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ROMANIAN MATHEMATICAL COMPETITIONS
2006

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FOREWORD

The thirteenth volume of the Romanian Mathematical Contests booklet consists, as usual, of two parts. In the first part we present the problems given at the district and final round of the Romanian National Olympiad along with those given at the selection tests for the Romanian Teams, junior and senior. We collected some of the problems considered by the problem selection committee at different stages of the Olympiad.

The second part provides full solutions to the problems, with emphasis on those given at the selection tests for the IMO. We hope that in this way we contribute to the development of the so-called problem solving community in the world.

Most of the problems are new or, to our knowledge, do not have equivalent statements in the mathematical olympiad literature. We thank the large number of teachers in mathematics, mathematicians and students who contributed during the year with more than one thousand problems, so our selection process was not easy. Part of the problems come from some other sources: shortlisted problems from the IMO's and BMO's, various mathematical journals, or the large variety of Web sites.

We thank the Ministry of Education and Research for permanent involvement in supporting the Olympiads and the participation of our teams in international events.

Special thanks are due to SOFTWIN, Volvo Romania, Medicover and WBS – sponsors of the Romanian IMO team. Thanks are also due to "Gill Publisher" and the "Sigma Foundation" for constant support in the mathematical competitions. The students taking part at the final training camp look carefully on the manuscript and made important remarks. We thank them all.

Luminiţa Stafi from "The Theta Foundation" helped the editor in the process of producing this booklet.

Last, not least, we are grateful to the Board of the Institute of Mathematics "Simion Stoilow" in Bucharest, for constant technical support in the Mathematical Olympiads and involvement in the training seminars for students.

Bucharest, June 22$^{\text{nd}}$, 2006 Mircea Becheanu and Radu Gologan

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PART ONE

PROBLEMS

DISTRICT ROUND

March 11^{th}, 2006

7^{th} GRADE

Problem 1. Let $n > 1$ be an integer. Prove that the number $\sqrt{11\ldots144\ldots4}$ (digit "1" occurs n times and digit "4" occurs $2n$ times) is an irrational number.

Cecilia Deaconescu, Piteşti

Problem 2. In triangle ABC, $\angle ABC = 2 \cdot \angle ACB$. Prove that:

a) $AC^2 = AB^2 + AB \cdot BC$;

b) $AB + BC < 2 \cdot AC$.

Gh. Bumbăcea, Buşteni

Problem 3. A set M containing 4 positive integers is called *connected*, if for every x in M at least one of the numbers $x-1$, $x+1$ belongs to M. Let U_n be the number of *connected* subsets of the set $\{1, 2, \ldots, n\}$.

a) Evaluate U_7.

b) Determine the least n for which $U_n \geqslant 2006$.

Lucian Dragomir, Oţelul Roşu

Problem 4. Let ABC be an isosceles triangle, with $AB = AC$. Let D be the midpoint of the side BC, M the midpoint of the line segment AD and let N be the projection of D on BM. Prove that $\angle ANC = 90^\circ$.

Marcel Chiriţa, Bucureşti

8^{th} GRADE

Problem 1. Let ABC be a right triangle (with $A = 90°$). Two perpendiculars on the triangle's plane are erected at points A and B, and the points M and N are considered on these perpendiculars, on the same side of the plane, such that $BN < AM$. It is known that $AC = 2a$, $AB = a\sqrt{3}$, $AM = a$ and that the angle between the planes MNC and ABC equals $30°$. Find:

a) the area of triangle MNC;

b) the distance from the point B to the plane MNC.

Gianina Busuioc, Niculai Solomon

Problem 2. For each positive integer n, denote by $u(n)$ the largest prime number less than or equal to n and by $v(n)$ the smallest prime number greater than n. Prove that

$$\frac{1}{u(2)v(2)} + \frac{1}{u(3)v(3)} + \frac{1}{u(4)v(4)} + \cdots + \frac{1}{u(2010)v(2010)} = \frac{1}{2} - \frac{1}{2011}.$$

Nicolae Stăniloiu, Bocşa

Problem 3. Prove that there exist infinitely many irrational numbers x and y such that $x + y = xy \in \mathbb{N}$.

Claudiu Ştefan Popa, Iaşi

Problem 4. a) Prove that one can assign to each of the vertices of a cube one of the numbers 1 or -1 such that the product of the numbers assigned to the vertices of each face equals -1.

b) Prove that such an assignment is impossible in the case of a regular hexagonal prism.

Cecilia Deaconescu, Piteşti

9^{th} GRADE

Problem 1. Let x, y, z be positive real numbers. Prove that the following inequality holds:

$$\frac{1}{x^2 + yz} + \frac{1}{y^2 + zx} + \frac{1}{z^2 + xy} \leqslant \frac{1}{2}\left(\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}\right).$$

Traian Tămîian

Problem 2. The entries of a 9×9 array are all the numbers from 1 to 81. Prove that there exists $k \in \{1, 2, 3, \ldots, 9\}$ such that the product of the numbers in the line k differs from the product of the numbers in the column k.

Marius Ghergu, Slatina

Problem 3. Let $ABCD$ be a convex quadrilateral. Let M and N be the midpoints of the line segments AB and BC, respectively. The line segments AN and BD intersect at E and the line segments DM and AC intersect at F. Prove that if $BE = \frac{1}{3}BD$ and $AF = \frac{1}{3}AC$, then $ABCD$ is a parallelogram.

Gh. Iurea, Iaşi

Problem 4. For each positive integer n, denote by $p(n)$ the largest prime number less than or equal to n and by $q(n)$ the smallest prime number greater than n. Prove that

$$\sum_{k=2}^{n} \frac{1}{p(k)q(k)} < \frac{1}{2}.$$

Nicolae Stăniloiu, Bocşa

10^{th} GRADE

Problem 1. Consider the real numbers $a, b, c \in (0, 1)$ and $x, y, z \in (0, \infty)$, such that

$$a^x = bc, \quad b^y = ca, \quad c^z = ab.$$

Prove that

$$\frac{1}{2+x} + \frac{1}{2+y} + \frac{1}{2+z} \leqslant \frac{3}{4}.$$

Cezar Lupu, Bucureşti

Problem 2. Let ABC be a triangle and consider the points $M \in (BC)$, $N \in (CA)$, $P \in (AB)$ such that $\frac{AP}{PB} = \frac{BM}{MC} = \frac{CN}{NA}$. Prove that if MNP is an equilateral triangle, then ABC is an equilateral triangle as well.

I.V. Maftei, A. Schier, Bucureşti

Problem 3. A prism is called *binary* if one can assign to each of its vertices a number from the set $\{-1, +1\}$, in such a way that the product of the numbers assigned to the vertices of every face equals -1.

a) Prove that the number of vertices of every *binary* prism is divisible by 8.

b) Prove that a prism with 2000 vertices is *binary*.

Cecilia Deaconescu, Piteşti

Problem 4. a) Find two sets X, Y such that $X \cap Y = \emptyset$, $X \cup Y = \mathbb{Q}_+^*$ and $Y = \{a \cdot b \mid a, b \in X\}$.

b) Find two sets U, V such that $U \cap V = \emptyset$, $U \cup V = \mathbb{R}$ and $V = \{x + y \mid x, y \in U\}$.

Marius Cavachi, Constanţa

11th GRADE

Problem 1. Let $x > 0$ be a real number and let A be a 2×2 matrix with real entries, such that

$$\det(A^2 + xI_2) = 0.$$

Prove that $\det(A^2 + A + xI_2) = x$.

Vasile Pop, Cluj

Problem 2. Let $n, p \geqslant 2$ be integer numbers and let A be a $n \times n$ real matrix such that $A^{p+1} = A$.

a) Prove that $\operatorname{rank}(A) + \operatorname{rank}(I_n - A^p) = n$.

b) Prove that if p is a prime number, then

$$\operatorname{rank}(I_n - A) = \operatorname{rank}(I_n - A^2) = \cdots = \operatorname{rank}(I_n - A^{p-1}).$$

Marius Ghergu, Slatina

Problem 3. The sequence of real numbers $(x_n)_{n \geqslant 0}$ satisfies

$$(x_{n+1} - x_n)(x_{n+1} + x_n + 1) \leqslant 0, \quad n \geqslant 0.$$

a) Prove that the sequence is bounded.

b) Can such a sequence be divergent?

Mihai Bălună, Bucureşti

Problem 4. We say that a function $f : \mathbb{R} \to \mathbb{R}$ has the property (P) if for every real x,

$$\sup_{t \leqslant x} f(t) = x.$$

a) Give an example of a function having property (P) which is discontinuous at every real point.

b) Prove that if f is continuous and has property (P) then f is the identical function.

Mihai Piticari, Câmpulung

12th GRADE

Problem 1. Let $f_1, f_2, \ldots, f_n : [0, 1] \to (0, \infty)$ be continuous functions and let σ be a permutation of the set $\{1, 2, \ldots, n\}$. Prove that

$$\prod_{i=1}^{n} \int_0^1 \frac{f_i^2(x)}{f_{\sigma(i)}(x)}\,dx \geqslant \prod_{i=1}^{n} \int_0^1 f_i(x)\,dx.$$

Cezar Lupu, Mihai Piticari

Problem 2. Let $G = \{A \in \mathcal{M}_2(\mathbb{R}) \mid \det(A) = \pm 1\}$ and $H = \{A \in \mathcal{M}_2(\mathbb{C}) \mid \det A = 1\}$. Prove that, under matrix multiplication, G and H are non-isomorphic groups.

Marius Cavachi, Constanţa

Problem 3. Let A be a finite commutative ring with at least two elements. Prove that for any positive integer $n \geqslant 2$, there exists a polynomial $f \in A[X]$ of degree n, with no roots in A.

Marian Andronache, Bucureşti

Problem 4. Let $\mathcal{F} = \{f : [0, 1] \to [0, \infty) \mid f \text{ continuous}\}$ and let $n \geqslant 2$ be a positive integer. Determine the least real constant c, such that

$$\int_0^1 f(\sqrt[n]{x})\,dx \leqslant c \int_0^1 f(x)\,dx$$

for all $f \in \mathcal{F}$.

Gh. Iurea, Iaşi

FINAL ROUND

April 15[th], 2006

7th GRADE

Problem 1. Consider the triangle ABC and points M, N belonging to the sides AB, BC respectively, such that $\frac{2\cdot CN}{BC} = \frac{AM}{AB}$. Let P be a point on AC. Prove that the lines MN and NP are perpendicular if and only if PN bisects the angle $\angle MPC$.

Marcel Teleucă

Problem 2. A square of side n is divided into n^2 unit squares each colored red, yellow or green. Find the minimum value of n such that for any such coloring we can find a row or a column containing at least three squares of the same color.

Mircea Fianu

Problem 3. In the acute triangle ABC angle C equals $45°$. Points A_1 and B_1 are the foots of the perpendiculars from A and B respectively. Denote by H the ortocenter of ABC. Points D and E are situated on the segments AA_1 and BC, respectively, such that $A_1D = A_1E = A_1B_1$. Prove that:

a) $A_1B_1 = \sqrt{\frac{A_1B^2 + A_1C^2}{2}}$;

b) $CH = DE$.

Claudiu-Ştefan Popa

Problem 4. Let A be a set of nonnegative integers containing at least two elements and such that for any $a, b \in A$, $a > b$, we have $\frac{[a,b]}{a-b} \in A$. Prove that the set A contains exactly two elements.

($[a, b]$ denotes the least common multiple of a and b).

Marius Ghergu

8th GRADE

Problem 1. Consider a convex poliedra with 6 faces each of them being a circumscribed quadrilaterals. Prove that all faces are circumscribed quadrilaterals.

G. Rene

Problem 2. Given a positive integer n, prove that there exists an integer k, $k \geqslant 2$ and numbers $a_1, a_2, \ldots, a_k \in \{-1, 1\}$ such that

$$n = \sum_{1 \leqslant i < j \leqslant k} a_i a_j.$$

Gheorghe Iurea

Problem 3. Let $ABCDA_1B_1C_1D_1$ be a cube and let P be a variable point on the side $[AB]$. The plane through P, perpendicular to AB meets AC_1 at Q. Let M and N be the midpoints of the segments A_1P and BQ, respectively.

a) Prove that the lines MN and BC_1 are perpendicular if and only if P is the midpoint of AB.

b) Find the minimal value of the angle between the lines MN and BC_1.

Petre Simion

Problem 4. Consider real numbers a, b, c contained in the interval $[\frac{1}{2}, 1]$. Prove that

$$2 \leqslant \frac{a+b}{1+c} + \frac{b+c}{1+a} + \frac{c+a}{1+b} \leqslant 3.$$

Mircea Lascu

9th GRADE

Problem 1. Find the maximum value of

$$(x^3 + 1)(y^3 + 1),$$

for $x, y \in \mathbb{R}$ such that $x + y = 1$.

Dan Schwarz

Problem 2. Consider quadrilaterals $ABCD$ inscribed in a circle of radius r, such that there is a point P on side CD for which $CB = BP = PA = AB$.

a) Prove that there is a configuration of points A, B, C, D, P for which the above configuration is possible.

b) Prove that for any such configuration we also have $PD = DA = r$.

Virgil Nicula

Problem 3. Consider the triangles ABC and DBC such that $AB = BC$, $DB = DC$ and $\angle ABD = 90°$. Let M be the midpoint of BC. Points E, F, P are such that $E \in (AB)$, $P \in (MC)$, $C \in (AF)$ and $\angle BDE = \angle ADP = \angle CDF$. Prove that P is the midpoint of EF and $DP \perp EF$.

Manuela Prajea

Problem 4. A table tennis competition takes place during 4 days, the number of participants being $2n$, $n \geqslant 5$. Every participant plays exactly one game daily (it is possible that a pair of participants meet more times). Prove that such a competition can end with exactly one winner and exactly three players on the second place and such that there is no player losing all four matches. How many participants have won a single match and how many exactly two, in the given above conditions?

Radu Gologan

10^{th} GRADE

Problem 1. Consider a set M with n elements and let $\mathcal{P}(M)$ denote all subsets of M. Find all functions $f : \mathcal{P}(M) \to \{0, 1, 2, \ldots, n\}$, satisfying the following two conditions:

a) $f(A) \neq 0$, for any $A \neq \emptyset$, and

b) $f(A \cup B) = f(A \cap B) + f(A \Delta B)$, for any $A, B \in \mathcal{P}(M)$, where $A \Delta B = (A \cup B) \setminus (A \cap B)$.

Vasile Pop

Problem 2. Prove that for $a, b \in \left(0, \frac{\pi}{4}\right)$ we have

$$\frac{\sin^n a + \sin^n b}{(\sin a + \sin b)^n} \geqslant \frac{\sin^n 2a + \sin^n 2b}{(\sin 2a + \sin 2b)^n}.$$

Iurie Boreico

Problem 3. Prove that the sequence given by $a_n = \lfloor n\sqrt{2} \rfloor + \lfloor n\sqrt{3} \rfloor$, $n \in \mathbb{N}$, contains infinitely many odd numbers and infinitely many even numbers.

Marius Cavachi

Problem 4. Given $n \in \mathbb{N}$, $n \geqslant 2$, find n disjoint sets A_i, $1 \leqslant i \leqslant n$, in the plane, such that:

a) for any disk $\mathcal{C}$ and any $i \in \{1, 2, \ldots .n\}$, we have $A_i \cap \operatorname{Int}(\mathcal{C}) \neq \emptyset$, and

b) for any line d and for any $i \in \{1, 2, \ldots .n\}$, the projection of A_i on d is not all of d.

Severius Moldoveanu, Costel Chiteş

11^{th} GRADE

Problem 1. A is a two by two matrix with complex entries. Denote by A^* its adjoint (the matrix formed by the cofactors of the transpose). Prove that if there is an integer $m \geqslant 1$ such that $(A^*)^m = 0_n$, then $(A^*)^2 = 0_n$.

Marian Ionescu

Problem 2. A matrix $B \in \mathcal{M}_n(\mathbb{C})$ will be called a *pseudo-inverse* of a matrix $A \in \mathcal{M}_n(\mathbb{C})$ if $A = ABA$ and $B = BAB$.

a) Prove that any square matrix has at least one pseudo-inverse.

b) Characterize the class of matrices with a unique pseudo-inverse.

Marius Cavachi

Problem 3. Consider two systems of points in the plane: $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ having different centroids. Prove that there is a point P in the plane such that

$$PA_1 + PA_2 + \cdots + PA_n = PB_1 + PB_2 + \cdots + PB_n.$$

Marius Cavachi

Problem 4. Consider a function $f : [0, \infty) \to \mathbb{R}$, with the property that for any $x > 0$, the sequence $(f(nx))_{n \geqslant 0}$ is increasing.

a) If f is also continuous on $[0, 1]$, does it follow that it is increasing?

b) What if f is continuous on $\mathbb{Q}_+$?

Gheorghe Grigore

12th GRADE

Problem 1. Let K be a finite field. Prove that the following statements are equivalent:

a) $1+1=0$;

b) for any $f \in K[X]$ with $\deg f \geqslant 1$ the polynomial $f(X^2)$ is reducible.

Marian Andronache

Problem 2. Prove that

$$\lim_{n\to\infty} n\left(\frac{\pi}{4} - n\int_0^1 \frac{x^n}{1+x^{2n}}\mathrm{d}x\right) = \int_0^1 f(x)\mathrm{d}x,$$

where $f(x) = \frac{\operatorname{arctg} x}{x}$, for $x \in (0,1]$ and $f(0) = 1$.

Dorin Andrica, Mihai Piticari

Problem 3. Let G be a group with n elements ($n \geqslant 2$) and let p be the smallest prime factor of n. Suppose G has a unique subgroup H with p elements. Prove that H is contained in the center of G. (The center of G is the set $Z(G) = \{a \in G \mid ax = xa, \forall x \in G\}$.)

Ion Savu

Problem 4. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(x)\mathrm{d}x = 0.$$

Prove that there is $c \in (0,1)$ such that

$$\int_0^c x f(x)\mathrm{d}x = 0.$$

Cezar şi Tudorel Lupu

SELECTION TESTS FOR THE BMO AND IMO ROMANIAN TEAMS

FIRST SELECTION TEST

Problem 1. Let ABC and AMN be two similar triangles with the same orientation, such that $AB = AC$, $AM = AN$, and having disjoint interiors. Let O be the circumcenter of the triangle MAB. Prove that the points O, C, N, A are concyclic if and only if the triangle ABC is equilateral.

Valentin Vornicu

Problem 2. Let $p \geqslant 5$ be a prime number. Find the number of irreducible polynomials in $\mathbb{Z}[X]$, of the form

$$x^p + px^k + px^l + 1, \quad k > l,\ k, l \in \{1, 2, \ldots, p-1\}.$$

The Editors

Problem 3. Let a, b be positive integers such that for any positive integer n we have $a^n + n \mid b^n + n$. Prove that $a = b$.

IMO Shortlist 2005

Problem 4. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $|a_i| \leqslant 1$ for all $i = 1, 2, \ldots, n$, and $a_1 + a_2 + \cdots + a_n = 0$.

(a) Prove that there exists $k \in \{1, 2, \ldots, n\}$ such that

$$|a_1 + 2a_2 + \cdots + ka_k| \leqslant \frac{2k+1}{4}.$$

(b) Prove that for $n > 2$ the bound above is the best possible.

Radu Gologan, Dan Schwarz

SECOND SELECTION TEST

Problem 5. Let $\{a_n\}_{n\geqslant 1}$ be a sequence given by $a_1 = 1$, $a_2 = 4$, and for all integers $n > 1$

$$a_n = \sqrt{a_{n-1}a_{n+1} + 1}.$$

(a) Prove that all the terms of the sequence are positive integers.

(b) Prove that the number $2a_na_{n+1} + 1$ is a perfect square for all integers $n \geqslant 1$.

Valentin Vornicu

Problem 6. Let ABC be a triangle with $\angle ABC = 30°$. Consider the closed discs of radius $AC/3$ centered at A, B and C. Does there exist an equilateral triangle whose three vertices lie one each in each of the three discs?

Radu Gologan, Dan Schwarz

Problem 7. Determine the pairs of positive integers (m, n) for which there exists a set A such that for x, y positive integers, if $|x - y| = m$, then at least one of the numbers x, y belongs to the set A, while if $|x - y| = n$, then at least one of the numbers x, y does not belong to the set.

Adapted by the Editors from AMM

Problem 8. Let x_i, $1 \leqslant i \leqslant n$ be real numbers. Prove that

$$\sum_{1\leqslant i<j\leqslant n} |x_i + x_j| \geqslant \frac{n-2}{2}\sum_{i=1}^{n} |x_i|.$$

Adapted by the Editors from Putnam

THIRD SELECTION TEST

Problem 9. The circle of center I is inscribed in the convex quadrilateral $ABCD$. Let M and N be points on the segments AI and CI respectively, such that $\angle MBN = \frac{1}{2}\angle ABC$. Prove that $\angle MDN = \frac{1}{2}\angle ADC$.

Problem 10. Let A be a point exterior to a circle $\mathcal{C}$. Two lines through A meet the circle $\mathcal{C}$ at points B and C, respectively at D and E (with D between A and E). The parallel through D to BC meets the second time the circle $\mathcal{C}$ at F. The line AF meets $\mathcal{C}$ again at G, and the lines BC and EG meet at M. Prove that

$$\frac{1}{AM} = \frac{1}{AB} + \frac{1}{AC}.$$

Bogdan Enescu

Problem 11. Let γ be the incircle of the triangle $A_0A_1A_2$. In what follows, indices are reduced modulo 3. For each $i \in \{0, 1, 2\}$, let γ_i be the circle through A_{i+1} and A_{i+2}, and tangent to γ; let T_i be the tangency point of γ_i and γ; and finally, let P_i be the point where the common tangent at T_i to γ_i and γ meets the line $A_{i+1}A_{i+2}$. Prove that

(a) the points P_0, P_1 and P_2 are collinear;

(b) the lines A_0T_0, A_1T_1 and A_2T_2 are concurrent.

AMM

Problem 12. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geqslant a^2 + b^2 + c^2.$$

Vasile Cârtoaje

FOURTH SELECTION TEST

Problem 13. Given $r, s \in \mathbb{Q}$, determine all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x + f(y)) = f(x + r) + y + s$$

for all $x, y \in \mathbb{Q}$.

Vasile Pop, Dan Schwarz

Problem 14. Find all positive integers m, n, p, q such that $p^m q^n = (p+q)^2+1$.

Adrian Stoica

Problem 15. Let $n > 1$ be an integer. A set $S \subset \{0, 1, \ldots, 4n - 1\}$ is called *sparse* if for any $k \in \{0, 1, \ldots, n - 1\}$ the following two conditions are satisfied:

(1) the set $S \cap \{4k - 2, 4k - 1, 4k, 4k + 1, 4k + 2\}$ has at most two elements;

(2) the set $S \cap \{4k + 1, 4k + 2, 4k + 3\}$ has at most one element.

Prove that the set $\{0, 1, \ldots, 4n - 1\}$ has exactly $8 \cdot 7^{n-1}$ sparse subsets.

AMM

Problem 16. Let p, q be two integers, $q \geqslant p \geqslant 0$. Let $n \geqslant 2$ be an integer and $a_0 = 0$, $a_1 \geqslant 0$, $a_2, \dots, a_{n-1}$, $a_n = 1$ be real numbers such that

$$a_k \leqslant \frac{a_{k-1} + a_{k+1}}{2}, \quad k = 1, 2, \dots, n-1.$$

Prove that

$$(p+1)\sum_{k=1}^{n-1} a_k^p \geqslant (q+1)\sum_{k=1}^{n-1} a_k^q.$$

Călin Popescu

FIFTH SELECTION TEST

Problem 17. Let $k \geqslant 1$ be an integer and $n = 4k+1$. Let $A = \{a^2 + nb^2 \mid a, b \in \mathbb{Z}\}$. Prove that there exist integers x, y such that $x^n + y^n \in A$ and $x + y \notin A$.

