51^{st} International Mathematical Olympiad
Astana, Kazakhstan 2010

Problems with Solutions

Contents

Problems

Problem 1. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that the equality

$$f\big(\lfloor x \rfloor y\big) = f(x)\big\lfloor f(y) \big\rfloor$$

holds for all $x, y \in \mathbb{R}$. (Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

Problem 2. Let I be the incentre of triangle ABC and let Γ be its circumcircle. Let the line AI intersect Γ again at D. Let E be a point on the arc $\widehat{BDC}$ and F a point on the side BC such that

$$\angle BAF = \angle CAE < \tfrac{1}{2}\angle BAC.$$

Finally, let G be the midpoint of the segment IF. Prove that the lines DG and EI intersect on Γ.

Problem 3. Let $\mathbb{N}$ be the set of positive integers. Determine all functions $g\colon \mathbb{N} \to \mathbb{N}$ such that

$$\big(g(m) + n\big)\big(m + g(n)\big)$$

is a perfect square for all $m, n \in \mathbb{N}$.

Problem 4. Let P be a point inside the triangle ABC. The lines AP, BP and CP intersect the circumcircle Γ of triangle ABC again at the points K, L and M respectively. The tangent to Γ at C intersects the line AB at S. Suppose that $SC = SP$. Prove that $MK = ML$.

Problem 5. In each of six boxes $B_1, B_2, B_3, B_4, B_5, B_6$ there is initially one coin. There are two types of operation allowed:

Type 1: Choose a nonempty box B_j with $1 \le j \le 5$. Remove one coin from B_j and add two coins to B_{j+1}.

Type 2: Choose a nonempty box B_k with $1 \le k \le 4$. Remove one coin from B_k and exchange the contents of (possibly empty) boxes B_{k+1} and B_{k+2}.

Determine whether there is a finite sequence of such operations that results in boxes B_1, B_2, B_3, B_4, B_5 being empty and box B_6 containing exactly $2010^{2010^{2010}}$ coins. (Note that $a^{b^c} = a^{(b^c)}$.)

Problem 6. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers. Suppose that for some positive integer s, we have

$$a_n = \max\{a_k + a_{n-k} \mid 1 \le k \le n-1\}$$

for all $n > s$. Prove that there exist positive integers ℓ and N, with $\ell \le s$ and such that $a_n = a_\ell + a_{n-\ell}$ for all $n \ge N$.

Solutions

Problem 1. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that the equality

$$f\big(\lfloor x \rfloor y\big) = f(x)\big\lfloor f(y)\big\rfloor \tag{1}$$

holds for all $x, y \in \mathbb{R}$. (Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

Answer. $f(x) = \text{const} = C$, where $C = 0$ or $1 \le C < 2$.

Solution 1. First, setting $x = 0$ in (1) we get

$$f(0) = f(0)\lfloor f(y)\rfloor \tag{2}$$

for all $y \in \mathbb{R}$. Now, two cases are possible.

Case 1. Assume that $f(0) \neq 0$. Then from (2) we conclude that $\lfloor f(y)\rfloor = 1$ for all $y \in \mathbb{R}$. Therefore, equation (1) becomes $f(\lfloor x \rfloor y) = f(x)$, and substituting $y = 0$ we have $f(x) = f(0) = C \neq 0$. Finally, from $\lfloor f(y)\rfloor = 1 = \lfloor C \rfloor$ we obtain that $1 \le C < 2$.

Case 2. Now we have $f(0) = 0$. Here we consider two subcases.

Subcase 2a. Suppose that there exists $0 < \alpha < 1$ such that $f(\alpha) \neq 0$. Then setting $x = \alpha$ in (1) we obtain $0 = f(0) = f(\alpha)\lfloor f(y)\rfloor$ for all $y \in \mathbb{R}$. Hence, $\lfloor f(y)\rfloor = 0$ for all $y \in \mathbb{R}$. Finally, substituting $x = 1$ in (1) provides $f(y) = 0$ for all $y \in \mathbb{R}$, thus contradicting the condition $f(\alpha) \neq 0$.

