Stanford PhD Qualifying Exam in Algebra
Fall 2004 (Morning Session)

General Instructions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. Classify all finite groups of order $147 = 3 \cdot 7^2$.

2. Let $A, B \in \mathrm{Mat}_n(\mathbb{R})$ be a pair of commuting matrices.

(a) Suppose that A and B are both nilpotent. Show that they have a nonzero common nullvector.

(b) Suppose that n is odd. Show that A and B have a common eigenvector. (It is no longer assumed that they are nilpotent.)

3. (a) Find the minimal polynomial of $\sqrt{4+\sqrt{7}}$ over $\mathbb{Q}$.

(b) Find the Galois group of that polynomial's splitting field over $\mathbb{Q}$.

Hint: Check that $\sqrt{4+\sqrt{7}} = \frac{1}{2}(\sqrt{2}+\sqrt{14})$.

4. Recall the following definitions: If R is a commutative ring and $\mathfrak{a}$ is an ideal, then the *radical* $r(\mathfrak{a}) = \{x \in R \,|\, x^n \in \mathfrak{a}\}$. An ideal $\mathfrak{q}$ is *primary* if $xy \in \mathfrak{q}$ implies that either $x \in \mathfrak{q}$ or $y \in r(\mathfrak{q})$. Prove that if $r(\mathfrak{q})$ is a maximal ideal, then $\mathfrak{q}$ is primary.

5. Let G be a nonabelian group of order pq where p and q are distinct primes such that $p < q$.

(a) Show that p divides $q-1$, and show that the number of conjugacy classes of G is exactly $p + \frac{q-1}{p}$.

(b) Determine the number and degrees of the irreducible complex characters of G.

Stanford PhD Qualifying Exam in Algebra
Fall 2004 (Afternoon Session)

General Instructions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. Let $G = \langle x \rangle$ be a cyclic group of order 2^n, and let $R = \mathbb{F}_2[G]$ be the group algebra.

(a) Show that

$$J = \left\{ \sum_{i=0}^{2^n-1} a_i x^i \mid \sum a_i = 0 \right\}$$

is a nilpotent ideal in the commutative ring R, and deduce that

$$\Gamma = 1 + J = \left\{ \sum_{i=0}^{2^n-1} a_i x^i \mid \sum a_i = 1 \right\}$$

is an abelian group of order 2^{2^n-1}.

(b) Consider $\Gamma^{2^k} = \{u^{2^k} \mid u \in \Gamma\}$. Show that

$$|\Gamma^{2^k}| = \begin{cases} 2^{2^{n-k}-1} & \text{if } k \leqslant n; \\ 1 & \text{if } k \geqslant n. \end{cases}$$

(c) There is enough information in this fact to determine the structure of Γ. Illustrate this by determining the structure of Γ when $n = 4$.

2. Let $r > 0$, and let q be a prime power. If $a \in \mathbb{F}_{q^r}$ let $T(a) : \mathbb{F}_{q^r} \longrightarrow \mathbb{F}_{q^r}$ be the map $T(a)x = ax$. Regarding $\mathbb{F}_{q^r}$ as a r-dimensional vector space over $\mathbb{F}_q$, we may think of $T(a)$ as an element of $\mathrm{GL}(r, \mathbb{F}_q)$.

(a) Show that the composite $\det \circ T$ coincides with the norm map $\mathbb{F}_{q^r} \longrightarrow \mathbb{F}_q$.

(b) Show that if $b \in \mathbb{F}_q^\times$, then there exists $a \in \mathbb{F}_{q^r}^\times$ such that $\det\, T(a) = b$.

3. Let G be a finite group and H a subgroup. Let $\rho : H \longrightarrow \mathrm{GL}_n(\mathbb{C})$ be an irreducible representation. Show that if ρ_1 and ρ_2 are extensions of ρ to G, and if the characters χ_1 and χ_2 of ρ_1 and ρ_2 are the same, then $\rho_1(g) = \rho_2(g)$ for all $g \in G$.

4. Let A be a Noetherian local commutative ring with maximal ideal $\mathfrak{m}$. Assume that $\mathfrak{m}$ is principal. Show that every nonzero ideal of A is of the form $\mathfrak{m}^k$ for some k.

5. Let $\zeta = e^{2\pi i/40}$ and let $K = \mathbb{Q}(\zeta)$. (a) Determine the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

(b) Find all quadratic extensions of $\mathbb{Q}$ contained in K. Express them in the form $\mathbb{Q}(\sqrt{D})$ for $D \in \mathbb{Z}$.