AMM

Problem 18. Let m and n be positive integers and let S be a subset with $(2^m - 1)n + 1$ elements of the set $\{1, 2, \dots, 2^m n\}$. Prove that S contains $m+1$ distinct numbers $a_0, a_1, \dots, a_m$ such that $a_{k-1} \mid a_k$ for all $k = 1, 2, \dots, m$.

AMM

Problem 19. Let $x_1 = 1, x_2, x_3, \dots$ be a sequence of real numbers such that for all $n \geqslant 1$ we have

$$x_{n+1} = x_n + \frac{1}{2x_n}.$$

Prove that

$$\lfloor 25x_{625} \rfloor = 625.$$

The Editors

Problem 20. Let ABC be an acute triangle with $AB \neq AC$. Let D be the foot of the altitude from A to BC and let ω be the circumcircle of the triangle ABC. Let ω_1 be the circle that is tangent to AD, BD and ω. Let ω_2 be the circle that is tangent to AD, CD and ω. Finally, let ℓ be the common internal tangent to ω_1 and ω_2 that is not AD.

Prove that ℓ passes through the midpoint of BC if and only if $2BC = AB + AC$.

SELECTION TESTS FOR THE JUNIOR BALKAN MATHEMATICAL OLYMPIAD

FIRST SELECTION TEST

Problem 1. Let ABC be a rightangle triangle at C and consider points D, E on the sides BC, CA, respectively, such that $\frac{BD}{AC} = \frac{AE}{CD} = k$. Lines BE and AD intersect at point O. Show that $\angle BOD = 60^\circ$ if and only if $k = \sqrt{3}$.

Marcel Chiriţă

Problem 2. Consider five points in the plane such that each triangle with vertices at three of those points has area at most 1. Prove that the five points can be covered by a trapezoid of area at most 3.

Marcel Chiriţă

Problem 3. For any positive integer n let $s(n)$ be the sum of its digits in decimal representation. Find all numbers n for which $s(n)$ is the largest proper divisor of n.

Laurenţiu Panaitopol

SECOND SELECTION TEST

Problem 4. Prove that $\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ba} \geqslant a + b + c$, for all positive real numbers a, b, and c.

Problem 5. Consider a circle C of center O and let A, B be points on the circle with $\angle AOB = 90^\circ$. Circles $C_1(O_1)$ and $C_2(O_2)$ are internally tangent to C at points A, B, respectively, and – moreover – are tangent to themselves. Circle $C_3(O_3)$, located inside the angle $\angle AOB$, is externally tangent to C_1, C_2 and internally tangent to C. Prove that O, O_1, O_2, O_3 are the vertices of a rectangle.

Problem 6. A 7×7 array is divided into 49 unit squares. Find all integers $n \in \mathbb{N}^*$ for which n checkers can be placed on the unit squares so that each row and each line contain an even number of checkers.

(0 is an even number, so empty rows or columns are not excluded. At most one checker is allowed inside a unit square.)

Dinu Şerbănescu

THIRD SELECTION TEST

Problem 7. Suppose $ABCD$ is a cyclic quadrilateral of area 8. Prove that if there exists a point O in the plane of the quadrilateral such that $OA+OB+OC+OD=8$, then $ABCD$ is an isosceles trapezoid (or a square).

Flavian Georgescu

Problem 8. Prove that

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2 \geqslant \frac{3}{2}\cdot\left(\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}\right),$$

for all positive real numbers a, b, and c.

Cezar Lupu

Problem 9. Find all real numbers a and b satisfying

$$2(a^2+1)(b^2+1)=(a+1)(b+1)(ab+1).$$

Valentin Vornicu

Problem 10. Show that the set of real numbers can be partitioned into subsets having two elements.

Dan Schwarz

FOURTH SELECTION TEST

Problem 11. Let $A=\{1,2,\ldots,2006\}$. Find the maximal number of subsets of A that can be chosen such that the intersection of any two such distinct subsets have 2004 elements.

Problem 12. Let ABC be a triangle and let A_1, B_1, C_1 be the midpoints of the sides BC, CA, AB, respectively. Show that if M is a point in the plane of the triangle such that

$$\frac{MA}{MA_1}=\frac{MB}{MB_1}=\frac{MC}{MC_1}=2,$$

then M is the centroid of the triangle.

Dinu Şerbănescu

Problem 13. Suppose a, b, c are positive real numbers which sum up to 1. Prove that

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geqslant 3(a^2+b^2+c^2).$$

Mircea Lascu

Problem 14. The set of positive integers is partitioned into subsets with infinitely many elements each. The following question arises: does there exist a subset in the partition such that any positive integer has a multiple in that subset?

a) Prove that if the number of subsets in the partition is finite, then the answer is "yes".

b) Prove that if the number of subsets in the partition is infinite, then the answer can be "no" (for some partition).

FIFTH SELECTION TEST

Problem 15. Let ABC be a triangle and D a point inside the triangle, located on the median from A. Show that if $\angle BDC = 180^\circ - \angle BAC$, then $AB\cdot CD = AC\cdot BD$.

Eduard Băzăvan

Problem 16. Consider the integers $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ with $a_k \neq b_k$ for all $k=1,2,3,4$. If

$$\{a_1,b_1\}+\{a_2,b_2\}=\{a_3,b_3\}+\{a_4,b_4\},$$

show that the number $|(a_1-b_1)(a_2-b_2)(a_3-b_3)(a_4-b_4)|$ is a square.

Note. For any sets A and B, we denote $A+B=\{x+y \mid x\in A,\ y\in B\}$.

Adrian Zahariuc

Problem 17. Let x, y, z be positive real numbers such that

$$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=2.$$

Prove that $8xyz \leqslant 1$.

Mircea Lascu

Problem 18. For a positive integer n denote by $r(n)$ the number having the digits of n in reverse order; for example, $r(2006) = 6002$. Prove that for any positive integers a and b the numbers $4a^2 + r(b)$ and $4b^2 + r(a)$ cannot be simultaneously perfect squares.

Marius Ghergu

SHORTLISTED PROBLEMS FOR THE 2006 OLYMPIAD

7th GRADE

Problem 1. The bisectors of the angles of the triangle ABC meet the sides BC, CA, AB in D, E, F respectively. Prove that

$$\frac{1}{AB \cdot CE}+\frac{1}{BC \cdot AF}+\frac{1}{CA \cdot BD}=\frac{1}{r \cdot R}$$

Problem 2. In a triangle ABC, $m(\angle BAC) = 110°, m(\angle ABC) = 50°$. Let D be an internal point such that $m(\angle DBC) = 20°$ and $m(\angle DCB) = 10°$. Find $m(\angle ADC)$.

Problem 3. The points M and N are taken on the sides AC, respectively AB of triangle ABC such that $MA = m \cdot MC$ and $NA = n \cdot NB$, where m, n are positive reals and $m + n = 2$. The straight lines BM and CN meet at P. Prove that area$(AMPN) \geqslant \frac{mn}{3}$area$(ABC)$.

Problem 4. Let $ABCDEF$ be a convex hexagon. Triangles Δ_1 şi Δ_2 will be called *opposite* if they are determined by consecutive vertices of the hexagon and have no common points. Prove that the straight lines joining the centroids of the three pairs of opposite triangles are concurrent.

Problem 5. Let $ABCDE$ be a convex pentagon. A straight line will be called *central* if it joins the centroid of the triangle determined by three consecutive vertices of the pentagon and the midpoint of the "opposite" side. Prove that the five central lines are concurrent.

Problem 6. Let a, b, c, d be four distinct positive integers whose product is a perfect square. Prove that the number $a^4 + b^4 + c^4 + d^4$ is the sum of five non-zero perfect squares.

Problem 7. Find three distinct positive integers with integral arithmetic, geometric and harmonic means. Same problem for $n \geqslant 4$ distinct positive integers.

Problem 8. Prove that three positive real numbers x, y, z satisfy the equality

$$\frac{x(y-z)}{y+z} + \frac{y(z-x)}{z+x} + \frac{z(x-y)}{x+y} = 0,$$

if and only if at least two of these numbers are equal.

Problem 9. A tennis competition lasted three days and had 20 participants. Every participant played a match each day (it is possible that the same pair of players met more than once). In the end there was only one winner and everybody had at least a victory. How many participants won exactly one match?

8th GRADE

Problem 10. Let m, n be integers such that $m > n > 3$. Prove that the roots x_1, x_2 of the equation $x^2 - mx + n = 0$ are integers if and only if the number $\lfloor mx_1 \rfloor + \lfloor mx_2 \rfloor$ is a perfect square.

Problem 11. Prove that if a, b, c are three positive real numbers then

$$\sum_{\text{cyc}} \frac{b+c}{a} \geqslant 3 + \frac{(a^2+b^2+c^2)(ab+bc+ca)}{abc(a+b+c)}.$$

Problem 12. Let a, b be positive integers such that $a < b$ and a is not a divisor of b. Solve the equation $a\lfloor x \rfloor - b\{x\} = 0$.

Problem 13. Consider the sets

$$A = \left\{ \sqrt{\frac{1}{a} + \frac{1}{b}} \mid a, b \in \mathbb{N}^*,\ a \neq b \right\}$$

and

$$B = \left\{ \sqrt{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}} \mid x, y, z \in \mathbb{N}^*,\ x > y > z \right\}.$$

Prove that $A \cap B$ contains infinitely many rational and infinitely many irrational numbers.

Problem 14. Prove that if a, b, c are positive real numbers then

$$\sum_{\text{cyc}} \frac{a^2}{3a^2 + b^2 + 2ac} \leqslant \frac{1}{2}.$$

Problem 15. Find all positive integers n and $x_1, x_2, \ldots, x_n$ such that

$$x_1 + x_2 + \cdots + x_n = 3n \quad \text{and} \quad \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1 + \frac{1}{4n}.$$

Problem 16. Let p, q be integers. Prove that if a set A has $p^2 - q$ elements then A cannot have exactly $q^2 - p$ subsets.

Problem 17. Find all integers x, y, z, t such that $x + y + z = t^2$ and $x^2 + y^2 + z^2 = t^3$.

Problem 18. The bases of a right prism $ABCDEFA_1B_1C_1D_1E_1F_1$ are regular hexagons. Prove that:

a) $AE_1 \perp B_1E$ if and only if $AA_1 = AB\sqrt{3}$;

b) if $AE_1 \perp B_1E$ then the distance between the straight lines AE_1 and B_1E is $\frac{\sqrt{42}}{14}AB$.

Problem 19. Consider a tetrahedron $ABCD$ of volume 2 and points M, N, P, Q, R, S on the edges AB, BC, CD, DA, AC and BD, respectively, such that the segments MP, NQ and RS be concurrent. Prove that the volume of the polyhedron $MNPQRS$ is at most 1.

Problem 20. The cube $ABCDA'B'C'D'$ has edges of length 2. The two triangles having as vertices the midpoints of the edges starting from B and C have centroids E and F respectively. Let $P = A'E \cap D'F$. Compute the cosine of the angle $\angle A'PD'$ and the distances from A' to the planes of the two triangles.

9th GRADE

Problem 21. Consider an integer $n \geqslant 2$ and positive real numbers $a_1, a_2, \ldots, a_{2n}$, with sum s. Prove that

$$\frac{a_1}{s + a_{n+1} - a_1} + \cdots + \frac{a_n}{s + a_{2n} - a_n} + \frac{a_{n+1}}{s + a_1 - a_{n+1}} + \cdots + \frac{a_{2n}}{s + a_n - a_{2n}} \geqslant 1$$

and determine the cases of equality.

Problem 22. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that, for every positive numbers x, y, z,

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \geqslant \sqrt{xy} + \sqrt{yz} + \sqrt{zx}.$$

Problem 23. Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\sum_{\text{cyc}} \frac{1}{a^2 + 2b^2 + 3} \leqslant \frac{1}{2}.$$

Problem 24. With each function $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ associate a function $\overline{f} : \{1, 2, \dots, n\} \to \{0, 1, \dots, n-1\}$, by letting

$$\overline{f}(k) = f(1) + \cdots + f(k) - n\left[\frac{f(1) + \cdots + f(k)}{n}\right],$$

for each $k = 1, 2, \dots, n$. Prove that a necessary and sufficient condition for a pair $(f, \overline{f})$ of one-to-one associated functions to exist is that n be even.

Problem 25. A function $f : [0, \infty) \to [0, \infty)$ will be said to have property $\mathcal{P}$ if

$$f(xf(y^2)) = f(y)f(f(x^2))$$

for all $x, y \in [0, \infty)$.

a) Show that there exist infinitely many functions which have property $\mathcal{P}$.

b) Prove that there exists an unique function with property $\mathcal{P}$, whose range contains an open interval centered at 1.

Problem 26. Find all integers x, y such that $x = \sqrt{y^2 - \sqrt{y^2 + x}}$.

Problem 27. Let a, b, c be three positive real numbers such that $a + b + c \geqslant \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Prove that

$$a + b + c \geqslant \frac{3}{a + b + c} + \frac{2}{abc}.$$

Problem 28. Let a, b, c be three positive real numbers such that $\sum \frac{1}{a} \leqslant 3$. Prove that

$$\sum \frac{a^2 + 1}{\sqrt{a^2 - a + 1}} \geqslant 6.$$

Problem 29. Given that the real numbers a, b, c satisfy $|ax^2 + bx + c| \leqslant 1$ for all $x \in [-1, 1]$ and all $\alpha \in [0, 1]$, prove that

$$\alpha(1 + \alpha)|b| + (1 - \alpha^2)|c| \leqslant 1 + \alpha^2.$$

Find the cases of equality.

Problem 30. An acute-angled triangle ABC has orthocenter H and altitudes (AM), (BN), (CP). Let Q and R be the midpoints of the segments (BH) and (CH), respectively, and let $U = MQ \cap AB$, $V = MR \cap AC$, $T = AH \cap PN$. Prove that:

a) $\frac{MH}{MA} = \frac{TH}{TA}$;

b) T is the orthocenter of triangle UAV.

Problem 31. Consider a triangle ABC, the point M on the side (BC) such that $\frac{MB}{MC} = \frac{c(b+c)}{b^2}$ and the point N on (AM) such that $\angle BNM = \angle BAC$ Prove that $2\angle CNM = \angle BAC$.

Problem 32. Let I be the incenter of triangle ABC and A_1, B_1, C_1 the incenters of triangles IBC, ICA, IAB, respectively. Prove that AA_1, BB_1, CC_1 are concurrent.

Problem 33. In a competition there were 18 teams. Each pair of teams met at most once, and within each group of 12 teams there were at least 6 matches. Find the minimum number of matches that have been played.

10^{th} GRADE

Problem 34. If $a_1, a_2, \dots, a_n \in \{-1, 1\}$ and $a_1 + a_2 + \cdots + a_n = 0$, prove that there exists $k \in \{1, 2, \dots, n\}$ such that

$$|a_1 + 2a_2 + \cdots + ka_k| \leqslant \left\lfloor \frac{k}{2} \right\rfloor.$$

Problem 35. Prove that $(2^{2n} + 2^{n+m} + 2^{2m})!$ is divisible by $(2^n!)^{2^n + 2^{m-1}} \cdot (2^m!)^{2^m + 2^{n-1}}$ for every $n, m \in \mathbb{N}^*$.

Problem 36. Let $a \in \mathbb{N}$, $a \geqslant 2$. Define the sequence $(x_n)_{n \geqslant 0}$ by

$$x_0 = \frac{a^2}{4},\ x_1 = \frac{a}{4}(2a^3 - 4a^2 - a + 4),\ x_{n+1} - (4a^2 - 2)x_n + x_{n-1} = 0$$

for $n \geqslant 1$. Prove that $2x_n - \frac{a^2-2}{2}$ is a perfect square for every $n \in \mathbb{N}$.

Problem 37. Solve the equation

$$\log_3\left(2^{\log_3(2^x+1)}+1\right) = \log_2\left(3^x - 1\right).$$

Problem 38. Consider $p, n \in \mathbb{N}^*$ and nonnegative integers $x_1, x_2, \ldots, x_n$. Prove that

$$2^{p-1}\left(\sum_{i=1}^n x_i\right)^p \leqslant \binom{p}{1}\sum_{i=1}^n x_i^{2p-1} + \binom{p}{3}\sum_{i=1}^n x_i^{2p-3} + \cdots + \binom{p}{2m+1}\sum_{i=1}^n x_i^{2p-2m-1},$$

where $m = \left\lfloor \dfrac{p-1}{2} \right\rfloor$.

Problem 39. Find all positive integers p, q such that p is prime, $p \geqslant q \geqslant q^2$ and

$$\binom{p^2}{q} - \binom{q}{p} = 1.$$

Problem 40. A quadrilateral $A_1A_2A_3A_4$ has an incircle of radius r.

a) Prove that there exist circles $\mathcal{C}_i = (A_i, r_i)$, centered at A_i and radii r_i, $i = 1, 2, 3, 4$, such that $\mathcal{C}_i$ is tangent to $\mathcal{C}_{i+1}$ (where $\mathcal{C}_5 = \mathcal{C}_1$).

b) If, in addition

$$\sum_{i=1}^4 \frac{1}{r_i} = \frac{4}{r},$$

prove that the quadrilateral is a square.

Problem 41. Consider a straight line d in space. For every n points A_1, $A_2, \ldots, A_n$ not outside d, the union of the halfplanes $\mathcal{S}_k = (dA_k,\ 1 \leqslant k \leqslant n$ will be called a *n-fan* if, when expressed in degrees, the measure of the dihedral angle between any two halfplanes $\mathcal{S}_i$ and $\mathcal{S}_j$, $1 \leqslant i < j \leqslant n$, is a positive integer.

a) Prove that every 91-fan has two perpendicular or two mutually estending halfplanes.

b) For each $1 \leqslant n \leqslant 360$, determine the number of n-fans containing two perpendicular or two prolongating halfplanes (two n-fans are considered to be identical if they can be obtained from one another through a rotation about d).

Problem 42. Show that if a, b, c are the lengths of the sides of a triangle, R is its circumradius and S is its area, then $a^2 + b^2 + c^2 = 4$

$$6R^2 + S^2 \geqslant 3.$$

Problem 43. Given a point P inside triangle ABC, let r_1, r_2, r_3, respectively, denote the inradii of triangles PBC, PCA and PAB. Prove that

$$\frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3} \geqslant 6(2 - \sqrt{3}).$$

Problem 44. Determine all complex numbers a, b, c such that $a^3 + b^3 + c^3 = 24$, $(a+b)(b+c)(c+a) = 64$, and $|a+b| = |b+c| = |c+a|$.

Problem 45. With reference to the standard notations in a triangle, prove that

$$s \leqslant \sqrt{\frac{3(m_a + m_b + m_c)}{2R}}.$$

11[th] GRADE

Problem 46. a) Prove that if a matrix $A \in \mathcal{M}_2(\mathbb{R})$ has the property that $\operatorname{rang}(A + XY) = \operatorname{rang}(A + YX)$ for every invertible matrices $X, Y \in \mathcal{M}_2(\mathbb{R})$, then there exists $a \in \mathbb{R}$ such that $A = aI_2$.

b) Let $A \in \mathcal{M}_n(\mathbb{R})$ $(n \geqslant 2)$ be a matrix which is not of the form aI_n, $a \in \mathbb{R}$. Prove that there exist $X, Y \in \mathcal{M}_n(\mathbb{R})$, with X invertible and $\operatorname{rang}(A + XY) < \operatorname{rang}(A + YX)$.

Problem 47. Determine the largest integer $n \geqslant 2$ with the following property: if $A \in \mathcal{M}_n(\mathbb{C})$, $A \neq \lambda I_n$, for any $\lambda \in \mathbb{C}$, then $B \in \mathcal{M}_n(\mathbb{C})$ and $AB = BA$ implies the existence of $a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$ such that $B = a_0 I_n + a_1 A + \cdots + a_{n-1}A^{n-1}$.

Problem 48. Let $n \geqslant 2$ be an integer. Find the largest integer $k \geqslant 1$ with the following property: for any k matrices $A_1, A_2, \ldots, A_k \in \mathcal{M}_n(\mathbb{C})$, if $I_n - A_1A_2\cdots A_k$ is invertible, then so is $I_n - A_{\tau(1)}A_{\tau(2)}\cdots A_{\tau(k)}$ for every permutation $\tau \in S_k$.

Problem 49. Consider a matrix $A \in \mathcal{M}_3(\mathbb{R})$ such that $\det(A^2 + I_3) = 0$. Prove that:

a) $\det(A+I_3)-\det(A-I_3)=4$;
b) $\operatorname{tr}(A^3)=\operatorname{tr}^3(A)$.

Problem 50. Let A, B, C be matrices with real entries and let

$$X = AB+BC+CA,\ \ Y = BA+CB+AC,\ \ Z = A^2+B^2+C^2.$$

Prove that

$$\det(2Z-X-Y)\geqslant 3\det(X-Y).$$

Problem 51. A function $f:\mathbb{R}\to\mathbb{R}_+$ is continuous and has an irrational period. Let $M=\max f$. Evaluate

$$\lim_{n\to\infty}\frac{f(1)f(2)\cdots f(n)}{M^n}.$$

Problem 52. A polynomial $p\in\mathbb{R}[X]$ has the following properties: $p(\mathbb{Q})\subset\mathbb{Q}$ and $p(\mathbb{R}\setminus\mathbb{Q})\subset\mathbb{R}\setminus\mathbb{Q}$. Prove that $\deg p = 1$.

Problem 53. Find all continuous functions $f:[0,1]\to\mathbb{R}$ with the following property: for every integer $n\geqslant 3$ and every arithmetic sequence $a_1, a_2, \dots, a_n$, the sequence $f(a_1), f(a_2), \dots, f(a_n)$ is a geometric sequence.

Problem 54. Let $f, g:\mathbb{R}\to\mathbb{R}$ be such that f has the intermediate value property and, for every $x\in\mathbb{R}$, the limit

$$\lim_{h\to 0}\frac{f(x+h)-g(x)}{h},$$

exists and is finite. Prove that $f=g$.

Problem 55. The function $f:(0,\infty)\to\mathbb{R}$ has the property that for every $a,b\in\mathbb{R}$, $a<b$, there exists $c\in(a,b)$ such that f is continuous at c. Given that $f(nx)<f((n+1)x)$ for every $x\in(0,\infty)$ and every $n\in\mathbb{N}^*$, prove that f is strictly increasing.

Problem 56. Let $f:(0,\infty)\to(0,\infty)$ be a twice differentiable function such that

$$f''(x)+f'(x)\geqslant f^2(x),$$

for all $x>0$. Prove that the limit $\lim\limits_{x\to\infty} f(x)$ exists and is finite. Evaluate the limit.

Problem 57. A function $f:\mathbb{R}\to\mathbb{R}$ satisfies the following condition: at every $x_0\in\mathbb{R}$:

$$\sup_{x<x_0}\frac{f(x)-f(x_0)}{x-x_0}=\inf_{x>x_0}\frac{f(x)-f(x_0)}{x-x_0}.$$

Prove that f is convex and differentiable.

Problem 58. The sequences $(a_n)_n$, $(b_n)_n$ and $(x_n)_n$ of positive numbers satisfy the conditions:

$$\lim_{n\to\infty} a_1a_2\cdots a_n=0;\quad \lim_{n\to\infty}\frac{b_n}{1-a_n}=0;\quad\text{and}\quad x_{n+1}\leqslant a_nx_n+b_n$$

for every $n\geqslant 1$. Prove that $\lim\limits_{n\to\infty} x_n=0$.

Problem 59. Consider an ellipse that is tangent to the sides of a rhombus $ABCD$ at their midpoints. Let A', B', C', D' respectively, denote the orthogonal projections of A, B, C, D onto a variable tangent to the ellipse. Prove that $AA'\cdot CC' = BB'\cdot DD'$.