Subcase 2b. Conversely, we have $f(\alpha) = 0$ for all $0 \le \alpha < 1$. Consider any real z; there exists an integer N such that $\alpha = \dfrac{z}{N} \in [0, 1)$ (one may set $N = \lfloor z \rfloor + 1$ if $z \ge 0$ and $N = \lfloor z \rfloor - 1$ otherwise). Now, from (1) we get $f(z) = f(\lfloor N \rfloor \alpha) = f(N)\lfloor f(\alpha)\rfloor = 0$ for all $z \in \mathbb{R}$.

Finally, a straightforward check shows that all the obtained functions satisfy (1).

Solution 2. Assume that $\lfloor f(y)\rfloor = 0$ for some y; then the substitution $x = 1$ provides $f(y) = f(1)\lfloor f(y)\rfloor = 0$. Hence, if $\lfloor f(y)\rfloor = 0$ for all y, then $f(y) = 0$ for all y. This function obviously satisfies the problem conditions.

So we are left to consider the case when $\lfloor f(a)\rfloor \neq 0$ for some a. Then we have

$$f(\lfloor x \rfloor a) = f(x)\lfloor f(a)\rfloor, \qquad \text{or} \quad f(x) = \frac{f(\lfloor x \rfloor a)}{\lfloor f(a)\rfloor}. \tag{3}$$

This means that $f(x_1) = f(x_2)$ whenever $\lfloor x_1 \rfloor = \lfloor x_2 \rfloor$, hence $f(x) = f(\lfloor x \rfloor)$, and we may assume that a is an integer.

Now we have

$$f(a) = f\left(2a \cdot \tfrac{1}{2}\right) = f(2a)\left\lfloor f\left(\tfrac{1}{2}\right)\right\rfloor = f(2a)\lfloor f(0)\rfloor;$$

this implies $\lfloor f(0)\rfloor \neq 0$, so we may even assume that $a = 0$. Therefore equation (3) provides

$$f(x) = \frac{f(0)}{\lfloor f(0)\rfloor} = C \neq 0$$

for each x. Now, condition (1) becomes equivalent to the equation $C = C\lfloor C \rfloor$ which holds exactly when $\lfloor C \rfloor = 1$.

Problem 2. Let I be the incentre of triangle ABC and let Γ be its circumcircle. Let the line AI intersect Γ again at D. Let E be a point on the arc $\widehat{BDC}$ and F a point on the side BC such that

$$\angle BAF = \angle CAE < \tfrac{1}{2}\angle BAC.$$

Finally, let G be the midpoint of the segment IF. Prove that the lines DG and EI intersect on Γ.

Solution 1. Let X be the second point of intersection of line EI with Γ, and L be the foot of the bisector of angle BAC. Let G' and T be the points of intersection of segment DX with lines IF and AF, respectively. We are to prove that $G = G'$, or $IG' = G'F$. By the Menelaus theorem applied to triangle AIF and line DX, it means that we need the relation

$$1 = \frac{G'F}{IG'} = \frac{TF}{AT} \cdot \frac{AD}{ID}, \qquad \text{or} \quad \frac{TF}{AT} = \frac{ID}{AD}.$$

Let the line AF intersect Γ at point $K \neq A$ (see Fig. 1); since $\angle BAK = \angle CAE$ we have $\widehat{BK} = \widehat{CE}$, hence $KE \parallel BC$. Notice that $\angle IAT = \angle DAK = \angle EAD = \angle EXD = \angle IXT$, so the points I, A, X, T are concyclic. Hence we have $\angle ITA = \angle IXA = \angle EXA = \angle EKA$, so $IT \parallel KE \parallel BC$. Therefore we obtain $\dfrac{TF}{AT} = \dfrac{IL}{AI}$.

Since CI is the bisector of $\angle ACL$, we get $\dfrac{IL}{AI} = \dfrac{CL}{AC}$. Furthermore, $\angle DCL = \angle DCB = \angle DAB = \angle CAD = \frac{1}{2}\angle BAC$, hence the triangles DCL and DAC are similar; therefore we get $\dfrac{CL}{AC} = \dfrac{DC}{AD}$. Finally, it is known that the midpoint D of arc BC is equidistant from points I, B, C, hence $\dfrac{DC}{AD} = \dfrac{ID}{AD}$.

Summarizing all these equalities, we get

$$\frac{TF}{AT} = \frac{IL}{AI} = \frac{CL}{AC} = \frac{DC}{AD} = \frac{ID}{AD},$$

as desired.

