E.R.I.Q lemma and applications

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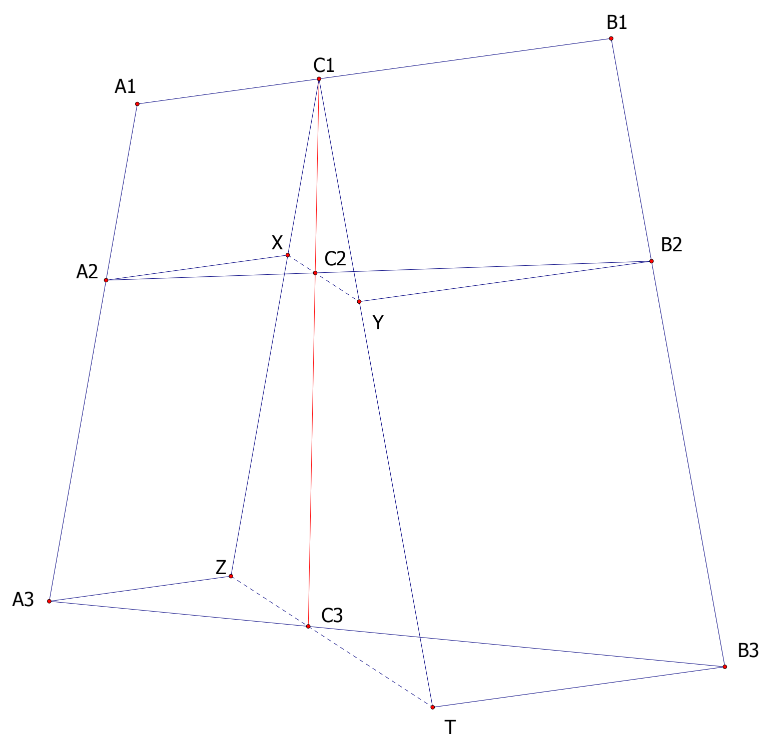
I/Preface

The E.R.I.Q (Equal-Ratio-In-Quadrilateral) lemma was named by vittasko at the webpage Mathlinks.ro.It's useful to prove the collinearity in elementary geometry.This little article only introduce some it's application.The following statement:

E.R.I.Q lemma:Let 2 distinct line $(\triangle 1);(\triangle 2);A_1;A_2;A_3\epsilon(\triangle 1);B_1;B_2;B_3\epsilon(\triangle 2)$ such that: $\frac{\overline{A_1A_2}}{\overline{A_1A_3}}=\frac{\overline{B_1B_2}}{\overline{B_1B_3}}=k.C_1\epsilon A_1B_1;C_2\epsilon A_2B_2;C_3\epsilon A_3B_3$ satisfy: $\frac{\overline{A_1C_1}}{\overline{C_1B_1}}=\frac{\overline{A_2C_2}}{\overline{C_2B_2}}=\frac{\overline{A_3C_3}}{\overline{C_3B_3}}$ Then $\overline{C_1;C_2;C_3}$ and $\frac{\overline{C_1C_2}}{\overline{C_1C_3}}=k$

Proof



+Let $X; Y; Z; T$ be the points such that $A_1C_1XA_2; A_1C_1ZA_3; B_1C_1YB_2; B_1C_1TB_3$ is parallelogram respectively

