=\ldots=x_{n-1}=0$. In this case $F(2,n-2,0,\ldots,0)=4n^2-8n+6$.

B) If $x_{n-1}=1$ then we have $x_1+\ldots+x_{n-2}=n-1$ and $x_1+2x_2+\ldots+(n-1)x_{n-2}==n-1$. But in $x_1+2x_2+\ldots+(n-1)x_{n-2}\geq x_1+\ldots+x_{n-2}$ the equality holds only for $x_2=\ldots=x_{n-2}=0$.It follows that $x_1=n-1$, $x_2=\ldots=x_{n-2}=0$. We deduce that $F(n-1,0,\ldots,0,1)==(n-1)(2n-1)+(n-1)(n+1)=3n^2-3n$.

But for $n\geq 3$ $4n^2-8n+6\geq 3n^2-3n$ and equality holds only for $n=3$, when $x_1=2$, $x_2=1$.

Concluding, $\min F(x_1,\ldots,x_{n-1})=3n^2-3n$ and equality holds only if $x_1=n-1$, $x_2=\ldots=x_{n-2}=0$ and $x_{n-1}=1$ for every $n\geq 3$.

Alternative solution.

We have:

$$\sum_{k=1}^{n-1} k^2x_k=\sum_{k=1}^{n-1}[(k-1)(k+1)+1]x_k=\sum_{k=1}^{n-1}x_k+\sum_{k=1}^{n-1}(k-1)(k+1)x_k\leq$$

$$\leq n+\sum_{k=1}^{n-1}(k-1)nx_k=n+n\sum_{k=1}^{n-1}(k-1)x_k=n+n(2n-2-n)=n^2-n.$$

Thus $\sum_{k=1}^{n-1}kx_k(2n-k)=2n(2n-2)-\sum_{k=1}^{n-1}k^2x_k\geq 2n(2n-2)-n^2+n=3n^2-3n$.

The inequality becomes equality if $x_1=n-1$, $x_2=\ldots=x_{n-2}=0$ and $x_{n-1}=1$.

Problem II

Let n,r be positive integers and A be a set of laticial points in the plane, such that in any open disc of radius r there exists a point from A. Show that for any coloring of the points from A using n colours, there exist four points which have the same colour and are the vertices of a rectangle.

Vasile Pop

Solution

We call the points of A to be A-points. In a square of side $L=4nr^2$, it is possible to inscribe $(2nr)^2=4n^2r^2$ disjoint discs of ray r (fig.1). Then in any such a square there are at least $4n^2r^2$ A-points. If one considers such a square of side L whose vertices are latticial points and sides are parallel with the coordinate axes, then all these A-points are situated on $L-1=4nr^2-1$ vertical segments. Because $\frac{4n^2r^2}{4nr^2-1}>n$, it follows by the pigeonhole principle, that some vertical segment contains n+1 A-points. Because these points are painted in n colors, once again by the pigeonhole principle, there exist two A-points having the same color.

If one considers an infinite horizontal ribbon of disjoint squares of dimensions LxL, one obtains infinitely many pairs of disjoint squares of A-points situated on the same vertical segment and identically painted (say red). These pairs of points can be distributed on $\binom{L-1}{2}$ pairs of horizontal lines in the interior of the ribbon. So, there exist two pairs of points painted in red, which are the vertices of a rectangle.

Problem III

Find all prime numbers for which the congruence $\alpha^{3pq} \equiv \alpha \mod 3pq$ holds for all integers α.

Proposed by Turkey for B.M.O

Solution

$\alpha^{3pq} \equiv \alpha \mod 3pq$ for all $\alpha \Rightarrow \alpha^{3pq} \equiv \alpha \mod 3$ for all α, in particular $2^{3pq-1} \equiv 1 \mod 3 \Rightarrow 2|(3pq-1) \Rightarrow$ p and q are odd.

$\alpha^{3pq} \equiv \alpha \mod 3pq$ for all $\alpha \Rightarrow \alpha^{3pq} \equiv \alpha \mod p$, and if u is a primitive root mod p, then $u^{3pq-1} \equiv 1 \mod p$ and $(p-1)|(3pq-1)$. Similarly$(q-1)|(3pq-1)$.