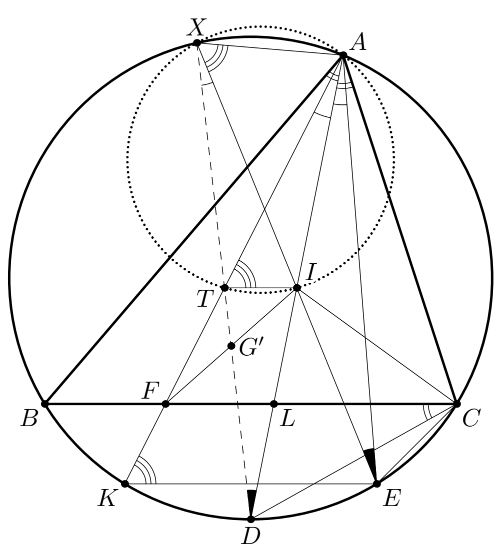

Fig. 1

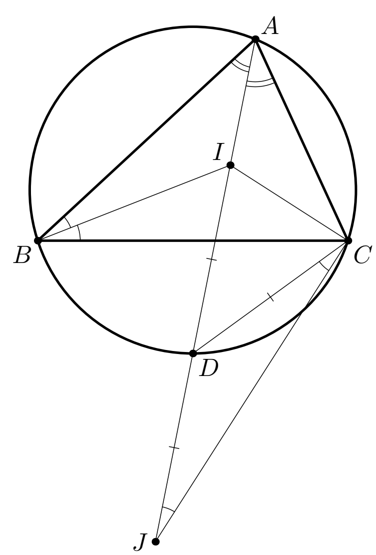

Fig. 2

Comment. The equality $\dfrac{AI}{IL}=\dfrac{AD}{DI}$ is known and can be obtained in many different ways. For instance, one can consider the inversion with center D and radius $DC = DI$. This inversion takes $\widehat{BAC}$ to the segment BC, so point A goes to L. Hence $\dfrac{IL}{DI}=\dfrac{AI}{AD}$, which is the desired equality.

Solution 2. As in the previous solution, we introduce the points X, T and K and note that it suffice to prove the equality

$$\frac{TF}{AT}=\frac{DI}{AD} \iff \frac{TF+AT}{AT}=\frac{DI+AD}{AD} \iff \frac{AT}{AD}=\frac{AF}{DI+AD}.$$

Since $\angle FAD = \angle EAI$ and $\angle TDA = \angle XDA = \angle XEA = \angle IEA$, we get that the triangles ATD and AIE are similar, therefore $\dfrac{AT}{AD}=\dfrac{AI}{AE}$.

Next, we also use the relation $DB = DC = DI$. Let J be the point on the extension of segment AD over point D such that $DJ = DI = DC$ (see Fig. 2). Then $\angle DJC = \angle JCD = \frac{1}{2}(\pi - \angle JDC) = \frac{1}{2}\angle ADC = \frac{1}{2}\angle ABC = \angle ABI$. Moreover, $\angle BAI = \angle JAC$, hence triangles ABI and AJC are similar, so $\dfrac{AB}{AJ}=\dfrac{AI}{AC}$, or $AB \cdot AC = AJ \cdot AI = (DI + AD) \cdot AI$.

On the other hand, we get $\angle ABF = \angle ABC = \angle AEC$ and $\angle BAF = \angle CAE$, so triangles ABF and AEC are also similar, which implies $\dfrac{AF}{AC}=\dfrac{AB}{AE}$, or $AB \cdot AC = AF \cdot AE$.

Summarizing we get

$$(DI + AD)\cdot AI = AB \cdot AC = AF \cdot AE \quad\Rightarrow\quad \frac{AI}{AE}=\frac{AF}{AD+DI} \quad\Rightarrow\quad \frac{AT}{AD}=\frac{AF}{AD+DI},$$

as desired.

Comment. In fact, point J is an excenter of triangle ABC.

Problem 3. Let $\mathbb{N}$ be the set of positive integers. Determine all functions $g\colon \mathbb{N}\to\mathbb{N}$ such that

$$\big(g(m)+n\big)\big(m+g(n)\big)$$

is a perfect square for all $m, n \in \mathbb{N}$.