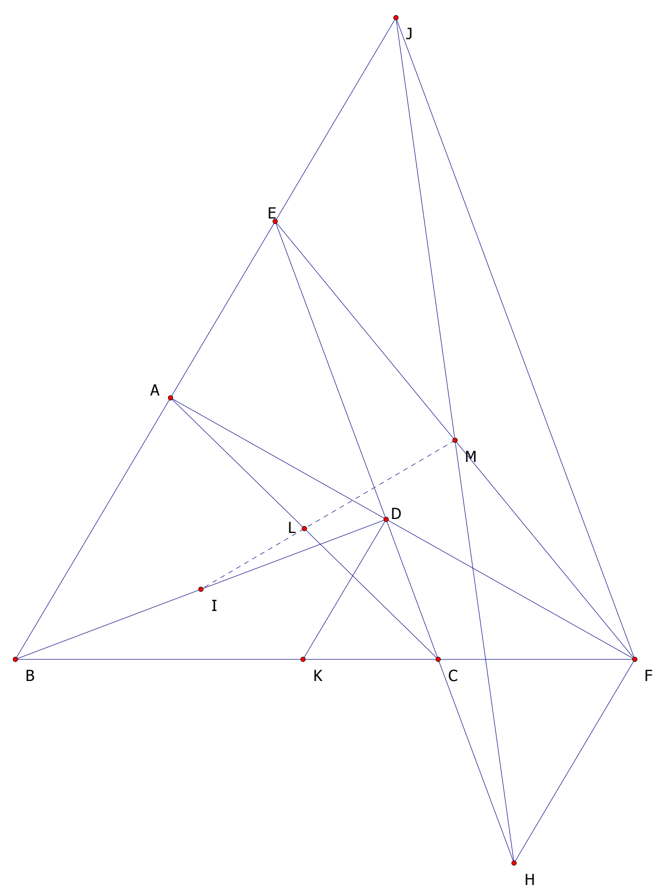
+Applying Thales's theorem $\overline{X; C_2; Y}; \overline{Z; C_3; T} \Rightarrow \overline{C_1; C_2; C_3}$ and $\frac{C_1C_2}{C_1C_3} = \frac{A_1A_2}{A_1A_3} = \frac{B_1B_2}{B_1B_3} = k$ .Out Proof is completed then

II/It's applycation in elementary geometry

Now,We begin by The following problem:

Problem 1: (The Gauss's line):Let ABCD be a quadrilateral.$E \equiv AB \cap CD; E \equiv AD \cap BC$ then the midpoint of $AC; BD; EF$ are collinear

Proof



+Let $L; I; M$ be midpoint of $AC; BD; EF$ respectively.Construct parallelogram $JEHF$ such that $J\epsilon AB; H\epsilon DC$.We'll have

$+\dfrac{DF}{AF} = \dfrac{FH}{AJ} = \dfrac{DK}{AB} \Rightarrow \dfrac{AB}{AJ} = \dfrac{DK}{FH} = \dfrac{DC}{CH}$.Applying E.R.I.Q lemma for 2 line $\overline{BAJ}$ and $\overline{DCH}$.We'll get $\overline{I; L; M}$(QED)

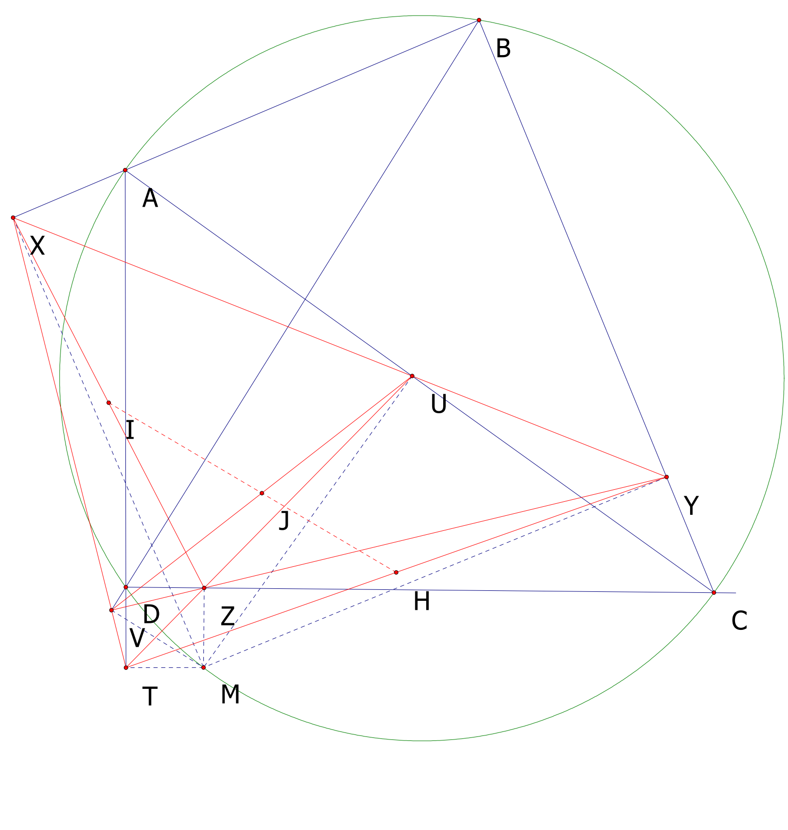
Problem 2:Let $ABCD$ incribed (O) and a point so-called M.Call $X; Y; Z; T; U; V$ are the projection of M onto $AB; BC; CD; DA; CA; BD$ respectively.Call $I; J; H$ are the midpoint of $XZ; UV; YT$ respectively.Prove that $\overline{N; P; Q}$