Problem 60. For each point L inside a given triangle ABC, consider the intersections E and F of the pairs of straight lines (AC, BL) and (AB, CL). Find the locus of L for which the quadrilateral $AELF$ has an inscribed circle.

12th GRADE

Problem 61. Suppose A is a ring such that $1+1+1+1+1+1=0$ and $x^4y^3=y^3x^4$ for all $x,y\in A$. Prove that A is commutative.

Problem 62. Let $\mathbb{Z}[\alpha]=\{a+\alpha b \mid a,b\in\mathbb{Z}\}$, where $\alpha\in\mathbb{C}\setminus\mathbb{Q}$ and $|\alpha|=1$. Prove that exactly two of the sets $\mathbb{Z}[\alpha]$ are rings under the usual operations with complex numbers.

Problem 63. Define a sequence $(a_n)_n$ by

$$a_n=\int_0^1\frac{x^{2n}}{1+x}\mathrm{d}x,\quad n\geqslant 1.$$

Prove that the sequence $(na_n)_n$ is convergent and find its limit.

Problem 64. Let $f : [0,1] \to \mathbb{R}_+$ be a continuous function with $f(1) = 1$ and let

$$a_n = \int_0^1 \frac{f(x)}{1+x^n}\,\mathrm{d}x, \quad n \geqslant 1.$$

Prove that

$$\lim_{n\to\infty} \frac{1}{n}\left(\int_0^1 f(x)\mathrm{d}x - a_n\right) = \ln 2.$$

Problem 65. Find all natural numbers n such that the integral

$$\int_0^n x[x]\{x\}\mathrm{d}x$$

is an integer.

Problem 66. Let $f : [0,\infty) \to \mathbb{R}$ be a continuous bounded function, with $f(0) = 0$.

a) Prove that $\lim\limits_{n\to\infty} \int_0^1 f(nx^n)\,\mathrm{d}x = 0$.

b) Evaluate $\lim\limits_{n\to\infty} \int_0^1 \sqrt{1+n^2x^{2n}}\,\mathrm{d}x$.

Problem 67. Prove that for any continuous function $f : [0,1] \to \mathbb{R}$,

$$\int_0^1 f(x)\,\mathrm{d}x \cdot \int_0^1 x^4 f(x)\,\mathrm{d}x \leqslant \frac{4}{15}\int_0^1 f^2(x)\mathrm{d}x.$$

Also, find the cases of equality.

Problem 68. Find all integrable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\int_0^{x+1/n} f(t)\,\mathrm{d}t = \int_0^x f(t)\,\mathrm{d}t + \frac{1}{n}f(x),$$

for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}^*$.

Problem 69. Let $f : [0,\infty) \to \mathbb{R}$ be a function such that $|f(x) - f(y)| \leqslant |x-y|$ for all $x, y \geqslant 0$. Prove that

$$\int_a^b f(x)\,\mathrm{d}x \leqslant bf(b) + \frac{b^2}{2} - af(a) - \frac{a^2}{2},$$

for all $a, b \in [0,\infty)$, $a < b$. For f differentiable, also consider the cases of equality.

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PART TWO

SOLUTIONS

PROBLEMS AND SOLUTIONS

DISTRICT ROUND

7^{th} GRADE

Problem 1. Let $n > 1$ be an integer. Prove that the number $\sqrt{11\ldots144\ldots4}$ (digit "1" occurs n times and digit "4" occurs $2n$ times) is an irrational number.

Solution. We have to prove that the number $11\ldots144\ldots4$ is not a square.

Let a be the n-digit number $11\ldots1$. We have $11\ldots144\ldots4 = a\cdot 10^{2n} + 4a\cdot 10^n + 4a = a(10^n+2)^2$.

Since the remainder of $a = 11\ldots1$ when divided by 4 equals 3, a is not a square, therefore neither is $11\ldots144\ldots4$.

Problem 2. In triangle ABC, we have $\angle ABC = 2\cdot\angle ACB$. Prove that:

a) $AC^2 = AB^2 + AB\cdot BC$;

b) $AB + BC < 2\cdot AC$.

Solution. a) Let BM be the angle bisector of $\angle ABC$.

The angle bisector theorem gives $\frac{AB}{BC} = \frac{AM}{MC}$, hence $\frac{AM}{AC} = \frac{AB}{AB+BC}$, which implies $AM = \frac{AB\cdot AC}{AB+BC}$.

Since $\angle ABM = \angle ACB$, it follows that $\Delta ABM \sim \Delta ACB$, therefore $\frac{AB}{AC} = \frac{AM}{AB}$, that is, $AM = \frac{AB^2}{AC}$.

It follows that $\frac{AB\cdot AC}{AB+BC} = \frac{AB^2}{AC}$ hence the conclusion.

b) Suppose that the parallel through A to BM intersects BC at P. We have $\angle APM = \angle MBC = \angle ABM = \angle PAB = \angle C$, hence $AB = BP$ and $AP = AC$. It follows that $AB + BC = PB + BC = PC < AP + AC = 2\cdot AC$, as needed.

Problem 3. A set M containing 4 positive integers is called *connected*, if for every x in M at least one of the numbers $x-1$, $x+1$ belongs to M. Let U_n be the number of *connected* subsets of the set $\{1, 2, \ldots, n\}$.

a) Evaluate U_7.

b) Determine the least n for which $U_n \geqslant 2006$.

Solution. Let $a < b < c < d$ be the elements of a connected set M. Since $a-1$ does not belong to the set, it follows that $a+1 \in M$, hence $b = a+1$. Similarly, since $d+1 \notin M$ we deduce that $d-1 \in M$, hence $c = d-1$. Therefore, a connected set has the form $\{a, a+1, d-1, d\}$, with $d - a > 2$.

a) There are 10 connected subsets of the set $\{1, 2, 3, 4, 5, 6, 7\}$:

$\{1,2,3,4\}$;$\{1,2,4,5\}$; $\{1,2,5,6\}$; $\{1,2,6,7\}$;

$\{2,3,4,5\}$; $\{2,3,5,6\}$; $\{2,3,6,7\}$;

$\{3,4,5,6\}$; $\{3,4,6,7\}$ and $\{4,5,6,7\}$.

b) Call $D = d - a + 1$ the *diameter* of the set $\{a, b = a+1, c = d-1, d\}$. Clearly, $D > 3$ and $D \leqslant n - 1 + 1 = n$. For $D = 4$ there are $n-3$ connected sets, for $D = 5$ there are $n-4$ connected sets, etc. Finally, for $D = n$ there is one connected set.

Adding up yields $U_n = 1 + 2 + 3 + \cdots + (n-3) = \frac{(n-3)(n-2)}{2}$.

Consequently, we have to find the least n such that $(n-3)(n-2) \geqslant 4012$. By inspection, we obtain $n = 66$.

Problem 4. Let ABC be an isosceles triangle, with $AB = AC$. Let D be the midpoint of the side BC, M the midpoint of the line segment AD and let N be the projection of D on BM. Prove that $\angle ANC = 90°$.

Solution. Consider the point S such that $ABDS$ is a parallelogram. Clearly, $ADCS$ is a rectangle and let R be the point of intersection of its diagonals. In the right triangle DNS the line segment NR is the median from the right angle and therefore $NR = \frac{1}{2}\cdot SD = \frac{1}{2}\cdot AC$.

Since R is the midpoint of AC and $NR = \frac{1}{2}\cdot AC$, it follows that the triangle ANC is a right triangle, as desired.

8th GRADE

Problem 1. Let ABC be a right triangle (with $A = 90°$). Two perpendiculars on the triangle's plane are erected at points A and B, and the points M and N are considered on these perpendiculars, on the same side of the plane, such that $BN < AM$. It is known that $AC = 2a$, $AB = a\sqrt{3}$, $AM = a$ and that the angle between the planes MNC and ABC equals $30°$. Find:

a) the area of triangle MNC;

b) the distance from the point B to the plane MNC.

Solution. a) The area of triangle ABC equals $a^2 \cdot \sqrt{3}$. On the other hand, we have area$[ABC]$ = area$[MNC] \cdot \cos\alpha$, where $\alpha = 30°$ is the angle between the planes MNC and ABC. It follows that area$[MNC] = 2 \cdot a^2$.

b) Suppose that the lines MN and AB intersect at P. Let T be the projection of the point A on PC. Using the theorem of the three perpendiculars, we obtain that $MT \perp PC$, hence $\angle MTA = \alpha = 30°$.

Since $AB = a$, in triangle MAT we find $AT = a\sqrt{3}$, so $\angle ACT = 60°$, hence $AP = 2a\sqrt{3}$. It follows that B is the midpoint of the line segment AP.

Project B on PC in Q. Using again the theorem of the three perpendiculars, we obtain $NQ \perp PC$. Then $BN = \frac{a}{2}$, $BQ = \frac{a\sqrt{3}}{2}$, $NQ = a$ and the altitude BS of the right triangle BNQ equals $\frac{BN \cdot BQ}{NQ} = \frac{a\sqrt{3}}{4}$. This is the requested distance.

Problem 2. For each positive integer n, denote by $u(n)$ the largest prime number less than or equal to n and by $v(n)$ the smallest prime number greater than n. Prove that

$$\frac{1}{u(2)v(2)} + \frac{1}{u(3)v(3)} + \frac{1}{u(4)v(4)} + \cdots + \frac{1}{u(2010)v(2010)} = \frac{1}{2} - \frac{1}{2011}.$$

Solution. Let p and q be consecutive prime numbers. Then there are $q - p$ numbers n such that $p \leqslant n < q$ and, for each such number, we have $u(n) = p$ and $v(n) = q$. It follows that the term $\frac{1}{pq}$ appears in the sum exactly $q - p$ times.

Since 2003 and 2011 are consecutive primes, the sum becomes

$$\frac{3-2}{2 \cdot 3} + \frac{5-3}{3 \cdot 5} + \cdots + \frac{2011 - 2003}{2003 \cdot 2011} =$$
$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{2003} - \frac{1}{2011} = \frac{1}{2} - \frac{1}{2011}.$$

Problem 3. Prove that there exist infinitely many irrational numbers x and y such that $x + y = xy \in \mathbb{N}$.

Solution. Let $n = x + y = xy$. Then $y = n - x$, so $n = x(n - x)$. We obtain

$$x = \frac{n \pm \sqrt{n^2 - 4n}}{2}.$$

Since for $n \geqslant 5$ we have

$$(n-3)^2 < n^2 - 4n < (n-2)^2,$$

it follows that the number $\sqrt{n^2 - 4n}$ is irrational. Consequently, we can choose $x = \frac{n + \sqrt{n^2 - 4n}}{2}$ and $y = \frac{n - \sqrt{n^2 - 4n}}{2}$.

Problem 4. a) Prove that one can assign to each of the vertices of a cube one of the numbers 1 or -1 such that the product of the numbers assigned to the vertices of each face equals -1.

b) Prove that such an assignment is impossible in the case of a regular hexagonal prism.

Solution. a) Let $ABCDA'B'C'D'$ be the cube. A possible labeling is the following: assign $+1$ to the vertices A, B, D, A' and -1 to the other vertices.

b) A contradiction is obtained by considering on one hand the product of the numbers assigned to all lateral faces and, on the other hand, the product of the numbers assigned to every second lateral face.

9th GRADE

Problem 1. Let x, y, z be positive real numbers. Prove that the following inequality holds:

$$\frac{1}{x^2 + yz} + \frac{1}{y^2 + zx} + \frac{1}{z^2 + xy} \leqslant \frac{1}{2}\left(\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}\right).$$

Solution. Using the AM-GM inequality we obtain $x^2 + yz \geqslant 2\sqrt{x^2 yz}$, and therefore

$$\frac{1}{x^2 + yz} \leqslant \frac{1}{2x\sqrt{yz}} = \frac{\sqrt{yz}}{2xyz}.$$

It follows that $\sum \frac{1}{x^2+yz} \leqslant \frac{1}{2xyz} \sum \sqrt{yz}$ but $\sum \sqrt{yz} \leqslant \sum \frac{y+z}{2} = \sum x$ (AM-GM again). The equality holds when $x = y = z$.

Problem 2. The entries of a 9×9 array are all the numbers from 1 to 81. Prove that there exists $k \in \{1, 2, 3, \ldots, 9\}$ such that the product of the numbers in the line k differs from the product of the numbers in the column k.

Solution. Suppose, by way of contradiction, that for each $i \in \{1, 2, \ldots, 9\}$, the product of the elements in line i equals the product of the elements in column i. Between 40 and 81 there are exactly 10 prime numbers, namely 41, 43, 47, 53, 59, 61, 67, 71, 73, and 79. We prove that these numbers belong to the main diagonal of the table. Indeed, if $40 < p < 81$ is a prime number, then it is the only multiple of p in the table. If p lies on line i, by the assumption it follows that it lies on column i, as well, that is, it lies on the main diagonal.

Therefore, on the main diagonal are all the 10 prime numbers, a contradiction.

Problem 3. Let $ABCD$ be a convex quadrilateral. Let M and N be the midpoints of the line segments AB and BC, respectively. The line segments AN and BD intersect at E and the line segments DM and AC intersect at F. Prove that if $BE = \frac{1}{3}BD$ and $AF = \frac{1}{3}AC$, then $ABCD$ is a parallelogram.

Solution. Denote $\overrightarrow{AB} = \overrightarrow{u}, \overrightarrow{BC} = \overrightarrow{v}, \overrightarrow{CD} = a\overrightarrow{u} + b\overrightarrow{v}$ with $a, b \in \mathbb{R}$. It follows that $\overrightarrow{AD} = (a+1)\overrightarrow{u} + (b+1)\overrightarrow{v}$, $\overrightarrow{AN} = \overrightarrow{u} + \frac{1}{2}\overrightarrow{v}$.

Since $\overrightarrow{BD} = 3\overrightarrow{BE}$, we obtain $\overrightarrow{AE} = \frac{2\overrightarrow{AB}+\overrightarrow{AD}}{3} = \frac{(a+3)}{3}\overrightarrow{u} + \frac{b+1}{3}\overrightarrow{v}$.

Because $\overrightarrow{AE}$ şi $\overrightarrow{AN}$ are collinear vectors, we deduce that $a - 2b + 1 = 0$. Similarly, we obtain $2a + b + 2 = 0$ hence $a = -1, b = 0$, that is, $\overrightarrow{CD} = -\overrightarrow{AB} = \overrightarrow{BA}$, whence $ABCD$ is a parallelogram.

Problem 4. For each positive integer n, denote by $p(n)$ the largest prime number less than or equal to n and by $q(n)$ the smallest prime number greater than n. Prove that

$$\sum_{k=2}^{n} \frac{1}{p(k)q(k)} < \frac{1}{2}.$$

Solution. Denote by $2 = p_1 < p_2 < \cdots < p_m < \cdots$ the sequence of the prime numbers. For $p_i \leqslant k < p_{i+1} - 1$ we have $p(k) = p_i$, $q(k) = p_{i+1}$.

Suppose that $p_m = q(n)$. Then

$$\sum_{k=2}^{n} \frac{1}{p(k)q(k)} \leqslant \sum_{k=2}^{p_m-1} \frac{1}{p(k)q(k)} = \sum_{i=1}^{m-1} \sum_{k=p_i}^{p_{i+1}-1} \frac{1}{p(k)q(k)} \sum_{i=1}^{m-1} \frac{p_{i+1}-p_i}{p_i p_{i+1}}$$
$$= \sum_{i=1}^{m-1} \left(\frac{1}{p_i} - \frac{1}{p_{i+1}} \right) = \frac{1}{2} - \frac{1}{p_m} < \frac{1}{2}.$$

10^{th} GRADE

Problem 1. Consider the real numbers $a, b, c \in (0, 1)$ and $x, y, z \in (0, \infty)$, such that

$$a^x = bc, \quad b^y = ca, \quad c^z = ab.$$

Prove that

$$\frac{1}{2+x} + \frac{1}{2+y} + \frac{1}{2+z} \leqslant \frac{3}{4}.$$

Solution. Denote $A = \log_{\frac{1}{2}} a$, $B = \log_{\frac{1}{2}} b$, $C = \log_{\frac{1}{2}} c$. Then $x = \frac{B+C}{A}$, $y = \frac{C+A}{B}$, $z = \frac{A+B}{C}$.

The inequality becomes

$$\sum \frac{1}{2 + \frac{B+C}{A}} \leqslant \frac{3}{2},$$

or, denoting $S = A + B + C$,

$$\sum \frac{A}{S+A} \leqslant \frac{3}{4}.$$

The latter is equivalent to

$$-\sum \frac{A}{S+A} \geqslant -\frac{3}{4} \text{ or } \sum \left(1 - \frac{A}{S+A}\right) \geqslant \frac{9}{4},$$

or

$$4S \sum \frac{1}{S+A} \geqslant 9,$$

which follows from the Cauchy-Schwartz inequality.

Problem 2. Let ABC be a triangle and consider the points $M \in (BC)$, $N \in (CA)$, $P \in (AB)$ such that $\frac{AP}{PB} = \frac{BM}{MC} = \frac{CN}{NA}$. Prove that if MNP is an equilateral triangle, then ABC is an equilateral triangle as well.

Solution. Let $\lambda = \frac{AP}{AB} = \frac{BM}{BC} = \frac{CN}{CA}$.

We use complex numbers and we choose the point M as origin. Furthermore, we can assume that the complex numbers corresponding to the points N and P are 1 and $\varepsilon = \cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}$, respectively.

Suppose that the complex numbers corresponding to the points A, B, C are a, b, c, respectively. We have then

$$\varepsilon = (1-\lambda)a + \lambda b, \quad 0 = (1-\lambda)b + \lambda c, \quad \text{and} \quad 1 = (1-\lambda)c + \lambda a.$$

It follows that $\frac{c-a}{b-a} = \varepsilon$. Therefore, $AC = AB$ and $A = \frac{\pi}{3}$.

Problem 3. A prism is called *binary* if one can assign to each of its vertices a number from the set $\{-1, +1\}$, in such a way that the product of the numbers assigned to the vertices of every face equals -1.

a) Prove that the number of vertices of every *binary* prism is divisible by 8.

b) Prove that there are binary prisms with 2000 entries.

Solution. a) Suppose the base of the prism is a polygon with n vertices. Then the product of the numbers assigned to the vertices of the lateral faces equals $(-1)^n$, but in the same time it must be equal to 1, since every vertex is counted twice. It follows that n is an even number.

Now, if $n = 4k+2$, for some k, then we consider the product of the numbers assigned to the vertices of every second lateral face. We obtain $(-1)^{2k+1} = -1$. This equals the product of all numbers, that is 1, which is a contradiction. This proves the result.

b) Label the vertices $A_1, A_3, A_5, \ldots, A_{997}$ with -1 and label the rest of the base vertices with 1. For the upper base, label all with 1, except A_{999}, labelled -1.

Problem 4. a) Find two sets X, Y such that $X \cap Y = \emptyset$, $X \cup Y = \mathbb{Q}_+^*$ and $Y = \{a \cdot b \mid a, b \in X\}$.

b) Find two sets U, V such that $U \cap V = \emptyset$, $U \cup V = \mathbb{R}$ and $V = \{x + y \mid x, y \in U\}$.

Solution. a) As an example, we can choose X as the set of all products of the type $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are distinct prime numbers, α_i are integers and $\sum_{i=1}^{n} \alpha_i$ is odd. Finally, we set $Y = \mathbb{Q}_+^* \setminus X$.

b) Choose

$$U = \bigcup_{k \in \mathbb{Z}} [3k+1, 3k+2) \quad \text{and} \quad V = \mathbb{R} \setminus U.$$

It is not difficult to check that these sets satisfy the requested conditions.

11th GRADE

Problem 1. Let $x > 0$ be a real number and let A be a 2×2 matrix with real entries, such that

$$\det(A^2 + xI_2) = 0.$$

Prove that

$$\det(A^2 + A + xI_2) = x.$$

Solution. We have $\det(A + \mathrm{i}\sqrt{x}I_2) \cdot \det(A - \mathrm{i}\sqrt{x}I_2) = 0$; therefore, denoting by d the determinant of A and by t its trace, it results $d = x$ and $t = 0$, hence $A^2 + xI_2 = O_2$. It follows that $\det(A^2 + A + xI_2) = \det(A) = x$.

Problem 2. Let $n, p \geqslant 2$ be integer numbers and let A be a $n \times n$ real matrix such that $A^{p+1} = A$.

a) Prove that $\operatorname{rank}(A) + \operatorname{rank}(I_n - A^p) = n$.

b) Prove that if p is a prime number, then

$$\operatorname{rank}(I_n - A) = \operatorname{rank}(I_n - A^2) = \cdots = \operatorname{rank}(I_n - A^{p-1}).$$

Solution. a) The Sylvester inequality yields $\operatorname{rank}(A) + \operatorname{rank}(I_n - A^p) \leqslant \operatorname{rank}(A(I_n - A^p)) + n = n$.

On the other hand, $\operatorname{rank}(A) + \operatorname{rank}(I_n - A^p) \geqslant \operatorname{rank}(A^p) + \operatorname{rank}(I_n - A^p) \geqslant \operatorname{rank}(A^p + (I_n - A^p)) = n$.

b) Observe that if $k, m \in \mathbb{N}^*$ and $k \mid m$ then $\operatorname{rank}(I_n - A^k) \geqslant \operatorname{rank}(I_n - A^m)$. Indeed, $I_n - A^m$ can be written as a product of two matrices, one of them being $I_n - A^k$, and $\operatorname{rank}(XY) \leqslant \operatorname{rank}(X)$ for all matrices X, Y.

Let $k \in \mathbb{N}, 1 \leqslant k \leqslant p-1$. We have $A^{kp+1} = A$ for all $k \in \mathbb{N}$. Since p is a prime number, the remainders of the numbers $p+1, 2p+1, \ldots, kp+1$ when divided by k are pairwise distinct. Therefore, one of these numbers, say $t = qp+1$, is divisible by k. Thus, $\operatorname{rank}(I_n - A) \geqslant \operatorname{rank}(I_n - A^k) \geqslant \operatorname{rank}(I_n - A^t) = \operatorname{rank}(I_n - A^{pq+1}) = \operatorname{rank}(I_n - A)$.

Problem 3. The sequence of real numbers $(x_n)_{n\geqslant 0}$ satisfies

$$(x_{n+1} - x_n)(x_{n+1} + x_n + 1) \leqslant 0, \quad n \geqslant 0.$$

a) Prove that the sequence is bounded.

b) Can such a sequence be divergent?

Solution. a) The hypothesis implies $x_{n+1}^2 + x_{n+1} \leqslant x_n^2 + x_n$, whence the sequence $y_n = x_n^2 + x_n$ is decreasing.

Since (y_n) is clearly bounded from below, it is a convergent sequence. Therefore, (x_n) is bounded.

b) The answer is "yes"; an example is the sequence $x_n = \frac{-1+(-1)^n}{2}$.

Problem 4. We say that a function $f : \mathbb{R} \to \mathbb{R}$ has the property (P) if for every real x,

$$\sup_{t\leqslant x} f(t) = x.$$

a) Give an example of a function having property (P) which is discontinuous at every real point.

b) Prove that if f is continuous and has property (P) then f is the identical function.

Solution. a) An example is

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}; \\ x-1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

b) Observe that $\sup\limits_{t\leqslant x} f(t) = \sup\limits_{y\leqslant t\leqslant x} f(t)$, for all $y \leqslant x$.

Since f is continuous, for each $n \in \mathbb{N}^*$, there exists $x_n < x$, such that

$$|f(t) - f(x)| < \frac{1}{n},$$

for all $t \in [x_n, x]$. Consequently,

$$\Big| \sup_{x_n\leqslant t\leqslant x} f(t) - f(x) \Big| \leqslant \frac{1}{n},$$

that is, $|x - f(x)| \leqslant \frac{1}{n}$, for all $n \in \mathbb{N}^*$. It follows that $f(x) = x$.