Answer. All functions of the form $g(n) = n + c$, where $c \in \mathbb{N} \cup \{0\}$.

Solution. First, it is clear that all functions of the form $g(n) = n + c$ with a constant nonnegative integer c satisfy the problem conditions since $\big(g(m)+n\big)\big(g(n)+m\big) = (n+m+c)^2$ is a square.

We are left to prove that there are no other functions. We start with the following

Lemma. Suppose that $p \mid g(k) - g(\ell)$ for some prime p and positive integers k, ℓ. Then $p \mid k - \ell$.

Proof. Suppose first that $p^2 \mid g(k) - g(\ell)$, so $g(\ell) = g(k) + p^2 a$ for some integer a. Take some positive integer $D > \max\{g(k), g(\ell)\}$ which is not divisible by p and set $n = pD - g(k)$. Then the positive numbers $n + g(k) = pD$ and $n + g(\ell) = pD + \big(g(\ell) - g(k)\big) = p(D + pa)$ are both divisible by p but not by p^2. Now, applying the problem conditions, we get that both the numbers $\big(g(k)+n\big)\big(g(n)+k\big)$ and $\big(g(\ell)+n\big)\big(g(n)+\ell\big)$ are squares divisible by p (and thus by p^2); this means that the multipliers $g(n) + k$ and $g(n) + \ell$ are also divisible by p, therefore $p \mid \big(g(n)+k\big) - \big(g(n)+\ell\big) = k - \ell$ as well.

On the other hand, if $g(k)-g(\ell)$ is divisible by p but not by p^2, then choose the same number D and set $n = p^3D - g(k)$. Then the positive numbers $g(k)+n = p^3D$ and $g(\ell)+n = p^3D + \big(g(\ell)-g(k)\big)$ are respectively divisible by p^3 (but not by p^4) and by p (but not by p^2). Hence in analogous way we obtain that the numbers $g(n)+k$ and $g(n)+\ell$ are divisible by p, therefore $p \mid \big(g(n)+k\big) - \big(g(n)+\ell\big) = k-\ell$.
□

We turn to the problem. First, suppose that $g(k) = g(\ell)$ for some $k, \ell \in \mathbb{N}$. Then by Lemma we have that $k - \ell$ is divisible by every prime number, so $k - \ell = 0$, or $k = \ell$. Therefore, the function g is injective.

Next, consider the numbers $g(k)$ and $g(k+1)$. Since the number $(k+1) - k = 1$ has no prime divisors, by Lemma the same holds for $g(k+1) - g(k)$; thus $|g(k+1) - g(k)| = 1$.

Now, let $g(2) - g(1) = q$, $|q| = 1$. Then we prove by induction that $g(n) = g(1) + q(n-1)$. The base for $n = 1, 2$ holds by the definition of q. For the step, if $n > 1$ we have $g(n+1) = g(n) \pm q = g(1) + q(n-1) \pm q$. Since $g(n) \neq g(n-2) = g(1) + q(n-2)$, we get $g(n) = g(1) + qn$, as desired.

Finally, we have $g(n) = g(1) + q(n-1)$. Then q cannot be -1 since otherwise for $n \geq g(1) + 1$ we have $g(n) \leq 0$ which is impossible. Hence $q = 1$ and $g(n) = (g(1) - 1) + n$ for each $n \in \mathbb{N}$, and $g(1) - 1 \geq 0$, as desired.

Problem 4. Let P be a point inside the triangle ABC. The lines AP, BP and CP intersect the circumcircle Γ of triangle ABC again at the points K, L and M respectively. The tangent to Γ at C intersects the line AB at S. Suppose that $SC = SP$. Prove that $MK = ML$.

Solution 1. We assume that $CA > CB$, so point S lies on the ray AB.

From the similar triangles $\triangle PKM \sim \triangle PCA$ and $\triangle PLM \sim \triangle PCB$ we get $\dfrac{PM}{KM} = \dfrac{PA}{CA}$ and $\dfrac{LM}{PM} = \dfrac{CB}{PB}$. Multiplying these two equalities, we get

$$\frac{LM}{KM} = \frac{CB}{CA} \cdot \frac{PA}{PB}.$$

Hence, the relation $MK = ML$ is equivalent to $\dfrac{CB}{CA} = \dfrac{PB}{PA}$.