$\frac{3pq-1}{p-1} = 3q + \frac{3q-1}{p-1}$ is an integer $\Rightarrow \frac{3q-1}{p-1}$ is an integer. If $p=q$ then $p=q=3$; but $4^3 \equiv 1 \mod 9 \Rightarrow 4^9 \equiv 1 \mod 27 \Rightarrow 4^{27} \equiv 1$ and 4^{27} is not congruent 4 mod 9.

So $p \neq q$ and we can suppose $q>p$. Then $q \geq p+2 \Rightarrow \frac{3q-1}{p-1} < 3 \Rightarrow$

$\frac{3q-1}{p-1} = 2 \Rightarrow q = \frac{3p+1}{2}$.

Then, as $\frac{3q-1}{p-1}$ is an integer, $\frac{9}{2} + \frac{10}{2(p-1)}$ is an integer $\Rightarrow$ $(p-1)|10 \Rightarrow p=11$ and $q=17$.

Finally, for $n=3\cdot 11\cdot 17=561$, observing that 2, 10 and 16 divide 560, and using Fermat's and Chinese Remainder Theorem, we verify that required condition is satisfied.

Let $n \geq 3$ be an integer and $p \geq 2n-3$ be a prime number. Let M be a set of n points in the plane such that no three points are colinear and $f:M \to \{0,1,\dots,p-1\}$ be a function such that:

(i) only one point of M has the value 0.

(ii) if the points A,B,C are distinct points of M and C(ABC) is the circumscribed circle of the triangle ABC then

$$\sum_{P \in M \cap C(ABC)} f(P) \equiv 0 \pmod{p}.$$

Show that all the points of M are on a circle.

Marian Andronache, Ion Savu

Solution

Let X be the point of value 0.

We will first prove that if every circle that passes through X and through two points of M contains a third point of M then all the points of M are on a circle. Indeed, consider an inversion I of pole X. Then the set $N=I(M \setminus \{X\})$ has the property :any straight line which contains two points of N contains also a third point of N. If not all the points of N are collinear then there is a triangle ABC which has the vertices from N and whose altitude AA' is smaller or equal than all altitudes of the triangles with vertices from N. But BC contains a third point D from N and, since at least one of the angles $\angle ABD$, $\angle ACD$, $\angle ADB$, $\angle ADC$ is not acute, the corresponding altitude is smaller than AA_1. This contradiction shows that all points of N are collinear, whence all the points of M are on a circle.

Suppose now that not all the points of M are on a circle. Then there exists a circle which passes through X and only two other points A,B of M. Let $f(A)=i$ and $f(B)=p-i$ ($f(A)+f(B)\equiv 0$-from the hypothesis). Let a be a number of the circles which pass through X,A and other points of M, b the number of circles that pass through X,B and other points of M and S the sum of the values of the points of M. By "adding" the circles which pass through X and A one gets $S+(a-1)i\equiv 0$; in the same way $S(b-1)(p-i)\equiv 0$. I follows that $i(a+b-2)\equiv 0$, whence $a+b\equiv 2$. But $1\leq a,b\leq n-2 \Rightarrow 2\leq a+b\leq 2n-4<p \Rightarrow a+b=2 \Rightarrow a=b=1$, which contradicts the hypothesis that not all the points of M are on the circle C(XAD).

THE FOURTH SELECTION EXAMINATION
FOR THE 37-th IMO
BUCHAREST, March 28, 1996

Problem I.

Let $x_1, x_2, \ldots, x_n, x_{n+1}$ be positive reals such that $x_1+x_2+\ldots+x_n=x_{n+1}$.

Prove that $\sum_{i=1}^{n}\sqrt{x_i(x_{n+1}-x_i)} \le \sqrt{\sum_{i=1}^{n} x_{n+1}(x_{n+1}-x_i)}$.

