Contributor Profiles: John G. Heuver

John G. Heuver was born in 1934 in Olst, the Netherlands where he became a teacher. He taught for three years at elementary school and then for six years at a vocational school for agriculture students between the ages of 12 and 16. In the meantime he acquired certificates in Mathematics and English as a requirement for teaching at the secondary school level.

In 1967 he immigrated to Canada and came to Calgary, where he obtained a B.Ed. degree at the University of Calgary with a major in Mathematics. His choice at that time was to settle down somewhere beyond Calgary or Edmonton, so he ended up in Grande Prairie in 1970 but only planned to stay for at most one year. But the wide-open spaces of Alberta had their own attraction. Except for the first six weeks at a junior high school, he taught mathematics from then on at the Grande Prairie Composite High School until 1997 when he retired. Over that period of time the city's population increased from 10 000 to over 50 000.

During his many years teaching high-school mathematics he witnessed quite a few curriculum changes, from teaching about probabilities with throwing dice and drawing cards from a deck (which was rather straightforward to explain to the students), to explaining statistics using the normal curve (a more difficult concept to convey, and often utilizing contrived data).

John is critical of the argument for teaching a topic merely because it represents a so-called practical application, and of the treacherous pitfalls of removing real-world constraints from real-world problems, such as modeling exponential growth rates for bacteria that are not allowed to expire.

John says he owes his involvement with problem solving in mathematical journals to Murray Klamkin, who once in the seventies gave a session at a teacher's convention in Grande Prairie. He had obtained a subscription to the *American Mathematical Monthly* and afterwards found a problem of Murray's regarding an inequality involving the edges of a tetrahedron, which he was able to solve. This caught his fancy, and the rest is history. A subsequent reference in the *Monthly* led him to ***Crux***.

After retiring he has found more time to work on mathematical problems. In 1999, with the help of a carpenter, he built a new cabin on Sturgeon Lake, where he visits frequently and even in the winter time since it has heat and water.

SKOLIAD No. 125

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by **1 Oct, 2010**. A copy of ***CRUX with Mayhem*** will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

The deadline for Skoliad 124 solutions in the previous issue (***CRUX with MAYHEM*** Vol. 36, No. 3) is **1 Sept, 2010** NOT 1 July, 2010; our apologies.

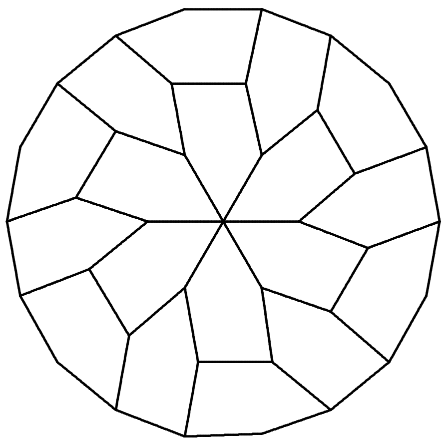
Our contest for this month is the Baden-Württemberg Mathematics Contest, 2009. Our thanks go to the Landeswettbewerb Mathematik Baden Württemberg for providing this contest and for permission to publish it.

La rédaction souhaite remercier Rolland Gaudet, de Collège universitaire de Saint-Boniface, Winnipeg, MB, d'avoir traduit ce concours.

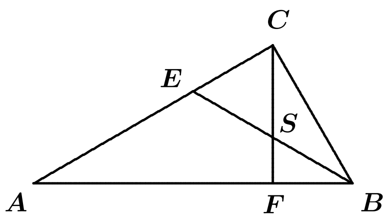
Concours mathématique Baden-Württemberg 2009

1. Déterminer tous les entiers naturels n tels que la somme de n et de ses chiffres décimaux est 2010.

2. Un polygone régulier à 18 côtés est découpé en pentagones congrus, tel qu'illustré. Déterminer les angles internes d'un tel pentagone.



3. Dans la figure à droite, $\triangle ABE$ est isocèle avec base AB, $\angle BAC = 30°$, et $\angle ACB = \angle AFC = 90°$. Déterminer le ratio entre la surface du $\triangle ESC$ et la surface du $\triangle ABC$.



4. À partir de deux nombres non nuls z_1 et z_2, soit z_n égal à $\frac{z_{n-1}}{z_{n-2}}$ pour $n > 2$. Alors $z_1, z_2, z_3, \ldots$ forment une suite. Démontrer que si on multiplie n'importe quels 2009 termes consécutifs de cette suite, le produit fait lui-même partie de la suite.

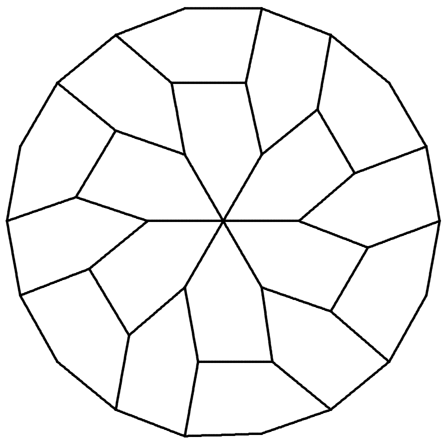
5. Soit $\triangle ABC$ un triangle isocèle tel que $\angle ACB = 90°$. Un cercle avec centre C coupe AC en D et BC en E. Tracer la ligne AE. La perpendiculaire à AE passant par C coupe la ligne AB en F, tandis que la perpendiculaire à AE passant par D coupe la ligne AB en G. Démontrer que la longueur de BF égale la longueur de GF.

6. Une machine choisit un des diviseurs de 2009^{2010} de façon aléatoire et vous misez sur le chiffre en position unitaire de ce diviseur. Sur quel chiffre misez-vous ?

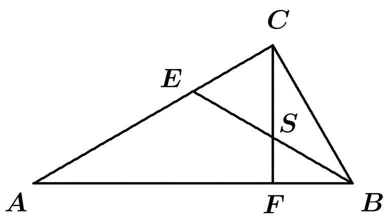
Baden-Württemberg Mathematics Contest 2009

1. Find all natural numbers n such that the sum of n and the digit sum of n is 2010.

2. A regular 18-gon can be cut into congruent pentagons as in the figure below. Determine the interior angles of such a pentagon.



3. In the figure on the right, $\triangle ABE$ is isosceles with base AB, $\angle BAC = 30°$, and $\angle ACB = \angle AFC = 90°$. Find the ratio of the area of $\triangle ESC$ to the area of $\triangle ABC$.



4. Given two nonzero numbers z_1 and z_2, let z_n be $\frac{z_{n-1}}{z_{n-2}}$ for $n > 2$. Then z_1, z_2, z_3, $\ldots$ form a sequence. Prove that if you multiply any 2009 consecutive terms of the sequence, then the product is itself a member of the sequence.

5. Let $\triangle ABC$ be an isosceles triangle such that $\angle ACB = 90^\circ$. A circle with centre C cuts AC at D and BC at E. Draw the line AE. The perpendicular to AE through C cuts the line AB at F, and the perpendicular to AE through D cuts the line AB at G. Show that the length of BF equals the length of GF.

6. A gaming machine randomly selects a divisor of 2009^{2010} and displays its ones digit. Which digit should you gamble on?

Next we give the solutions to the World Youth Mathematics Intercity Competition, Individual Contest, Part I, 2005, given in Skoliad 119 at [2009 : 354–356].

1. The sum of a four-digit number and its four digits is 2005. What is this four-digit number?

Solution by Ian Chen, student, Centennial Secondary School, Coquitlam, BC.

Let n denote the desired number. Surely $n \leq 2005$. Since the sum of three digits is at most 27, the digit sum of n is at most 29. Therefore $n \geq 1976$.

Let d represent a digit, and let S be the sum of n and its digits.

If $n = 2000 + d$, then $S = 2000 + 2 + 2d$ which is even and thus cannot equal 2005.

If $n = 1990 + d$, then $S = 2009 + 2d$ which is too large.

If $n = 1980 + d$, then $S = 1998 + 2d$ which is even and thus cannot equal 2005.

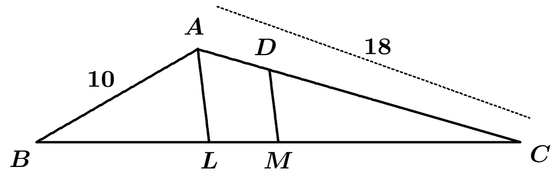
If $n = 1970 + d$, then $S = 1987 + 2d$. Solving $S = 1987 + 2d = 2005$ yields that $d = 9$.

Hence, $n = 1979$.

Also solved by MICHAEL CHEUNG, student, Port Moody Secondary School, Port Moody, BC; LENA CHOI, student, École Banting Middle School, Coquitlam, BC; TIMOTHY CHU, student, R.C. Palmer Secondary School, Richmond, BC; VINCENT CHUNG, student, Burnaby North Secondary School, Burnaby, BC; WEN-TING FAN, student, Burnaby North Secondary School, Burnaby, BC; KRISTIAN HANSEN, student, Burnaby North Secondary School, Burnaby, BC; and LISA WANG, student, Port Moody Secondary School, Port Moody, BC.

2. In triangle ABC, $AB = 10$ and $AC = 18$. M is the midpoint of BC, and the line through M parallel to the bisector of $\angle CAB$ cuts AC at D. Find the length of AD.

Solution by Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.



Let L denote the point on BC such that AL is the bisector of $\angle CAB$. The Sine Law in $\triangle ABL$ yields that $\frac{BL}{\sin\angle BAL} = \frac{10}{\sin\angle ALB}$, and therefore $BL = 10\left(\frac{\sin\angle BAL}{\sin\angle ALB}\right)$.

Likewise, using the Sine Law in $\triangle ALC$ yields $\frac{CL}{\sin\angle CAL} = \frac{18}{\sin\angle ALC}$. Thus, $CL = 18\left(\frac{\sin\angle CAL}{\sin\angle ALC}\right)$. But it is also true that $\angle CAL = \angle BAL$ and $\angle ALC = 180^\circ - \angle ALB$, so $CL = 18\left(\frac{\sin\angle BAL}{\sin\angle ALB}\right)$.

Let z denote the fraction $\frac{\sin\angle BAL}{\sin\angle ALB}$. Then $BL = 10z$ and $CL = 18z$. Therefore, $BC = 28z$ and $CM = 14z$. As $\triangle ACL$ is similar to $\triangle DCM$, it follows that $\frac{DC}{AC} = \frac{CM}{CL}$, so $\frac{DC}{18} = \frac{14z}{18z}$, so $DC = 14$. Hence, $AD = 4$.

3. Let x, y, z be positive numbers such that $x+y+xy = 8$, $y+z+yz = 15$, and $z + x + zx = 35$. Find the value of $x + y + z + xy$.

Solution by Vincent Chung, student, Burnaby North Secondary School, Burnaby, BC.

Since $x + y + xy = 8$, it follows that $x(1 + y) = 8 - y$, so $x = \frac{8-y}{y+1}$. Likewise, since $y + z + yz = 15$, it follows that $z(1 + y) = 15 - y$, so $z = \frac{15-y}{y+1}$. Substituting these into the third given equation yields that

$$\frac{15-y}{y+1} + \frac{8-y}{y+1} + \left(\frac{15-y}{y+1}\right)\left(\frac{8-y}{y+1}\right) = 35\,,$$

so

$$\frac{23-2y}{y+1} + \frac{120-23y+y^2}{(y+1)^2} = 35$$

and $(23-2y)(y+1) + 120 - 23y + y^2 = 35(y+1)^2$. Therefore,

$$23y + 23 - 2y^2 - 2y + 120 - 23y + y^2 = 35y^2 + 70y + 35\,,$$

so $0 = 36y^2+72y-108 = 36(y^2+2y-3) = 36(y-1)(y+3)$. Thus, $y = 1$ or $y = -3$. Since y is given to be positive, $y = 1$, and, thus, $x = \frac{8-y}{y+1} = \frac{7}{2}$ and $z = \frac{15-y}{y+1} = 7$. Hence $x + y + z + xy = \frac{7}{2} + 1 + 7 + \frac{7}{2}\cdot 1 = 15$.

Also solved by MICHAEL CHEUNG, student, Port Moody Secondary School, Port Moody, BC.