Denote by E the foot of the bisector of angle B in triangle ABC. Recall that the locus of points X for which $\dfrac{XA}{XB} = \dfrac{CA}{CB}$ is the Apollonius circle Ω with the center Q on the line AB, and this circle passes through C and E. Hence, we have $MK = ML$ if and only if P lies on Ω, that is $QP = QC$.

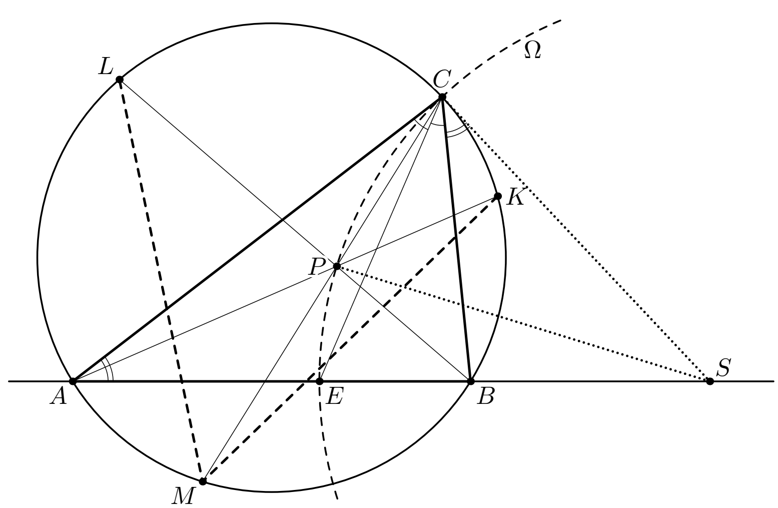

Fig. 1

Now we prove that $S = Q$, thus establishing the problem statement. We have $\angle CES = \angle CAE + \angle ACE = \angle BCS + \angle ECB = \angle ECS$, so $SC = SE$. Hence, the point S lies on AB as well as on the perpendicular bisector of CE and therefore coincides with Q.

Comment. In this solution we proved more general fact: $SC = SP$ if and only if $MK = ML$.

Solution 2. As in the previous solution, we assume that S lies on the ray AB.

Let P be an arbitrary point inside both the circumcircle ω of the triangle ABC and the angle ASC, the points K, L, M defined as in the problem.

Let E and F be the points of intersection of the line SP with ω, point E lying on the segment SP (see Fig. 2).

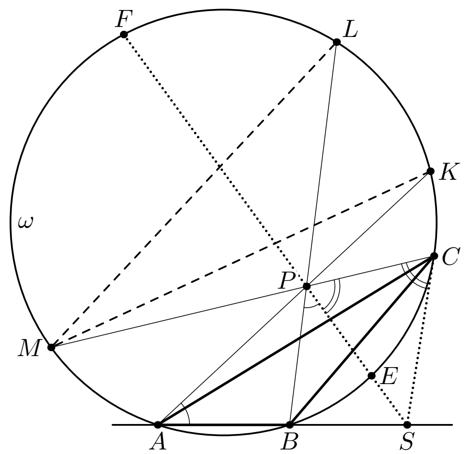

Fig. 2

We have $SP^2 = SC^2 = SA \cdot SB$, so $\dfrac{SP}{SB} = \dfrac{SA}{SP}$, and hence $\triangle PSA \sim \triangle BSP$. Then $\angle BPS = \angle SAP$. Since $2\angle BPS = \overset{\frown}{BE} + \overset{\frown}{LF}$ and $2\angle SAP = \overset{\frown}{BE} + \overset{\frown}{EK}$ we have

$$\overset{\frown}{LF} = \overset{\frown}{EK}. \tag{4}$$

On the other hand, from $\angle SPC = \angle SCP$ we have $\overset{\frown}{EC} + \overset{\frown}{MF} = \overset{\frown}{EC} + \overset{\frown}{EM}$, or

$$\overset{\frown}{MF} = \overset{\frown}{EM}. \tag{5}$$

From (4) and (5) we get $\overset{\frown}{MFL} = \overset{\frown}{MF} + \overset{\frown}{FL} = \overset{\frown}{ME} + \overset{\frown}{EK} = \overset{\frown}{MEK}$ and hence $MK = ML$. The claim is proved.