Mircea Becheanu

Solution

$$\sum_{i=1}^{n} x_{n+1}(x_{n+1}-x_i) = x_{n+1}^2\sum_{i=1}^{n}\left(1-\frac{x_i}{x_{n+1}}\right) = x_{n+1}^2\left(n-\sum_{i=1}^{n}\frac{x_i}{x_{n+1}}\right) =$$
$$=(n-1)x_{n+1}^2$$

The inequality becomes $\sum_{i=1}^{n}\sqrt{x_i(x_{n+1}-x_i)} \le x_{n+1}\sqrt{n-1}$ which is the same as $\sum_{i=1}^{n}\sqrt{\frac{x_i}{x_{n+1}}\left(1-\frac{x_i}{x_{n+1}}\right)\frac{1}{n-1}} \le 1$.

Since $\sqrt{\frac{x_i}{x_{n+1}}\left(1-\frac{x_i}{x_{n+1}}\right)\frac{1}{n-1}} \le \frac{1}{2}\left(\frac{x_i}{x_{n+1}}+\frac{1-\frac{x_i}{x_{n+1}}}{n-1}\right)$ it follows that $\sum_{i=1}^{n}\sqrt{\frac{x_i}{x_{n+1}}\left(1-\frac{x_i}{x_{n+1}}\right)\frac{1}{n-1}} \le \frac{1}{2}\sum_{i=1}^{n}\frac{x_i}{x_{n+1}}+\frac{1}{2(n-1)}\left(n-\sum_{i=1}^{n}\frac{x_i}{x_{n+1}}\right)=$
$\frac{1}{2}+\frac{1}{2(n-1)}(n-1) = 1.$

Alternative Solution

By Cauchy - Schwartz inequality we have

$\sum_{i=1}^{n}\sqrt{x_i(x_{n+1}-x_i)} \le \sqrt{n\sum_{i=1}^{n}x_i(x_{n+1}-x_i)} = \sqrt{n(x_{n+1}^2-\sum_{i=1}^{n}x_i^2)}$. But, again by Cauchy - Schwartz inequality, $\sum_{i=1}^{n}x_i^2 \ge \frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2 = \frac{1}{n}x_{n+1}^2$. Hence

$$\sqrt{n(x_{n+1}^2-\sum_{i=1}^{n}x_i^2)} \le \sqrt{nx_n^2 - x_{n+1}^2} = \sqrt{\sum_{i=1}^{n} x_{n+1}(x_{n+1}-x_i)}\ .$$

Problem II

Let x,y,z be real numbers. Prove that the following conditions are equivalent:

i) $x>0, y>0, z>0$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\le 1$.

ii) for every quadrilateral with sides a,b,c,d, $a^2x+b^2y+c^2z>d^2$.

Laurenţiu Panaitopol

Solution

(i)⇒(ii). $a^2x+b^2y+c^2z \ge (a^2x+b^2y+c^2z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \ge$

$$\ge \left(a\sqrt{x}\cdot\frac{1}{\sqrt{x}}+b\sqrt{y}\cdot\frac{1}{\sqrt{y}}+c\sqrt{z}\cdot\frac{1}{\sqrt{z}}\right)^2 = (a+b+c)^2 > d^2.$$

(ii)⇒(i). If $x\le 0$ then, by taking a quadrilateral with sides $a=n$, $b=1$, $c=1$, $d=n$, we get $y+z>n^2(1-x)$, which ,for large n, is impossible. therefore $x>0$ and in the same way $y,z>0$.

Using now a quadrilateral with sides $a=\frac{1}{x}$, $b=\frac{1}{y}$, $c=\frac{1}{z}$, $d=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{1}{n}$ (where n is sufficiently large), one has $\frac{1}{x^2}\cdot x+\frac{1}{y^2}\cdot y+\frac{1}{z^2}\cdot z > \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{1}{n}\right)^2$, that is $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} >$ $> \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{1}{n}\right)^2$ for every sufficiently large n, whence $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \ge$ $\ge \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2$ and therefore $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \le 1$.

Problem III

Let $n\in N^*$ and D be a set of n concentric circles of a plane. Prove that if the function f:D→D satisfies: $d(f(A),f(B))\ge d(A,B)$ for every $A,B\in D$ then $d(f(A),f(B))=d(A,B)$ for every $A,B\in D$.

Dinu Şerbănescu

Solution

We will denote by A' the point $f(A)$.