Proof

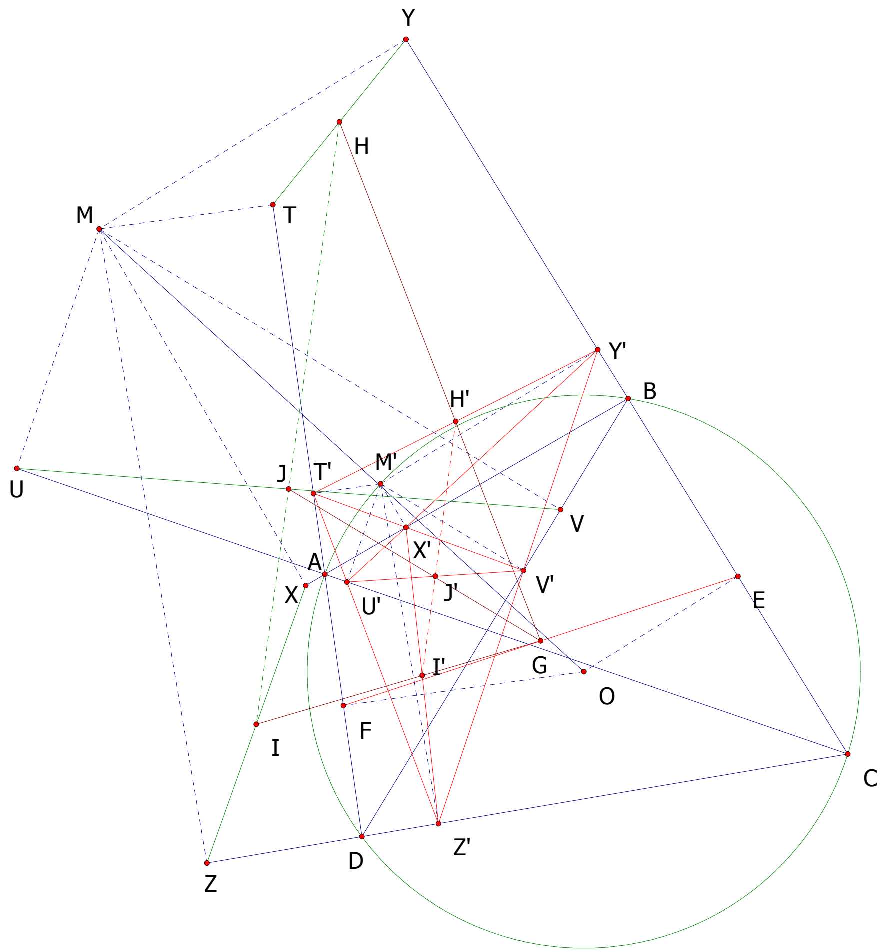
There are three case for consideration

+Case 1: $M \equiv O$.This case make the problem become trivial

+Case 2:M lies on (O).According to the Simson's line then $XYZTUV$ become a complete quadrilateral and we can conclude that $\overline{IJH}$ is the Gauss's line of $XYZTUV$ (QED)



+ Case 3: M not coincide O and not lies on (O)



+Let OM meet (O) at M'.Call X', Y', Z', T', U', V' are the projections of M' onto AB, BC, CD, DA, AC, BD.For the same reason at Case 2,We'll have I', J', H' are collinear (With I', J', H' are the midpoint of $X'Z', U'V', Y'T'$ respectively

+Let $E; F$ be the midpoint of $BC; AD$ respectively and G be the centroid of quadrilateral $ABCD \Rightarrow G$ is the midpoint of EF.We'll have :