12th GRADE

Problem 1. Let $f_1, f_2, \ldots, f_n : [0,1] \to (0,\infty)$ be continuous functions and let σ be a permutation of the set $\{1, 2, \ldots, n\}$. Prove that

$$\prod_{i=1}^{n} \int_0^1 \frac{f_i^2(x)}{f_{\sigma(i)}(x)}\,\mathrm{d}x \geqslant \prod_{i=1}^{n} \int_0^1 f_i(x)\,\mathrm{d}x.$$

Solution. Since $f_i(x) > 0$ for $x \in [0,1]$, $i = 1, 2, \ldots, n$, we can use Cauchy-Schwartz inequality:

$$\left(\int_0^1 \frac{f_i^2(x)}{f_{\sigma(i)}(x)}\,\mathrm{d}x\right)\left(\int_0^1 f_{\sigma(i)}(x)\,\mathrm{d}x\right) \geqslant \left(\int_0^1 f_i(x)\,\mathrm{d}x\right)^2,$$

for each $i = 1, 2, \ldots, n$. Taking the product of these inequalities yields the result.

Problem 2. Let $G = \{A \in \mathcal{M}_2(\mathbb{R}) \mid \det(A) = \pm 1\}$ and $H = \{A \in \mathcal{M}_2(\mathbb{C}) \mid \det A = 1\}$. Prove that, under matrix multiplication, G and H are non-isomorphic groups.

Solution. It is not difficult to show that G and H are groups. If they were isomorphic, then the equation $X^2 = I_2$ should have the same number of solutions in both groups. Cayley theorem implies that this equation has exactly two solutions in H, namely $\pm I_2$.

Since the equation has other solutions in $G \setminus H$, e.g.,

$$X = \begin{pmatrix} 0 & a \\ 1/a & 0 \end{pmatrix}, \quad a \in \mathbb{C}^*,$$

it follows that G and H are not isomorphic.

Problem 3. Let A be a finite commutative ring having at least two elements. Prove that for every positive integer $n \geqslant 2$, there exists a polynomial $f \in A[X]$, with $\deg f = n$, having no roots in A.

Solution. Observe that the function $\varphi : A \to A$, $\varphi(x) = x^n - x$, is not one-to-one, since $\varphi(0) = 0 = \varphi(1)$.

Because A is a finite set, it follows that φ is not onto either.

Therefore, one can find $a \in A \setminus \operatorname{Im} \varphi$. But then, the polynomial $f = X^n - X - a$ has no roots in A.

Problem 4. Let $\mathcal{F} = \{f : [0,1] \to [0,\infty) \mid f \text{ continuous}\}$ and let $n \geqslant 2$ be a positive integer. Determine the least real constant c, such that

$$\int_0^1 f(\sqrt[n]{x})\mathrm{d}x \leqslant c \int_0^1 f(x)\mathrm{d}x$$

for every $f \in \mathcal{F}$.

Solution. Substitute $\sqrt[n]{x} = t$ to obtain

$$\int_0^1 f(\sqrt[n]{x})\mathrm{d}x = n \int_0^1 t^{n-1} f(t)\mathrm{d}t \leqslant \int_0^1 f(t)\mathrm{d}t,$$

hence $c \leqslant n$.

For $p > 0$, the function $f_p : [0,1] \to [0,1]$, $f_p(x) = x^p$, belongs to $\mathcal{F}$.

$\int_0^1 x^{\frac{p}{n}}\mathrm{d}x \leqslant c\int_0^1 x^p \mathrm{d}x$ implies $\frac{n}{n+p} \leqslant \frac{c}{p+1}$, therefore $c \geqslant \frac{pn+n}{p+n}$.

Finally, $c \geqslant \lim\limits_{p\to\infty} \frac{pn+n}{p+n} = n$, that is, $c \geqslant n$. Consequently, $c = n$.

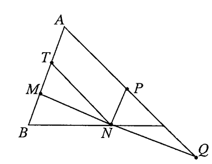
PROBLEMS AND SOLUTIONS

FINAL ROUND

7^{th} GRADE

Problem 1. Consider the triangle ABC and points M, N belonging to the sides AB, BC respectively, such that $\frac{2 \cdot CN}{BC} = \frac{AM}{AB}$. Let P be a point on AC. Prove that the lines MN and NP are perpendicular if and only if PN bisects the angle $\angle MPC$.

Solution. Let T be the intersection point of the parallel to AC which contains N with line AB. From $\frac{CN}{BC} = \frac{AT}{AB}$ we get $AM = 2 \cdot AT$, thus T is the midpoint of AM.



Denote by Q the intersection point of MN and AC. In triangle PMQ the point N is the midpoint of MQ.

In the triangle PMQ, PN is a median, thus PN is perpendicular to MN if and only if PN bisects the angle $\angle MPC$.

Problem 2. A square of side n is divided into n^2 unit squares each colored red, yellow or green. Find the minimum value of n such that for any such coloring we can find a row or a column containing at least three squares of the same color.

Solution. The number is 7. For $n = 7$, at least 17 squares have the same color by the PGH principle ($49 = 3 \cdot 16 + 1$).

As $17 = 7 \cdot 2 + 3$, we get, by the same principle, that among the 7 rows there is one containing three squares of the same color. The same argument works for columns.

The fact that for $n = 6$ the result is no more valid is given by the following example.

r	g	a	r	g	a
g	a	r	q	a	r
a	r	g	a	r	g
r	g	a	r	g	a
g	a	r	g	a	r
a	r	g	a	r	g

The same table can be used to find counterexamples for any $n \leqslant 6$.

Problem 3. In the acute triangle ABC angle C equals $45°$. Points A_1 and B_1 are the foots of the perpendiculars from A and B respectively. Denote by H the ortocenter of ABC. Points D and E are situated on the segments AA_1 and BC, respectively, such that $A_1D = A_1E = A_1B_1$. Prove that:

a) $A_1B_1 = \sqrt{\frac{A_1B^2 + A_1C^2}{2}}$;

b) $CH = DE$.

Solution. a) As the triangle ABC is acute, we have $\angle ABC > 45°$, so the midpoint M of BC is situated on the segment A_1C. We get $B_1M = \frac{BC}{2} = \frac{A_1B + A_1C}{2}$ and $A_1M = MB - A_1B = \frac{BC}{2} - A_1B = \frac{A_1C - A_1B}{2}$.

In the right triangle MA_1B_1 we also have

$$A_1B_1^2 = A_1M^2 + B_1M^2 = \left(\frac{A_1B + A_1C}{2}\right)^2 + \left(\frac{A_1C - A_1B}{2}\right)^2 = \frac{A_1B^2 + A_1C^2}{2},$$

thus

$$A_1B_1 = \sqrt{\frac{A_1B^2 + A_1C^2}{2}}.$$

b) As the right triangle DA_1E is isosceles we have succesively

$$DE = A_1E \cdot \sqrt{2} = A_1B_1 \cdot \sqrt{2} = \sqrt{A_1B^2 + A_1C^2} = \sqrt{A_1B^2 + A_1A^2} = AB.$$

The equality of triangles AA_1B and CA_1H implies $AB = CH$, and, as a consequence $CH = DE$.

Problem 4. Let A be a set of nonnegative integers containing at least two elements and such that for any $a, b \in A$, $a > b$, we have $\frac{[a,b]}{a-b} \in A$. Prove that the set A contains exactly two elements.

($[a, b]$ denotes the least common multiple of a and b).

Solution. We begin by proving that A is finite. For, if $b = \min A$ and $a \in A \setminus \{b\}$, then from $(a-b)|[a,b]$ we get $(a-b)|ab$. As $(a-b)|(a-b)$ we get $(a-b)|ab - b(a-b)$, thus $a - b|b^2$, and, in turn, $a \leqslant b + b^2$. But $a \in A$ was arbitrarily chosen, so A is finite.

Put $a = \max A$ and $b = \min A$. If $d = (a, b)$, then $b = dx, a = dy$, with $x, y \in \mathbb{N}^*$ and $(x, y) = 1$. Then $\frac{[a,b]}{a-b} = \frac{xy}{y-x} \in \mathbb{N}^*$. As x, y and $x - y$ are mutually coprime, we deduce $y - x = 1$ or $y = x + 1$, implying $a = d(x+1)$ and $b = dx$. Then $\frac{[a,b]}{a-b} = x(x+1) \in A$, from which $b \leqslant x(x+1) \leqslant a$ or $d \in \{x, x+1\}$.

First case. $d = x$.

We have $a = x(x+1)$ and $b = x^2$. We show that A has no other elements.

By contradiction if $c = \min(A \setminus \{b\})$, we get, as before, $d', z \in \mathbb{N}^*$ such that $a = d'(z+1)$ and $c = d'z$. Then $\frac{[a,c]}{a-c} = z(z+1) \in A$. As $z(z+1) \neq x^2 = b$, or $c \leqslant z(z+1) \leqslant a$ we obtain $d' \in \{z, z+1\}$.

If $d' = z$, then $a = z(z+1)$ and, as $a = x(x+1)$, we would have $x = z$, a contradiction (this would leads to $b = c$).

If $d' = z + 1$, then $a = (z+1)^2$ and $a = x(x+1)$, a contradiction. Thus, in this case A has exactly two elements.

Second case. $d = x + 1$.

We have $a = (x+1)^2$ and $b = x(x+1)$. As in the previous case it is easy to show that A has no other elements.

8^{th} GRADE

Problem 1. Consider a convex poliedra with 6 faces each of them being a circumscribed quadrilateral. Prove that all faces are circumscribed quadrilaterals.

Solution. It is known that a quadrilateral is circumscriptible if the sums of opposite sides are the same. If 5 of the faces of the convex body are circumscribed quadrilaterals, we can suppose that the body is $ABCDA'B'C'D'$ with quadrilaterals $ABCD$ and $A'B'C'D'$ opposite. Denote by x, y, z, t and x', y', z', t' and a, b, c, d the sides AB, BC, CD, DA and $A'B', B'C', C'D', D'A'$ and AA', BB', CC', DD' respectively. Suppose all faces except $A'B'C'D'$ are circumscribed quadrilaterals. Then

$$\begin{aligned} x' + y' &= (x - a - b) + (z - c - d) = (x + z) - a - b - c - d \\ &= (t + y) - a - b - c - d = t' + y', \end{aligned}$$

thus $A'B'C'D'$ is also circumscribed.

Problem 2. Given a positive integer n, prove that there exists an integer k, $k \geqslant 2$ and numbers $a_1, a_2, \ldots, a_k \in \{-1, 1\}$ such that

$$n = \sum_{1 \leqslant i < j \leqslant k} a_i a_j.$$

Solution. Consider the identity

$$(a_1 + a_2 + \cdots + a_k)^2 = a_1^2 + a_2^2 + \cdots + a_k^2 + 2 \sum_{1 \leqslant i < j \leqslant k} a_i a_j.$$

Thus, the problem amounts to finding an integer $k \geqslant 2$, and $a_1, a_2, \ldots, a_k \in \{-1, 1\}$ such that $2n = (a_1 + a_2 + \cdots + a_k)^2 - (a_1^2 + a_2^2 + \cdots + a_k^2) = (a_1 + a_2 + \cdots + a_k)^2 - k$.

Let m be the number of ones in the sequence $a_1, a_2, \ldots, a_k$ and $p = k - m$ the number of minus ones. We have $2n = (m - p)^2 - k$, or, denoting by $l = m - p$, $2n = l^2 + l - 2m$. We have to find $l, m \in \mathbb{N}$.

Take $l \in \mathbb{N}$, $l \geqslant 2$ with $l^2 + l \geqslant 2n$ and $m = \frac{l^2 + l - 2n}{2} \in \mathbb{N}$.

Then $k = -l + 2m = l^2 - 2n$ satisfies the given condition with $a_1 = a_2 = \cdots = a_m = 1$ and the remaining ones equal to -1.

Problem 3. Let $ABCDA_1B_1C_1D_1$ be a cube and let P be a variable point on side $[AB]$. The plane through P, perpendicular to AB meets AC at Q. Let M and N be the midpoints of the segments A_1P and BQ, respectively.

a) Prove that the lines MN and BC_1 are perpendicular if and only if P is the midpoint of AB.

b) Find the minimal value of the angle between the lines MN and BC_1.

Solution. a) Denote by O the center of the square BCC_1B_1. If P is the midpoint of AB, then Q is the midpoint of AC_1, thus $PBOQ$ is a parallelogram. This means that the points P, N and O are collinear and MN is parallel to A_1O. As the triangle A_1BC_1 is equilateral, we get $A_1O \perp BC_1$, thus $MN \perp BC_1$.

For the converse, MN is perpendicular to BC_1, and, as BC_1 is also perpendicular to A_1O we have that $A_1O \parallel MN$, or $BC_1 \perp (A_1OP)$. But as BC_1 is not perpendicular to OP, we must have $A_1O \parallel MN$. This means that N is the midpoint of OP.

It follows that $PBOQ$ is a parallelogram and as a consequence of the fact that Q is the midpoint of AC_1, we get that P is also the midpoint of AB.

b) Let U be the point where the parallel through Q to AB meets the line BC_1. As $QPBU$ is a parallelogram we get $PN = NU$, thus MN bisects the sides A_1P and A_1U of the triangle. As a consequence, the angle between MN and BC_1 equals the angle between A_1U and BC_1. The triangle A_1BC_1 is equilateral. This implies that the angle between the lines A_1U and BC_1 is at least $60°$. Equality occurs for $P = A$ or $P = B$.

Problem 4. Consider real numbers a, b, c contained in the interval $[\frac{1}{2}, 1]$. Prove that

$$2 \leqslant \frac{a+b}{1+c} + \frac{b+c}{1+a} + \frac{c+a}{1+b} \leqslant 3.$$

Solution. We begin by proving the lefthand-side inequality. Since $a, b \geqslant \frac{1}{2}$, we have $a + b \geqslant 1$, thus

$$\frac{a+b}{1+c} \geqslant \frac{a+b}{a+b+c}$$

and the like.

Summing up the three we obtain

$$2 = \frac{(a+b)+(b+c)+(c+a)}{a+b+c} \leqslant \frac{a+b}{1+c} + \frac{b+c}{1+a} + \frac{c+a}{1+b}.$$

For the second inequality, observe that the considered expression can be written

$$\sum\left(\frac{a}{1+c}+\frac{c}{1+a}\right).$$

As $a, c \leqslant 1$, we have $\frac{a}{1+c} \leqslant \frac{a}{a+c}$ and $\frac{c}{1+a} \leqslant \frac{c}{c+a}$, so

$$\frac{a}{1+c}+\frac{c}{1+a} \leqslant \frac{a}{a+c}+\frac{c}{c+a}=1$$

and the like. Summing up the three we get the desired result.

9th GRADE

Problem 1. Find the maximum value of

$$(x^3+1)(y^3+1),$$

for $x, y \in \mathbb{R}$ such that $x+y=1$.

Solution. Put $xy=t$; as $x+y=1$ we get $(x^3+1)(y^3+1)=t^3-3t+2$.

From $x+y=1$ we obtain $t=xy \leqslant \left(\frac{x+y}{2}\right)^2=\frac{1}{4}$. It is easy to prove that $t^3-3t+2 \leqslant 4$ for $t \leqslant \frac{1}{4}$, with equality if and only if $t=-1$.

We infer that $(x^3+1)(y^3+1) \leqslant 4$ for $x, y \in \mathbb{R}$ with $x+y=1$ and $(\phi^3+1)(-1/\phi^3+1)=4$, where ϕ is one of the roots of $z^2-z-1=0$.

Remark. In fact, for $x, y \in \mathbb{R}$, we have

$$[x^3+(x+y)^3][y^3+(x+y)^3] \leqslant 4(x+y)^6,$$

with equality if and only if $x^2+3xy+y^2=0$.

Problem 2. Consider the triangles ABC and DBC such that $AB=BC$, $DB=DC$ and $\angle ABD=90°$. Let M be the midpoint of BC. Points E, F, P are such that $E \in (AB)$, $P \in (MC)$, $C \in (AF)$ and $\angle BDE=\angle ADP=\angle CDF$. Prove that P is the midpoint of EF and $DP \perp EF$.

Solution. Put $u=\angle BDE=\angle MDP=\angle CDF$. In the right triangles DBE, DMP, DCP we have

$$\cos\angle BDE=\frac{BD}{DE}, \quad \cos\angle MDP=\frac{DM}{DP}, \quad \cos\angle CDF=\frac{DC}{DF}.$$

Thus $BD=DE\cos u$, $DM=DP\cos u$, $DC=DF\cos u$.

Moreover, $\angle BDC=\angle EDF$ and $\angle BDM=\angle EDP$. We get from here that the triangles DBC and DEF are similar and points M, F correspond to each other.

Point M is the midpoint of BC, implying that P is the midpoint of EF. As $DM \perp BC$ we conclude that $DP \perp EF$.

Problem 3. Consider quadrilaterals $ABCD$ inscribed in a circle of radius r, such that there is a point P on sidea CD for which $CB=BP=PA=AB$.

a) Prove that there is a configuration of points A, B, C, D, P for which the above configuration is possible.

b) Prove that for any such configuration we also have $PD=DA=r$.

Solution. a) Consider a chord AB such that $AB < r\sqrt{3}$ and P in the interior of the circle such that triangle ABC is equilateral. Let C be a point on the circle such that $BP=BC$ and $AC \cap BP \neq \emptyset$. Line PC meets again the circle at D. The configuration thus obtained fulfils the conditions in the statement.

b) Let $\angle BPC=\angle BCP=x$. As the triangle BPC is isosceles, we get $\angle PBC=180°-2x$. The quadrilateral $ABCD$ is cyclic, $\angle BCD=x$, thus $\angle DAB=180°-x$. Therefore, $\angle DAP=120°-x, \angle ABC=240°-2x$, and consequently, $\angle ADC=2x-60°$. In the triangle ADP we have $\angle APD=120°-x$, thus $DA=DP$.

Triangles ABD and PBD are equal, thus $\angle ABD=\angle PBD=30°$. Moreover, as $\angle ABD=30°$ we get $DP=r$.

Problem 4. A table tennis competition takes place during 4 days, the number of participants being $2n$, $n \geqslant 5$. Every participant plays exactly one game daily (it is possible that a pair of participants meet more times). Prove that such a competition can end with exactly one winner and exactly three players on the second place and such that there is no player losing all four matches. How many participants have won a single game and how many exactly two, under the above conditions?

Solution. Denote by n_k the number of participants that won exactly k games,

$0 \leqslant k \leqslant 4$. Under the given conditions we have

$$n_0 = 0, \quad n_1 + n_2 + n_3 + n_4 = 2n \geqslant 10. \tag{1}$$

The total number of games is $4n$, thus

$$4n = 1 \cdot n_1 + 2 \cdot n_2 + 3 \cdot n_3 + 4 \cdot n_4 \text{ (counting the winners)} \tag{2}$$

$$4n = 3 \cdot n_1 + 2 \cdot n_2 + 1 \cdot n_3 + 0 \cdot n_4 \text{ (counting the loosers)} \tag{3}$$

thus $2n_1 = 2n_3 + 4n_4$. Substituing in (1) we obtain

$$n_2 + 2n_3 + 3n_4 = 2n. \tag{4}$$

The other conditions of the problem will imply

n_4	n_3	n_2	n_1		$n_2 + 2n_3 + 3n_4$
0	0	1	3		1
0	1	0	3		2
1	0	0	3		3
0	1	3			5
1	0	3			6

giving a contradiction.

It remains the case $n_4 = 1, n_3 = 3$, which implies $n_2 = 2n - 9$, $n_1 = 5$.

For a model, denote by a the winner; by b_1, b_2, b_3 those on the second place; by c one of the $2n-9$ winners of exactly two games and by d_1, d_2, d_3, d_4, d_5 those five with only one won game. The remaining $2n - 10$ players having won two games, will be denoted (for $n > 5$) by $c_1, \ldots, c_{2n-10}$. Finally, by xy we will mean that x won the game against y.

Day						
1	ab_1	cd_2	b_2d_3	b_3d_4	d_1d_5	c_ic_{i+1}
2	ab_2	b_1d_1	cd_3	b_3d_4	d_2d_5	c_ic_{i+1}
3	ab_3	b_1d_1	b_2d_2	d_5c	d_3d_4	$c_{i+1}c_i$
4	ac	b_1d_1	b_2d_2	b_3d_3	d_4d_5	$c_{i+1}c_i$,

where $i = 1, 3, \ldots, 2n - 11$.

10th GRADE

Problem 1. Consider a set M with n elements and let $\mathcal{P}(M)$ denote all subsets of M. Find all functions $f : \mathcal{P}(M) \to \{0, 1, 2, \ldots, n\}$, satisfying the following two conditions:

a) $f(A) \neq 0$, for any $A \neq \emptyset$, and

b) $f(A \cup B) = f(A \cap B) + f(A \Delta B)$, for any $A, B \in \mathcal{P}(M)$, where $A \Delta B = (A \cup B) \setminus (A \cap B)$.

Solution. From condition b) we obtain

$$f(\emptyset \cup \emptyset) = f(\emptyset \cap \emptyset) + f(\emptyset \Delta \emptyset),$$

giving $f(\emptyset) = 0$.

By b), for $A, B \in \mathcal{P}(M)$, with $A \subsetneq B$, we get

$$f(B) = f(A \cup B) = f(A) + f(B \setminus A).$$

From a) we have $f(B \setminus A) \neq 0$, thus $f(B) > f(A)$.

Consequently, for any permutation $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of the set M we have the sequence of inequalities

$$0 = f(\emptyset) < f(\{\alpha_1\}) < f(\{\alpha_1, \alpha_2\}) < \cdots < f(\{\alpha_1, \alpha_2, \ldots, \alpha_n\}).$$

Since $f(A) \in \{0, 1, \ldots, n\}$, for all $A \in \mathcal{P}(M)$, it follows that

$$f(\{\alpha_1, \alpha_2, \ldots, \alpha_j\}) = j,$$

for any $j \in \{1, 2, \ldots, n\}$.

Consequently, $f(A) = |A|$, for any $A \in \mathcal{P}(M)$. It is easy to see that this function fulfils the given conditions.

Problem 2. Prove that for $a, b \in \left(0, \frac{\pi}{4}\right)$ we have

$$\frac{\sin^n a + \sin^n b}{(\sin a + \sin b)^n} \geqslant \frac{\sin^n 2a + \sin^n 2b}{(\sin 2a + \sin 2b)^n}.$$

Solution. Let $n \in \mathbb{N}^*$ and consider the function $f_n : [0,\infty) \to \mathbb{R}$, $f_n(x) = (\frac{1}{2} - t)^n + (\frac{1}{2} + t)^n$ which, as

$$f_n(t) = s \sum_{k=0}^{\left[\frac{n}{2}\right]} C_n^{2k} \left(\frac{1}{2}\right)^{n-2k} t^{2k}, \quad t \geqslant 0$$

is increasing. Let $x_1, x_2, y_1, y_2 \in (0,1)$ such that $x_1 + x_2 = y_1 + y_2 = y_1 + y_2 = 1$ and $x_1 x_2 \leqslant y_1 y_2$. We show that $x_1^n + x_2^n \geqslant y_1^n + y_2^n$, for any $n \in \mathbb{N}^*$. By symmetry, we may suppose $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$. Then, denoting $t = \frac{1}{2} - x_1 = x_2 - \frac{1}{2} \geqslant 0$ and $s = \frac{1}{2} - y_1 = y_2 - \frac{1}{2} \geqslant 0$, we have $t \geqslant s \geqslant 0$. Thus

$$x_1^n + x_2^n = f_n(t) \geqslant f_n(s) = y_1^n + y_2^n, \quad \text{for any } n \in \mathbb{N}^*.$$

For $a, b \in \left(0, \frac{\pi}{4}\right)$, put

$$x_1 = \frac{\sin a}{\sin a + \sin b}, \quad x_2 = \frac{\sin b}{\sin a + \sin b},$$

$$y_1 = \frac{\sin 2a}{\sin 2a + \sin 2b}, \quad y_2 = \frac{\sin 2b}{\sin 2a + \sin 2b}.$$

Clearly, $x_1, x_2, y_1, y_2 \in (0,1)$ and $x_1 + x_2 = y_1 + y_2$. The inequality $x_1 x_2 \leqslant y_1 y_2$ is equivalent to

$$(\cos a - \cos b)^2 (\cos^2 a + \cos^2 b + \cos a \cos b - 1) \geqslant 0,$$

which is true for any $a, b \in \left(0, \frac{\pi}{4}\right)$. This ends the proof.