Let $D=D_1\cup D_2\cup\ldots\cup D_n$ with center O and radii $r_1<r_2<\ldots<r_n$. It is obvious that f takes diametrically opposed points from D_n into diametrically opposed points from D_n.

Let now A, B, C $\in D_n$ such that A and C are diametrically opposed. Hence $A'B'^2+B'C'^2 \geq AB^2+BC^2 = AC^2 = A'C'^2$ it follows that $OB'^2 = =\frac{1}{2}\left(A'B'^2+B'C'^2\right)-\frac{1}{4}A'C'^2 \geq r_n^2$ and therefore $B'\in D_n$ and AB=A'B', BC=B'C'. This proves that $f(D_n)\subset D_n$ and the restriction $f|D_n$ is an isometry. If one takes A, X, Y, Z $\in D_n$ such that AX=AY=A'Z and X≠Y it follows that A'X' = A'Y'=A'Z and X'≠Y', therefore X'=Z of Y'=Z, whence $f(D_n)=D_n$.

Since f is clearly injective and $f(D_n)=D_n$ one gets in the same way that $f(D_{n-1})=D_{n-1}$, $f(D_{n-2})=D_{n-2}$, …, $f(D_1)=D_1$ and all the restrictions $f|D_i$ are isometrics.

Let us now take $A\in D_k$, $B\in D_p$ $1\leq k<p\leq n$ such that $O\in(AB)$. One gets $A'B'\geq AB = r_k+r_p = OA'+OB'$ and therefore $O\in(A'B')$, whence A'B'=AB. Finally, if $O\notin AB$ let $A_1\in D_k$, $B_1\in D_p$ be such that $O\in(A_1B)$ and $O\in(AB_1)$. It follows that $AA_1=A'A_1'$, $BB_1=B'B_1'$, $AB_1=A'B_1'$, $BA_1=B'A_1'$, the isosceles trapezoids AA_1B_1B and $A'A_1'B_1'B'$ are congruent and therefore AB=A'B'(this argument also holds in the case $A\in(OB)$).

Problem IV

Let $n\geq 3$ be an integer and $X\subset\{1,2,3,\ldots,n^3\}$ be a set with $3n^2$ elements. Prove that one can find nine pairwisely distinct numbers a_1,a_2,a_3, b_1,b_2,b_3, c_1,c_2,c_3 from X such that the system

$$a_1x+a_2y+a_3z=0$$
$$b_1x+b_2y+b_3z=0$$
$$c_1x+c_2y+c_3z=0$$

has a solution (x_0,y_0,z_0) with x_0,y_0,z_0 integers and $x_0y_0z_0\neq 0$.

Marius Cavachi

Solution

Let $x_1<x_2<\ldots<x_{3n^2}$ be the elements of X and $X_1=\{x_1,x_2,\ldots,x_{n^2}\}$, $X_2=\{x_1,x_2,\ldots,x_{2n^2}\}$, $X_3=\{x_1,x_2,\ldots,x_{3n^2}\}$. For every $(a,b,c)\in X_1\times X_2\times X_3$ let $f(a,b,c)=(b-a,c-b)$. This defines a function f: $X_1\times X_2\times X_3 \rightarrow Y\subset\{1,\ldots,n^3\}\times\{1,\ldots,n^3\}$, where Y is the set of pairs (p,q) with $p+q\leq n^3$.

Since $X_1 \times X_2 \times X_3$ has $(n^2)^3 = n^6$ elements and Y has $\sum_{p=1}^{n^3-1}(n^3-p) =$

$n^6 - n^3 - \frac{(n^3-1)n^3}{2} = \frac{n^6-n^3}{2} < \frac{n^6}{2}$ elements, there exist three different triples (a_1,b_1,c_1), (a_2,b_2,c_2), (a_3,b_3,c_3) such that $a_1-b_1 = a_2-b_2 = a_3-b_3 = k$ and $c_1-b_1 = c_2-b_2 = c_3-b_3 = p$. The elements a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 are also pairwisely distinct because $a_1, a_2, a_3 \in X_1$; $b_1, b_2, b_3 \in X_2$; $c_1, c_2, c_3 \in X_3$; $a_1=a_2 \Rightarrow b_1=b_2 \Rightarrow c_1=c_2$; $b_1=b_2 \Rightarrow c_1=c_2 \Rightarrow a_1=a_2$; $c_1=c_2 \Rightarrow b_1=b_2 \Rightarrow a_1=a_2$.