While our solver's brute force solution shows admirable stamina, a more elegant solution is also possible: If $x + y + xy = 8$, then $x + y + xy + 1 = 9$, and now the left-hand side can be factored: $(x + 1)(y + 1) = 9$. Similarly the other two given equations yield that $(y + 1)(z + 1) = 16$ and that $(z + 1)(x + 1) = 36$. Multiplying the last two of these equations and dividing by the first yields that

$$\frac{(y+1)(z+1)^2(x+1)}{(x+1)(y+1)} = \frac{16 \cdot 36}{9},$$

so $(z + 1)^2 = 64$, so $z + 1 = \pm 8$, so $z = 7$ or $z = -9$. Again, z is positive, so $z = 7$. It now follows from the first of the given equations that $x + y + z + xy = 8 + 7 = 15$.

4. The number of mushrooms gathered by 11 boys and n girls is n^2+9n-2, with each person gathering exactly the same number. Determine the positive integer n.

Solution by Wen-Ting Fan, student, Burnaby North Secondary School, Burnaby, BC.

Each of the $n+11$ children must gather $\frac{n^2+9n-2}{n+11}$ mushrooms. Now $n^2 + 9n - 2 = (n + 11)(n - 2) + 20$, so the number of mushrooms is $n - 2 + \frac{20}{n+11}$. This must be an integer, so $n + 11$ must divide 20. Since n is nonnegative, $n = 9$.

Also solved by MICHAEL CHEUNG, student, Port Moody Secondary School, Port Moody, BC; TIMOTHY CHU, student, R.C. Palmer Secondary School, Richmond, BC; VINCENT CHUNG, student, Burnaby North Secondary School, Burnaby, BC; and LISA WANG, student, Port Moody Secondary School, Port Moody, BC.

One can use polynomial division *to find that $n^2 + 9n - 2 = (n + 11)(n - 2) + 20$, or you can use guess and check: If $n^2 + 9n - 2 = (n + 11)P + R$, then P must contain an n to get n^2 on the other side. Thus $n^2 + 9n - 2 = (n + 11)(n + ?) + R$. The question mark must be -2 to get $9n$ on the other side, so $R = 20$ follows.*

5. The positive integer x is such that both x and $x + 99$ are squares of integers. Find the sum of all such integers x.

Solution by Ellen Chen, student, Burnaby North Secondary School, Burnaby, BC.

Say $x = n^2$ and $x + 99 = m^2$. Then $99 = m^2 - n^2 = (m+n)(m-n)$, so 99 is written as the product of two integers. This is only possible in three ways:

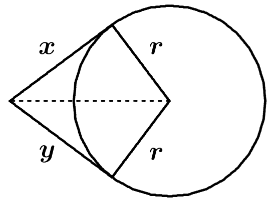
$m+n$	$m-n$	m	n	$x=n^2$
99	1	50	49	2401
33	3	18	15	225
11	9	10	1	1
Sum:				2627

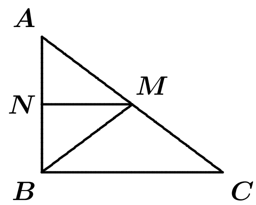
Also solved by TIMOTHY CHU, student, R.C. Palmer Secondary School, Richmond, BC; WEN-TING FAN, student, Burnaby North Secondary School, Burnaby, BC; KRISTIAN HANSEN, student, Burnaby North Secondary School, Burnaby, BC; and LISA WANG, student, Port Moody Secondary School, Port Moody, BC.

6. The side lengths of a right triangle are all positive integers, and the length of one of the legs is at most 20. The ratio of the circumradius to the inradius of this triangle is $5 : 2$. Determine the maximum value of the perimeter of this triangle.

Solution by the editors.

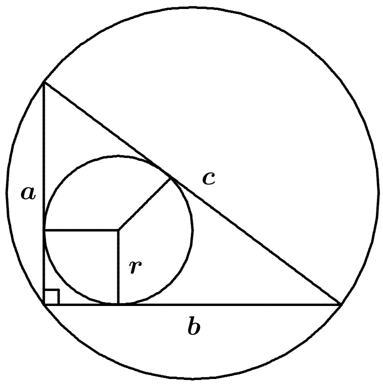
First let us review a few facts from geometry. The angle between a tangent to a circle and the radius to the point of tangency is $90°$. Therefore you can use the Pythagorean Theorem in each of the two triangles in the figure: The square of the length of the dotted line equals both $x^2 + r^2$ and $y^2 + r^2$. Therefore $x = y$, that is, *intersecting tangents are equal.*





Consider the right-angled triangle $\triangle ABC$. Let M be the midpoint of AC, and let N be the midpoint of AB. Then MN is parallel to BC, so $\triangle ANM$ is also right-angled. Using the Pythagorean Theorem in $\triangle ANM$ and in $\triangle BNM$ it follows that $AM = BM$. Thus *M is the centre of the circle through A, B, and C.*

Now we can attack the problem. You have just seen that since the triangle is right-angled, its hypotenuse is a diameter for the circumscribed circle, whose radius is therefore $c/2$. Let r be the radius of the inscribed circle. Note that two of the radii in the figure together with parts of the left and bottom sides of the triangle form a square. Therefore, the length of the remaining part of the left side is $a-r$ and the length of the remaining part of the bottom side is $b - r$. Since intersecting tangents are equal, this means that $c = a-r+b-r$. Thus $r = (a + b - c)/2$.



Since the ratio of the circumradius to the inradius is $5 : 2$,

$$\frac{c/2}{(a+b-c)/2} = \frac{5}{2}.$$

Therefore, $\dfrac{c}{a+b-c} = \dfrac{5}{2}$, so $2c = 5a + 5b - 5c$, so $c = \dfrac{5}{7}(a + b)$. By the Pythagorean Theorem, $a^2 + b^2 = c^2 = \dfrac{25}{49}(a + b)^2 = \dfrac{25}{49}(a^2 + 2ab + b^2)$. Hence, $49a^2 + 49b^2 = 25a^2 + 50ab + 25b^2$, so $24a^2 - 50ab + 24b^2 = 0$, so $2(4a - 3b)(3a - 4b) = 0$. Thus $a : b = 3 : 4$ or $a : b = 4 : 3$. Either way the given triangle is a 3–4–5 triangle.

The shortest side is given to be at most 20. The largest multiple of 3 less than or equal to 20 is 18. Thus, the sides are 18, 24, and 30, and the maximum value of the perimeter is 72.

7. Let α be the larger root of $(2004x)^2 - 2003 \cdot 2005x - 1 = 0$ and β be the smaller root of $x^2 + 2003x - 2004 = 0$. Determine the value of $\alpha - \beta$.

Solution by Timothy Chu, student, R.C. Palmer Secondary School, Richmond, BC.

The constant term of a quadratic polynomial is the product of its roots. Both polynomials have negative constant terms, so both must have one positive and one negative root. Since $2003 \cdot 2005 = (2004 - 1)(2004 + 1) = 2004^2 - 1$ and $2004^2 - (2004^2 - 1) - 1 = 0$, one of the roots of the first polynomial is 1. Since the other root is negative, $\alpha = 1$. The second polynomial is easily factored as $(x - 1)(x + 2004)$, whence $\beta = -2004$. Therefore $\alpha - \beta = 2005$.

Also solved by WEN-TING FAN, student, Burnaby North Secondary School, Burnaby, BC.

To see that the constant term of a quadratic polynomial is indeed the product of its roots, consider that $(x-a)(x-b) = x^2 - (a+b)x + ab$. *A similar property holds for higher degree polynomials.*

Once you realise that $2003 \cdot 2005 = 2004^2 - 1$, *the first polynomial is also easy to factor as* $(2004^2x + 1)(x - 1)$.

8. Let a be a positive number such that $a^2 + \frac{1}{a^2} = 5$. Determine the value of $a^3 + \frac{1}{a^3}$.

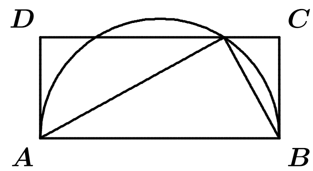
Solution by the editors.

Since $\left(a + \frac{1}{a}\right)^2 = a^2 + 2 + \frac{1}{a^2}$, it follows from the given equation that $\left(a + \frac{1}{a}\right)^2 = 7$, and so $a + \frac{1}{a} = \sqrt{7}$ since a is positive. Similarly,

$$\begin{aligned} \left(a + \frac{1}{a}\right)^3 &= \left(a + \frac{1}{a}\right)^2 \left(a + \frac{1}{a}\right) = \left(a^2 + 2 + \frac{1}{a^2}\right)\left(a + \frac{1}{a}\right) \\ &= a^3 + 2a + \frac{1}{a} + a + \frac{2}{a} + \frac{1}{a^3} = a^3 + 3\left(a + \frac{1}{a}\right) + \frac{1}{a^3}. \end{aligned}$$

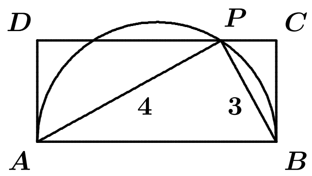
Therefore, $a^3 + \frac{1}{a^3} = \left(a + \frac{1}{a}\right)^3 - 3\left(a + \frac{1}{a}\right) = (\sqrt{7})^3 - 3\sqrt{7} = 4\sqrt{7}$.

9. In the figure, $ABCD$ is a rectangle with $AB = 5$ such that the semicircle with diameter AB cuts CD at two points. If the distance from one of them to A is 4, find the area of $ABCD$.



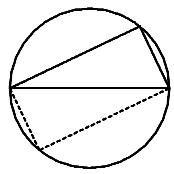
Solution by Lena Choi, student, École Banting Middle School, Coquitlam, BC.

Since AB is a diameter and P is on the circle, $\angle APB = 90°$. Since $AP = 4$ and $AB = 5$, it follows that $BP = 3$. Hence the area of $\triangle ABP$ is $\frac{3 \cdot 4}{2} = 6$. If you instead use AB as the base of the triangle, then the height equals the length of BC. Therefore, the area of the rectangle is twice the area of the triangle, so the area of the rectangle is 12.



Also solved by KRISTIAN HANSEN, student, Burnaby North Secondary School, Burnaby, BC.

Our solver used the fact that if P is on the circle with diameter AB, then $\angle APB = 90°$. To prove this fact, rotate the triangle around the centre of the circle to obtain the dotted part in the figure on the right. By construction, the four sided polygon is a parallelogram. Since both diagonals are diameters and therefore equal, the parallelogram must be a rectangle, whence $\angle APB = 90°$.



10. Let a be $9\left(n\left(\frac{10}{9}\right)^n - 1 - \frac{10}{9} - \left(\frac{10}{9}\right)^2 - \cdots - \left(\frac{10}{9}\right)^{n-1}\right)$ where n is a positive integer. If a is an integer, determine the maximum value of a.

Solution by Kristian Hansen, student, Burnaby North Secondary School, Burnaby, BC.

The sum of the geometric series is

$$1 + \frac{10}{9} + \left(\frac{10}{9}\right)^2 + \cdots + \left(\frac{10}{9}\right)^{n-1} = \frac{1 - \left(\frac{10}{9}\right)^n}{1 - \frac{10}{9}} = -9\left(1 - \left(\frac{10}{9}\right)^n\right).$$

Therefore,

$$\begin{aligned} a &= 9\left(n\left(\frac{10}{9}\right)^n + 9\left(1 - \left(\frac{10}{9}\right)^n\right)\right) \\ &= 9\left((n-9)\left(\frac{10}{9}\right)^n + 9\right) \\ &= 9(n-9)\left(\frac{10}{9}\right)^n + 81\,. \end{aligned}$$

For this to be an integer, either $n = 1$ or $n = 9$. (If $n > 1$, then the denominator contains too many copies of 9 except when $n = 9$ and the numerator is zero by a lucky miracle.) If $n = 1$, then $a = 1$; if $n = 9$, then $a = 81$. The larger of these is 81, which is the maximum value of a.

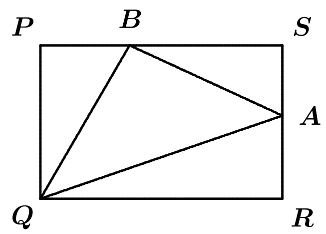
11. In a two-digit number, the tens digit is greater than the ones digit. The product of these two digits is divisible by their sum. What is this two-digit number?

Solution by Michael Cheung, student, Port Moody Secondary School, Port Moody, BC.

Any (two-digit) multiple of ten satisfies the condition. Otherwise, if the number contains the digit 1 and the digit d, the condition is that d is divisible by $d+1$ which is impossible. This leaves just 28 numbers to consider: 32, 42, 43, 52, 53, 54, 62, 63, 64, 65, 72, 73, 74, 75, 76, 82, 83, 84, 85, 86, 87, 92, 93, 94, 95, 96, 97, and 98. These are easily checked one by one; only 63 works out. Thus the solutions are 10, 20, 30, 40, 50, 60, 63, 70, 80, and 90.