Problem 5. In each of six boxes $B_1, B_2, B_3, B_4, B_5, B_6$ there is initially one coin. There are two types of operation allowed:

Type 1: Choose a nonempty box B_j with $1 \leq j \leq 5$. Remove one coin from B_j and add two coins to B_{j+1}.

Type 2: Choose a nonempty box B_k with $1 \leq k \leq 4$. Remove one coin from B_k and exchange the contents of (possibly empty) boxes B_{k+1} and B_{k+2}.

Determine whether there is a finite sequence of such operations that results in boxes B_1, B_2, B_3, B_4, B_5 being empty and box B_6 containing exactly $2010^{2010^{2010}}$ coins. (Note that $a^{b^c} = a^{(b^c)}$.)

Answer. Yes. There exists such a sequence of moves.

Solution. Denote by $(a_1, a_2, \ldots, a_n) \to (a'_1, a'_2, \ldots, a'_n)$ the following: if some consecutive boxes contain $a_1, \ldots, a_n$ coins, then it is possible to perform several allowed moves such that the boxes contain $a'_1, \ldots, a'_n$ coins respectively, whereas the contents of the other boxes remain unchanged.

Let $A = 2010^{2010^{2010}}$, respectively. Our goal is to show that

$$(1, 1, 1, 1, 1, 1) \to (0, 0, 0, 0, 0, A).$$

First we prove two auxiliary observations.

Lemma 1. $(a, 0, 0) \to (0, 2^a, 0)$ for every $a \geq 1$.

Proof. We prove by induction that $(a, 0, 0) \to (a-k, 2^k, 0)$ for every $1 \leq k \leq a$. For $k = 1$, apply Type 1 to the first box:

$$(a, 0, 0) \to (a-1, 2, 0) = (a-1, 2^1, 0).$$

Now assume that $k < a$ and the statement holds for some $k < a$. Starting from $(a-k, 2^k, 0)$, apply Type 1 to the middle box 2^k times, until it becomes empty. Then apply Type 2 to the first box:

$$(a-k, 2^k, 0) \to (a-k, 2^k - 1, 2) \to \cdots \to (a-k, 0, 2^{k+1}) \to (a-k-1, 2^{k+1}, 0).$$

Hence,

$$(a, 0, 0) \to (a-k, 2^k, 0) \to (a-k-1, 2^{k+1}, 0). \qquad \square$$

Lemma 2. For every positive integer n, let $P_n = \underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n}$ (e.g. $P_3 = 2^{2^2} = 16$). Then $(a, 0, 0, 0) \to (0, P_a, 0, 0)$ for every $a \geq 1$.

Proof. Similarly to Lemma 1, we prove that $(a, 0, 0, 0) \to (a-k, P_k, 0, 0)$ for every $1 \leq k \leq a$.

For $k = 1$, apply Type 1 to the first box:

$$(a, 0, 0, 0) \to (a-1, 2, 0, 0) = (a-1, P_1, 0, 0).$$

Now assume that the lemma holds for some $k < a$. Starting from $(a-k, P_k, 0, 0)$, apply Lemma 1, then apply Type 1 to the first box:

$$(a-k, P_k, 0, 0) \to (a-k, 0, 2^{P_k}, 0) = (a-k, 0, P_{k+1}, 0) \to (a-k-1, P_{k+1}, 0, 0).$$

Therefore,

$$(a, 0, 0, 0) \to (a-k, P_k, 0, 0) \to (a-k-1, P_{k+1}, 0, 0). \qquad \square$$

Now we prove the statement of the problem.