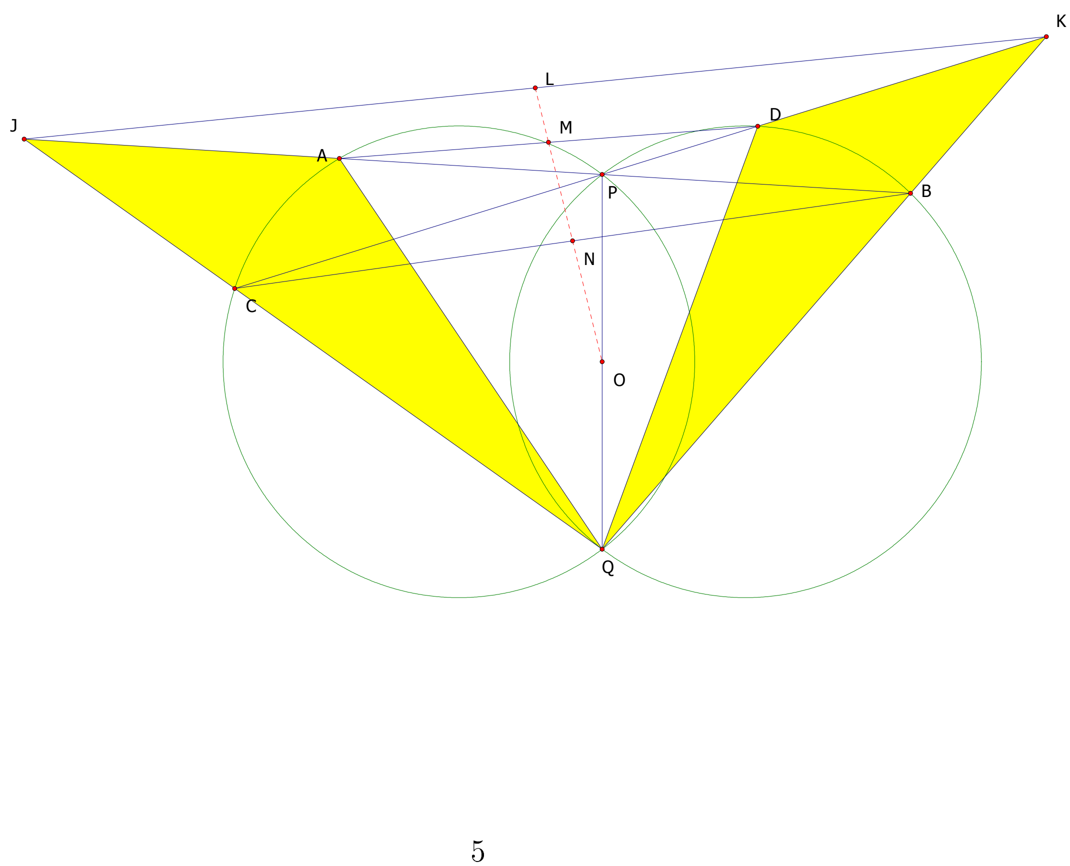
$+\dfrac{YY'}{YE}=\dfrac{MM'}{MO}=\dfrac{TT'}{TF}$.Applying E.R.I.Q Lemma above we'll get $\overline{H,H',G}$ and $\dfrac{GH'}{GH}=\dfrac{EY'}{EY}=\dfrac{OM'}{OM}=k$

+Anagolously, We'll get $\overline{I,I',G};\overline{J,J',G}$ and $\dfrac{GI'}{GI}=\dfrac{GJ'}{GJ}=\dfrac{GH'}{GH}=k(i)$.Morever, $\overline{I',J',H'}(ii)$

+From $(i);(ii)\Rightarrow\overline{I,J,H}$(QED)

Problem 3: Let 2 equal circle $(O_1);(O_2)$ meet each other at $P;Q$.O be the midpoint of PQ.2 line through P meet 2 circle at $A;B;C;D(A;C\epsilon(O_1);B;D\epsilon(O_2))$.$M;N$ be midpoint of $AD;BC$.Prove that $\overline{M;N;O}$