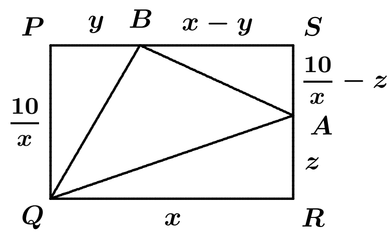
Also solved by TIMOTHY CHU, student, R.C. Palmer Secondary School, Richmond, BC.

12. In the figure, $PQRS$ is a rectangle of area 10. A is a point on RS and B is a point on PS such that the area of triangle QAB is 4. Determine the smallest possible value of $PB + AR$.



Solution by Vincent Chung, student, Burnaby North Secondary School, Burnaby, BC.

Label the lengths as in the figure. Since the area of $\triangle QAB$ is 4, the areas of the remaining three triangles must add up to 6. That is,



$$\frac{(\frac{10}{x} - z)(x-y)}{2} + \frac{10y}{2x} + \frac{xz}{2} \;=\; 6\,.$$

Multiplying by 2 and expanding yields

$$10 - \frac{10y}{x} - xz + yz + \frac{10y}{x} + xz = 12\,,$$

so $yz = 2$.

The smallest possible value of $PB + AR = y + z$ subject to the constraint that $yz = 2$ is obtained when $y = z$. Then $y = z = \sqrt{2}$ and $PB + AR = 2\sqrt{2}$.

Also solved by KRISTIAN HANSEN, student, Burnaby North Secondary School, Burnaby, BC.

This issue's prize of one copy of ***CRUX with MAYHEM*** for the best solutions goes to Timothy Chu, student, R.C. Palmer Secondary School, Richmond, BC.

We congratulate our solvers on their success with a rather difficult contest and hope that they and other readers will continue to submit solutions to our problems.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Ian VanderBurgh (University of Waterloo). The other staff members are Monika Khbeis (Our Lady of Mt. Carmel Secondary School, Mississauga, ON) and Eric Robert (Leo Hayes High School, Fredericton, NB).

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **15 septembre 2010**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

M438. *Proposé par l'Équipe de Mayhem.*

Trouver toutes les paires de nombres réels (x, y) telles que

$$x^2 + (y^2 - y - 2)^2 \;=\; 0\,.$$

M439. *Proposé par Eric Schmutz, Université Drexel, Philadelphia, PA, É-U.*

Trouver l'entier positif x pour lequel on a $\frac{1}{\log_2 x} + \frac{1}{\log_5 x} = \frac{1}{100}$.

M440. *Proposé par l'Équipe de Mayhem.*

On donne un trapèze $ABCD$ avec AB parallèle à DC et AD perpendiculaire à AB. Si $AB = 20$, $BC = 5x$, $CD = x^2 + 3x$ et $DA = 3x$, trouver la valeur de x.

M441. *Proposé par Katherine Tsuji et Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Quel est le nombre maximal de rois non menaçants qu'on peut placer sur un échiquier $n \times n$? (Un «roi» est une pièce d'échecs qu'on peut déplacer d'une seule case horizontalement, verticalement ou diagonalement.)

M442. *Proposed by Carl Libis, Université Cumberland, Lebanon, TN, É-U.*

Dans le tableau carré suivant

$$\begin{bmatrix} 1 & 2 & \cdots & n-1 & n \\ n+1 & n+2 & \cdots & 2n-1 & 2n \\ \vdots & \vdots & & \vdots & \vdots \\ (n-1)n+1 & (n-1)n+2 & \cdots & n^2-1 & n^2 \end{bmatrix}$$

construit en écrivant sur n lignes consécutives la liste des nombres de 1 à n^2, déterminer la somme des nombres sur chaque diagonale. Comparer cette somme à la «constante magique» obtenue en réarrangeant les n^2 éléments pour former un carré magique.

M443. *Proposé par Neculai Stanciu, École secondaire George Emil Palade, Buzău, Roumanie.*

On note $\lfloor x \rfloor$ le plus grand entier n'excédant pas x. Ainsi, $\lfloor 3.1 \rfloor = 3$ et $\lfloor -1.4 \rfloor = -2$. On désigne par $\{x\}$ la partie fractionnaire du nombre réel x (c'est-à-dire $\{x\} = x - \lfloor x \rfloor$). Par exemple, $\{3.1\} = 0.1$ et $\{-1.4\} = 0.6$. Trouver tous les nombres réels positifs x tels que

$$\left\{\frac{2x+3}{x+2}\right\} + \left\lfloor \frac{2x+1}{x+1} \right\rfloor \;=\; \frac{14}{9}\,.$$

M444. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit a et b deux nombres réels. Montrer que

$$\sqrt{a^2+b^2+6a-2b+10} \;+\; \sqrt{a^2+b^2-6a+2b+10} \;\geq\; 2\sqrt{10}\,.$$

....

M438. *Proposed by the Mayhem Staff.*

Find all pairs of real numbers (x, y) such that

$$x^2 + (y^2 - y - 2)^2 \;=\; 0\,.$$

M439. *Proposed by Eric Schmutz, Drexel University, Philadelphia, PA, USA.*

Determine the positive integer x for which $\frac{1}{\log_2 x} + \frac{1}{\log_5 x} = \frac{1}{100}$.

M440. *Proposed by the Mayhem Staff.*

In trapezoid $ABCD$, AB is parallel to DC and AD is perpendicular to AB. If $AB = 20$, $BC = 5x$, $CD = x^2 + 3x$, and $DA = 3x$, determine the value of x.

M441. *Proposed by Katherine Tsuji and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

What is the maximum number of non-attacking kings that can be placed on an $n \times n$ chessboard? (A "king" is a chess piece that can move horizontally, vertically, or diagonally from one square to an adjacent square.)

M442. *Proposed by Carl Libis, Cumberland University, Lebanon, TN, USA.*

Consider the square array

$$\begin{bmatrix} 1 & 2 & \cdots & n-1 & n \\ n+1 & n+2 & \cdots & 2n-1 & 2n \\ \vdots & \vdots & & \vdots & \vdots \\ (n-1)n+1 & (n-1)n+2 & \cdots & n^2-1 & n^2 \end{bmatrix}$$

formed by listing the numbers 1 to n^2 in order in consecutive rows. Determine the sum of the numbers on each diagonal. How does this sum compare to the "magic constant" that would be obtained if the n^2 entries were re-arranged to form a magic square?

M443. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Let $\lfloor x \rfloor$ denote the greatest integer not exceeding x. For example, $\lfloor 3.1 \rfloor = 3$ and $\lfloor -1.4 \rfloor = -2$. Let $\{x\}$ denote the fractional part of the real number x (that is, $\{x\} = x - \lfloor x \rfloor$). For example, $\{3.1\} = 0.1$ and $\{-1.4\} = 0.6$. Find all positive real numbers x such that

$$\left\{ \frac{2x+3}{x+2} \right\} + \left\lfloor \frac{2x+1}{x+1} \right\rfloor = \frac{14}{9}.$$

M444. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a and b be real numbers. Prove that

$$\sqrt{a^2+b^2+6a-2b+10} + \sqrt{a^2+b^2-6a+2b+10} \geq 2\sqrt{10}.$$

Mayhem Solutions

M381. Correction. *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all of the solutions to the equation

$$\frac{1}{x-1} + \frac{2}{x-2} + \frac{6}{x-6} + \frac{7}{x-7} = x^2 - 4x - 4.$$

Solution by Sonthaya Senamontree, Thesaban 2 Mukkhamontree School, Udonthani, Thailand.

From the given equation

$$\begin{aligned}
\frac{1}{x-1} + \frac{2}{x-2} + \frac{6}{x-6} + \frac{7}{x-7} &= x^2 - 4x - 4\,; \\
\left(\frac{1}{x-1}+1\right) + \left(\frac{2}{x-2}+1\right) + \left(\frac{6}{x-6}+1\right) + \left(\frac{7}{x-7}+1\right) &= x^2 - 4x\,; \\
\frac{x}{x-1} + \frac{x}{x-2} + \frac{x}{x-6} + \frac{x}{x-7} &= x^2 - 4x\,.
\end{aligned}$$

Since x is a common factor of both sides, then $x = 0$ is a solution. We can continue by assuming that $x \neq 0$ and dividing by x to obtain

$$\begin{aligned}
\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-6} + \frac{1}{x-7} &= x - 4\,; \\
\left(\frac{1}{x-1} + \frac{1}{x-7}\right) + \left(\frac{1}{x-2} + \frac{1}{x-6}\right) &= x - 4\,; \\
\frac{2x-8}{(x-1)(x-7)} + \frac{2x-8}{(x-2)(x-6)} &= x - 4\,; \\
\frac{2x-8}{x^2-8x+7} + \frac{2x-8}{x^2-8x+12} &= x - 4\,.
\end{aligned}$$

Since $x = 4$ makes both sides 0, then $x = 4$ is a solution. We can continue by assuming that $x \neq 4$ and dividing by $x - 4$ to obtain:

$$\frac{2}{x^2-8x+7} + \frac{2}{x^2-8x+12} = 1\,,$$

and then make the substitution $a = x^2 - 8x$ to obtain

$$\begin{aligned}
\frac{2}{a+7} + \frac{2}{a+12} &= 1\,; \\
2(a+12) + 2(a+7) &= (a+7)(a+12)\,; \\
2a + 24 + 2a + 14 &= a^2 + 19a + 84\,; \\
0 &= a^2 + 15a + 46\,.
\end{aligned}$$

The quadratic formula yields $a = \dfrac{-15 \pm \sqrt{15^2 - 4(1)(46)}}{2} = \dfrac{-15 \pm \sqrt{41}}{2}$.

Since $a = x^2 - 8x$, then

$$\begin{aligned} x^2 - 8x &= \frac{-15 \pm \sqrt{41}}{2}\,; \\ x^2 - 8x + 16 &= \frac{17 \pm \sqrt{41}}{2}\,; \\ (x-4)^2 &= \frac{17 \pm \sqrt{41}}{2}\,; \\ x &= 4 \pm \sqrt{\frac{17 \pm \sqrt{41}}{2}}\,. \end{aligned}$$

Therefore, $x = 0$ or $x = 4$ or $x = 4 \pm \sqrt{\dfrac{17 \pm \sqrt{41}}{2}}$, with all four combinations of signs being possible.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; G.C. GREUBEL, Newport News, VA, USA; KONSTANTINOS AL. NAKOS, Agrinio, Greece; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON.

M401. *Proposed by the Mayhem Staff.*

Graham and Vazz were marking out a new lawn at ***CRUX*** Headquarters. Graham said: "If you make the lawn 9 metres longer and 8 metres narrower, the area will be the same". Vazz said: "If you make it 12 metres shorter and 16 metres wider, the area will still be the same". What are the dimensions of the lawn?

Solution by Jaclyn Chang, student, Western Canada High School, Calgary, AB.

Let x be the length of the lawn and y be the width of the lawn. Thus, the area of the lawn is xy. We can translate Graham's and Vazz's statements into equations.

According to Graham, $xy = (x+9)(y-8) = xy - 8x + 9y - 72$, and so $8x - 9y = -72$.

According to Vazz, $xy = (x-12)(y+16) = xy + 16x - 12y - 192$, and so $16x - 12y = 192$ or $8x - 6y = 96$.

Subtracting the first linear equation from the second one, we obtain $3y = 168$, or $y = 56$. We can substitute $y = 56$ into either equation to obtain $x = 54$.

Therefore, the lawn is 54 m long and 56 m wide.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; WINDA KIRANA, student, SMPN 8, Yogyakarta, Indonesia; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; MRIDUL SINGH, student, Kendriya Vidyalaya School, Shillong, India; MRINAL SINGH, student, Kendriya Vidyalaya School, Shillong, India; JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON; and GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia. There were two incorrect solutions submitted.

M402. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Determine all ordered pairs (a, b) of positive integers such that

$$a^b b^a + a^b + b^a = 89 .$$

Solution by Winda Kirana, student, SMPN 8, Yogyakarta, Indonesia and Gusnadi Wiyoga, student, SMPN 8, Yogyakarta, Indonesia, independently.

Since $a^b b^a + a^b + b^a = 89$, then we have that $a^b b^a + a^b + b^a + 1 = 90$, or $(a^b + 1)(b^a + 1) = 90$.