First apply Type 1 to box 5, then apply Type 2 to boxes B_4, B_3, B_2 and B_1 in this order. Then apply Lemma 2 twice:

$$(1,1,1,1,1,1) \to (1,1,1,1,0,3) \to (1,1,1,0,3,0) \to (1,1,0,3,0,0) \to (1,0,3,0,0,0) \to$$
$$\to (0,3,0,0,0,0) \to (0,0,P_3,0,0,0) = (0,0,16,0,0,0) \to (0,0,0,P_{16},0,0).$$

We already have more than A coins in box B_4, since

$$A \le 2010^{2010^{2010}} < (2^{11})^{2010^{2010}} = 2^{11\cdot 2010^{2010}} < 2^{2010^{2011}} < 2^{(2^{11})^{2011}} = 2^{2^{11\cdot 2011}} < 2^{2^{2^{15}}} < P_{16}.$$

To decrease the number of coins in box B_4, apply Type 2 to this stack repeatedly until its size decreases to $A/4$. (In every step, we remove a coin from B_4 and exchange the empty boxes B_5 and B_6.)

$$(0,0,0,P_{16},0,0) \to (0,0,0,P_{16}-1,0,0) \to (0,0,0,P_{16}-2,0,0) \to$$
$$\to \cdots \to (0,0,0,A/4,0,0).$$

Finally, apply Type 1 repeatedly to empty boxes B_4 and B_5:

$$(0,0,0,A/4,0,0) \to \cdots \to (0,0,0,0,A/2,0) \to \cdots \to (0,0,0,0,0,A).$$

Comment. Starting with only 4 boxes, it is not hard to check manually that we can achieve at most 28 coins in the last position. However, around 5 and 6 boxes the maximal number of coins explodes. With 5 boxes it is possible to achieve more than $2^{2^{14}}$ coins. With 6 boxes the maximum is greater than $P_{P_{2^{14}}}$.

Problem 6. Let $a_1, a_2, a_3, \dots$ be a sequence of positive real numbers. Suppose that for some positive integer s, we have

$$a_n = \max\{a_k + a_{n-k} \mid 1 \le k \le n-1\} \tag{6}$$

for all $n > s$. Prove that there exist positive integers ℓ and N, with $\ell \le s$ and such that $a_n = a_\ell + a_{n-\ell}$ for all $n \ge N$.

Solution 1. First, from the problem conditions we have that each a_n $(n > s)$ can be expressed as $a_n = a_{j_1} + a_{j_2}$ with $j_1, j_2 < n$, $j_1 + j_2 = n$. If, say, $j_1 > s$ then we can proceed in the same way with a_{j_1}, and so on. Finally, we represent a_n in a form

$$a_n = a_{i_1} + \cdots + a_{i_k}, \tag{7}$$
$$1 \le i_j \le s, \quad i_1 + \cdots + i_k = n. \tag{8}$$

Moreover, if a_{i_1} and a_{i_2} are the numbers in (7) obtained on the last step, then $i_1 + i_2 > s$. Hence we can adjust (8) as

$$1 \le i_j \le s, \quad i_1 + \cdots + i_k = n, \quad i_1 + i_2 > s. \tag{9}$$

On the other hand, suppose that the indices $i_1, \dots, i_k$ satisfy the conditions (9). Then, denoting $s_j = i_1 + \cdots + i_j$, from (6) we have

$$a_n = a_{s_k} \ge a_{s_{k-1}} + a_{i_k} \ge a_{s_{k-2}} + a_{i_{k-1}} + a_{i_k} \ge \cdots \ge a_{i_1} + \cdots + a_{i_k}.$$

Summarizing these observations we get the following

Claim. For every $n > s$, we have

$$a_n = \max\{a_{i_1} + \cdots + a_{i_k} : \text{the collection } (i_1, \ldots, i_k) \text{ satisfies } (9)\}. \qquad \square$$

Now we denote

$$m = \max_{1 \le i \le s} \frac{a_i}{i}$$

and fix some index $\ell \le s$ such that $m = \dfrac{a_\ell}{\ell}$.

Consider some $n \ge s^2\ell + 2s$ and choose an expansion of a_n in the form (7), (9). Then we have $n = i_1 + \cdots + i_k \le sk$, so $k \ge n/s \ge s\ell + 2$. Suppose that none of the numbers $i_3, \ldots, i_k$ equals ℓ. Then by the pigeonhole principle there is an index $1 \le j \le s$ which appears among $i_3, \ldots, i_k$ at least ℓ times, and surely $j \ne \ell$. Let us delete these ℓ occurrences of j from $(i_1, \ldots, i_k)$, and add j occurrences of ℓ instead, obtaining a sequence $(i_1, i_2, i'_3, \ldots, i'_{k'})$ also satisfying (9). By Claim, we have