Problem 3. Prove that the sequence given by $a_n = \left[n\sqrt{2}\right] + \left[n\sqrt{3}\right]$, $n \in \mathbb{N}$, contains an infinity of odd nubers and an infinity of even numbers.

Solution. Let $x_n = \left[n\sqrt{2}\right]$, $y_n = \left[n\sqrt{3}\right]$, $n \in \mathbb{N}$. We have $x_{n+1} - x_n$, $y_{n+1} - y_n \in \{1, 2\}$, for all $n \in \mathbb{N}$.

Suppose, by way of contradiction, that there is $k \in \mathbb{N}$ such that all elements a_n, $n \geqslant k$, have the same parity. As $2 \leqslant a_{n+1} - a_n \leqslant 4$, for any $n \in \mathbb{N}$, we get that $a_{n+1} - a_n \in \{2, 4\}$, for all $n \geqslant k$.

If $a_{n+1} - a_n = 2$, then $x_{n+1} - x_n = y_{n+1} - y_n = 1$, and if $a_{n+1} - a_n = 4$, then $x_{n+1} - x_n = y_{n+1} - y_n = 2$.

Thus $y_n - x_n = y_{n+1} - x_{n+1}$, for all $n \geqslant k$, which gives $y_n - x_n = y_k - x_k$, for all $n \geqslant k$.

But $y_n - x_n > n\sqrt{3} - 1 - n\sqrt{2}$, for any n, so

$$n < \frac{y_k - x_k + 1}{\sqrt{3} - \sqrt{2}},$$

for all $n \geqslant k$, a contradiction.

Problem 4. Given $n \in \mathbb{N}$, $n \geqslant 2$, find n disjoint sets A_i, $1 \leqslant i \leqslant n$, in the plane, such that:

a) for any disk $\mathcal{C}$ and any $i \in \{1, 2, \ldots .n\}$, we have $A_i \cap \operatorname{Int}(\mathcal{C}) \neq \emptyset$; and

b) for any line d and any $i \in \{1, 2, \ldots .n\}$, the projection of A_i on d is not all of d.

Solution. There are a lot of natural ways to construct such sets. For example, take $p_1, p_2, \ldots, p_n$ square free positive integers and consider the sets

$$A_k = \left\{ \left(\frac{m_1}{q_1\sqrt{p_k}}, \frac{m_2}{q_2\sqrt{p_k}} \right) \mid m_1, m_2, q_1, q_2 \in \mathbb{Z}^* \right\}.$$

It is easy to see that any such set is countable and dense in the plane.

11th GRADE

Problem 1. A is a square matrix with complex entries. Denote by A^* its adjoint (the matrix formed by the cofactors of the transpose). Prove that if there is an integer $m \geqslant 1$ such that $(A^*)^m = 0_n$, then $(A^*)^2 = 0_n$.

Solution. From $\det(A^*)^m = 0$, we have $\det(A^*) = 0$ and $\det(A) = 0$. We claim that the rank of A^* is at most 1. For if rank$A \leqslant n-2$ then $A^* = 0_n$, and if rank $A = n-1$ then $AA^* = 0_n$ and by Sylvester's inequality $0 = \operatorname{rank}(AA^*) \geqslant \operatorname{rank}(A) + \operatorname{rank}(A^*) - n = \operatorname{rank}(A^*) - 1$. Suppose now that $m \geqslant 3$ (otherwise, there is nothing to prove). As A has at most rank 1 there is a row matrix $X \in \mathcal{M}_{1n}(\mathbb{C})$ and a column matrix $Y \in \mathcal{M}_{n1}(\mathbb{C})$ such that $A = YX$. Denoting $XY = a \in \mathbb{C}$ we obtain $0_n = A^m = Y(XY)^{m-1}X = a^{m-1}YX = a^{m-1}A$, implying $a = 0$ or $A = 0_n$, so $A^2 = aA = 0_n$.

Problem 2. A matrix $B \in \mathcal{M}_n(\mathbb{C})$ will be called a *pseudo-inverse* of a matrix $A \in \mathcal{M}_n(\mathbb{C})$ if $A = ABA$ and $B = BAB$.

a) Prove that any square matrix has at least one pseudo-inverse.

b) Characterize the class of matrices with a unique pseudo-inverse.

Solution. a) Denote by r the rank of A. This means that by elementary transformations, A goes over to a matrix that has zeroes everywhere except the first r entries on the main diagonal. That is, there exist two invertible matrices P and Q such that PAQ has 1 on the first r entries on the main diagonal and 0 elsewhere. We thus have

$$PAQ = D = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

This means that $B = QDP$ is a pseudo-inverse of A.

b) If $A = 0_n$ then $B = 0_n$, and, if A is invertible then $B = A^{-1}$. We claim that in any other situation A has an infinite number of pseudo-inverses. Such matrices can be obtained, for instance, by replacing D by any matrix of the form

$$D_x = \begin{pmatrix} 1 & 0 & \dots & x \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

which coincides with D, except that the $(1, n)$-entry is now any complex x.

Problem 3. Consider two systems of points in the plane: $A_1, A_2, \dots, A_n$ and $B_1, B_2, \dots, B_n$ having different centers of gravity. Prove that there is a point P in the plane such that

$$PA_1 + PA_2 + \dots + PA_n = PB_1 + PB_2 + \dots + PB_n.$$

Solution. Consider a Cartesian system of coordinates such that the two centroids have different x-coordinates. Suppose coordonates are $A_i(a_i, a_i')$ and $B(b_i, b_i')$. We are looking for P on Ox: $P(p, 0)$. Consider $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(p) = PA_1 + PA_2 + \dots + PA_n - (PB_1 + PB_2 + \dots + PB_n).$$

We have

$$\lim_{p\to\infty} f(p) = \lim_{p\to\infty} \sum_{k=1}^{n} \frac{2p(b_k - a_k) + a_k^2 - b_k^2 + a_k'^2 - b_k'^2}{\sqrt{(p-a_k)^2 + a_k'^2} + \sqrt{(p-b_k)^2 + b_k'^2}} = \sum_{k=1}^{n}(b_k - a_k)$$

and

$$\lim_{p\to-\infty} f(p) = -\sum_{k=1}^{n}(b_k - a_k).$$

Since f is continuous, by the intermediate value property, $f(p) = 0$ for some real p.

Remark. The condition on the centroids is necessary only for $n \geqslant 3$. For, if $B_1, B_2, \dots, B_n$ are the mid-points of $[A_1A_2], [A_2A_3], \dots, [A_nA_1]$, P can exist, unless the A_i are collinear.

Problem 4. Consider a function $f : [0, \infty) \to \mathbb{R}$, with the property: for any $x > 0$, the sequence $(f(nx))_{n\geqslant 0}$ is increasing.

a) If f is also continuous on $[0, 1]$, does it follow that it is increasing?

b) What if f is continuous on $\mathbb{Q}_+$?

Solution. a) The answer is "no". A counterexample is

$$f(x) = \begin{cases} x & \text{if } x \in [0,1] \cup \mathbb{Q}_+; \\ 2x - 1 & \text{if } x \in (1, \infty) \setminus \mathbb{Q}. \end{cases}$$

b) The answer is "yes". The following remark is an easy consequence of the hypothesis: if (r_n) is an increasing sequence of rational numbers and x is positive, then the sequence $(f(r_n x))$ is increasing.

Suppose, by contradiction, that for some $x < y$ we have $f(x) \geqslant f(y)$. Consider a rational number a in (x, y). Find an increasing sequence of rationals (q_n) and a decreasing sequence of rationals (r_n) such that $(q_n x) \to a$ and $(r_n y) \to a$. We get, by continuity

$$f(x) < \lim_{n\to\infty} f(q_n x) = f(a) = \lim_{n\to\infty} f(r_n y) < f(y),$$

a contradiction.

12th GRADE

Problem 1. Let K be a finite field. Prove that the following are equivalent:

a) $1+1=0$;

b) for any $f \in K[X]$ with $\deg f \geqslant 1$ the polynomial $f(X^2)$ is reducible.

Solution. To prove that a) implies b), consider $F : K \to K$, given by $F(x) = x^2$. If $F(x) = F(y)$ we get $x^2 = y^2$, thus $(x-y)(x+y) = 0$, so $(x-y)^2 = 0$, or $x = y$. This means that F is one-to-one, consequently onto (for K is finite).

If $f = \sum\limits_{k=0}^{n} a_k X^k$, then there exists $b_k \in K$ such that $a_k = b_k^2$ for $k = 0, 1, \ldots, n$. Thus

$$f(X^2) = \sum_{k=0}^{n} b_k^2 X^{2k} = \sum_{k=0}^{n} b_k^2 X^{2k} + 2\sum_{i<j} b_i b_j X^{i+j} = \left(\sum_{k=0}^{n} b_k X^k\right)^2.$$

Thus $f(X^2)$ is reducible.

Conversely, take $a \in K$ and let $f = X - a$. As $g = f(X^2) = X^2 - a$ is reducible, it has a root in K. This means that F is onto and thus one-to-one. As $F(1) = F(-1) = 1$, we get $1 = -1$.

Problem 2. Prove that

$$\lim_{n\to\infty} n\left(\frac{\pi}{4} - n\int_0^1 \frac{x^n}{1+x^{2n}}\mathrm{d}x\right) = \int_0^1 f(x)\mathrm{d}x,$$

where $f(x) = \frac{\operatorname{arctg} x}{x}$, for $x \in (0,1]$ and $f(0) = 1$.

Solution. Denote

$$I = n\left(\frac{\pi}{4} - n\int_0^1 \frac{x^n}{1+x^{2n}}\,\mathrm{d}x\right).$$

We have

$$I = n\left(\frac{\pi}{4} - \int_0^1 x(\operatorname{arctg} x^n)'\,\mathrm{d}x\right)$$
$$= n\left(\frac{\pi}{4} - x\operatorname{arctg} x^n\,\Big|_0^1 + \int_0^1 \operatorname{arctg} x^n\mathrm{d}x\right) = n\int_0^1 \operatorname{arctg} x^n\mathrm{d}x$$

$$I = n\int_0^1 x^n\frac{\operatorname{arctg} x^n}{x^n}\mathrm{d}x = n\int_0^1 x\cdot x^{n-1}\frac{\operatorname{arctg} x^n}{x^n}\mathrm{d}x$$
$$= \int_0^1 x\left(\int_0^{x^n}\frac{\operatorname{arctg} t}{t}\mathrm{d}t\right)'\,\mathrm{d}x$$
$$= x\int_0^{x^n}\frac{\operatorname{arctg} t}{t}\mathrm{d}t\,\Big|_0^1 - \int_0^1\left(\int_0^{x^n}\frac{\operatorname{arctg} t}{t}\mathrm{d}t\right)\mathrm{d}x$$
$$= \int_0^1\frac{\operatorname{arctg} t}{t}\mathrm{d}t - \int_0^1\left(\int_0^{x^n}\frac{\operatorname{arctg} t}{t}\mathrm{d}t\right)\mathrm{d}x.$$

Using the inequality $\operatorname{arctg} t \leqslant t$, $t \geqslant 0$, we obtain

$$0 \leqslant \int_0^1\left(\int_0^{x^n}\frac{\operatorname{arctg} t}{t}\mathrm{d}t\right)\mathrm{d}x \leqslant \int_0^1\left(\int_0^{x^n}\mathrm{d}t\right)\mathrm{d}x = \int_0^1 x^n\,\mathrm{d}x = \frac{1}{n+1},$$

which implies

$$\lim_{n\to\infty}\int_0^1\left(\int_0^{x^n}\frac{\operatorname{arctg} t}{t}\mathrm{d}t\right)\mathrm{d}x = 0,$$

whence the conclusion.

Problem 3. Let G be a group with n elements ($n \geqslant 2$) and let p be the smallest prime factor of n. Suppose G has a unique subgroup H with p elements. Prove that H is contained in the center of G. (The center of G is the set $Z(G) = \{a \in G \mid ax = xa,\ \forall x \in G\}$.)

Solution. For any $g \in G$ the set gHg^{-1} is a subgroup of order p of G. The hypothesis this implies $gHg^{-1} = H$. Let $g \in G$ and $f : H \to H$, given by $f(x) = gxg^{-1}$. As $f(e) = e$ and f is bijective, it follows that the restriction of f to $H \setminus \{e\}$ is a permutation of the set $H \setminus \{e\}$. Thus $f^{(p-1)!} = 1_H$ and consequently $f^{(n)} = 1_H$.

As $(n, (p-1)!) = 1$ we get $f = 1_H$, thus $gx = xg$ for any $x \in H$.

Alternative solution 1. As the order p of H is prime, we deduce that H is cyclic:

$$H = \langle h\rangle = \{h, \ldots, h^{p-1}, h^p = e\}, \tag{1}$$

where e is the identity of G. It suffices to show that $gh = hg$, for any $g \in G$. Let $g \in G$ and

$$K = \{k : k \in \{1, \ldots, n\} \text{ and } g^k h = hg^k\}.$$

The set K is non-empty, for $n \in K$: $g^n = e$, thus $g^n h = eh = h = he = hg^n$. We shall prove that 1 is an element of K, thus concluding the proof. Let m be the smallest element of K. By minimality, m divides n and

$$K = \left\{km : k = 1, \ldots, \frac{n}{m}\right\}. \tag{2}$$

On the other hand, gHg^{-1} is a subgroup of order p of G. We get $gHg^{-1} = H$, i.e. $gH = Hg$. From (1) and $gH = Hg$, we deduce the existence of $k \in \{1, \ldots, p-1\}$, such that $gh = h^k g$. Consequently,

$$g^{p-1}h = h^{k^{p-1}} g^{p-1} = hg^{p-1},$$

for $k^{p-1} \equiv 1 \pmod p$, and $h^p = e$. Thus $p - 1$ is in K. By (2) m divides $p - 1$. By p is the smallest prime factor of n, we conclude that n and $p - 1$ are mutually prime, so $m = 1$.

Alternative solution 2. Let $f : G \to H$, be given by $f(x) = xax^{-1}$ with $a \in H \setminus \{e\}$, fixed. Consider $C(a) = \{x \in G \mid ax = xa\}$ and denote by q the number of elements in $C(a)$. As $f(x) = f(y)$ means $xax^{-1} = yay^{-1}$, or $y^{-1}xa = ay^{-1}x$, or $y^{-1}x \in C(a)$, from where $x \in yC(a)$, we conclude that for any $b \in \operatorname{Im} f$ the set $\{x \in G \mid f(x) = b\}$ has q elements. Thus $\operatorname{Im} f$ has $\frac{n}{q}$ elements. As $e \notin \operatorname{Im} f$ we get $\frac{n}{q} \leqslant p - 1$ and as $\frac{n}{q} | n$, we obtain $\frac{n}{q} = 1$, that is $C(a) = G$. As a consequence $ax = xa$ for any $x \in G$, meaning $a \in Z(G)$. As $e \in Z(G)$ we get $H \subset Z(G)$.

Remarks. The last proof can be translated as:

Because H is the unique subgroup with p elements of G, H is normal in G. Let $a \in H \setminus \{e\}$. G acts on H by conjugacy, thus

$$|G| = |\operatorname{Stab}(a)| \cdot |\operatorname{Orb}(a)|.$$

As $e \notin \operatorname{Orb}(a)$ we get $|\operatorname{Orb}(a)| \leqslant p - 1$ thus $|\operatorname{Orb}(a)| = 1$. It results $\operatorname{Stab}(a) = G$, or $a \in Z(G)$.

The simplest example of a noncomutative group that satisfies the given condition the multiplicative group of quaternions $\{\pm 1, \pm i, \pm j, \pm k\}$; multiplication is completely given by $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. The only subgroup of order 2 is the center $\{\pm 1\}$.

The conclusion still holds if we consider any prime factor p of n with n and $p - 1$ coprime.

Problem 4. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(x)\mathrm{d}x = 0.$$

Prove that there is $c \in (0,1)$ such that

$$\int_0^c xf(x)\mathrm{d}x = 0.$$

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Solution. Consider $F(x) = \int_0^x tf(t)\mathrm{d}t$ defined on $[0,1]$. By the l'Hospital rule it is easily seen that

$$\lim_{x \to 0} \frac{F(x)}{x^2} = 0. \tag{1}$$

Take $a \in (0,1)$ and integrate by parts

$$\int_a^1 f(x)\mathrm{d}x = \frac{1}{x} \cdot F(x)|_a^1 + \int_a^1 F(x) \cdot \frac{1}{x}\mathrm{d}x.$$

Taking limits as $a \to 0$ and using (1), we obtain

$$0 = \int_0^1 f(x)\mathrm{d}x = \frac{1}{x} \cdot F(x)|_0^1 + \int_0^1 F(x) \cdot \frac{1}{x}\mathrm{d}x = F(1) + \int_0^1 F(x) \cdot \frac{1}{x}\mathrm{d}x.$$

The last relation actually ends the proof: if $F(1) > 0$ there will be a point x_0 such that $F(x_0) > 0$ and, similarly for $F(1) < 0$. The intermediate value property gives the desired c where $F(c) = 0$. It is an easily seen that c can be choosen in $(0,1)$.

Remark. The result stil holds for a Lebesgue integrable f such that $\int_0^1 f(x)\mathrm{d}x = 0$ and an increasing function $g : [0,1] \to \mathbb{R}$, continuous at 0 and such that $g(0) = 0$. There is a c such that $\int_0^c f(x)g(x)\mathrm{d}x = 0$. Details will be given elsewhere.

PROBLEMS AND SOLUTIONS

IMO AND BMO SELECTION TESTS

Problem 1. Let ABC and AMN be two similar triangles with the same orientation, such that $AB = AC$, $AM = AN$, and having disjoint interiors. Let O be the circumcenter of the triangle MAB. Prove that the points O, C, N, A are concyclic if and only if the triangle ABC is equilateral.

Solution. Let $\alpha = \angle BAC = \angle MAN$. We consider the rotation of center A and angle α; from the hypothesis we infer that B is mapped onto C, and M is mapped onto N. This means that the triangle BAM is transformed into the triangle CAN, and thus O is mapped onto O', the circumcenter of the triangle CAN. Moreover, $\angle OAO' = \alpha$ and $OA = O'A$.

The condition that O, C, N, A lie on the same circle is equivalent to $O'O = O'A$ (as already O' is the circumcenter of the triangle CAN). But then the triangle $O'AO$ is equilateral, therefore $\alpha = 60°$, and the triangles ABC and AMN are also equilateral. The above reasoning works both ways, so the problem is solved.

Problem 2. Let $p \geqslant 5$ be a prime number. Find the number of irreducible polynomials in $\mathbb{Z}[X]$, of the form

$$x^p + px^k + px^l + 1, \quad k > l, \; k, l \in \{1, 2, \ldots, p-1\}.$$

Solution. Let $f_{k,l}(x) = x^p + px^k + px^l + 1$, $k > l$, $k, l \in \{1, \ldots, p-1\}$. If the numbers k and l have different parities, then $f_{k,l}(-1) = 0$. For $f_{k,l}(x)$ to be irreducible in $\mathbb{Z}[X]$ it is required that the numbers k and l have the same parity; then

$$f_{k,l}(x-1) = x^p + pxg(x) + p((-1)^k + (-1)^l) = x^p + pxg(x) \pm 2p,$$

where $g(x) \in \mathbb{Z}[X]$.

By Eisenstein's criterion, $f_{k,l}(x-1)$ is irreducible in $\mathbb{Z}[X]$, thus $f_{k,l}(x)$ is irreducible in $\mathbb{Z}[X]$.

Therefore the number of polynomials $f_{k,l}(x)$, irreducible in $\mathbb{Z}[X]$, is equal to the number of pairs (k, l), in which k, l are distinct numbers, of the same parity, in the set $\{1, 2, \ldots, p-1\}$. The number of such pairs is

$$2\binom{\frac{p-1}{2}}{2} = \frac{(p-1)(p-3)}{4}.$$

Remarks. The idea of making the transformation $x \mapsto x - 1$ is suggested by the method used by Gauss to prove the irreducibility of the p-th order cyclotomic polynomial

$$\phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1.$$

Problem 3. Let a, b be positive integers such that for any positive integer n we have $a^n + n \mid b^n + n$. Prove that $a = b$.

Solution. Assume that $b \neq a$. Taking $n = 1$ shows that $a + 1$ divides $b + 1$, hence $b \geqslant a$. Let $p > b$ be a prime and let n be a positive integer such that

$$n \equiv 1 \pmod{p-1} \quad \text{and} \quad n \equiv -a \pmod{p}.$$

Such an n exists by the Chinese Remainder theorem (without the Chinese Remainder theorem, one could notice that $n = (a+1)(p-1) + 1$ has this property).

By Fermat's little theorem, $a^n = a(a^{p-1} \cdots a^{p-1}) \equiv a \pmod{p}$, and therefore $a^n + n \equiv 0 \pmod{p}$. So p divides the number $a^n + n$, hence also $b^n + n$. However, by Fermat's little theorem again, we have analogously $b^n + n \equiv b - a \pmod{p}$. We are therefore led to the conclusion $p \mid b - a$, which is a contradiction.

Remarks. The first thing coming to mind is to show that a and b share the same prime divisors. This is easily established by using Fermat's little theorem or Wilson's theorem. However, we know of no solution which uses this fact in any meaningful way.

For the conclusion to remain true, it is not sufficient that $a^n + n \mid b^n + n$ holds for infinitely many n. Indeed, take $a = 1$ and any $b > 1$. The given divisibility relation holds for all positive integers n of the form $p-1$, where $p > b$ is a prime, but $a \neq b$.

Problem 4. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $|a_i| \leqslant 1$ for all $i = 1, 2, \ldots, n$, and $a_1 + a_2 + \cdots + a_n = 0$.

(a) Prove that there exists $k \in \{1, 2, \ldots, n\}$ such that

$$|a_1 + 2a_2 + \cdots + ka_k| \leqslant \frac{2k+1}{4}.$$

(b) Prove that for $n > 2$ the bound above is the best possible.

Solution. (a) We may suppose that $a_1 > 0$, otherwise if $a_1 = 0$, $k = 1$ is a solution, and if $a_1 < 0$ we can work with the set $\{-a_i : i = \overline{1,n}\}$.

Let us define $s_0 = 0$, and $s_k = \sum\limits_{i=1}^{k} ia_i$, for $k = 1, 2, \ldots, n$. Then $a_k = \frac{s_k - s_{k-1}}{k}$, which implies

$$0 = \sum_{k=1}^{n} a_k = \frac{s_n}{n} + \sum_{k=1}^{n} \frac{s_k}{k(k+1)}.$$

Because $s_1 = a_1 > 0$ it follows that there exists (a smallest) k such that $s_k < 0$. Then

$$k \geqslant |ka_k| = |s_k - s_{k-1}| = s_{k-1} - s_k = |s_{k-1}| + |s_k|.$$

If at the same time $|s_{k-1}| > \frac{2(k-1)+1}{4} = \frac{2k-1}{4}$, and $|s_k| > \frac{2k+1}{4}$, it follows that

$$|s_{k-1}| + |s_k| > \frac{2k-1}{4} + \frac{2k+1}{4} = k,$$

in contradiction with the previous relation.