Since a and b are positive integers, then $a^b + 1$ and $b^a + 1$ are both positive integer divisors of 90 and each of these divisors is larger than 1.

We make a table of the possible values of a^b and b^a:

a^b+1	2	3	5	6	9	10	15	18	30	45
b^a+1	45	30	18	15	10	9	6	5	3	2

a^b	1	2	4	5	8	9	14	17	29	44
b^a	44	29	17	14	9	8	5	4	2	1

If $a^b = 2$, then $a = 2$ and $b = 1$, which does not give $b^a = 29$. If $a^b = 4$, then $(a, b) = (4, 1)$ or $(a, b) = (2, 2)$, neither of which gives $b^a = 17$. If $a^b = 5$, then $a = 5$ and $b = 1$, which does not give $b^a = 14$. Similar reasoning shows that a^b cannot be 14, 17, or 29.

If $b^a = 44$, then $b = 44$ and $a = 1$, which does give $a^b = 1$. Thus, $(a, b) = (1, 44)$ is a solution. Similarly, $(a, b) = (44, 1)$ is a solution from the last row.

If $a^b = 8$, then $(a, b) = (8, 1)$ or $(a, b) = (2, 3)$. The second of these gives $b^a = 9$, so $(a, b) = (2, 3)$ is a solution, as is $(a, b) = (3, 2)$ from the following row.

Therefore, the solutions are $(a, b) = (1, 44)$, $(44, 1)$, $(2, 3)$, $(3, 2)$.

Also solved by CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and JIXUAN WANG, student, Don Mills Collegiate Institute, Toronto, ON. There were six incorrect solutions submitted.

All of the incorrect solutions missed the cases $(a, b) = (44, 1)$ and $(a, b) = (1, 44)$.

M403. *Proposed by Matthew Babbitt, home-schooled student, Fort Edward, NY, USA.*

Jason wrote a computer program that tests if an integer greater than 1 is prime. His devious sister Alice has edited the code so that if the input is odd, the probability that the program gives the correct output is 52% and if the input is even, the probability that the program gives the correct output is 98%. Jason tests the program by inputting two random integers each greater than 1. What is the probability that both outputs are correct?

Solution by Jixuan Wang, student, Don Mills Collegiate Institute, Toronto, ON.

The probability that the first random input is even is 0.5, in which case there is a 98% chance that the output is correct. The probability that the first random input is odd is 0.5, in which case there is a 52% chance that the output is correct. Thus, the probability that the first output is correct is $(0.5)(0.98) + (0.5)(0.52) = 0.75$.

The probability that the second output is correct is also 0.75. Therefore, the probability that both outputs are correct is $(0.75)^2 = 0.5625 = 9/16$.

Also solved by JACLYN CHANG, student, Western Canada High School, Calgary, AB; CARL LIBIS, Cumberland University, Lebanon, TN, USA; and RICARD PEIRÓ, IES "Abastos", Valencia, Spain.

M404. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

A store sells copies of a certain item at $\$x$ each, or at a items for $\$y$, or at b items for $\$z$, where a and b are positive integers satisfying $1 < a < b$ and x, y, and z are positive real numbers. To make "a items for $\$y$" a sensible bargain, $\$y$ should be less than buying a separate items; in other words we should have $y < ax$. To make "b items for $\$z$" also a sensible bargain, we could insist on one of two conditions:

(a) $\frac{z}{b} < \frac{y}{a}$; that is, the average price of an item under the "b items for $\$z$" deal is less than under the "a items for $\$y$" deal.

(b) Whenever we can write $b = qa + r$ for nonnegative integers q and r, then $z < qy + rx$ holds; that is, it should always cost more to buy b items by buying a combination of a items plus individual items, than by choosing the "b items for $\$z$" deal.

Show that if condition (a) is true, then condition (b) is also true. Give an example to show that condition (b) could be true while condition (a) is false.

Solution by the proposer.

First, we prove by contradiction that if condition (a) is true, then condition (b) is true.

Suppose that $\frac{z}{b} < \frac{y}{a}$; that is, assume that $az < by$. Assume that (b) is not true; that is, that there exist nonnegative integers q and r with $b = qa+r$ but with $z \geq qy + rx$.

Then $az \geq aqy + arx$, so $aqy + arx \leq az < by = y(qa + r) = aqy + ry$. Therefore, $arx < ry$. Since $r \geq 0$ and the inequality is not true if $r = 0$, then $r > 0$, so $ax < y$, which contradicts the given information.

Therefore, if condition (a) is true, then condition (b) is true.

If $a = 3$, $b = 5$, $x = 2$, $y = 3$, and $z = 6$, then $1 < a < b$ and $y < ax$, but $\frac{z}{b} > \frac{y}{a}$, so (a) is not true. But condition (b) is true, since the only ways to write $b = 5$ in the form $b = qa + r$ are $5 = 0(3) + 5$ and $5 = 1(3) + 2$, which gives $qy + rx = 0(3) + 5(2) = 10 > 6 = z$ and $qy + rx = 1(3) + 2(2) = 7 > 6 = z$, so condition (b) is true.

M405. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Determine a closed form expression for the sum

$$17 \;+\; 187 \;+\; 1887 \;+\; 18887 \;+\; \cdots \;+\; 188\ldots87\,,$$

where the last term contains exactly n 8's.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.

We note first that $17(1) = 17$ and $17(11) = 187$ and $17(111) = 1887$. Then $18887 = 17000 + 1887 = 17(1000 + 111) = 17(1111)$. We can continue this argument inductively to show that the integer $188\ldots87$ (containing n copies of 8) is equal to $17(11\ldots1)$ (containing n copies of 1 inside the parentheses).

Therefore,

$$\begin{aligned}
&\big(17 + 187 + 1887 + 18887 + \cdots + (188\ldots87)\big)\\
&= 17\big(1 + 11 + 111 + 1111 + \cdots + (11\ldots1)\big)\\
&\qquad \text{(where the last integer consists of } n+1 \text{ digits all equal to } 1)\\
&= \frac{17}{9}\big(9 + 99 + 999 + 9999 + \cdots + (99\ldots9)\big)\\
&= \frac{17}{9}\big((10-1) + (10^2-1) + (10^3-1) + \cdots + (10^{n+1}-1)\big)\\
&= \frac{17}{9}\big(10(1 + 10 + 10^2 + \cdots + 10^n) - (n+1)\big)\\
&= \frac{17}{9}\left(10\left(\frac{10^{n+1}-1}{9}\right) - (n+1)\right)\\
&= \frac{17}{81}(10^{n+2} - 10 - 9n - 9)\\
&= \frac{17}{81}(10^{n+2} - 9n - 19)\,.
\end{aligned}$$

Also solved by LUIS J. BLANCO (student) and ANGEL PLAZA, University of Las Palmas de Gran Canaria, Spain; JOAQUÍN GÓMEZ REY, IES Luis Buñuel, Alcorcón, Madrid, Spain; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; PEDRO HENRIQUE O. PANTOJA, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA. There were three incorrect solutions submitted.

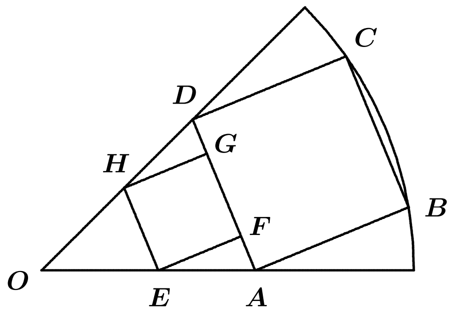
M406. *Proposed by Constantino Ligouras, student, E. Majorana Scientific High School, Putignano, Italy.*

Square $ABCD$ is inscribed in one-eighth of a circle of radius 1 and centre O so that there is one vertex on each radius and two vertices B and C on the arc. Square $EFGH$ is inscribed in $\triangle DOA$ so that E and H lie on the radii, and F and G lie on AD. In problem M295 [2007 : 200, 202; solution 2008 : 203-204], we saw that the area of square $ABCD$ is $\frac{2-\sqrt{2}}{3}$. Determine the area of square $EFGH$.

Solution by Ricard Peiró, IES "Abastos", Valencia, Spain, modified by the editor.

In problem M295, we saw that $AD^2 = \frac{2-\sqrt{2}}{3}$.
Since $\tan 45° = 1$, then

$$1 = \tan 45° = \frac{2\tan 22.5°}{1-\tan^2 22.5°}.$$



Setting $u = \tan 22.5°$, we have that $1-u^2 = 2u$, or $u^2+2u-1 = 0$. Using the quadratic formula, we obtain

$$\begin{aligned} u &= \frac{-2 \pm \sqrt{2^2-4(1)(-1)}}{2} \\ &= \frac{-2\pm\sqrt{8}}{2} \\ &= -1 \pm \sqrt{2}. \end{aligned}$$

Since $u = \tan 22.5° > 0$, then $\tan 22.5° = \sqrt{2}-1$.

Let x be the side length of square $EFGH$. Then $EF = FG = x$.

By symmetry, $AF = DG$, so $AF = \frac{AD-FG}{2} = \frac{AD-x}{2}$. Since $\triangle DOA$ is isosceles, then $\angle DAO = \frac{1}{2}(180° - 45°) = 67.5°$. Since $\triangle EFA$ is right-angled, then $\angle FEA = 90° - 67.5° = 22.5°$. Therefore,

$$\begin{aligned} \tan 22.5° &= \frac{AF}{EF}; \\ \sqrt{2}-1 &= \frac{AD-x}{2x}; \\ (2\sqrt{2}-2)x &= AD - x; \\ (2\sqrt{2}-1)x &= AD; \\ x &= \frac{AD}{2\sqrt{2}-1}. \end{aligned}$$

Therefore x^2, the area of square $EFGH$, is equal to

$$\begin{aligned} \frac{AD^2}{(2\sqrt{2}-1)^2} &= \frac{2-\sqrt{2}}{3}\cdot\frac{1}{9-4\sqrt{2}} \\ &= \frac{(2-\sqrt{2})(9+4\sqrt{2})}{3[9^2-4^2(2)]} \\ &= \frac{10-\sqrt{2}}{147}. \end{aligned}$$

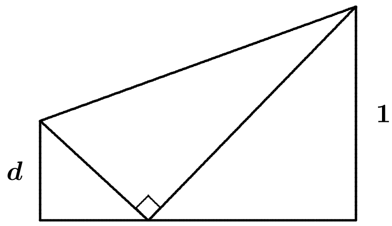
Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; and GEOFFREY A. KANDALL, Hamden, CT, USA.

Problem of the Month

Ian VanderBurgh

A popular type of geometry problem involves folding paper. A folding problem usually involves a sheet of paper of specific dimensions and the method of folding. We are then asked to determine one or more lengths in the resulting configuration.

Problem (UK Intermediate Challenge 1999) A rectangular sheet of paper with sides 1 and $\sqrt{2}$ has been folded once as shown, so that one corner just meets the opposite long edge. What is the value of the length d?

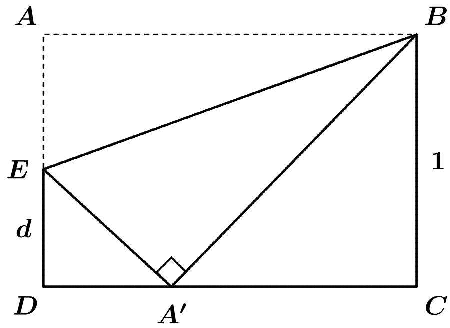


Feel free to actually try this out! If you're in the UK, you'll have a much easier time finding a sheet of paper with dimensions in the ratio $\sqrt{2} : 1$.

How should we start? One of the very first problem solving strategies that we learn is "draw a diagram". This strategy should almost always be extended very slightly by adding the clause "...and label it carefully". As it turns out, this is the key to solving this problem.

Solution We redraw the given diagram by adding the "phantom" edges of the paper (the dotted lines) and labelling the relevant points on the diagram.

We then label as many lengths as we possibly can. I suggest that you follow along by labelling each new length that we determine. Make sure that you understand why each length is what it is before moving on to the next step. Since the paper has length $\sqrt{2}$, then $AB = DC = \sqrt{2}$.



Can you see another length that equals $\sqrt{2}$? In fact, $A'B = \sqrt{2}$ since this is the folded image of AB.

Can you determine the length of AE in terms of d? Since $AD = 1$ and $ED = d$, then $AE = 1 - d$.