$$a_{i_1} + \cdots + a_{i_k} = a_n \ge a_{i_1} + a_{i_2} + a_{i'_3} + \cdots + a_{i'_{k'}},$$

or, after removing the coinciding terms, $\ell a_j \ge j a_\ell$, so $\dfrac{a_\ell}{\ell} \le \dfrac{a_j}{j}$. By the definition of ℓ, this means that $\ell a_j = j a_\ell$, hence

$$a_n = a_{i_1} + a_{i_2} + a_{i'_3} + \cdots + a_{i'_{k'}}.$$

Thus, for every $n \ge s^2\ell + 2s$ we have found a representation of the form (7), (9) with $i_j = \ell$ for some $j \ge 3$. Rearranging the indices we may assume that $i_k = \ell$.

Finally, observe that in this representation, the indices $(i_1, \ldots, i_{k-1})$ satisfy the conditions (9) with n replaced by $n - \ell$. Thus, from the Claim we get

$$a_{n-\ell} + a_\ell \ge (a_{i_1} + \cdots + a_{i_{k-1}}) + a_\ell = a_n,$$

which by (6) implies

$$a_n = a_{n-\ell} + a_\ell \qquad \text{for each } n \ge s^2\ell + 2s,$$

as desired.

Solution 2. As in the previous solution, we involve the expansion (7), (8), and we fix some index $1 \le \ell \le s$ such that

$$\frac{a_\ell}{\ell} = m = \max_{1 \le i \le s} \frac{a_i}{i}.$$

Now, we introduce the sequence (b_n) as $b_n = a_n - mn$; then $b_\ell = 0$.

We prove by induction on n that $b_n \le 0$, and (b_n) satisfies the same recurrence relation as (a_n). The base cases $n \le s$ follow from the definition of m. Now, for $n > s$ from the induction hypothesis we have

$$b_n = \max_{1 \le k \le n-1}(a_k + a_{n-k}) - nm = \max_{1 \le k \le n-1}(b_k + b_{n-k} + nm) - nm = \max_{1 \le k \le n-1}(b_k + b_{n-k}) \le 0,$$

as required.

Now, if $b_k = 0$ for all $1 \le k \le s$, then $b_n = 0$ for all n, hence $a_n = mn$, and the statement is trivial. Otherwise, define

$$M = \max_{1 \le i \le s} |b_i|, \quad \varepsilon = \min\{|b_i| : 1 \le i \le s,\ b_i < 0\}.$$

Then for $n > s$ we obtain

$$b_n = \max_{1\le k\le n-1}(b_k + b_{n-k}) \ge b_\ell + b_{n-\ell} = b_{n-\ell},$$

so

$$0 \ge b_n \ge b_{n-\ell} \ge b_{n-2\ell} \ge \cdots \ge -M.$$

Thus, in view of the expansion (7), (8) applied to the sequence (b_n), we get that each b_n is contained in a set

$$T = \{b_{i_1} + b_{i_2} + \cdots + b_{i_k} : \; i_1, \dots, i_k \le s\} \cap [-M, 0]$$

We claim that this set is finite. Actually, for any $x \in T$, let $x = b_{i_1} + \cdots + b_{i_k}$ $(i_1, \dots, i_k \le s)$. Then among b_{i_j}'s there are at most $\frac{M}{\varepsilon}$ nonzero terms (otherwise $x < \frac{M}{\varepsilon} \cdot (-\varepsilon) < -M$). Thus x can be expressed in the same way with $k \le \frac{M}{\varepsilon}$, and there is only a finite number of such sums.

Finally, for every $t = 1, 2, \dots, \ell$ we get that the sequence

$$b_{s+t}, b_{s+t+\ell}, b_{s+t+2\ell}, \dots$$

is non-decreasing and attains the finite number of values; therefore it is constant from some index. Thus, the sequence (b_n) is periodic with period ℓ from some index N, which means that

$$b_n = b_{n-\ell} = b_{n-\ell} + b_\ell \qquad \text{for all } n > N + \ell,$$

and hence

$$a_n = b_n + nm = (b_{n-\ell} + (n-\ell)m) + (b_\ell + \ell m) = a_{n-\ell} + a_\ell \qquad \text{for all } n > N + \ell,$$

as desired.