(b) We will treat separately the cases when n is odd and n is even.

CASE I. n odd ($n > 1$). We take

$$\{a_i\}_{i=1,n} = \left\{\frac{3}{4}, \frac{1}{4}, -1, 1, -1, 1, -1, \cdots\right\}.$$

Then we obtain the sequence

$$\{s_i\}_{i=1,n} = \left\{\frac{3}{4}, \frac{5}{4}, -\frac{7}{4}, \frac{9}{4}, -\frac{11}{4}, \frac{13}{4}, -\frac{15}{4}, \ldots\right\}.$$

CASE II. n even ($n > 2$). We take

$$\{a_i\}_{i=1,n} = \left\{1, \frac{1}{8}, -1, -\frac{1}{8}, 1, -1, 1, -1, \cdots\right\}.$$

Then we obtain the sequence

$$\{s_i\}_{i=1,n} = \left\{1, \frac{5}{4}, -\frac{7}{4}, -\frac{9}{4}, \frac{11}{4}, -\frac{13}{4}, \frac{15}{4}, -\frac{17}{4}, \ldots\right\}.$$

It turns out that, for $n = 3$ and $n = 4$, these sequences are also unique in disproving the possibility of lowering the bound.

Problem 5. Let $\{a_n\}_{n \geqslant 1}$ be a sequence given by $a_1 = 1$, $a_2 = 4$, and for all integers $n > 1$

$$a_n = \sqrt{a_{n-1}a_{n+1} + 1}.$$

(a) Prove that all the terms of the sequence are positive integers.

(b) Prove that the number $2a_na_{n+1} + 1$ is a perfect square for all integers $n \geqslant 1$.

Solution. (a) We rewrite the recurrence relation as

$$a_{n+1} = \frac{a_n^2 - 1}{a_{n-1}}$$

and we want to prove using induction that $a_k \in \mathbb{N}$ for all $k \leqslant n$ implies $a_{n+1} \in \mathbb{N}$. For this we require the following, stronger, statement: $a_k \in \mathbb{N}$, $\forall k \leqslant n$, and in plus $\gcd(a_k, a_{k-1}) = 1$ for all $k \leqslant n$. The initial two steps $n = 2$ and $n = 3$ are easily calculated, so we suppose that $n \geqslant 4$.

The relation $a_n = \frac{a_{n-1}^2 - 1}{a_{n-2}}$ implies that $a_{n+1} = \frac{a_{n-1}^4 - 2a_{n-1}^2 + 1 - a_{n-2}^2}{a_{n-2}^2 a_{n-1}}$.

As $a_{n-1}a_{n-3} + 1 = a_{n-2}^2$, we infer that $a_{n-1} \mid a_{n-2}^2 - 1$, therefore $a_{n-1} \mid a_{n-1}^4 - 2a_{n-1}^2 + 1 - a_{n-2}^2$. On the other hand, $a_{n-2}^2 \mid (a_{n-1}^2 - 1)^2 = a_{n-2}^2 a_n^2$, and as $\gcd(a_{n-2}, a_{n-1}) = 1$ we obtain $a_{n+1} \in \mathbb{N}$. From $a_{n+1}a_{n-1} - 1 = a_n^2$ we obtain that $\gcd(a_n, a_{n+1}) = 1$, so the induction is complete.

(b) Taking small values for n we notice that

$$2a_na_{n+1} + 1 = (a_{n+1} - a_n)^2,$$

and we will prove this relation using induction. The first step, $n = 1$, is trivial. Let $n \geqslant 2$. The relation implies

$$2a_na_{n+1} = a_{n+1}^2 - 2a_na_{n+1} + a_n^2 - 1 = a_{n+1}(a_{n+1} - 2a_n) + a_{n+1}a_{n-1},$$

therefore, by dividing both sides above with $a_{n+1} > 0$, we obtain the equivalent relation

$$4a_n = a_{n+1} + a_{n-1} \Rightarrow a_{n+1} = 4a_n - a_{n-1}.$$

We only have to prove that $a_{n+2} = 4a_{n+1} - a_n$. But $a_{n+2} = \frac{a_{n+1}^2-1}{a_n}$ so we require

$$4a_{n+1}a_n - a_n^2 = a_{n+1}^2 - 1 \Leftrightarrow 2a_na_{n+1} + 1 = (a_{n+1} - a_n)^2,$$

which is our induction hypothesis, and we are done.

ALTERNATIVE SOLUTION. (a) First notice that the sequence may be extended to the left with $x_0 = 0$ (this helps with having the recurrence relation also available for $n = 1$). Now, writing two consecutive (squared) recurrence relations yields, for $n \geqslant 1$, $x_n^2 = x_{n+1}x_{n-1} + 1$ and $x_{n+1}^2 = x_{n+2}x_n + 1$, so by subtracting, $x_{n+1}(x_{n+1} + x_{n-1}) = x_n(x_{n+2} + x_n)$, that is,

$$\frac{x_{n+2} + x_n}{x_{n+1}} = \frac{x_{n+1} + x_{n-1}}{x_n},$$

thus having constant value $(x_2+x_0)/x_1 = 4$, whence $x_{n+1} = 4x_n - x_{n-1}$ (clearly $x_n \neq 0$ for $n \geqslant 1$).

Therefore, once the first two terms are given as integers, so will all following terms be.

(b) We have, for $n \geqslant 1$, $0 = x_{n+1}(x_{n+1} - 4x_n + x_{n-1}) = x_{n+1}^2 - 4x_{n+1}x_n + x_{n+1}x_{n-1} = x_{n+1}^2 - 4x_{n+1}x_n + x_n^2 - 1 = (x_{n+1} - x_n)^2 - (2x_nx_{n+1} + 1)$, hence $2x_nx_{n+1} + 1 = (x_{n+1} - x_n)^2$ and therefore a perfect square.

Remarks. The first solution entails a longer and more ardous process, which fails to provide the linear recurrence that turns to be instrumental in proving (b).

One may (similarly) easily obtain that $3x_n^2 + 1 = (x_{n+1} - 2x_n)^2$, therefore a perfect square, thus falling over a solution family for the Pell equation $y^2 - 3x^2 = 1$.

This type of sequences and the way to attack them is quite well-known, see [A. Engel], [A. Neguţ], [V. Vornicu].

Problem 6. Let ABC be a triangle with $\angle ABC = 30°$. Consider the closed discs of radius $AC/3$ centered at A, B and C. Does there exist an equilateral triangle whose three vertices lie one each in each of the three discs?

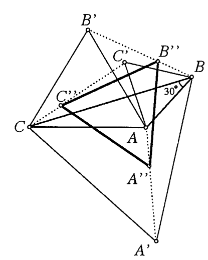
Solution. We will start with the following

LEMMA. Given two points A_1, A_2 and two closed discs centered at A_1, A_2 of radii r_1, r_2 respectively, the locus of the third vertex of an equilateral triangle with the two other vertices lying in each of the two discs and located in one of the halfplanes determined by the line A_1A_2 is the closed disc of radius $r_1 + r_2$ centered at A_3 which forms with A_1 and A_2 an equilateral triangle.

To be self-contained, we present a proof using complex numbers. Let lower-case letters represent the affixes of capital-letter points. Then $a_3 = a_1 + \omega(a_2 - a_1) = \bar{\omega}a_1 + \omega a_2$, where $\omega = \cos 60° + \mathrm{i}\sin 60°$ is the primitive 6-th root of 1, hence $|\omega| = 1$, $\bar{\omega} = 1 - \omega$ and $\omega\bar{\omega} = 1$ (if in the other halfplane, we may use the symmetrical relation $a_3 = \omega a_1 + \bar{\omega}a_2$).

Take now $a_1' = a_1 + \alpha_1$, $a_2' = a_2 + \alpha_2$; then $a_3' = \bar{\omega}a_1' + \omega a_2' = a_3 + (\bar{\omega}\alpha_1 + \omega\alpha_2) = a_3 + \alpha_3$. For $|\alpha_1| \leqslant r_1$, $|\alpha_2| \leqslant r_2$ we get $|\alpha_3| = |\bar{\omega}\alpha_1 + \omega\alpha_2| \leqslant r_1 + r_2$.

Conversely, for any α_3 with $|\alpha_3| \leqslant r_1 + r_2$ one can take $\alpha_1 = \frac{r_1}{r_1+r_2}\omega\alpha_3$, $\alpha_2 = \frac{r_2}{r_1+r_2}\bar{\omega}\alpha_3$ and get $|\alpha_1| \leqslant r_1$, $|\alpha_2| \leqslant r_2$ and $\bar{\omega}\alpha_1 + \omega\alpha_2 = \alpha_3$.



Returning to the original problem and denoting by C', A', B' the points that form an equilateral triangle with (A, B), (B, C), (C, A) respectively, in the same halfplane with C, A, B respectively, one easily establishes that $AC = AA' = CC' = BB'$. This is because A lies on the perpendicular bisector of CA', C lies on the perpendicular bisector of AC', while $\angle ABC = 30°$ and $\angle AB'C = 60°$ implies that B' is the circumcenter of the triangle ABC, hence $B'B = B'C = AC$.

By the lemma, the points A'' on AA' such that $AA'' = AA'/3$, C'' on CC' such that $CC'' = CC'/3$ and B'' on BB' such that $BB'' = BB'/3$ are the only points available as vertices for an equilateral triangle like that asked for, as the discs of radii $\frac{2}{3}AC$ centered at A', C', B' are in fact tangent to the discs of radii $\frac{1}{3}AC$ centered at A, C, B respectively.

We have proved that there exists such a triangle, and in fact that the triangle is unique.

Problem 7. Determine the pairs of positive integers (m, n) for which there exists a set A such that for x, y positive integers, if $|x - y| = m$, then at least one of the numbers x, y belongs to the set A, while if $|x - y| = n$, then at least one of the numbers x, y does not belong to the set.

Solution. For k positive integer, we will denote by $\nu(k)$ the exponent of 2 in the decomposition in prime factors of k. We shall prove that the pairs (m, n) that fulfill the hypothesis are the ones for which $\nu(m) = \nu(n)$.

Let us suppose that a set A with the properties in the hypothesis exists; then for $a \in A$ we have $a+n \notin A$ thus $(a+n)+m \in A$ which means $((a+n)+m)-n \notin A$, therefore $a + m \notin A$.

Analogously, for $b \notin A$ we have $b + m \in A$ thus $(b + m) + n \notin A$ which means that $((b + m) + n) - m \in A$, therefore $b + n \in A$.

Therefore, for x, y with $|x - y| \in \{m, n\}$, one of them belongs to A and the other one does not, so the problem's statement is symmetric in m, n.

Through a simple induction we obtain that $a+km$ and $a+kn$ both belong to the same set (A or its complementary) as a for k even, and to different sets for k odd.

Let us suppose now that $\nu(m) \neq \nu(n)$. Without loss of generality we may suppose that $\nu(m) > \nu(n)$. Then $m = 2^{\nu(m)} \cdot m'$, $n = 2^{\nu(n)} \cdot n'$, with m', n' odd positive integers. Let $a \in A$ and $b = a + 2^{\nu(m)} \cdot m'n'$; because $b = a + n'm$ and n' is odd, it follows that $b \notin A$, but because $b = a + 2^{\nu(m)-\nu(n)} \cdot m'n$, and $\nu(m) - \nu(n) > 0$ we have that $b \in A$, contradiction.

Finally, for $\nu(m) = \nu(n) = \nu$, let us consider for example

$$a = \bigcup_{r=0}^{2^\nu - 1} \{a \in \mathbb{N} : a \equiv r \pmod{2^{\nu+1}}\}.$$

It is easy to verify that this set fulfills the conditions in the statement.

Problem 8. Let x_i, $1 \leqslant i \leqslant n$ be real numbers. Prove that

$$\sum_{1 \leqslant i < j \leqslant n} |x_i + x_j| \geqslant \frac{n-2}{2} \sum_{i=1}^{n} |x_i|.$$

Solution. The inequality above is equivalent with

$$\sum_{1 \leqslant i,j \leqslant n} |x_i + x_j| \geqslant n \sum_{i=1}^{n} |x_i|.$$

We may suppose that (if necessary by reindexing the variables x_i)

$$x_i \geqslant 0, \quad 1 \leqslant i \leqslant k, \quad x_j < 0, \quad k+1 \leqslant j \leqslant n.$$

Let $p = \sum_{i=1}^{k} x_i$ and $m = -\sum_{j=k+1}^{n} x_j$. We may suppose that $(n-k)p \geqslant km$, otherwise we work with $-x_i$ instead. We have

$$\sum_{1 \leqslant i,j \leqslant k} |x_i + x_j| = \sum_{1 \leqslant i,j \leqslant k} (x_i + x_j) = 2kp$$

$$\sum_{k+1 \leqslant i,j \leqslant n} |x_i + x_j| = -\sum_{k+1 \leqslant i,j \leqslant n} (x_i + x_j) = 2(n-k)m$$

$$\sum_{\substack{1 \leqslant i \leqslant k \\ k+1 \leqslant j \leqslant n}} |x_i + x_j| \geqslant \left| \sum_{\substack{1 \leqslant i \leqslant k \\ k+1 \leqslant j \leqslant n}} (x_i + x_j) \right| = |(n-k)p - km|$$

$$\sum_{\substack{k+1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant k}} |x_i + x_j| \geqslant \left| \sum_{\substack{k+1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant k}} (x_i + x_j) \right| = |(n-k)p - km|.$$

Therefore,

$$\sum_{1\leqslant i,j\leqslant n}|x_i+x_j|\geqslant 2kp+2(n-k)m+2(n-k)p-2km \text{ and } n\sum_{i=1}^{n}|x_i|=n(p+m).$$

But

$$2kp+2(n-k)m+2(n-k)p-2km = 2np+2(n-2k)m = n(p+m)+(np+nm-4km).$$

Finally,

$$np+nm-4km\geqslant m\left(\frac{nk}{n-k}+n-4k\right)=m\cdot\frac{(n-2k)^2}{n-k}\geqslant 0.$$

As an aside, it is illuminating that $2kp+2(n-k)m\geqslant n(p+m)$, equivalent to $(2k-n)(p-m)\geqslant 0$ is not necessarily true (e.g., for $2k>n$, $p<m$ take $n=3$, $k=2$, $x_1=x_2=1$, $x_3=-3$). We need the "little" extra brought in by $2|(n-k)p-km|$ in order to prove the inequality true.

Equality will occur if and only if $x_i=0$, for all i, or for $n=2k$, $\sum_{i=1}^{k}x_i=-\sum_{j=k+1}^{2k}x_j$ (that is $p=m$), and, moreover, $|x_i+x_j|=0$ for all $1\leqslant i\leqslant k$, $k+1\leqslant j\leqslant n$, that is $x_1=x_2=\cdots=x_k=a>0$, $x_{k+1}=\cdots=x_n=-a$.

ALTERNATIVE SOLUTION. From the obvious relation, for $a,b\in\mathbb{R}$,

$$|a|+|b|-|a+b|=\begin{cases}0 & \text{if } ab\geqslant 0,\\ 2\min(|a|,|b|) & \text{if } ab<0,\end{cases}$$

we get

$$\begin{aligned}\sum_{1\leqslant i,j\leqslant n}(|x_i|+|x_j|-|x_i+x_j|)&=\sum_{x_ix_j<0}2\min(|x_i|,|x_j|)\\&=4\sum_{x_i>0>x_j}\min(x_i,-x_j)\\&\leqslant 4\sum_{x_i>0>x_j}\sqrt{-x_ix_j}\\&=4\left(\sum_{x_i>0}\sqrt{x_i}\right)\left(\sum_{0>x_j}\sqrt{-x_j}\right)\end{aligned}$$

$$\begin{aligned}&\leqslant\left(\sum_{x_i>0}\sqrt{x_i}+\sum_{0>x_j}\sqrt{-x_j}\right)^2 \quad \text{(AM-GM)}\\&=\left(\sum_{1\leqslant k\leqslant n}\sqrt{|x_k|}\right)^2\\&\leqslant n\sum_{1\leqslant k\leqslant n}|x_k| \quad \text{(CBS)},\end{aligned}$$

hence

$$\sum_{1\leqslant i,j\leqslant n}|x_i+x_j|+n\sum_{1\leqslant k\leqslant n}|x_k|\geqslant\sum_{1\leqslant i,j\leqslant n}(|x_i|+|x_j|)=2n\sum_{1\leqslant k\leqslant n}|x_k|,$$

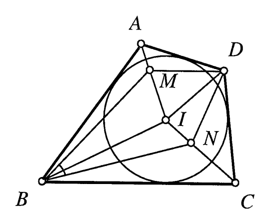
which is equivalent to the stated inequality.

Problem 9. The circle of center I is inscribed in the convex quadrilateral $ABCD$. Let M and N be points on the segments AI and CI respectively, such that $\angle MBN=\frac{1}{2}\angle ABC$. Prove that $\angle MDN=\frac{1}{2}\angle ADC$.

Solution. Denote by $\angle A$, $\angle B$, $\angle C$, $\angle D$ the angles of $ABCD$. Since BI is the bisector of $\angle ABC$ and $\angle MBN=\frac{1}{2}\angle B$, we may put

$$\alpha_1=\angle ABM=\angle IBN,\quad \alpha_2=\angle MBI=\angle NBC.$$

We also put $\angle ADM=\beta_1$, $\angle IDN=\beta_2$, $\angle NDC=\beta_4$.



Consider triangles AMB and MBI. From the sine theorem it follows that

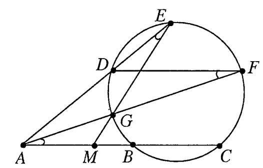
$$\frac{AM}{\sin\alpha_1}=\frac{BM}{\sin\left(\frac{A}{2}\right)},\quad \frac{MI}{\sin\alpha_2}=\frac{BM}{\sin\angle BIM}.$$

Thus
$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{AM\cdot\sin\left(\frac{A}{2}\right)}{MI\cdot\sin\angle BIA}.$$
Similarly, from considering triangles AMD and MID we have
$$\frac{\sin\beta_1}{\sin\beta_2} = \frac{AM\cdot\sin\left(\frac{A}{2}\right)}{MI\cdot\sin\angle DIA}.$$
Hence
$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{\sin\beta_1}{\sin\beta_2}\cdot\frac{\sin\angle DIA}{\sin\angle BIA}. \qquad (*)$$
Similarly considering triangles IBN and NBC, and then triangles IDN and NDC, we get
$$\frac{\sin\alpha_2}{\sin\alpha_1} = \frac{\sin\beta_4}{\sin\beta_3}\cdot\frac{\sin\angle DIC}{\sin\angle BIC}. \qquad (**)$$
Now consider triangles ABI and DIC:
$$\begin{aligned}\angle BIA &= 180^\circ - \frac{\angle A}{2} - \frac{\angle B}{2}\\ &= \left(\frac{\angle A}{2}+\frac{\angle B}{2}+\frac{\angle C}{2}+\frac{\angle D}{2}\right) - \frac{\angle A}{2} - \frac{\angle B}{2}\\ &= \frac{\angle C}{2}+\frac{\angle D}{2} = 180^\circ - \angle DIC.\end{aligned}$$

It follows that $\sin\angle BIA = \sin\angle DIC$ and similarly $\sin\angle AID = \sin\angle BIC$. Multiplying $(*)$ and $(**)$ we obtain
$$\frac{\sin\beta_1}{\sin\beta_2} = \frac{\sin\beta_3}{\sin\beta_4}.$$
Since $\beta_1+\beta_2 = \beta_3+\beta_4 = \frac{\angle D}{2} < \frac{\pi}{2}$, and the function $f(\beta) = \frac{\sin\beta}{\sin\left(\frac{\angle D}{2}-\beta\right)}$ is increasing for $\beta\in\left(0,\frac{\angle D}{2}\right)$, we conclude that $\beta_1=\beta_3$, and $\beta_2=\beta_4$, which means $\angle MDN = \frac{1}{2}\angle D$, and the problem is solved.

Problem 10. Let A be a point exterior to a circle $\mathcal{C}$. Two lines through A meet the circle $\mathcal{C}$ at points B and C, respectively at D and E (with D between A and E). The parallel through D to BC meets the second time the circle $\mathcal{C}$ at F. The line AF meets $\mathcal{C}$ again at G, and the lines BC and EG meet at M. Prove that
$$\frac{1}{AM} = \frac{1}{AB}+\frac{1}{AC}.$$

Solution. Since lines DF and AC are parallel, it follows that the angle $\angle DFA = \angle CAF$. On the other hand, $\angle DFA = \angle DEG$ because both angles subtend the arc DG. Thus $\angle CAF = \angle DEG$, whence triangles AMG and EMA are similar, which implies $\frac{AM}{MG} = \frac{EM}{AM}$, that is $AM^2 = MG\cdot ME$.



By the power of a point theorem, $MG\cdot ME = MB\cdot MC$, whence $AM^2 = MB\cdot MC = (AB-AM)(AC-AM) = AB\cdot AC - AM(AB+AC)+AM^2$, which yields $AM(AB+AC) = AB\cdot AC$, equivalent to the required relation.

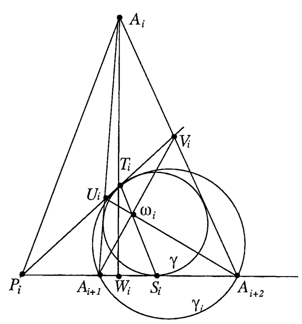
Remarks. The condition "D between A and E" is required, not so much because AF may be tangent to $\mathcal{C}$ (in which case $G\equiv F$, but the result holds), but because the parallel through D to BC may be tangent to $\mathcal{C}$ (in which case $D\equiv F$, whence $E\equiv G$, and the line EG does not exist). Otherwise, except this degenerate case, the result holds also when E between A and D.

Problem 11. Let γ be the incircle of the triangle $A_0A_1A_2$. In what follows, indices are reduced modulo 3. For each $i\in\{0,1,2\}$, let γ_i be the circle through A_{i+1} and A_{i+2}, and tangent to γ; let T_i be the tangency point of γ_i and γ; and finally, let P_i be the point where the common tangent at T_i to γ_i and γ meets the line $A_{i+1}A_{i+2}$. Prove that

(a) the points P_0, P_1 and P_2 are collinear;

(b) the lines A_0T_0, A_1T_1 and A_2T_2 are concurrent.

Solution. (a) Consider the power of P_i relative to the circles $A_0A_1A_2$, γ_i and γ: the first equals $P_iA_{i+1}\cdot P_iA_{i+2}$; the second equals $P_iA_{i+1}\cdot P_iA_{i+2} = P_iT_i^2$; and the third equals $P_iT_i^2$. Consequently, P_i lies on the radical axis of the circles $A_0A_1A_2$ and γ and we are done.



(b) Let us make the following notations: $S_i = \gamma \cap A_{i+1}A_{i+2}$, $W_i = A_iT_i \cap A_{i+1}A_{i+2}$, U_i for the intersection of the line A_iA_{i+1} and the common tangent at T_i, V_i for the intersection of the line A_iA_{i+2} and the common tangent at T_i, and $X(ABCD)$ for the cross-ratio of the rays XA, XB, XC, XD.

