Real Analysis Quals—Fall 2004

Part I.

Answer all problems

1 Prove that if $1 \leq p < \infty$ and $f \in L^p(\mathbb{R})$, then the map $\tau \mapsto f_\tau$, where $f_\tau(x) = f(x-\tau)$, is continuous from $\mathbb{R}$ into $L^p(\mathbb{R})$ (endowed with the norm topology).

2 Let $(X, \mathcal{B}, \mu)$ be a measure space, μ finite or infinite. For a measurable real-valued f on $(X, \mathcal{B}, \mu)$ and $n \in \mathbb{Z}$ write

$$\mathbf{m}_n = \mathbf{m}_n(f) = \mu(\{x \colon 2^{n-1} \leq |f(x)| < 2^n\}).$$

Give a condition, stated in terms of $\{\mathbf{m}_n\}$, which is necessary and sufficient for $f \in L^p$, where $1 \leq p < \infty$.

3 Let $(X, \mathcal{B}, \mu)$ be a finite measure space, $\mu(X) = a$.

a. Prove that $L^{p_1}(X, \mathcal{B}, \mu) \subset L^{p_0}(X, \mathcal{B}, \mu)$ for $1 \leq p_0 \leq p_1 < \infty$, and for $f \in L^{p_1}(X, \mathcal{B}, \mu)$,

$$\|f\|_{p_0} \leq a^{(\frac{1}{p_0} - \frac{1}{p_1})} \|f\|_{p_1}$$

Hint: Reduce to, and prove, the case $a = 1$.

b. Let $(X, \mathcal{B}, \mu)$ be a finite measure space, $f_n \in L^2(X, \mathcal{B}, \mu)$, $\|f_n\|_{L^2} \leq 1$, f measurable, and $f_n \to f$ in measure. Prove:

b.1. $f \in L^2(X, \mathcal{B}, \mu)$.

b.2. $f_n \to f$ in the L^p-norm for every $p < 2$.

b.3. $f_n \to f$ weakly (in L^2).

4 Denote by $C^\infty([0,1])$ the space of all the infinitely differentiable complex-valued functions on the interval $[0,1]$. Define convergence in $C^\infty([0,1])$ as follows: $f_n \to f$ in C^∞ if $f_n^{(j)} \to f^{(j)}$ uniformly for every $j \in \mathbb{N}$.

a. Prove that this convergence is equivalent to convergence in the metric

$$d(f,g) = \sum_{j=0}^{\infty} 2^{-j} \frac{\|(f-g)^{(j)}\|_\infty}{1 + \|(f-g)^{(j)}\|_\infty}$$

b. Prove that the topology so defined *cannot* be defined by a norm.

5 Let $f \in L^1(\mathbb{R})$ and assume that $\hat{f}(\xi) = \int e^{-ix\xi} f(x)dx = 0$ for $|\xi| > 1$. Prove that f is equal a.e. to the restriction to $\mathbb{R}$ of an entire function $f(z)$, $z = x + iy$, such that $\lim_{|y|\to\infty} e^{-|y|} f(x+iy) = 0$.

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Part II.

Answer all problems

6 Let $\{v_n\}$ be a sequence of vectors in a Hilbert space $\mathcal{H}$, and assume that $v_n \to v$ in the weak topology.

a. Prove that $\|v_n\| = O(1)$.

b. Prove also that $v_n \to v$ in norm if, and only if $\|v_n\| \to \|v\|$.

7 Let (X, ρ) be a complete metric space. A set $E \subset X$ has *the property of Baire* if $E = (G \setminus P_1) \cup P_2$, where G is assumed open and P_j, $j = 1, 2$ meager (first category). We denote by $\mathcal{PB}$ the set of all such E.

a. Prove: *$\mathcal{PB}$ is the sigma algebra generated by all the open sets and all the meager sets in X.*
Hint: The boundary of a closed set is non-dense.

b. If E is a second category Borel set then it is residual (has meager complement) on some nonempty open subset.

c. If $E \subset \mathbb{R}$ is a second category Borel set then $E - E = \{x - y : x, y \in E\}$ contains an interval $(-\delta, \delta)$, $\delta > 0$.

8 **a.** Let $B \subset C^1([0,1])$ be a subspace of dimension $N + 1$. Show that there exists $f \in B$ such that $\sup_{0\le x\le 1}|f(x)| = 1$ and $\sup_{0\le x\le 1}|f'(x)| \ge 2N$.
Hint: There is a nontrivial $g \in B$ which vanishes on $E = \{\frac{2j-1}{2N}\}_{j=1}^{N}$.

b. Prove that a subspace $B \subset C^1([0,1])$ which is closed under uniform convergence is finite dimensional.
Hint: Show that $\|f'\|_\infty \le K\|f\|_\infty$ for some constant K and all $f \in B$.

9 We denote by $\mathcal{H}\text{-}\dim E$ the Hausdorff dimension of a (closed) set $E \in \mathbb{R}^k$, that is the number α_0 such that $\begin{cases} H^\alpha(E) = \infty & \text{for } \alpha < \alpha_0, \\ H^\alpha(E) = 0 & \text{for } \alpha > \alpha_0, \end{cases}$
where $H^\alpha(E)$ denotes the Hausdorff measure in dimension α of the set E.

a. Let F be a Lipschitz map from a compact set $E \subset \mathbb{R}^k$ into $\mathbb{R}^l$. Show that $F(E)$ is closed and $\mathcal{H}\text{-}\dim E \ge \mathcal{H}\text{-}\dim F(E)$.

b. Prove that $\mathcal{H}\text{-}\dim(\mathsf{bdry}(D)) \ge 1$ for every non-empty bounded open set $D \subset \mathbb{R}^2$.

10 Assume that α is an irrational multiple of π. Let μ be a finite Borel measure on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and assume that the measure ν defined for all Borel sets $A \subset \mathbb{T}$ by: $\nu(A) = \mu(A - \alpha) - \mu(A)$ is absolutely continuous (with respect to Lebesgue).

Prove that μ is absolutely continuous (Lebesgue).
Hint: Write $\mu = \mu_{ac} + \mu_s$, the decomposition of μ to its absolutely continuous and singular parts.