Can you find another line segment of length $1 - d$? Since AE becomes $A'E$ after folding, then $A'E = 1 - d$.

Can you see any triangles where we know two of the three side lengths? In $\triangle A'CB$, we have $A'B = \sqrt{2}$ and $BC = 1$.

How can we determine the third side length of $\triangle A'CB$? This triangle is right-angled at C, so we can use the Pythagorean Theorem to conclude that $A'C^2 = A'B^2 - BC^2 = (\sqrt{2})^2 - 1^2 = 1$; since $A'C > 0$, then $A'C = \sqrt{1} = 1$.

Can we use this to determine another length? Yes! Since $DC = \sqrt{2}$ and $A'C = 1$, then $A'D = \sqrt{2} - 1$. Now $\triangle EDA'$ is right-angled at D. We know one of the three side lengths, namely, $A'D = \sqrt{2} - 1$, and we know the other two side lengths in terms of d, namely, $ED = d$ and $EA' = 1 - d$.

What should we do to try to solve for d? Let's apply the Pythagorean Theorem again. (Spoiler alert: There is a better way! If you are uncomfortable squaring expressions like $1 - d$ or have never even done this before, skip down to just after the end of the solution for a simpler approach.) We obtain

$$\begin{aligned} A'E^2 &= ED^2 + A'D^2\,; \\ (1-d)^2 &= d^2 + (\sqrt{2}-1)^2\,; \\ 1 - 2d + d^2 &= d^2 + 2 - 2\sqrt{2} + 1\,; \\ -2d &= 2 - 2\sqrt{2}\,; \\ d &= \sqrt{2} - 1\,. \end{aligned}$$

Therefore, $d = \sqrt{2} - 1$. ∎

My apologies for the spoiler alert above. We were on such a roll that I didn't want to interrupt our Pythagorean flow.

Do you see a different approach that we could have taken? You may note that $A'D = ED = \sqrt{2} - 1$. Can you see a reason why this should be the case? Let's go back and do some angle-chasing.

Triangle $A'CB$ has sides of lengths 1, 1, and $\sqrt{2}$. What are its angles? Since it is isosceles and right-angled, then $\angle BA'C = \angle A'BC = 45^\circ$. Thus,

$$\angle DA'E = 180^\circ - \angle EA'B - \angle BA'C = 180^\circ - 90^\circ - 45^\circ = 45^\circ\ .$$

What does that say about $\triangle A'DE$? This tells us that this is also isosceles and right-angled! (If you're not convinced, calculate $\angle DEA'$.) Therefore, $ED = A'D$ and we know that $A'D = \sqrt{2} - 1$. This allows us to conclude that $d = ED = \sqrt{2} - 1$, as required.

This gives us two different ways of handling this problem. Knowing two different approaches is really useful, because it means that if we don't see one of the approaches in a problem that we're working on, we might just see the other.

For those of you wanting more of a challenge, here's a follow-up problem to work on:

> A rectangular sheet of paper $ABCD$ has $AB = 8$ and $BC = 6$. The paper is folded so that corner A coincides with the midpoint, M, of DC. What is the length of the fold?

THE OLYMPIAD CORNER

No. 286

R.E. Woodrow

We start this number with translations of a number of Olympiads from South America. My thanks go to Bill Sands, Canadian Team Leader to the IMO in Vietnam, for collecting them for our use and to Leda Sanchez, Executive Assistant to the Vice Provost (International), for helping with the translation. The first set are the problems of the XV Olimpíada Matemática Rioplatense 2006, Nivel 2.

XV OLIMPÍADA MATEMÁTICA RIOPLATENSE
San Isidra, 9–10 December 2006
Nivel 2

1. Let ABC be a right triangle with right angle at A. Consider all the isosceles triangles XYZ with right angle at X, where X lies on the segment BC, Y lies on AB, and Z is on the segment AC. Determine the locus of the medians of the hypotenuses YZ of such triangles XYZ.

2. Carlitos listed all the subsets of $\{1, 2, \ldots, 2006\}$ in which the difference between the number of even numbers and the number of odd numbers is a multiple of 3. How many subsets did Carlitos list?

3. A finite number of (possibly overlapping) intervals on a line are given. If the rightmost $1/3$ of each interval is deleted, an interval of length 31 remains. If the leftmost $1/3$ of each interval is deleted, an interval of length 23 remains. Let M and m be the maximum and minimum of the lengths of an interval in the collection, respectively. How small can $M - m$ be?

4. Let $a_1, a_2, \ldots, a_n$ be positive numbers. The sum of all the products a_ia_j with $i < j$ is equal to 1. Show that there is a number among them such that the sum of the remaining numbers is less than $\sqrt{2}$.

5. A circle Γ is tangent to the sides AB and AC of triangle ABC at E and F, respectively. Let BF and EC intersect at X, let Γ intersect AX at H, and let EH and FH intersect BC at Z and T, respectively. The lines ET and FZ intersect at Q. Show that Q lies on the line AX.

6. For each permutation $(x_1, x_2, \ldots, x_{99})$ of $\{1, 2, \ldots, 99\}$, let

$$L = |x_1 - x_2\sqrt{3}| + |x_2 - x_3\sqrt{3}| + \cdots + |x_{98} - x_{99}\sqrt{3}| + |x_{99} - x_1\sqrt{3}|\,.$$

Determine the maximum value of L. How many permutations give rise to this value of L?

Next from the same package are the problems of the XV Olimpíada Matemática Rioplatense 2006, Nivel 3. Again, thanks go to Bill Sands and to Leda Sanchez.

XV OLIMPÍADA MATEMÁTICA RIOPLATENSE 2006
San Isidra, 9–10 December 2006
Nivel 3

1. (a) For each $k \geq 3$, find a positive integer n that can be represented as the sum of exactly k mutually distinct positive divisors of n.

(b) Suppose that n can be expressed as the sum of exactly k mutually distinct positive divisors of n for some $k \geq 3$. Let p be the smallest prime divisor of n.

Show that

$$\frac{1}{p} + \frac{1}{p+1} + \cdots + \frac{1}{p+k-1} \geq 1\,.$$

2. Let $ABCD$ be a convex quadrilateral with $AB = AD$ and $CB = CD$. The bisector of $\angle BDC$ intersects BC at L, and AL intersects BD at M, and it is known that $BL = BM$. Determine the value of $2\angle BAD + 3\angle BCD$.

3. The numbers 1, 2, $\ldots$, 2006 are written around the circumference of a circle. One allowed operation is to exchange two adjacent numbers. After a sequence of such exchanges each number ends up 13 positions to the right of its initial position.

If the 2006 numbers 1, 2, $\ldots$, 2006 are partitioned into 1003 distinct pairs, then show that in at least one of the operations two numbers of one of the pairs are exchanged.

4. The acute triangle ABC with $AB \neq AC$ has circumcircle Γ, circumcentre O and orthocentre H. The midpoint of BC is M and the extension of the median AM intersects Γ at N. The circle of diameter AM intersects Γ again at A and P.

Show that the lines AP, BC, and OH are concurrent if and only if $AH = HN$.

5. Consider a finite number of lines in the plane no two of which are parallel and no three of which are concurrent. These lines divide the plane into finite and infinite regions. In each finite region we write 1 or -1. In one operation, we can choose any triangle made of three of the lines (which may be cut by other lines in the collection) and multiply by -1 each of the numbers in the triangle. Determine if it is always possible to obtain 1 in all the regions by successively applying this operation, regardless of the initial distribution of the numbers 1 and -1.

6. Consider an infinite sequences $\{x_n\}_{n=1}^{\infty}$ of positive integers that satisfies the recurrence

$$x_{n+2} \;=\; \gcd(x_{n+1}, x_n) + 2006$$

for each positive integer n, where $\gcd(u, v)$ is the greatest common divisor of the integers u and v.

Does there exist a sequence of this type which contains exactly 10^{2006} distinct numbers?

Continuing with this theme we have the problems of the 21st Olimpiada Iberoamericana de Matemática. Premer Dia, 2006. Thanks again go to Bill Sands and Leda Sanchez for making them available to the *Corner*.

21 OLIMPIADA IBEROAMERICANA DE MATEMÁTICA
Guayaquil, 26-27 September 2006

1. In the scalene triangle ABC with $\angle BAC = 90^\circ$, the tangent line to the circumcircle at at A intersects the line BC at M. Let S and R be the points where the incircle of ABC touches AC and AB, respectively. The line RS intersects the line BC at N. The lines AM and SR meet at U. Show that triangle UMN is isosceles.

2. Let a_1, a_2, $\dots$, a_n be real numbers. Let d be the difference between the smallest and the largest of them, and let $s = \sum_{i<j} |a_i - a_j|$. Show that

$$(n-1)d \;\leq\; s \;\leq\; \frac{n^2 d}{4}$$

and determine the conditions under which equality holds in each inequality.

3. The numbers 1, 2, $\dots$, n^2 are placed in the cells of an $n \times n$ board, one number per cell. A coin is initially placed in the cell containing the number n^2. The coin can move to any of the cells which share a side with the cell it currently occupies.

First, the coin travels from the cell containing the number 1 to the cell containing the number n^2, using the smallest possible number of moves. Then the coin travels from the cell containing the number 1 to the cell containing the number 2 using the smallest possible number of moves, and then from the cell containing the number 3, and continuing until the coin returns to the initial cell, taking a shortest route each time it travels. The complete trip takes N steps. Determine the smallest and largest possible values of N.

4. Determine all pairs (a, b) of positive integers such that $2a + 1$ and $2b - 1$ are relatively prime and $a + b$ divides $4ab + 1$.

5. The circle Γ is inscribed in quadrilateral $ABCD$ with AD and CD tangent to Γ at P and Q, respectively. If BD intersects Γ at X and Y and M is the midpoint XY, prove that $\angle AMP = \angle CMQ$.

6. Let n be an odd positive integer, and let P_0 and P_1 be two consecutive vertices of a regular n-gon. For each $k \geq 2$ define P_k to be the vertex of the n-gon that lies on the perpendicular bisector of $P_{k-1}P_{k-2}$. Determine all n for which the sequence P_0, P_1, P_2, $\ldots$ covers all the vertices of the n-gon.

As the last problem set for this *Corner* we give the XVIII Olimpiada de Matematica de Paises del Cono Sur. Again, many thanks to Bill Sands and Leda Sanchez.

XVIII OLIMPIADA DE MATEMÁTICA DE PAISES DEL CONO SUR
Atlántida, June 14–15, 2007

1. Find all pairs (x, y) of nonnegative integers that satisfy

$$x^3y + x + y = xy + 2xy^2 .$$

2. Given are 100 positive integers whose sum equals their product. Determine the minimum number of 1's that may occur among the 100 numbers.

3. Let ABC be an acute triangle with altitudes AD, BE, CF where D, E, F lie on BC, AC, AB, respectively. Let M be the midpoint of BC. The circumcircle of triangle AEF cuts the line AM at A and X. The line AM cuts the line CF at Y. Let Z be the point of intersection of AD and BX. Show that the lines YZ and BC are parallel.

4. Some cells of a 2007×2007 table are coloured. The table is "charrua" if none of the rows and none of the columns are completely coloured.

(a) What is the maximum number k of coloured cells that a charrua table can have?

(b) For such k, calculate the number of distinct charrua tables that exist.

5. Let $ABCDE$ be a convex pentagon that satisfies the following:

(i) There is a circle Γ tangent to each of the sides.

(ii) The lengths of the sides are all positive integers.

(iii) At least one of the sides of the pentagon has length 1.

(iv) The side AB has length 2.

Let P be the point of tangency of Γ with AB.

(a) Determine the lengths of the segments AP and BP.

(b) Give an example of a pentagon satisfying the given conditions.

6. Show that for each positive integer n, there is a positive integer k such that the decimal representation of each of the numbers k, $2k$, $\ldots$, nk contains all of the digits 0, 1, 2, $\ldots$, 9.

Next we look at the solutions to problems of the 55th Czech and Slovak Mathematical Olympiad 2006 given at [2009 : 81–82].

1. (P. Novotný) A sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers is defined for $n \geq 1$ by $a_{n+1} = a_n + b_n$, where b_n is obtained from a_n by reversing its digits (the number b_n may start with zeroes). For instance if $a_1 = 170$, then $a_2 = 241$, $a_3 = 383$, $a_4 = 766$, $\ldots$. Decide whether a_7 can be a prime number.

Solution by Titu Zvonaru, Comănești, Romania, modified by the editor.