By Brianchon's theorem, $A_{i+1}V_i \cap S_iT_i \cap A_{i+2}U_i = \omega_i$. Now,

$$A_i(P_iW_iA_{i+1}A_{i+2}) = A_i(P_iT_iU_iV_i) = \omega_i(P_iT_iU_iV_i) = \omega_i(P_iS_iA_{i+2}A_{i+1}).$$

This yields

$$\frac{W_iA_{i+2}}{W_iA_{i+1}} = \left(\frac{P_iA_{i+2}}{P_iA_{i+1}}\right)^2 \cdot \frac{S_iA_{i+1}}{S_iA_{i+2}},$$

so

$$\prod_{i=0}^{2} \frac{W_iA_{i+2}}{W_iA_{i+1}} = \underbrace{\left(\prod_{i=0}^{2} \frac{P_iA_{i+2}}{P_iA_{i+1}}\right)^2}_{=1 \quad (1)} \cdot \underbrace{\prod_{i=0}^{2} \frac{S_iA_{i+1}}{S_iA_{i+2}}}_{=1 \quad (2)} = 1,$$

where (1) holds by Menelaus' theorem using (a), and (2) holds because $S_iA_{i+1} = S_{i+2}A_{i+1}$ as tangents to γ from A_{i+1}. The conclusion follows by the converse to Ceva's theorem.

ALTERNATIVE SOLUTION TO (b). Denote by $\triangle T$ the triangle made by the three common tangents; its sides are the tangents through the vertices of the triangle $T_0T_1T_2$ to its circumscribed circle, whence by Lemoine's theorem, their intersections with the sides of triangle $T_0T_1T_2$ are collinear (Lemoine's line). Now, Desargues' theorem shows that $\triangle T$ and triangle $T_0T_1T_2$ are perspective. By (a), Desargues' theorem similarly shows that $\triangle T$ and triangle $A_0A_1A_2$ are perspective. But the relation (for triangles) of being perspective is transitive, hence triangles $A_0A_1A_2$ and $T_0T_1T_2$ are perspective and therefore the lines A_iT_i are concurrent.

It is obvious, from both solutions, that this problem calls for projective methods to be used.

Problem 12. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geqslant a^2 + b^2 + c^2.$$

Solution. Let $x = ab + bc + ca$. From the well-known inequalities

$$(a+b+c)^2 \geqslant 3(ab+bc+ca) \quad \text{and} \quad (ab+bc+ca)^2 \geqslant 3abc(a+b+c)$$

we get $0 < x \leqslant 3$ and $abc \leqslant \frac{x^2}{9}$. Since

$$a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 9 - 2x,$$

and

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 - 2\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) = \frac{x^2}{a^2b^2c^2} - \frac{6}{abc},$$

the inequality becomes

$$x^2 - 6abc \geqslant (9 - 2x)a^2b^2c^2.$$

Therefore, we have

$$\begin{aligned} x^2 - 6abc - (9-2x)a^2b^2c^2 &\geqslant x^2 - \frac{2x^2}{3} - \frac{x^4(9-2x)}{81} \\ &= \frac{x^2(2x^3 - 9x^2 + 27)}{81} \\ &= \frac{x^2(x-3)^2(2x+3)}{81} \\ &\geqslant 0. \end{aligned}$$

Equality occurs if and only if $a = b = c = 1$.

ALTERNATIVE SOLUTION. Let us make the notation

$$f(x) = \frac{1}{x^2} + \frac{1}{x} + 1 + x,$$

then

$$\frac{1}{x^2} - x^2 = (1-x)f(x).$$

Let $E = \sum\left(\frac{1}{a^2} - a^2\right) = \sum(1-a)f(a)$. Because of the symmetry of the relation we may assume without loss of generality $a \geqslant b \geqslant c$; then using $1 - c = (a-1) + (b-1)$ and $1 - a = (c-1) + (b-1)$ we get

$$E = (1-b)\left(f(b) - f(c)\right) + (1-a)(f(a) - f(c)), \qquad (*)$$

and

$$E = (1-b)(f(b) - f(a)) + (1-c)(f(c) - f(a)). \qquad (**)$$

Now, for $x \leqslant y$, we will show that $f(x) \geqslant f(y)$:

$$f(x) - f(y) = \frac{x-y}{x^2y^2}(x^2y^2 - xy - x - y).$$

For $x + y = k$ we have $x + y + xy \geqslant 3\sqrt[3]{x^2y^2}$ by AM-GM. In order to have $3\sqrt[3]{x^2y^2} \geqslant x^2y^2$ we need $3^3 \geqslant x^4y^4$, but $k \geqslant 2\sqrt{xy}$ yields $k^8 \geqslant 2^8x^4y^4$, so $3^3 \geqslant \frac{k^8}{2^8}$ is enough, which reduces to $k \leqslant 2\sqrt[8]{27}$.

But for x, y being any of a, b, c we will have $x + y < 3$ so the above inequality is fulfilled, as $3 < 2\sqrt[8]{27}$.

This shows that f is decreasing. All that is left now is to use relation $(*)$ for $b \geqslant 1$ and relation $(**)$ for $b \leqslant 1$.

Remarks. Yet another solution would be to homogenize the equation and use "brute force" along with AM-GM and Muirhead, as in one of the solutions of IMO 2005, Problem 3.

The real beautiful thing to say is that if instead of 3 variables we think of the inequality with n variables

$$\sum_{i=1}^{n} \frac{1}{x_i^2} \geqslant \sum_{i=1}^{n} x_i^2, \quad \text{for } x_i > 0 \quad \text{with } \sum_{i=1}^{n} x_i = n,$$

this holds up to $n = 10$. For $n \geqslant 11$ it fails, e.g., for $x_1 = \cdots = x_{10} = 0.6$, $x_{11} = 5$, and $x_i = 1$, for all $i \geqslant 12$. The proof for $4 \leqslant n \leqslant 10$ involves mixed variables (Sturm-type) techniques and is probably worth a short article by itself.

Problem 13. Given $r, s \in \mathbb{Q}$, determine all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x + f(y)) = f(x + r) + y + s$$

for all $x, y \in \mathbb{Q}$.

Solution. Denote $g(x) = f(x) - r - s$. The functional equation becomes

$$\begin{aligned} g(x + g(y)) &= g(x + f(y) - r - s) \\ &= f((x + r - s) + f(y)) - r - s \\ &= f(x - s) + (y - s) - r - s \\ &= g(x - s) + y + s, \end{aligned}$$

and

$$g^2(x + g(y)) = g(y + s + g(x - s)) = g(y) + x - s + s = x + g(y),$$

hence $g^2 = \mathrm{id}$, on elements of the form $x + g(y)$. By fixing $y = y_0$, the set

$$\{x + g(y_0) : x \in \mathbb{Q}\} = \mathbb{Q},$$

hence $g^2 = \mathrm{id}$ on all of $\mathbb{Q}$, so g is one-to-one.

Then, by replacing y by $g(y)$ (any y is in the image)

$$g(x + y) = g(x + g(g(y))) = g(x - s) + g(y) + s.$$

Obviously, $g(y + x) = g(y - s) + g(x) + s$, hence $g(x) - g(x - s) = g(y) - g(y - s) = k_g$ a constant. This gives

$$g(x + y) = g(x) + g(y) + (s - k_g).$$

For $x = y = 0$: $s - k_g = -g(0)$, so $g(x + y) = g(x) + g(y) - g(0)$. Consider all solutions g having $g(0) = z$ fixed; then for any two such solutions g_1, g_2, denoting

$h = g_1 - g_2$ we get $h(x+y) = h(x) + h(y)$, so $h(x) = \lambda x$ (with $\lambda = h(1)$). On the other hand, $g_0(x) \equiv z$ is obviously a particular solution, hence the general solution g is $g(x) = h(x) + g_0(x) = \lambda x + z$.

Checking this into the main relation for g, we get

$$g(x + g(y)) = g(x + \lambda y + z) = \lambda x + \lambda^2 y + \lambda z + z$$

and

$$g(x-s) + y + s = \lambda x - \lambda s + s + z + y + s = \lambda x + y - \lambda s + s + z,$$

hence $\lambda^2 = 1$, $\lambda z = -\lambda s + s = s(1-\lambda)$. We have therefore two solutions:

$\lambda = 1$; $z = 0$ gives $g(x) = x$;

$\lambda = -1$; $z = -2s$ gives $g(x) = -x - 2s$.

These lead to the following solutions for f:

$$f(x) = x + r + s \quad \text{and} \quad f(x) = -x + r - s.$$

Remarks. As it is, the method of solving an equation

$$\varphi(x + y + a) = \varphi(x) + \varphi(y) + a,$$

is refreshingly reminiscent of solving Cauchy, combined with the theory of homogenizing plus a particular solution.

Problem 14. Find all positive integers m, n, p, q such that

$$p^m q^n = (p+q)^2 + 1.$$

Solution. Clearly we have $p \mid q^2+1$ and $q \mid p^2+1$. Now, if we assume $p = q$, it follows $p \mid p^2 + 1$, so $p = q = 1$ which is not a solution.

We may therefore assume without loss of generality $p < q$. But $p = 1$ leads to $q = 2$, and $p = 1$, $q = 2$ is again no solution, therefore $2 \leqslant p$. We have

$$p^m q^n = (p+q)^2 + 1 < 4q^2 \leqslant p^2 q^2 < pq^3 \leqslant p^m q^3,$$

so $n < 3$. The case $n = 2$ leads to $p^m < 4$, so $m = 1$, $p = 2$ or $p = 3$. For $p = 3$ we get $3q^2 = (3+q)^2 + 1$, impossible, while for $p = 2$ we get $2q^2 = (2+q)^2 + 1$, whence $q = 5$, a solution.

The case $n = 1$ leads to $p^m < 4q$. Now, from $q \mid p^2 + 1$, if $q = p^2 + 1$ then $p \mid q^2 + 1 = p^4 + 2p^2 + 2$ implies $p = 2$, whence $q = 5$, but $2^m \cdot 5^1 = (2+5)^2 + 1 = 50$ is impossible, so $q \leqslant \dfrac{p^2+1}{2}$, whence $p^m < 2(p^2+1)$, therefore $p = 2$ and $m \leqslant 3$, or $m \leqslant 2$. But $p = 2$ leads to $q \leqslant \dfrac{5}{2}$, so $q \leqslant 2$, impossible, hence truly $m \leqslant 2$.

For $m = 1$ the equation writes

$$pq = (p+q)^2 + 1,$$

clearly impossible.

For $m = 2$ we get $p^2 < 4q$ which combined with $q \mid p^2 + 1$ leads to the possibilities $p^2 + 1 = q, 2q, 3q, 4q$. But the case $p^2 + 1 = q$ was dismissed in the above, while $p^2 + 1$ cannot be congruent with 0 modulo 3, nor 4, so the only case left is $p^2 + 1 = 2q$. But then, from $p \mid q^2 + 1$ we get $4p \mid p^4 + 2p^2 + 5$, so $p \mid 5$, i.e., $p = 5$, $q = 13$, which checks as a solution.

Therefore, the only solutions are

$$(m, n, p, q) \in \{(2, 1, 5, 2),\ (2, 1, 5, 13),\ (1, 2, 2, 5),\ (1, 2, 13, 5)\}.$$

If anyone reaches the equation $p^2 q = (p+q)^2 + 1$ (dealing with the preamble in any different manner), yet another way to solve this is to consider it as a polynomial in q whose discriminant $\Delta = (2p-p^2)^2 - 4p^2 - 4 = (p^2-2p-2)^2 - 8p - 8$ should be non-negative, which implies $p \geqslant 5$, and should also be a perfect square. But the next lower perfect square is $(p^2 - 2p - 3)^2 = (p^2 - 2p - 2)^2 - 2(p^2 - 2p - 2) + 1$, thus we must have $-2(p^2 - 2p - 2) + 1 \geqslant -8p - 8$, that is $2p^2 - 12p - 13 \leqslant 0$, which implies $p \leqslant 6$.

Finally, we only have to check it for two values, $p = 5$ and $p = 6$, and only the former provides a solution (see above).

This fruitful combination of divisibility constrains which lead to inequalities is also reminscent of the BMO 2005, Problem 2.

ALTERNATIVE SOLUTION. The "proof from the Book" solution makes use of the following known

Lemma. The equation $x^2 + y^2 + 1 = kxy$ has (infinitely many) solutions in positive integers if and only if $k = 3$.

Indeed, assume $k \neq 3$, then a solution x_0, y_0 cannot have $x_0 = y_0$; without loss of generality we may take $x_0 < y_0$. One then also has

$$(kx_0 - y_0)^2 + x_0^2 + 1 = k(kx_0 - y_0)x_0,$$

as it may be readily verified. But $kx_0y_0 = x_0^2 + y_0^2 + 1 > y_0^2$ implies $kx_0 > y_0$, while $kx_0y_0 = x_0^2 + y_0^2 + 1 < 2y_0^2$ implies $kx_0 < 2y_0$, therefore $0 < x_0(kx_0 - y_0) < x_0y_0$.

Take $x_1 = \min\{x_0, kx_0 - y_0\}$, $y_1 = \max\{x_0, kx_0 - y_0\}$; one has $0 < x_1y_1 < x_0y_0$, which by Fermat's infinite descent method leads to a contradiction.

Conversely, for $k = 3$, we have the infinite family of solutions $(1,1)$, $(1,2)$, $(2,5)$, $(5,13), \ldots, (x_n, y_n), \ldots$, with $x_{n+1} = y_n$, $y_{n+1} = 3y_n - x_n$.

For an alternative solution to the lemma, using Pell equation techniques, see [A. Gica, L. Panaitopol].

Back to the original problem, the stated equation may be written as

$$p^2 + q^2 + 1 = (p^{m-1}q^{n-1} - 2)pq.$$

According to the lemma above, in order to have positive integer solutions, $p^{m-1}q^{n-1} - 2 = 3$, that is, $p^{m-1}q^{n-1} = 5$.

This quickly leads to the solutions $(m, n, p, q) \in \{(2,1,5,2), (2,1,5,13), (1,2,2,5), (1,2,13,5)\}$.

Remarks. While in English the expression "positive integer n" means an integer $n > 0$, in Romanian the verbatim translation of the expression includes the case $n = 0$; this leads to a case which was not meant to be considered, but which is worth noticing to be also true: for $mn = 0$ the equation is equivalent to $x^r = y^2 + 1$ which has no solutions, either by invoking Catalan's theorem, or by direct proof in Gauss' integer ring $\mathbb{Z}[i]$ using parity arguments.

Problem 15. Let $n > 1$ be an integer. A set $S \subset \{0, 1, \ldots, 4n-1\}$ is called *sparse* if for any $k \in \{0, 1, \ldots, n-1\}$ the following two conditions are satisfied:

(1) the set $S \cap \{4k-2, 4k-1, 4k, 4k+1, 4k+2\}$ has at most two elements;
(2) the set $S \cap \{4k+1, 4k+2, 4k+3\}$ has at most one element.
Prove that the set $\{0, 1, \ldots, 4n-1\}$ has exactly $8 \cdot 7^{n-1}$ sparse subsets.

Solution. It is enough to have available a set of 7 elements (at the "end" of the set) in order to write some recurrence relations.

$$[4n-3 \quad \underbrace{4n-2 \quad 4n-1] \quad 4n \quad [4n+1 \quad 4n+2} \quad 4n+3].$$

Denote by T_n the total number of sparse sets, by A_n the number of sparse sets that contain one of the "last" two elements $(4n-1, 4n-2)$ and by B_n the number of sparse sets that contain none of these two elements (no sparse set may contain both because of condition (2)).

Then

$$\begin{aligned} A_{n+1} &= T_n \ (4n+3 \text{ and no other element } \geqslant 4n) \\ &\quad + T_n \ (4n+3 \text{ and } 4n) \\ &\quad + T_n \ (4n+2 \text{ and no other element } \geqslant 4n) \\ &\quad + B_n \ (4n+2 \text{ and } 4n) \\ &= 3T_n + B_n, \end{aligned}$$

and

$$\begin{aligned} B_{n+1} &= T_n \ (\text{no elements } \geqslant 4n) \\ &\quad + T_n \ (4n+1, \text{ but not } 4n) \\ &\quad + T_n \ (4n, \text{ but not } 4n+1) \\ &\quad + B_n \ (\text{both } 4n \text{ and } 4n+1) \\ &= 3T_n + B_n, \end{aligned}$$

hence $A_{n+1} = B_{n+1}$ and $T_{n+1} = A_{n+1} + B_{n+1} = 6T_n + 2B_n$.

Now it is enough to calculate A_1 and B_1; clearly $\{2\}$, $\{0,2\}$, $\{3\}$, $\{0,3\}$ are A_1 and $\emptyset$, $\{0\}$, $\{1\}$, $\{0,1\}$ are B_1 so $A_1 = B_1 = 4$. Therefore $T_1 = 8$, and $T_{n+1} = 6T_n + 2B_n = 7T_n$ for $n \geqslant 1$, hence $T_n = 8 \cdot 7^{n-1}$.

Remarks. The problem becomes more challenging if we work with remainders modulo $4n$ rather than with their set of representants. In this case the result is 7^n, but writing recurrence relations is far from being obvious. Counting a related one-to-one set of differently defined subsets will do the trick.

Problem 16. Let p, q be two integers, $q \geqslant p \geqslant 0$. Let $n \geqslant 2$ be an integer and $a_0 = 0$, $a_1 \geqslant 0$, $a_2, \ldots, a_{n-1}$, $a_n = 1$ be real numbers such that

$$a_k \leqslant \frac{a_{k-1} + a_{k+1}}{2}, \quad k = 1, 2, \ldots, n-1.$$

Prove that

$$(p+1)\sum_{k=1}^{n-1} a_k^p \geqslant (q+1)\sum_{k=1}^{n-1} a_k^q.$$

Solution. It immediately follows that

$$0 = a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n = 1$$

and

$$0 \leqslant a_1 = a_1 - a_0 \leqslant a_2 - a_1 \leqslant \cdots \leqslant a_n - a_{n-1} = 1 - a_{n-1}.$$

A useful observation is that it suffices to prove the inequality for $q = p+1$, as we may then extend it step-by-step to $p+2$, $p+3$, ...

Let $S(m, r) = \sum\limits_{k=1}^{m} a_k^r$. By Abel's summation formula we get

$$S(n, p+1) = \sum_{k=1}^{n} a_k \cdot a_k^p = a_n S(n, p) - \sum_{k=1}^{n-1} (a_{k+1} - a_k) S(k, p).$$

Since $a_n = 1$, this yields

$$S(n-1, p+1) = S(n-1, p) - \sum_{k=1}^{n-1} (a_{k+1} - a_k) S(k, p).$$

Now, $a_j - a_{j-1} \leqslant a_{k+1} - a_k$, for $j = 1, 2, \ldots, k$, so

$$(a_{k+1} - a_k) S(k, p) \geqslant \sum_{j=1}^{k} (a_j - a_{j-1}) a_j^p.$$

But

$$(a_j - a_{j-1}) a_j^p \geqslant \frac{a_j^{p+1} - a_{j-1}^{p+1}}{p+1},$$

for

$$p a_j^{p+1} + a_{j-1}^{p+1} \geqslant (p+1) a_{j-1} a_j^p$$

by the weighted AM-GM inequality.

Therefore

$$\begin{aligned} S(n-1, p+1) &\leqslant S(n-1, p) - \frac{1}{p+1} \sum_{k=1}^{n-1} \sum_{j=1}^{k} (a_j^{p+1} - a_{j-1}^{p+1}) \\ &= S(n-1, p) - \frac{1}{p+1} S(n-1, p+1), \end{aligned}$$

where we used $a_0 = 0$ and the telescoping of the inner sum.

Consequently,

$$(p+2) S(n-1, p+1) \leqslant (p+1) S(n-1, p).$$

Equality occurs for $p = q$, or $a_1 = \cdots = a_{n-1} = 0$, or $p = 0$, $q = 1$ and $a_k = \dfrac{k}{n}$.

Problem 17. Let k be a positive integer and $n = 4k+1$. Let $A = \{a^2 + nb^2 : a, b \in \mathbb{Z}\}$. Prove that there exist integers x, y such that $x^n + y^n \in A$ and $x + y \notin A$.

Solution. It looks natural, in order to facilitate further factorizations, to take $y = qx$. Since squares are congruent to 0 or 1 modulo 4, and $n \equiv 1 \pmod 4$, any number of the form $a^2 + nb^2$ is congruent to 0, 1 or 2, but not 3, modulo 4. In order to have $x + y \notin A$ it is then enough to have $x + y \equiv 3 \pmod 4$, that is, $(q+1)x \equiv 3 \pmod 4$, therefore we need take q even. Now, as $n > 1$, $x = 1 + q^n \equiv 1 \pmod 4$, so we need take $q + 1 \equiv 3 \pmod 4$, that is $q \equiv 2 \pmod 4$. It remains to find x such that $x^n + y^n = (1 + q^n) x^n$ is of the form $a^2 + nb^2$.

Simply taking $x = 1 + q^n$ yields $x^n + y^n = (1 + q^n)^{n+1} = \left((1 + q^n)^{\frac{n+1}{2}}\right)^2$, of the form $a^2 + nb^2$, where $b = 0$.

It is rather more difficult to find an x such that neither $a = 0$, nor $b = 0$. Take any integers u, v such that $u^2 + v^2 \equiv 1 \pmod 4$ and take $x = (1 + q^n)(u^2 + nv^2)$.

Then $x \equiv 1 \pmod 4$, ensuring $x + y \notin A$, and

$$\begin{aligned} x^n + y^n &= (1+q^n)^{n+1}(u^2+nv^2)^n \\ &= \left(u(u^2+nv^2)^{\frac{n-1}{2}}(1+q^n)^{\frac{n+1}{2}}\right)^2 \\ &\quad + n\left(v(u^2+nv^2)^{\frac{n-1}{2}}(1+q^n)^{\frac{n+1}{2}}\right)^2. \end{aligned}$$

Finally, the requirements for the exhibited solution are $q \equiv 2 \pmod 4$ and $u + v \equiv 1 \pmod 2$.

This result seems to have been a Sophie Germain conjecture.

Clearly, this method also works for any odd exponent $m > 1$ (instead of n), but not for $n \equiv 3 \pmod 4$.

Problem 18. Let m and n be positive integers and let S be a subset with $(2^m - 1)n + 1$ elements of the set $\{1, 2, \ldots, 2^m n\}$. Prove that S contains $m + 1$ distinct numbers $a_0, a_1, \ldots, a_m$ such that $a_{k-1} \mid a_k$ for all $k = 1, 2, \ldots, m$.

Solution. We shall prove a stronger statement: such a set S will contain a subset $\{x, 2^{i_1}x, \ldots, 2^{i_m}x\}$ with $m + 1$ elements (obviously fulfilling the original assertion). Assume this statement false and take among all sets for which it fails one with $\min(S)$ maximal. As S has $2^m n - (n-1)$ elements, it follows $S \cap \{1, 2, \ldots, n\} \neq \emptyset$, whence $1 \leqslant x = \min(S) \leqslant n$, so $2^m x \leqslant 2^m n$. Therefore, the set $\{x, 2x, \ldots, 2^m x\}$ cannot be included in S, so there is some $1 \leqslant i \leqslant m$ such that $2^i x \notin S$. Take $S' = (S \cup \{2^i x\}) \setminus \{x\}$; obviously $\min(S') > \min(S)$, hence there will exist a subset $A' = \{y, 2^{j_1}y, \ldots, 2^{j_m}y\} \subset S'$. As this subset cannot be included in S it follows that some $2^j y = 2^i x$, $j \in \{0, j_1, j_2, \ldots, j_m\}$. But $x < y$ so $i > j$ and thus $y = 2^{i-j}x$, so the set $A = (A' \setminus \{2^i x\}) \cup \{x\} \subset S$, but is clearly one of the forbidden subsets for S, contradiction.

Remarks. Note that the set $\{n+1, \ldots, 2^m n\}$ has $2^m n - n$ elements, and cannot contain a subset with $m + 1$ elements that fulfills the original assertion. Therefore the size of the sets S cannot be taken any lower than stated.

This generalizes a famous (and early and folklore nowadays) result of Erdös (for $m = 1$).

Other ways of proving this result are available: induction on n or m, direct consideration of subsets of elements having the same maximal odd factor (and pigeonhole principle) etc.