Finally, it is easy to check that the system has the solution $x_0=a_2-a_3$, $y_0=a_3-a_1$, $z_0=a_1-a_2$.

THE 13-th BALKAN MATHEMATICAL OLYMPIAD

BACAU, ROMANIA, APRIL 30, 1996

Problem I

Let O, G be the circumcentre and the barycentre of a triangle ABC, respectively. If R is the circumradius and r is the inradius of ABC, show that

$$OG \leq \sqrt{R(R-2r)}$$

proposed by Greece

Solution

Using Leibniz's relation, it is known that

$$OG^2 = R^2 - (a^2+b^2+c^2)/9$$

Then, the given inequality is equivalent with:

$$a^2+b^2+c^2 \geq 18\,Rr$$

From abc= 4Rrp, it follows:

$$Rr = \frac{abc}{2(a+b+c)}$$

Hence, the given inequality is equivalent with:

$$(a+b+c)(a^2+b^2+c^2) \geq 9abc.$$

This inequality is a consequence of the mean inequalities:

$$a+b+c \geq 3\sqrt[3]{abc}, \text{ and}$$

$$a^2+b^2+c^2 \geq 3\sqrt[3]{a^2b^2c^2}.$$

The equalities both hold if a=b=c, hence ABC is equilateral.

Remark: It is known the Euler's equality:

$$OI^2 = R^2 - 2Rr,$$

where I is the incenter of ABC. Then the problem is to prove that $OG \leq OI$.

Problem II

Let p>5 be a prime number and $X = \{\, p-n^2 \mid n \in N^* \text{ and } n^2 < p \,\}$. Prove that X contains two different elements x, y such that $x \neq 1$ and x divides y.

proposed by Albania

Solution

There are two cases to be considered:

a) if $1 \in X$, then $p = n^2+1$, where n is even. Then

$$y = p-1 = p-1^2 = n^2 \in X$$

and 2n divides n^2. But $2n = n^2+1 - (n-1)^2 = p - (n-1)^2$.
Hence $x = p-(n-1)^2 = 2n \in X$ and it is obvious that x and y satisfy the question.

b) $1 \notin X$. Let $n = [\sqrt{p}]$ be the least positive integer such that $n^2 \leq p$.
We have:

$$n^2 + 1 < p < (n+1)^2$$

Denote $x = p - n^2$, $x > 1$. Because p is a prime number, we also have $p < n^2 + 2n$ and $p \neq n^2 + n$. Therefore $x - n \neq 0$ and $0 < x < 2n$, which gives $0 < |x-n| < n$.
We may consider $y = p - (x-n)^2 \in X$ and from

$$y = p - n^2 + 2nx - x^2 = x(1 + 2n - x)$$

we deduce $x|y$.

Problem III

Let ABCDE be a convex pentagon. Denote by M, N, P, Q, R the midpoints of the segments AB, BC, CD, DE, EA, respectively. If the segments AP, BQ, CR, DM have a common point, prove that this point also belongs to the segment EN.

proposed by Yugoslavia

Solution

First solution:

It is easy to prove that a point O belongs to the median XX' of the triangle XYZ if and only if O is an interior point of XYZ and $\sigma(XOY) = \sigma(XOZ)$, where σ denotes the area of the triangle.
Now, let O be the common point of the segments AP, BQ, CR and DM. It follows that: $\sigma(BOE) = \sigma(BOD) = \sigma(AOD) = \sigma(AOC) = \sigma(COE)$.
On the other hand,
$O \in [CR]$ implies $O \in \text{int}\sphericalangle BCE$, $O \in [BQ]$ implies $O \in \text{int}\sphericalangle EBC$ and we deduce $O \in \text{int}\Delta BCE$.