The answer is that a_7 cannot be a prime number.
We use the following lemmas:

Lemma 1. If a_n has an even number of digits, then 11 divides $a_n + b_n$.

Proof: Let $a_n = d_1 d_2 \ldots d_{2k}$, $b_n = d_{2k} \ldots d_2 d_1$ be the decimal representations of a_n and b_n. Modulo 11 we have

$$\begin{aligned} a_n + b_n & \\ =\; & \left(d_1 10^{2k-1} + \cdots + d_{2k-1}10 + d_{2k}\right) + \left(d_{2k}10^{2k-1} + \cdots + d_2 10 + d_1\right) \\ =\; & d_1[(11-1)^{2k-1} + 1] + d_2[(11-1)^{2k-2} + (11-1)] + \cdots \\ & + d_{2k-1}[(11-1) + (11-1)^{2k-2}] + d_{2k}[1 + (11-1)^{2k-1}] \\ \equiv\; & d_1(-1+1) + d_2(1-1) + \cdots + d_{2k}(1-1) \\ \equiv\; & 0 \pmod{11} . \end{aligned}$$ ∎

Lemma 2. If a_n is divisible by 11, then b_n is divisible by 11.

Proof: If a_n has an even number of digits this follows from Lemma 1. If a_n has an odd number of digits, then as in the proof of Lemma 1 we deduce that $a_n - b_n \equiv 0 \pmod{11}$, and the result follows. ∎

Clearly, if a_n has k digits, then a_{n+1} has at most $k+1$ digits.

Suppose for the sake of contradiction that 11 does not divide a_7. Then it follows from Lemma 1 and Lemma 2 that a_1 has an odd number of digits and that a_2, a_3, $\ldots$, a_6 each have the same number of digits as a_1 (otherwise the first a_i after a_1 with more digits than a_1 has an even number of digits, implying that 11 divides a_7).

Let f and ℓ be the first and last digits of a_1. Then, in order not to have an increase in the number of digits, the first digits of a_1, a_2, a_3, a_4, a_5 are f, $f+\ell$, $2(f+\ell)$, $4(f+\ell)$, $8(f+\ell)$; and then a_6 has one more digit than a_1 (since $f+\ell \geq 1$), a contradiction.

Therefore, a_7 is divisible by 11, and since it is easy to see that $a_7 > 11$, this means that a_7 is not prime.

2. (J. Šimša) Let m and n be positive integers such that

$$(x+m)(x+n) \;=\; x+m+n$$

has at least one integer solution. Prove that $\dfrac{1}{2} < \dfrac{m}{n} < 2$.

Solved by Michel Bataille, Rouen, France; and Oliver Geupel, Brühl, NRW, Germany. We give Bataille's version.

Let $p(x) = x^2 + (m+n-1)x + (mn-m-n)$. Since

$$\begin{aligned}&(m+n-1)^2 - 4(mn-m-n)\\ &\quad=\; (m+n+1)^2 - 4mn \;\geq\; (2\sqrt{mn}+1)^2 - 4mn \;>\; 0\,,\end{aligned}$$

the equation $p(x) = 0$ has two distinct solutions, one of which is an integer (from the hypothesis). As the sum of the solutions is the integer $1-m-n$, the other solution is an integer as well. We denote these solutions by a, a' with $a' < a$. Also, we note that $p(-m) = -n < 0$, $p(-n) = -m < 0$ so that $-m$ and $-n$ are between a and a' and in particular,

$$m,\; n \;\geq\; 1-a\,. \qquad (1)$$

Lastly, we observe that $a \leq 0$, since $p(x) \neq 0$ for $x \geq 1$ (if $x \geq 1$, then $x^2 \geq x$, $x(m+n) \geq m+n$ and so $p(x) \geq mn > 0$). Now, we rewrite $p(a) = 0$ as

$$\big(m-(1-a)\big)\big(n-(1-a)\big) \;=\; 1-a\,.$$

From $1-a \geq 1$ and the inequalities (1), we see that $m-(1-a)$ and $n-(1-a)$ are divisors d, d' of $1-a$ with d, $d' \geq 1$ and $dd' = 1-a$. Thus, $m = dd'+d$, $n = dd'+d'$ and so

$$2m-n \;=\; dd'+2d-d' \;=\; d'(d-1)+2d \;\geq\; 2d \;>\; 0$$

with $2n-m > 0$ deduced similarly. The desired inequalities follow.

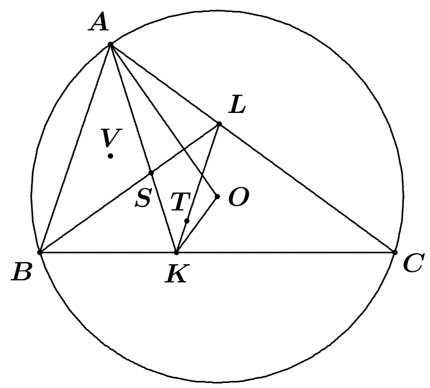
3. (T. Jurík) Triangle ABC is not equilateral, and the angle bisectors at A and B intersect the sides BC and AC at K and L, respectively. Let S be the incentre, O be the circumcentre, and V be the orthocentre of triangle ABC. Prove that the following statements are equivalent:

(a) The line KL is tangent to the circumcircles of triangles ALS, BVS, and BKS.

(b) The points A, B, K, L, and O are concyclic.

Solution by Titu Zvonaru, Comăneşti, Romania.

We denote by $\Gamma(XYZ)$ the circumcircle of $\triangle XYZ$, and let $\alpha = \angle BAC$, $\beta = \angle CBA$, and $\gamma = \angle ACB$.



Suppose first that (a) is true.

Since KL is tangent to $\Gamma(ALS)$ and AK is the bisector of $\angle BAC$, we have

$$\begin{aligned} \angle KLS &= \angle LAS = \angle SAB \\ \iff \quad \angle KLB &= \angle KAB\,, \end{aligned}$$

hence,

$$\text{points } A, B, K, L \text{ are concyclic}\,. \qquad (1)$$

We also deduce that

$$\angle LBK = \angle KAL \iff \alpha = \beta\,. \qquad (2)$$

Suppose that KL is tangent to $\Gamma(BVS)$ at T. Taking the power of point L with respect to $\Gamma(BVS)$ and $\Gamma(BKS)$ we obtain $LT^2 = LS \cdot LB = LK^2$, hence KL is tangent to $\Gamma(BVS)$ at K, that is, the quadrilateral $VBKS$ is cyclic and

$$\angle VBK + \angle VSK = 180^\circ\,.$$

By (2) we know that the points V, S, and C are collinear, so that

$$\angle VBK = \angle KSC \iff 90^\circ - \gamma = \frac{\alpha}{2} + \frac{\gamma}{2} \iff \alpha + 3\gamma = 180^\circ\,.$$

Since $\alpha = \beta$, $\alpha + \beta + \gamma = 180^\circ$ and $\alpha + 3\gamma = 180^\circ$, hence

$$\alpha = \beta = 72^\circ\,, \qquad \gamma = 36^\circ\,. \qquad (3)$$

Using (3), we deduce that

$$\begin{aligned} \angle OBK &= \frac{180^\circ - \angle BOC}{2} = \frac{180^\circ - 2\alpha}{2} = 18^\circ \\ \angle KAO &= \angle KAC - \angle OAC = \frac{\alpha}{2} - \frac{180^\circ - \angle AOC}{2} \\ &= 36^\circ - 18^\circ = 18^\circ\,, \end{aligned}$$

hence,

$$\text{quadrilateral } AOKB \text{ is cyclic}\,. \qquad (4)$$

By (1) and (4) it follows that the statement (b) is true.

Conversely, suppose that the statement (b) is true, so that the points A, B, K, L and O are concyclic.

Since $ABKL$ is cyclic, $\angle LAK = \angle LBK$ is equivalent to $\alpha = \beta$, which is equivalent to $LK \| AB$; it follows that $\angle SLK = \angle SBA = \angle SAL$ and $\angle SKL = \angle SAB = \angle SBK$, and hence

$$KL \text{ is tangent to } \Gamma(ALS) \text{ and to } \Gamma(BKS)\,. \qquad (5)$$

Since $ABKO$ is cyclic, we have that $\angle OBK = \angle KAO$ is equivalent to $90^\circ - \alpha = \frac{\alpha}{2} - (90^\circ - \beta)$; but $\alpha = \beta$, hence $90^\circ - \alpha = \frac{\alpha}{2} - 90^\circ + \alpha$ is equivalent to $\alpha = \beta = 72^\circ$ and $\gamma = 36^\circ$.

Since $\alpha = \beta$, we deduce that

$$\angle SKL \;=\; \angle SAB \;=\; \frac{\alpha}{2} \;=\; 36^\circ \;=\; \angle SBK \qquad (6)$$

and $\angle VBK = 90^\circ - \gamma = 54^\circ$; $\angle KSC = \frac{\alpha}{2} + \frac{\gamma}{2} = 54^\circ$, hence

$$\text{the quadrilateral } SVBK \text{ is cyclic}\,. \qquad (7)$$

By (6) and (7)

$$LK \text{ is tangent to } \Gamma(BVS) \text{ at point } K\,. \qquad (8)$$

By (5) and (8) it follows that the statement (a) is true.

4. (J. Švrček) A segment AB is given in the plane. Find the locus of the centroids of all acute triangles ABC for which the following holds: the vertices A and B, the orthocentre V, and the centre S of the incircle of the triangle ABC are concyclic.

Solved by Oliver Geupel, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give Geupel's solution.

Let A' and B' be points on AB such that $3AA' = 3BB' = AB$. Let σ denote the region which is the open strip between the two perpendiculars to AB through A' and B'. Let Γ_1 and Γ_2 denote the two circular arcs joining A' and B' with peripheral angles of 60°. We will prove that the locus of the centroids G are the two sub-arcs of Γ_1 and Γ_2 which lie inside σ (see the figure on the next page).

Let C be any point such that $\triangle ABC$ is acute. Let $AA^\star$ and $BB^\star$ be the altitudes of $\triangle ABC$ passing through A and B. Since the points C, $B^\star$, V, and $A^\star$ are concyclic, we have

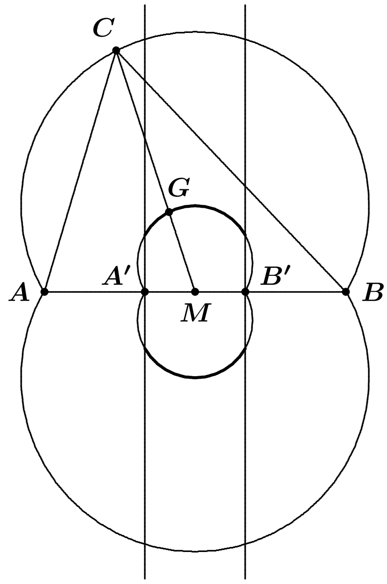
$$\angle AVB \;=\; \angle A^\star VB^\star \;=\; 180^\circ - \angle ACB\,.$$

On the other hand,

$$\begin{aligned}\angle ASB &= 180^\circ - \frac{1}{2}(\angle BAC + \angle ABC) \\ &= 90^\circ + \frac{1}{2}\angle ACB\,.\end{aligned}$$

The points A, B, V, and S are concyclic if and only if $\angle AVB = \angle ASB$, equivalently $180^\circ - \angle ACB = 90^\circ + \frac{1}{2}\angle ACB$, that is, $\angle ACB = 60^\circ$.

Therefore, the locus of C is the union of the two circular arcs joining A and B that have peripheral angles of 60°, restricted to the region which is the open strip between the perpendiculars to AB through A and B. Finally, if M is the mid-point of AB, then $MC = 3MG$, that is, the locus of G is homothetic to the locus of C with M as the centre of the homothety and ratio $\frac{1}{3}$.



5. (M. Panák) Find all triples (p, q, r) of distinct prime numbers such that

$$p|(q+r)\,, \qquad q|(r+2p)\,, \qquad r|(p+3q)\,.$$

Solved by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; and Titu Zvonaru, Comănești, Romania. We give Manes' solution.

The triples (p, q, r) of distinct primes satisfying the above divisibility conditions are $(5, 3, 2)$, $(2, 11, 7)$, and $(2, 3, 11)$.

