Ph.D. Qualifying Exam problems, Real Analysis

June 2005, part I.

1 (Quickies)

a. Let $(B, ||\cdot||)$ be a normed space, and $A: B \to B$ an invertible linear transformation such that $\|A^n\| \le c$ for some constant $c > 0$ and all $n \in \mathbb{Z}$. Prove that there is an equivalent norm on B with respect to which A is an isometry.

b. Let μ be a finite measure on $[-1, 1]$ and assume that $\int x^{kn} = 0$ for some integer k and all nonnegative integers n. Prove that if k is odd then $\mu = 0$. What can you say when k is even?

c. Prove that the space $C(0, 1)$ is not reflexive.
Hint: Identify the dual space $M(0, 1)$ of $C(0, 1)$ and show that the dual of $M(0, 1)$ contains elements that can not be identified with elements of $C(0, 1)$.

d. Prove that $C(0, 1)$ is not isomorphic—and in particular not isometric— to a uniformly convex Banach space.

2 Prove that a linear operator T on a Hilbert space $\mathcal{H}$ is *compact* if, and only if, it is the limit, in the norm topology of operators, of a sequence of operators of finite rank.

3 Suppose B is a Banach space and $K \subset B$ is a subset. Recall that its convex hull, ch(K), is the smallest convex subset of B which contains K.

a. Prove that if K is compact, then the closure of ch(K) is compact as well.

b. Show that the set of indicator functions $\{\mathbb{1}_{[\tau,\tau+\frac{1}{5}]}(t) : \tau \in \mathbb{T}\}$ is compact in $L^1(\mathbb{T})$, and its convex hull is not.

4 Let $A_j \subset [0, 1]$, for $j = 1, 2, \ldots, N$ be Lebesgue measurable, $\mu(A_j) \ge \frac{1}{2}$.
Let $0 < a < 1/2$ and denote $E_a = \{x : x \in A_j$ for more than aN values of $j\}$.
Prove that $\mu(E_a) \ge \frac{1-2a}{2(1-a)}$.
Show that the estimate $\mu(E_a) \ge \frac{1-2a}{2(1-a)}$ is best possible (if it is to apply to all N).
Hint: Consider $F = \sum_1^N \mathbb{1}_{A_j}$.

5 The Hausdorff-Young inequality on the line states that if $1 \le p \le 2$ and $1/p + 1/q = 1$, then

$$f \in L^p(\mathbb{R}) \quad \text{implies} \quad \|\hat{f}\|_{L^q(\hat{\mathbb{R}})} \le \|f\|_{L^p(\mathbb{R})}. \tag{1}$$

Prove that the converse is true: (1) *implies* $1/p + 1/q = 1$ *and* $1 \le p \le 2$.
Hint: For the first claim use scaling ($f_{\lambda,s} = \lambda^s f(\lambda x)$ for appropriate s, and its effect on the Fourier transform). For the second claim, show that if $\varphi_\lambda(x) = e^{i\lambda x^2}\mathbb{1}_{[-1,1]}$, then as $\lambda \to \infty$ $\|\hat{\varphi}_\lambda\|_\infty \to 0$ while $\|\hat{\varphi}_\lambda\|_2$ remains constant.

Ph.D. Qualifying Exam problems, Real Analysis

June 2005, part II.

6 (Quickies)

a. A distribution μ on $\mathbb{T}$ is *positive* if $\langle f, \mu\rangle \geq 0$ for every nonnegative $f \in C^\infty(\mathbb{T})$.
Show that a positive distribution is a measure.
Hint: Positivity implies: for real-valued f, $\langle f, \mu\rangle \leq \max f(t)\langle 1, \mu\rangle$.

b. Assume $f_n \in L^2[0,1]$, $n \in \mathbb{N}$, and $\|f_n\| \leq 1$.
Prove that $\mu(\{x \,|\, |f_n(x)| > n^{\frac{2}{3}}\}) < n^{-\frac{4}{3}}$, and conclude that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that the measure of the set $\{x : |f_n(x)| \leq n^{\frac{2}{3}} \text{ for } n > N\}$ exceeds $> 1-\varepsilon$.

7 Let $\{f_n\}$ be an orthonormal sequence in $L^2(0,1)$.
Prove that $S_n = \frac{1}{n}\sum f_m \to 0$ a.e.
Hints:

a. $\|S_n\|_{L^2} = \frac{1}{n}$. It follows that if $\sum \lambda_j^{-1} < \infty$, and in particular if $\lambda_j = [j\log^2 j]$, then $\sum |S_{\lambda_j}|^2$ converges a.e. and $S_{\lambda_j} \to 0$ a.e.

b. If $N \in (\lambda_j, \lambda_{j+1})$, then $S_N = \frac{\lambda_j}{N} S_{\lambda_j} + \frac{1}{N}\sum_{\lambda_j+1}^{N} f_m$. Use **6.b.** to estimate the last sum.

8 Let $B = \{(x,y,z) : r = \sqrt{x^2+y^2+z^2} \leq 1\}$, the unit ball in $\mathbb{R}^3$, $G = \mathbb{1}_B$ its indicator function, and $g(x) = \int G(x,y,z)\,dy\,dz$.

a. Compute g.

b. Show that $\hat{g}(\xi) = O\left(|\xi|^{-3}\right)$

c. Prove $\hat{G}(\xi,\eta,\zeta) = O\left((\xi^2+\eta^2+\zeta^2)^{-\frac{3}{2}}\right)$.

d. Let $F_n(x,y,z) = \sin^2(nr)G(x,y,z)$, and $f_n(x) = \int F_n(x,y,z)dy\,dz$.
Prove: $f_n \to \frac{1}{2}g$ uniformly as $n \to \infty$,

9 Let $f(x) = \sum 10^{-n}\cos 10^{2n}x$

a. Prove that f satisfies the Hölder $\frac{1}{2}$ condition:

$$|f(x+h) - f(x)| \leq \text{const} \cdot h^{\frac{1}{2}}.$$

b. Prove that f is nowhere differentiable.
Hint: For every x and every n find points y_n and z_n such that
$|x-y_n| \sim 10^{-2n}$, $|x-z_n| \sim 10^{-2n}$, $\frac{f(x)-f(y_n)}{x-y_n} > 10^{n-2}$, and $\frac{f(x)-f(y_n)}{x-y_n} < -10^{n-2}$.

c. Can a Lipschitz function be nowhere differentiable? (Justify your answer by quoting relevant standard theorems.)

10 Let B be a Banach space and S a linear map from B into $C([0,1])$, such that if $\{v_n\} \subset B$ and $\|v_n\|_B \to 0$ then $Sv_n(x) \to 0$ pointwise in $[0,1]$. Prove that S is bounded; in particular, the assumptions $\|v_n\|_B \to 0$ implies $Sv_n(x) \to 0$ *uniformly*.