Second solution:

Using complex numbers, let a, b, c, d, e be the afixes of the points A B C D E respectively. We may assume that the common point of the segments AP, BQ, CR and DM is the origin O of the complex plane. Then, from the hypotesis we deduce

$$\frac{c+d}{2a}, \frac{d+e}{2b}, \frac{e+a}{2c}, \frac{a+b}{2d} \in R$$

This is equivalent with:

$$c\bar{a}+d\bar{a}, d\bar{b}+e\bar{b}, e\bar{c}+a\bar{c}, a\bar{d}+b\bar{d} \in R$$

If we add these numbers, we obtain:

$$e\bar{b}+e\bar{c}+(c\bar{a}+a\bar{c})+(d\bar{a}+a\bar{d})+(d\bar{b}+b\bar{d}) \quad \in R.$$

This gives $b\bar{e}+c\bar{e} \in R$ and then $\frac{b+c}{2e} \in R$.

This condition is equivalent with the fact that E, O, N are colinear points.

Problem IV

Show that there exists a subset A of the set $\{1, 2, \ldots, 2^{1996}-1\}$ having the following properties: a) $1 \in A$ and $2^{1996}-1 \in A$; b) every element of A except 1 is the sum of two (not necessarily distinct) elements of A; c) the number of elements of A does not exceed 2012.

proposed by Romania

Solution

For a positive integer n denote f(n) the least number of elements of a set A, $A \subset \{1,2,\ldots,n\}$, and satisfying the conditions a),b). We shall prove that $f(2^{1996}-1) \leq 2012$.

First, note that the number f(n) has the following two properties:

1) $f(2^{n+1}-1) \leq f(2^n-1) + 2$. Indeed if $A \subset \{1,2,\ldots,2^{n+1}-1\}$ satisfies a) and b) and has $f(2^n-1)$ elements, then

$$B = A \cup \{2^{n+1}-2, 2^{n+1}-1\}$$

is a subset of $\{1,2,\ldots,2^{n+1}-1\}$ and satisfies a) and b).

This is because

$$2^{n+1}-2 = (2^n-1) + (2^n-1) \text{ and}$$
$$2^{n+1}-1 = 1 + (2^{n+1}-2).$$

Hence $f(2^{n+1}-1) \leq |B| = f(2^n-1) + 1$.

2) $f(2^{2n}-1) \leq f(2^n-1) + (n+1)$. If $A \subset \{1,2,\ldots,2^n-1\}$ satisfies a) and b) and has $f(2^n-1)$ elements then

$$B = A \cup \{2(2^n-1), 2^2(2^n-1), \ldots, 2^n(2^n-1), 2^{2n}-1\}$$

is a subset of $\{1,2,3, \ldots, 2^{2n} - 1\}$ witch satisfies a) because:

$$2^{j+1}(2^n - 1) = 2^j(2^n - 1), \text{ all } j = 0, 1, \ldots, n-1$$

and

$$2^{2n} - 1 = 2^n(2^n - 1) + (2^n - 1).$$

Hence $f(2^{2n} - 1) \le |B| = f(2^n - 1) + (n+1)$.

Now, we go down in applying the above two properties:

$$\begin{aligned}
f(2^{1996} - 1) &\le f(2^{998} - 1) + 999 \\
f(2^{998} - 1) &\le f(2^{499} - 1) + 500 \\
f(2^{499} - 1) &\le f(2^{498} - 1) + 2 \\
f(2^{498} - 1) &\le f(2^{249} - 1) + 250 \\
f(2^{249} - 1) &\le f(2^{248} - 1) + 2 \\
f(2^{248} - 1) &\le f(2^{124} - 1) + 125 \\
f(2^{124} - 1) &\le f(2^{62} - 1) + 63 \\
f(2^{62} - 1) &\le f(2^{31} - 1) + 32 \\
f(2^{31} - 1) &\le f(2^{30} - 1) + 2 \\
f(2^{30} - 1) &\le f(2^{15} - 1) + 16 \\
f(2^{15} - 1) &\le f(2^{14} - 1) + 2 \\
f(2^{14} - 1) &\le f(2^{7} - 1) + 8 \\
f(2^{7} - 1) &\le f(2^{6} - 1) + 2 \\
f(2^{6} - 1) &\le f(2^{3} - 1) + 4 \\
f(2^{3} - 1) &= 5
\end{aligned}$$

Adding these inequalities we obtain

$$f(2^{1996} - 1) \le 2012.$$