Note that q is an odd prime since $q = 2$ and $q \mid (r + 2p)$ implies $r + 2p$ is even, and so $r = 2$, a contradiction since p, q, and r are distinct. Assume that p and r are also odd. Then $q + r = pa$, $r + 2p = qb$, and $p + 3q = rc$ for some integers a, b, c where b is odd. Therefore, $b = 2d + 1$ for some integer d. Then $r = pa - q$ and $r + 2p = qb$ implies $p(a + 2) = q(b + 1)$. Therefore, $p \mid (b + 1) = 2(d + 1)$, so that $p \mid (d + 1)$. Multiplying the equation $r + 2p = q(2d + 1)$ by c and substituting $rc = p + 3q$ yields $p(1 + 2c) = 2q(d - 1)$, whence $p \mid (d - 1)$. Thus, $p \mid (d + 1)$ and $p \mid (d - 1)$ implies $p = 2$, a contradiction. Therefore, either p or r must equal 2.

Assume $r = 2$ with p and q odd primes. Then $p \mid (q + 2)$ implies either $p = q + 2$ or $p < q + 2$. If $p < q + 2$, then $q \mid (r + 2p) = 2(p + 1)$, so that $q \mid (p + 1)$. Since p and q are both odd, it follows that $q < p + 1 < q + 3$. Therefore, either $p + 1 = q + 1$ or $p + 1 = q + 2$, both of which are contradictions since p and q are distinct odd primes. Hence, $p = q + 2$. Then $q \mid (r + 2p) = 2(p + 1)$, and so $q \mid (p + 1) = q + 3$, whence $q = 3$ and $p = 5$. This yields the first triple $(5, 3, 2)$.

Finally, assume $p = 2$ with q and r odd primes. The divisibility conditions for this case are

$$\begin{aligned} q + r &= 2a\,, & (1)\\ r + 4 &= qb\,, & (2)\\ 3q + 2 &= rc\,, & (3) \end{aligned}$$

for some positive integers a, b, c with b and c odd. Assume $r < q$. Then $q \mid (r + 4)$ implies $q \leq r + 4$. Therefore, $r < q \leq r + 4$. Since q, r are odd primes, it follows that the only possible values for q are $q = r + 2$ and $q = r + 4$. If $q = r + 2$, then $q \mid (r + 2p) = r + 4$ implies $(r + 2) \mid (r + 4)$, a contradiction since $r > 0$. Therefore, $q = r+4$ so that in (3), $3(r+4)+2 = rc$ implies $r(c - 3) = 14$. Hence, $r \mid 14$ so that $r = 7$ and $q = r + 4 = 11$. Thus, the second triple is $(2, 11, 7)$.

On the other hand if $r > q$, let $r = q + 2k$ for some integer k. Note that $k > 1$ since $r = q + 2$ and $q \mid (r + 4) = 1 + 6$ imply $q = 3$ and $r = 5$. However, these values do not satisfy $r \mid (3q + 2)$. In (3), $3q + 2 = (q + 2k)c$ implies $q(3 - c) = 2(kc - 1) > 0$. Therefore, $2 \mid (3 - c) > 0$ and c is odd yield $c = 1$. Hence, $q \mid 6$, so that $q = 3$ and $r = 3q + 2 = 11$. This yields the last triple $(2, 3, 11)$.

Now we turn to the files for the April 2009 number of the *Corner* and solutions from our readers to problems of the Scientific and Technical Research Institute of Turkey, Team Selection Examination for the International Mathematical Olympiad given at [2009 : 144].

2. Let n be a positive integer. In how many different ways can a $2 \times n$ rectangle be partitioned into rectangles with sides of integer length?

Solution by Oliver Geupel, Brühl, NRW, Germany.

Consider the rectangle with vertices $(0, 0)$, $(0, 2)$, $(n, 0)$, and $(n, 2)$ in the Cartesian plane. A partition can be characterized by the set E of line segments $\langle (j, k), (j + 1, k)\rangle$ and $\langle (j, k), (j, k + 1)\rangle$ which constitute the borders of the small rectangles. We call a partition *type A* if $\langle (n - 1, 1), (n, 1)\rangle \in E$; we call it *type B* if $\langle (n - 1, 1), (n, 1)\rangle \notin E$. For each partition E, the set

$$\begin{aligned} E' = E - \;&\{\langle (k, n - 1), (k, n)\rangle,\ \langle (n, j), (n, j + 1)\rangle | 0 \leq k \leq 2, 0 \leq j \leq 1\}\\ &\cup\ \{\langle (n - 1, 0), (n - 1, 1)\rangle, \langle (n - 1, 1), (n - 1, 2)\rangle\} \end{aligned}$$

constitutes a partition of the $2 \times (n - 1)$ rectangle with vertices $(0, 0)$, $(0, 2)$, $(n - 1, 0)$, and $(n - 1, 2)$.

If E' is of type A, that means $\langle (n - 2, 1), (n - 1, 1)\rangle \in E'$, then there are five corresponding sets E possible, four of type A and one of type B; see Figure 1. Otherwise, if E' is of type B, then there are three corresponding sets E possible, one of type A and two of type B; see Figure 2.

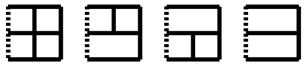
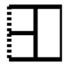
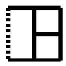
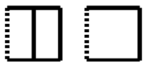
Figure 1 type A:  type B: 

Figure 2 type A:  type B: 

Let A_n and B_n denote the number of type A and type B partitions, respectively, and let $C_n = A_n + B_n$. We obtain $A_n = 4A_{n-1} + B_{n-1}$ and $B_n = A_{n-1} + 2B_{n-1}$ for $n \geq 2$. For $n \geq 3$ we derive

$$\begin{aligned} C_n &= A_n + B_n = 5A_{n-1} + 3B_{n-1} \\ &= 23A_{n-2} + 11B_{n-2} = 6C_{n-1} - 7C_{n-2}\,. \end{aligned}$$

The initial values $C_1 = 2$ and $C_2 = 8$ are easy to check. We have obtained a linear recursion for C_{n-1} which can be solved with repertoire methods, thus yielding the desired number of partitions

$$C_n = \frac{2+\sqrt{2}}{2}\left(3+\sqrt{2}\right)^{n-1} + \frac{2-\sqrt{2}}{2}\left(3-\sqrt{2}\right)^{n-1}\,.$$

3. Let x, y, z be positive real numbers with $xy + yz + zx = 1$. Prove that

$$\frac{27}{4}(x+y)(y+z)(z+x) \geq \left(\sqrt{x+y} + \sqrt{y+x} + \sqrt{z+x}\right)^2 \geq 6\sqrt{3}\,.$$

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and Titu Zvonaru, Comănești, Romania. We give Bataille's write-up.

From the constraint, we have

$$\begin{aligned} (x+y)(y+z) &= y^2+1\,, \\ (y+z)(z+x) &= z^2+1\,, \\ (z+x)(x+y) &= x^2+1\,, \end{aligned}$$

so that the right inequality can be rewritten as

$$x+y+z+\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1} \geq 3\sqrt{3}\,. \tag{1}$$

Now, $(x+y+z)^2 = x^2+y^2+z^2+2 \geq xy+yz+zx+2 = 3$, hence

$$x+y+z \geq \sqrt{3}\,. \tag{2}$$

Also, the function $f(t) = \sqrt{t^2+1}$ is a convex function (its second derivative satisfies $f''(t) = (t^2+1)^{-3/2} > 0$). Thus,

$$\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1} \geq 3\sqrt{\left(\frac{x+y+z}{3}\right)^2+1}$$

and using (2) we obtain

$$\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1} \;\geq\; 2\sqrt{3}\,. \tag{3}$$

Adding (2) and (3) yields (1). As for the left inequality, it is equivalent to

$$\frac{1}{\sqrt{x^2+1}}+\frac{1}{\sqrt{y^2+1}}+\frac{1}{\sqrt{z^2+1}} \;\leq\; \frac{3\sqrt{3}}{2}\,. \tag{4}$$

The constraint allows us to write $x=\tan\frac{\alpha}{2}$, $y=\tan\frac{\beta}{2}$, $z=\tan\frac{\gamma}{2}$ where α, β, γ are the angles of a triangle. Then, (4) can be rewritten as

$$\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2} \;\leq\; \frac{3\sqrt{3}}{2}\,,$$

which holds because from the concavity of $\cos$ on $\left(0,\frac{\pi}{2}\right)$ we have

$$\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2} \;\leq\; 3\cos\left(\frac{\alpha+\beta+\gamma}{6}\right) \;=\; \frac{3\sqrt{3}}{2}\,.$$

5. Given a circle with diameter AB and a point Q on the circle different from A and B, let H be the foot of the perpendicular dropped from Q to AB. Prove that if the circle with centre Q and radius QH intersects the circle with diameter AB at C and D, then CD bisects QH.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; and Geoffrey A. Kandall, Hamden, CT, USA. We give the version of Amengual Covas.

Let O be the centre of the circle on AB as diameter, and let Q' be the point on this circle diametrically opposite to Q.

Let the common chord CD of the two given circles intersects QH and QO at points M and N, respectively.

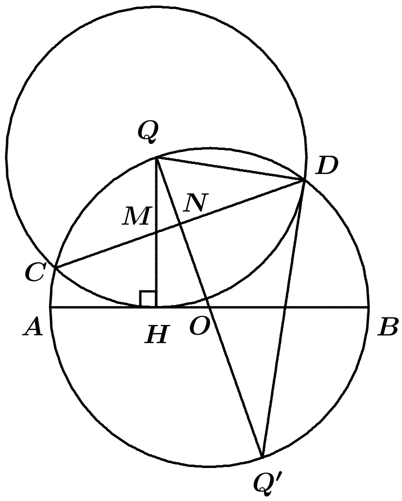
Since this common chord is perpendicular to the line of centres QO, we see that, in right triangle DQQ', DN is the altitude to the hypotenuse.

By a standard mean proportion we then have

$$QD^2 \;=\; QQ'\cdot QN\,,$$

that is,

$$QH^2 \;=\; 2QO\cdot QN\,.$$



Since $\triangle QNM$ is similar to $\triangle QHO$, we also have $\dfrac{QM}{QN} = \dfrac{QO}{QH}$, and hence $QM \cdot QH = QO \cdot QN$.

Therefore, $QM \cdot QH = \frac{1}{2}QH^2$; whence $QM = \frac{1}{2}QH$, as required.

Next we will look at solutions for the Scientific and Technical Research Institute of Turkey XIII, National Mathematical Olympiad, Second Round, given at [2009 : 145].

1. Let a, b, c, and d be real numbers. Prove that

$$\sqrt{a^4+c^4}+\sqrt{a^4+d^4}+\sqrt{b^4+c^4}+\sqrt{b^4+d^4} \ \geq\ 2\sqrt{2}(ab+bc)\,.$$

Solved by Arkady Alt, San Jose, CA, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's contribution.

The stated inequality is incorrect. A simple counterexample is given by $a = b = c = 1$ and $d = 0$. We prove the following correct version:

$$\sqrt{a^4+c^4}+\sqrt{a^4+d^4}+\sqrt{b^4+c^4}+\sqrt{b^4+d^4} \ \geq\ 2\sqrt{2}(ab+cd)\,. \qquad (1)$$

By the AM–GM Inequality and the Cauchy–Schwarz Inequality, we have

$$\begin{aligned}\sqrt{a^4+c^4}+\sqrt{b^4+d^4} &\geq 2\sqrt[4]{(a^4+c^4)(b^4+d^4)}\\ &\geq 2\sqrt{a^2b^2+c^2d^2}\,. \qquad (2)\end{aligned}$$

Since $2(a^2b^2+c^2d^2)-(ab+cd)^2 = (ab-cd)^2 \geq 0$ we have

$$\begin{aligned}\sqrt{2(a^2b^2+c^2d^2)} &\geq ab+cd\,;\\ 2\sqrt{a^2b^2+c^2d^2} &\geq \sqrt{2}(ab+cd)\,. \qquad (3)\end{aligned}$$

From (2) and (3) we obtain

$$\sqrt{a^4+c^4}+\sqrt{b^4+d^4} \ \geq\ \sqrt{2}(ab+cd)\,. \qquad (4)$$

Similarly, we have

$$\sqrt{a^4+d^4}+\sqrt{b^4+c^4} \ \geq\ \sqrt{2}(ab+dc)\,. \qquad (5)$$

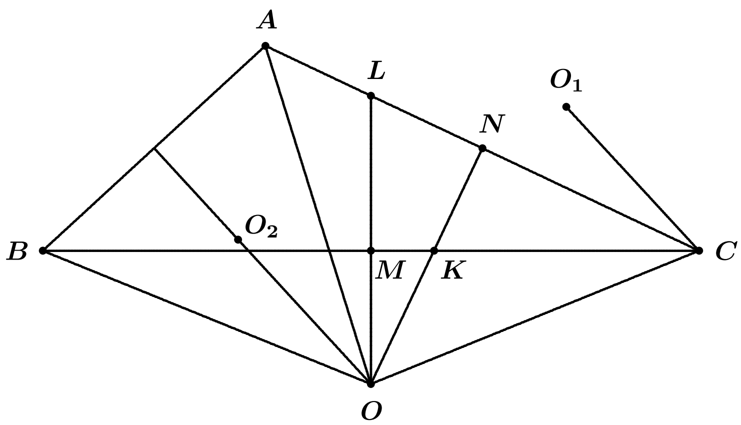
Adding (4) and (5), inequality (1) follows.