Problem 19. Let $x_1 = 1, x_2, x_3, \ldots$ be a sequence of real numbers such that for all $n \geqslant 1$ we have

$$x_{n+1} = x_n + \frac{1}{2x_n}.$$

Prove that

$$\lfloor 25x_{625} \rfloor = 625.$$

Solution. Anyone provided with a pocket calculator and a good deal of patience (unless one runs into rounding problems), could prove this ... just kidding folks!

We will prove that, for all $n \geqslant 1$, $n \leqslant \sqrt{n}x_n < n + \frac{1}{8}H_n$, where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Squaring the recurrence relation yields $x_{n+1}^2 = x_n^2 + 1 + 1/(4x_n^2)$. As $x_1^2 = 1$ we will prove by simple induction that $x_n^2 \geqslant n$: $x_{n+1}^2 \geqslant n + 1 + 1/(4x_n^2) > n + 1$; then $\sqrt{n}x_n \geqslant n$. Now, by iterating the squared relation:

$$\begin{aligned} x_n^2 &= x_{n-1}^2 + 1 + \frac{1}{4x_{n-1}^2} = \cdots = x_1^2 + (n-1) + \sum_{k=1}^{n-1} \frac{1}{4x_k^2} \\ &\leqslant n + \frac{1}{4}\sum_{k=1}^{n-1}\frac{1}{k} < n + \frac{1}{4}H_n < \left(\sqrt{n} + \frac{1}{8\sqrt{n}}H_n\right)^2, \end{aligned}$$

so $\sqrt{n}x_n < n + \frac{1}{8}H_n$, as claimed.

All we need now is to show that $H_{625} < 8$ (again, the pocket calculator ...!). It is well known that $H_n \leqslant 1 + \ln n$, so it is enough to show that $\ln 625 < 7$, or $e^7 > 5^4$. This follows from $e^7 > 2.6\left(\frac{5}{2}\right)^6 = 5^4 \cdot \frac{65}{64} > 5^4$. Alternatively, one can prove that for $n \geqslant 2k - 1$, $H_n \leqslant (H_{2k-1} - H_{k-1}) + H_{\lfloor n/2 \rfloor}$, by replacing each $\frac{1}{2m+1}$ by $\frac{1}{2m}$ for $k \leqslant m \leqslant \frac{n-1}{2}$. Taking $k = 10$, (and repeatedly using the same type of majoration) we get

$$\delta = H_{19} - H_9 = \frac{1}{10} + \cdots + \frac{1}{19} < \frac{1}{5} + \cdots + \frac{1}{9} < \frac{1}{5} + \frac{1}{3} + \frac{1}{4} = \frac{47}{60}.$$

Now

$$H_{625} \leqslant \delta + H_{312} \leqslant 2\delta + H_{156} \leqslant 3\delta + H_{78}$$
$$\leqslant 4\delta + H_{39} \leqslant 5\delta + H_{19} = 6\delta + H_9$$

and

$$H_9 = 1 + \frac{1}{2} + \cdots + \frac{1}{9} < 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) + \frac{1}{3} + \frac{1}{4} = 2 + \frac{52}{60} < 3.$$

Therefore

$$H_{625} < 6 \cdot \frac{47}{60} + 3 = 7 \cdot 7 < 8, \text{ as wished for.}$$

Remarks. While $\sqrt{n}x_n \geqslant n$ may be proven through other methods than the squaring of the relation (with various degrees of difficulty or success), the upper asymptotic bound seems to be out of reach without resorting to it.

Of course, one would be happy to have $\sqrt{n}x_n < n+1$ for all n, but this is just not true; in fact one can prove that

$$\lim_{n\to\infty} \left(\sqrt{n}x_n - \left(n + \frac{1}{8}\ln n\right)\right)$$

exists and is finite, thus spoiling any pretense to obtain the above wishful bound.

This idea (of squaring such recurrence relations) easily solves related problems like $x_{n+1} = x_n - \frac{1}{x_n}$ [A. Neguţ] or $x_{n+1} = x_n + \frac{n}{x_n}$ [A. Gica, L. Panaitopol].

Problem 20. Let ABC be an acute triangle with $AB \neq AC$. Let D be the foot of the altitude from A to BC and let ω be the circumcircle of the triangle ABC. Let ω_1 be the circle that is tangent to AD, BD and ω. Let ω_2 be the circle that is tangent to AD, CD and ω. Finally, let ℓ be the common internal tangent to ω_1 and ω_2 that is not AD.

Prove that ℓ passes through the midpoint of BC if and only if $2BC = AB + AC$.

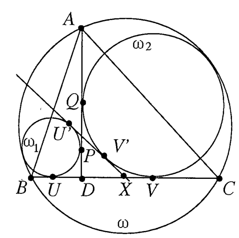
Solution. Let ω_1 be tangent to AD, BD and ω at P, U and S respectively, and let ω_2 be tangent to AD, CD and ω at Q, V, and T respectively. Let ℓ be tangent to ω_1 and ω_2 at U' and V' respectively, and meeting BC at X.

Without loss of generality, assume that $b > c$. Note that $XU + XV = UV = r_1 + r_2$, and $XU - XV = XU' - XV' = U'V' = PQ = r_1 - r_2$, so $XU = r_2$ and $XV = r_1$. Then X is the midpoint of BC if and only if $BX = BU + UX = c\cos B - r_1 + r_2 = \frac{a}{2}$, or

$$r_2 - r_1 = \frac{b^2 - c^2}{2a}.$$

By Casey's theorem on points A, B, C, and circle ω_1,

$$AC \cdot BU + BC \cdot AP = AB \cdot CU,$$



or

$$b(c\cos B - r_1) + a\left(\frac{2K}{a} - r_1\right) = c(b\cos C + r_1),$$

which implies that

$$r_1 = \frac{bc(\cos B - \cos C) + 2K}{a + b + c},$$

where K denotes the area of the triangle ABC.

Similarly,

$$r_2 = \frac{bc(\cos C - \cos B) + 2K}{a + b + c},$$

so

$$r_2 - r_1 = \frac{2bc(\cos C - \cos B)}{a + b + c} = \frac{(b-c)(b+c-a)}{a}.$$

Since $b \neq c$,

$$\frac{(b-c)(b+c-a)}{a} = \frac{b^2 - c^2}{2a}$$

if and only if $2a = b + c$.

PROBLEMS AND SOLUTIONS

JUNIOR BMO SELECTION TESTS

Problem 1. Let ABC be a rightangle triangle at C and consider points D, E on the sides BC, CA, respectively, such that $\frac{BD}{AC} = \frac{AE}{CD} = k$. Lines BE and AD intersect at point O. Show that $\angle BOD = 60°$ if and only if $k = \sqrt{3}$.

Solution. Consider the rectangle $ACDP$. The hypothesis rewrites as $\frac{BD}{DP} = \frac{AE}{AP} = k$, so $\angle APE = \angle BPD$ and $\angle APD = \angle EPB$. Moreover, $\frac{AP}{PE} = \frac{PD}{PB}$, hence $\Delta PAD \sim \Delta PEB$.

It follows that $\angle DAP = \angle PEB$, so $APOE$ is cyclic and hence $\angle BOD = \angle AOE = \angle APE$.

The claim is proved by the following chain of equivalences: $\angle BOD = 60° \Leftrightarrow \tan\angle BOD = \sqrt{3} \Leftrightarrow \tan\angle APE = \sqrt{3} \Leftrightarrow \frac{AE}{AP} = \sqrt{3} \Leftrightarrow k = \sqrt{3}$.

Problem 2. Consider five points in the plane such that each triangle with vertices at three of those points has area at most 1. Prove that the five points can be covered by a trapezoid of area at most 3.

Solution. Denote A, B, C, D, E the given points and suppose ABC is the triangle of maximal area. The distance from D to BC can not exceed the distance from A to BC, hence D – and similarly E – are located between the parallel through A to BC and its mirror image across BC. Apply the same argument to AB and AC, to deduce that the five poinys must lie in a triangular (bounded) region $A_1B_1C_1$ whose median triangle is ABC.

Since D and E lie in at most two of the triangles A_1BC, AB_1C, ABC_1, one of the trapezoids $ABA_1B_1, BCB_1C_1, CAC_1A_1$ must contains the points $ABCDE$.

And since the area of such a trapezoid is 3 times the area of ABC, hence at most 3, the conclusion follows.

Problem 3. For any positive integer n let $s(n)$ be the sum of its digits in decimal representation. Find all numbers n for which $s(n)$ is the largest proper divisor of n.

Solution. The numbers are 18 and 27.

Let k be the number of digits of n in decimal representation. Notice that:

(1) $n = p \cdot s(n)$, where p is prime, so any prime divisor of $s(n)$ is greater than of equal to p;

(2) $s(n)^2 \geqslant n$, so $10^{k-1} \leqslant n \leqslant s(n)^2 \leqslant (9k)^2$, hence $k \leqslant 4$.

Consider the following cases:

a) If $k = 4$, then $n = \overline{abcd}$, $n \leqslant s(n)^2 \leqslant 36^2 = 1296$, so $a = 1$. Then $s(n) \leqslant 28$, thus $n \leqslant 28^2 < 1000$, false.

b) If $k \leqslant 3$, then $\overline{abc}$, so $9(11 \cdot a + b) = (p-1)(a+b+c)$.

If 9 divides $p-1$, since $p < a+b+c = 27$ we get $p = 19$. Next $9a = b + 2c$, hence $a \leqslant 3$. As $a+b+c \geqslant 23$ – see (1) – we have no solution.

If 9 does not divide $p-1$, from $3|a+b+c$ and (1) we get $p = 2$ or $p = 3$.

For $p = 3$ we have $n = 3(a+b+c)$, so $a = 0$ and $10 \cdot b + c = 3(b+c)$. Consequently, $7b = 2c$ and $n = 27$.

For $p = 2$ we get $n = 2(a+b+c)$, so $a = 0$ and $8b = c$, hence $n = 18$.

Problem 4. Prove that $\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ba} \geqslant a+b+c$, for all positive real numbers a, b, and c.

Solution. The inequality can be rewritten as $a^4 + b^4 + c^4 \geqslant abc(a+b+c)$. A successive application of the well-known inequality $x^2 + y^2 + z^2 \geqslant xy + yz + zx$ yields the desired result:

$$a^4 + b^4 + c^4 \geqslant a^2b^2 + b^2c^2 + c^2a^2 = (ab)^2 + (bc)^2 + (ca)^2 \geqslant abc(a+b+c).$$

Problem 5. Consider a circle C of center O and let A, B be points on the circle with $\angle AOB = 90°$. Circles $C_1(O_1)$ and $C_2(O_2)$ are internally tangent to

C at points A, B, respectively, and – moreover – are tangent to themselves. Circle $C_3(O_3)$, located inside the angle $\angle AOB$, is externally tangent to C_1, C_2 and internally tangent to C. Prove that O, O_1, O_2, O_3 are the vertices of a rectangle.

Solution. Let R, r_1, r_2 be the radii of the circles C, C_1, C_2 and let $r = R - r_1 - r_2$. Consider the point P so that OO_1PO_2 is a rectangle. The tangency conditions yield $OO_1 = R - r_1, OO_2 = R - r_2$ and $O_1O_2 = r_1 + r_2 = R - r$. It is sufficient to prove that C_3 is the circle of radius r centered at P.

To prove this, notice that $O_1P = OO_2 = R - r_2 = r + r_1$, $O_2P = OO_1 = R - r_1 = r + r_2$, and $OP = O_1O_2 = R - r$, so the three tangency conditions are fulfilled.

Problem 6. A 7×7 array is divided into 49 unit squares. Find all integers $n \in \mathbb{N}^*$ for which n checkers can be placed on the unit squares so that each row and each line contain an even number of checkers.

(0 is an even number, so empty rows or columns are not excluded. At most one checker is allowed inside a unit square.)

Solution. One can place $4, 6, \ldots, 40, 42$ checkers under the given conditions.

We start by noticing that n is the sum of 7 even numbers, hence n is also even. One can place at most 6 checkers on a row, hence $n \leqslant 6 \cdot 7 = 42$.

The key step is to use $2k \times 2k$ squares filled completely with checkers and $(2k+1) \times (2k+1)$ squares having checkers on each unit square except for one diagonal. Notice that all these squares satisfy the required conditions, and moreover, we may glue together several such squares under the conditions in the statement.

We describe below the configurations of n checkers for any even n between 4 and 42.

For $4, 8, 12, 16, 20, 24, 28, 32$ or 36 checkers use $1, 2, 3, 4, 5, 6, 7, 8$ or 9, 2×2 squares; notice that all fit inside the 7×7 array!.

For 6 checkers consider a 3×3 square, except for one diagonal; then adding 2×2 squares we get configurations of $10, 14, 18, 22, \ldots, 38$ checkers.

For 40 checkers use a 5×5 and five 2×2 squares.

For 42 checkers we complete the 7×7 array but a diagonal.

Finally, notice that 2 checkers cannot be placed under the conditions in the statement.

Problem 7. Suppose $ABCD$ is a cyclic quadrilateral of area 8. Prove that if there exists a point O in the plane of the quadrilateral such that $OA + OB + OC + OD = 8$, then $ABCD$ is an isosceles trapezoid (or a square).

Solution. Let α be the measure of the angle determined by the diagonals.

Since $8 = OA + OB + OC + OD \geqslant AC + BD \geqslant 2 \cdot \sqrt{AC \cdot BD} \geqslant 2 \cdot \sqrt{AC \cdot BD \cdot \sin\alpha} = 2\sqrt{2S} = 8$, we get $AC = BD = 1$ and $\alpha = 90°$. The claim follows by a simple arc subtraction.

Problem 8. Prove that

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2 \geqslant \frac{3}{2} \cdot \left(\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b}\right),$$

for all positive real numbers a, b, and c.

Solution. Let $\frac{a}{b} = x, \frac{b}{c} = y$ and $\frac{c}{a} = z$. The inequality can be rewritten successively: $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx \geqslant \frac{3}{2}\left(x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \Leftrightarrow x^2 + y^2 + z^2 + 2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geqslant \frac{3}{2}\left(x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \Leftrightarrow 2(x^2 + y^2 + z^2) + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geqslant 3(x + y + z)$.

From AM-GM inequality we get

$$2x^2 + \frac{1}{x} = x^2 + x^2 + \frac{1}{x} \geqslant 3\sqrt[3]{x^2 \cdot x^2 \cdot \frac{1}{x}} = 3x.$$

Summing up the resulting inequalities and its analogues in x and y we get the conclusion.

Problem 9. Find all real numbers a and b satisfying

$$2(a^2 + 1)(b^2 + 1) = (a + 1)(b + 1)(ab + 1).$$

Solution. Consider the given equation as quadratic in a:

$$a^2(b^2 - b + 2) - a(b + 1)^2 + 2b^2 - b + 1 = 0.$$

The discriminant is $\Delta = -(b-1)^2(7b^2-2b+7)$, hence we have solutions only for $b=1$. It follows that $a=1$.

ALTERNATIVE SOLUTION. Apply the Cauchy-Schwarz inequality, to get $2(a^2+1) \geqslant (a+1)^2, 2(b^2+1) \geqslant (b+1)^2$ and $(a^2+1)(b^2+1) \geqslant (ab+1)^2$. Multiplying, we obtain $2(a^2+1)(b^2+1) \geqslant (a+1)(b+1)(ab+1)$, hence the equality case occurs in all inequalities, so $a=b=1$.

Problem 10. Show that the set of real numbers can be partitioned into subsets having two elements.

Solution. For example, consider $\mathbb{R} \setminus \mathbb{Z}$ partitionated into doubletons $\{-x, x\}$ and $\mathbb{Z}$ into doubletons $\{2n, 2n+1\}$.

Another example: split $\mathbb{R}$ into disjoint intervals $[2n, 2n+1)$, with $n \in \mathbb{Z}$. Then take the pairs $(x, x+1)$ from each interval $[2n, 2n+1)$, with $x \in [2n, 2n+1)$.

Problem 11. Let $A = \{1, 2, \ldots, 2006\}$. Find the maximal number of subsets of A that can be chosen such that the intersection of any two such distinct subsets have 2004 elements.

Solution. The required number is 2006, the number of the subsets having 2005 elements.

To begin with, notice that each subset must have at least 2004 elements. If there exist a set with exactly 2004 elements, then this is unique and moreover, only 2 other subsets may be chosen.

If no set has 2004 elements, then we can choose among the 2006 subsets with 2005 elements and the set A with 2006 elements. But if A is among the chosen subsets, then any intersection will have more than 2004 elements, false. The claim is thus proved.

Problem 12. Let ABC be a triangle and let A_1, B_1, C_1 be the midpoints of the sides BC, CA, AB, respectively. Show that if M is a point in the plane of the triangle such that

$$\frac{MA}{MA_1} = \frac{MB}{MB_1} = \frac{MC}{MC_1} = 2,$$

then M is the centroid of the triangle.

Solution. Let A_2, B_2, C_2 be the mirror images of M across A_1, B_1, C_1, respectively. The given condition shows that $MA = MA_2, MB = MB_2, MC = MC_2$. From the parallelograms $AMBC_2, BMCA_2, AMCB_2$ we derive that $MA = MA_2 = BC_2 = B_2C, MB = MB_2 = AC_2 = CA_1$ and $MC = MC_2 = BA_2 = AB_2$. It follows that MA_2BC_2, MA_2CB_2 and MB_2AC_2 are also parallelograms, therefore A, M and A_2 are collinear. The conclusion is now clear.

Problem 13. Suppose a, b, c are positive real numbers which sum up to 1. Prove that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geqslant 3(a^2+b^2+c^2).$$

Solution. By Cauchy-Schwarz inequality we get:

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} = \frac{a^4}{ba^2} + \frac{b^4}{cb^2} + \frac{c^4}{ac^2} \geqslant \frac{(a^2+b^2+c^2)^2}{a^2b+b^2c+c^2a}.$$

It suffices to show that $a^2+b^2+c^2 \geqslant 3(a^2b+b^2c+c^2a)$ or, since $a+b+c=1$, that

$$(a+b+c)(a^2+b^2+c^2) \geqslant 3(a^2b+b^2c+c^2a).$$

The last inequality can be rewritten $\sum a(a-b)^2 \geqslant 0$, which is obvious.

ALTERNATIVE SOLUTION. Since $a+b+c = (a+b+c)^2 = 1$, the inequality can be successively rewrtitten as follows:

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} - (a+b+c) \geqslant 3(a^2+b^2+c^2) - (a+b+c)^2,$$

or

$$\sum\left(\frac{a^2}{b} - 2a + b\right) \geqslant \sum(a-b)^2,$$

that is,

$$\sum\frac{(a-b)^2}{b} \geqslant \sum(a-b)^2.$$

Since $a, b, c \leqslant 1$, we are done.

Problem 14. The set of positive integers is partitioned into subsets with infinitely many elements each. The following question arises: does there exist a subset in the partition such that any positive integer has a multiple in that subset?

a) Prove that if the number of subsets in the partition is finite, then the answer is "yes".

b) Prove that if the number of subsets in the partition is infinite, then the answer can be "no" (for some partition).

Solution. a) Let A_k be the partition classes, with $k = 1, 2, \ldots r$. Assuming that the answer is "no", there exist positive integers n_k, $k = 1, 2, \ldots r$, such that no multiple of n_k is in A_k. But $n_1 n_2 \ldots n_r$ lies in one of the sets A_k and is multiple of any n_k, false.

b) We exhibit a partition for which the answer is "no".

Let A_k be the set of all numbers written only with the first k primes at any positive power; moreover, put $1 \in A_1$. For any fixed k, the number $p_1 p_2 \ldots p_{k+1}$ has no multiples in A_k.

FIFTH SELECTION TEST

Problem 15. Let ABC be a triangle and D a point inside the triangle, located on the median from A. Show that if $\angle BDC = 180^\circ - \angle BAC$, then $AB \cdot CD = AC \cdot BD$.

Solution. Let E be the mirror image of D across the midpoint of the side BC. Notice that $DBEC$ is a parallelogram and $ABEC$ is cyclic. The equality of the areas of triangles ABE and ACE implies $AB \cdot BE = AC \cdot CE$. Noticing that $CE = BD$ and $BE = CD$, the conclusion follows.

Problem 16. Consider the integers $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ with $a_k \neq b_k$ for all $k = 1, 2, 3, 4$. If

$$\{a_1, b_1\} + \{a_2, b_2\} = \{a_3, b_3\} + \{a_4, b_4\},$$

show that the number $|(a_1 - b_1)(a_2 - b_2)(a_3 - b_3)(a_4 - b_4)|$ is a perfect square.

Note. For any sets A and B, we denote by $A + B = \{x + y \mid x \in A, y \in B\}$.

Solution. Without loss of generality, assume $a_k > b_k$, $k = \overline{1,4}$. Then $a_1 + a_2 = a_3 + a_4$ and $b_1 + b_2 = b_3 + b_4$. Two cases may occur:

i) $a_1 + b_2 = a_3 + b_4$ and $a_2 + b_1 = a_4 + b_3$. Subtracting we get $|a_2 - b_2| = |a_4 - b_4|$, $|a_1 - b_1| = |a_3 - b_3|$ and the claim follows.

ii) $a_1 + b_2 = a_4 + b_3$ and $a_2 + b_1 = a_3 + b_4$. By subtraction we obtain $|a_2 - b_2| = |a_3 - b_3|$, $|a_1 - b_1| = |a_4 - b_4|$, as needed.

Problem 17. Let x, y, z be positive real numbers such that

$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 2.$$

Prove that $8xyz \leqslant 1$.

Solution. Cancel out denominators to get $1 = xy + yz + zx + 2xyz$. By the AM-GM inequality we get $1 \geqslant 4\sqrt[4]{2x^3y^3z^3}$, so $1 \geqslant (8xyz)^3$. The conclusion follows.

Problem 18. For a positive integer n denote by $r(n)$ the number having the digits of n in reverse order; for example, $r(2006) = 6002$. Prove that for any positive integers a and b the numbers $4a^2 + r(b)$ and $4b^2 + r(a)$ cannot be simultaneously perfect squares.

Solution. Assume by contradiction that both $4a^2 + r(b)$ and $4b^2 + r(a)$ are perfect squares and let $b \leqslant a$. The number $r(b)$ has at most as many digits as b, so $r(b) < 10b \leqslant 10a$. It follows that

$$(2a)^2 < 4a^2 + 10a < (2a + 3)^2,$$

hence $4a^2 + r(b) = (2a + 1)^2$ or $(2a + 2)^2$, thus $r(b) = 4a + 1$ or $8a + 4$. Notice that $r(b) > a \geqslant b$, implying that a and b have the same number of digits. Then, as above, we get $r(a) \in \{4b + 1, 8b + 4\}$.

Three cases may occur:

1. $r(a) = 4b+1$ and $r(b) = 4a+1$. Subtracting we get $(r(a)-a)+(r(b)-b) = 3(b - a) + 2$, which is false since 9 divides $r(n) - n$ for any positive integer n.

2. $r(a) = 8b+4$ and $r(b) = 4a+1$ (the same reasoning applies to $r(b) = 8a+4$ and $r(a) = 4b+1$). Subtracting we obtain $(r(a)-a)+(r(b)-b) = 7b+3a+3$, so 3 divides b. Then 3 also divides $r(b) = 4a + 1$, so a and $r(a)$ both yield remainder 2 when divided by 3. This leads to a contradiction, for $r(a) = (8b + 3) + 1$.

3. $r(a) = 8b + 4$ and $r(b) = 8a + 4$. Then the last digit of both $r(a)$ and $r(b)$ is even, so at least 2. Hence the first digit of both a and b is greater than or equal to 2, so $8a + 4$ and $8b + 4$ have more digits than a and b. It follows that $r(a) < 8b + 4$ and $r(b) < 8a + 4$, a contradiction.

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