Proof



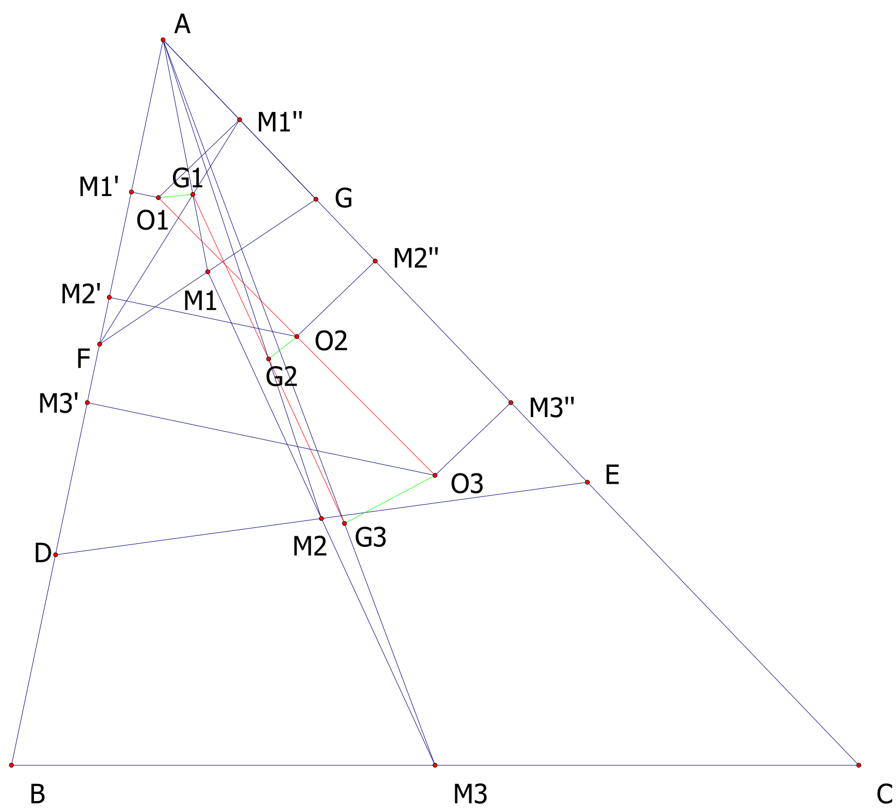
+$J \equiv AB \cap CQ; K \equiv CD \cap QB$.Let L be midpoint of KJ.It's follow that $\overline{ONL}(i)$ is the Gauss's line of complete quadrilateral $QBPCJK$

+It's easy to see $\triangle QCD \sim \triangle QAB; \triangle QAJ \sim \triangle QDK \Rightarrow \dfrac{JA}{DK} = \dfrac{AQ}{QD} = \dfrac{AB}{CD} \Rightarrow \dfrac{JA}{AB} = \dfrac{DK}{CD}$.Applying E.R.I.Q lemma we'll get $\overline{N; M; L}(ii)$

+From $(i); (ii)$ We'll have $\overline{M; N; O}$ (QED)

Problem 4: Let ABC be a triangle.$F; G$ be arbitrary point $AB; AC$.Take $D; E$ midpoint of $BF; CG$.Show that the center of nine-point circle of $\triangle ABC; \triangle ADE; \triangle AFG$ are collinear

Proof



+Let$M_1; M_2; M_3$ be midpoint of $FG; DE; BC$.$G_1; G_2; G_3$ be centroid of $\triangle AFG; ADE; ABC$

+Applying E.R.I.Q for 2 line $\overline{FDB}$ and $\overline{CEG}$.We'll get $\overline{M_1; M_2; M_3}$ and $\dfrac{M_1M_2}{M_1M_3} = \dfrac{1}{2}$.It's implies that $\overline{G_1; G_2; G_3}$ and $\dfrac{G_1G_2}{G_1G_3} = \dfrac{1}{2}$
+Let $M_1'; M_1''; M_2'; M_2''; M_3'; M_3''$ be the midpoint $AF; AG; AD; AE; AB; AC$ respectively and $O_1; O_2; O_3$ be the circumcenter of $\triangle AFG; \triangle ADE; \triangle ABC$
+It's easy to see that $M_1'M_2' = M_2'M_3' = \dfrac{FB}{2}; M_1''M_2'' = M_2''M_3'' = \dfrac{GC}{2}$ and $\overline{O_1; O_2; O_3}; \dfrac{O_1O_2}{O_1O_2} = \dfrac{1}{2}$
+Applying E.R.I.Q lemma for 2 line $\overline{G_1G_2G_3}; \overline{O_1O_2O_3}$.We'll get $\overline{E_1; E_2; E_3}$ are collinear ($E_1; E_2; E_3$ is the center of nine-point of $AFG; ADE; ABC$).We are done

THE END

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