2. In a triangle ABC with $|AB| < |AC| < |BC|$, the perpendicular bisector of AC intersects BC at K and the perpendicular bisector of BC intersects AC at L. Let O, O_1, and O_2 be the circumcentres of the triangles ABC, CKL, and OAB, respectively. Prove that OCO_1O_2 is a parallelogram.

Solution by Titu Zvonaru, Cománeşti, Romania.

As usual write $\angle BAC = \alpha$, $\angle CBA = \beta$, $\angle ACB = \gamma$ and $a = BC$, $b = CA$, $c = AB$.

Since $c < b < a$, it follows that $\gamma < \beta < \alpha$, and it is easy to see that $\beta < 90^\circ$ and $\alpha + \gamma > 90^\circ$. Let M and N be the midpoints of the sides BC and AC, respectively.



In $\triangle CML$ and $\triangle CNK$ we have

$$CL = \frac{a}{2\cos\gamma}; \qquad CK = \frac{b}{2\cos\gamma}.$$

Since $\frac{a}{CL} = \frac{b}{CK}$ and $\angle BCA = \angle LCK$, it follows that $\triangle CLK$ and $\triangle ABC$ are similar, hence $LK = \frac{c}{2\cos\gamma}$.

By the Law of Sines in $\triangle CKL$, we obtain

$$CO_1 = \frac{LK}{2\sin\gamma} = \frac{c}{4\sin\gamma\cos\gamma} = \frac{c}{2\sin 2\gamma}. \tag{1}$$

By the Law of Sines in $\triangle OAB$, we have

$$OO_2 = \frac{AB}{2\sin\angle AOB} = \frac{c}{2\sin 2\gamma}. \tag{2}$$

By (1) and (2) we have that $CO_1 = OO_2$.

If $\alpha \geq 90^\circ$, then $\angle CKL > 90^\circ$ and

$$\angle O_1CL = \frac{180^\circ - \angle LO_1C}{2} = 90^\circ - \left(\frac{360^\circ - 2\angle LKC}{2}\right) = \alpha - 90^\circ.$$

If $\alpha < 90^\circ$, then we obtain

$$\angle O_1CL = \frac{180^\circ - \angle LO_1C}{2} = 0^\circ - \frac{2\angle LKC}{2} = 90^\circ - \alpha.$$

In any case, it is easy to see that $\angle O_1CB = \alpha + \gamma - 90^\circ = 90^\circ - \beta$, hence $CO_1 \perp AB$. This implies that $CO_1 \| OO_2$, because O_2 belongs to the perpendicular bisector of AB. It follows that OCO_1O_2 is a parallelogram.

4. Find all triples (m, n, k) of nonnegative integers such that $5^m + 7^n = k^3$.

Solved by Oliver Geupel, Brühl, NRW, Germany; and by Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give Geupel's solution.

The unique solution is $(m, n, k) = (0, 1, 2)$.

For nonnegative integers i, we have

$$5^{2i} \equiv 1 \pmod 8\ , \qquad 5^{2i+1} \equiv -3 \pmod 8\ ,$$
$$7^{2i} \equiv 1 \pmod 8\ , \qquad 7^{2i+1} \equiv -1 \pmod 8\ , \qquad i^3 \not\equiv \pm 2, 4 \pmod 8\ .$$

Therefore, if m, n, and k are nonnegative integers with $5^m + 7^n = k^3$, then there are nonnegative integers s, t, and u such that $m = 2s$, $n = 2t+1$ and $k = 2u$; hence

$$25^s + 7 \cdot 49^t \;=\; 8u^3\,. \tag{1}$$

We claim that $3 \mid s$.

If $t = 0$, then $25^s \equiv 2 - u^3 \pmod 9$. For nonnegative integers i, it holds that $25^{3i+1} \equiv 7 \pmod 9$ and $25^{3i+2} \equiv 4 \pmod 9$. On the other hand, however, $2 - u^3 \equiv 1,\ 2,\ 3 \pmod 9$, hence $3 \mid s$.

Otherwise, if $t > 0$, then $25^s \equiv 8u^3 \pmod{49}$; hence $\gcd(u, 7) = 1$. By Euler's Totient Theorem, $25^{14s} \equiv (2u)^{42} \equiv (2u)^{\phi(49)} \equiv 1 \pmod{49}$. It is tedious but straightforward to check that $5^i \equiv 1 \pmod{49}$ if and only if $42 \mid i$. Thus, $3 \mid s$, which completes the proof of our claim.

Substituting $s = 3v$, we obtain from (1) that

$$7 \cdot 49^t \;=\; (2u)^3 - 25^{3v} \;=\; (2u - 25^v)\left((2u)^2 + 2u \cdot 25^v + 25^{2v}\right). \tag{2}$$

Therefore, there exists a nonnegative integer w such that

$$(2u)^2 + 2u \cdot 25^v + 25^{2v} \;=\; 7^{2t+1-w} \tag{3}$$

and $2u - 25^v = 7^w$; thus

$$(2u)^2 - 4u \cdot 25^v + 25^{2v} \;=\; 7^w. \tag{4}$$

From (3) and (4) it follows that $6u \cdot 25^v = 7^{2t+1-w} - 7^w$. If $w \geq 1$ then $7 \mid u$, and 7 would be a divisor of $2u - 7^w = 25^v$, which is impossible. Consequently, $w = 0$. It follows that $2u = 25^v + 1$; hence by (2):

$$\begin{aligned} 25^{3v} + 7 \cdot 49^t &= (25^v + 1)^3 \;=\; 25^{3v} + 3 \cdot 25^{2v} + 3 \cdot 25^v + 1\,, \\ 7 \cdot 49^t &= 3 \cdot 25^{2v} + 3 \cdot 25^v + 1\,, \end{aligned}$$

and thus $25^v \mid (7 \cdot 49^t - 1)$.

Now, if $v \geq 1$, then $5 \mid (7 \cdot 49^t - 1)$. However, the residues of $7 \cdot 49^t$ modulo 5 are ± 2, which is a contradiction. We conclude that $v = 0$ and therefore $u = 1$ and $(m, n, k) = (0, 1, 2)$.

5. Let a, b, and c be the side lengths of a triangle whose incircle has radius r. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \leq \frac{1}{4r^2}\,.$$

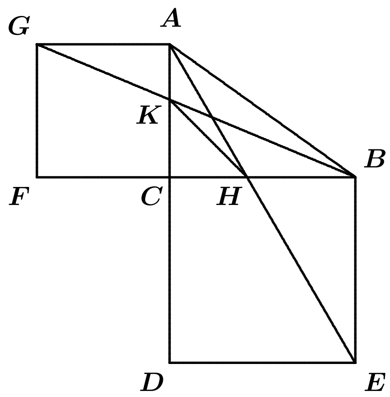
Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and Titu Zvonaru, Comăneşti, Romania. We give the comment and reference from Bataille.

This problem appears as Problem F.3019 in C2K — Century 2 of Kömal "1994–1997", Roland Eötvös Physical Society, Budapest, 1999. A solution can be found on page 125.

Next we turn to solutions to problems of the 2005 Australian Mathematical Olympiad given at [2009 : 146–147].

1. Let ABC be a right-angled triangle with the right angle at C. Let $BCDE$ and $ACFG$ be squares external to the triangle. Furthermore, let AE intersect BC at H, and let BG intersect AC at K. Find the size of $\angle DKH$.

Solved by Oliver Geupel, Brühl, NRW, Germany; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comăneşti, Romania. We give Kandall's solution.



Let $BC = a$ and $AC = b$. Triangle KCB is similar to triangle GFB and triangle HCA is similar to triangle EDA. Therefore,

$$\frac{KC}{b} = \frac{a}{a+b} \quad \text{and} \quad \frac{HC}{a} = \frac{b}{a+b}\,.$$

Consequently, $KC = HC = \dfrac{ab}{a+b}$, hence $\angle DKH = 45^\circ$.

3. Let n be a positive integer, and let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = n$. Prove that

$$\frac{a_1}{a_1^2+1} + \frac{a_2}{a_2^2+1} + \cdots + \frac{a_n}{a_n^2+1} \leq \frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_n+1}\,.$$

Solved by George Apostolopoulos, Messolonghi, Greece; Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and Henry Ricardo, Tappan, NY, USA. We give Ricardo's write-up.

We need two easily established facts: (a) $x+\frac{1}{x}\geq 2$ for positive x, and (b) $f(t)=\frac{1}{t+1}$ is a convex function for nonnegative t. Then for each k we have

$$\frac{a_k}{a_k^2+1} \;=\; \frac{1}{\left(\frac{a_k^2+1}{a_k}\right)} \;=\; \frac{1}{\left(a_k+\frac{1}{a_k}\right)} \;\leq\; \frac{1}{2}\,,$$

and so

$$\sum_{k=1}^{n}\frac{a_k}{a_k^2+1}\leq\frac{n}{2}=nf(1)=nf\left(\sum_{k=1}^{n}\frac{a_k}{n}\right)\leq\sum_{k=1}^{n}f(a_k)=\sum_{k=1}^{n}\frac{1}{a_k+1}\,.$$

It is easy to see that equality holds if and only if $a_k=1$ for each k.

4. Prove that for each positive integer n there exists a positive integer x such that $\sqrt{x+2004^n}+\sqrt{x}=\left(\sqrt{2005}+1\right)^n$.

Solved by Arkady Alt, San Jose, CA, USA; and Michel Bataille, Rouen, France. We give Bataille's version.

First we solve for x the given equation. Squaring yields

$$2\sqrt{x(x+2004^n)} \;=\; \left(\sqrt{2005}+1\right)^{2n}-2004^n-2x\,.$$

and squaring again yields

$$x \;=\; \frac{\left(\left(\sqrt{2005}+1\right)^{2n}-2004^n\right)^2}{4\left(\sqrt{2005}+1\right)^{2n}}\,.$$

Observing that $2004=\left(\sqrt{2005}+1\right)\left(\sqrt{2005}-1\right)$, we finally see that

$$x \;=\; \frac{1}{4}\left(\left(\sqrt{2005}+1\right)^n-\left(\sqrt{2005}-1\right)^n\right)^2$$

is the unique real solution to the given equation. To complete the proof, it is sufficient to show that for any positive integers n and a the number $A=\left((\sqrt{a}+1)^n-(\sqrt{a}-1)^n\right)^2$ is an integer multiple of 4.

From the Binomial Theorem, we have

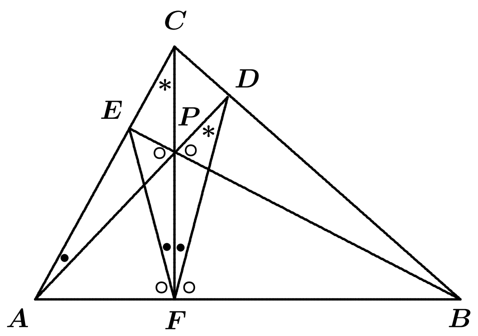
$$A \;=\; \left(\sum_{k=0}^{n}\binom{n}{k}(\sqrt{a}\,)^{n-k}(1+(-1)^{k+1})\right)^2 \;=\; \left(2\sum_{\substack{k=0\\ k\text{ odd}}}\binom{n}{k}(\sqrt{a}\,)^{n-k}\right)^2.$$

Now, if n is odd, then $n-k$ is even for each odd k and $\sum_{k \text{ odd}} \binom{n}{k}(\sqrt{a})^{n-k}$ is an integer so that A is an integer multiple of 4.

If n is even, then $2\sum_{k \text{ odd}} \binom{n}{k}(\sqrt{a})^{n-k} = 2(\sqrt{a})\cdot B$ for some integer B and $A = 4aB^2$ is an integer multiple of 4 as well.

6. Let ABC be a triangle. Let D, E, and F be points on the line segments BC, CA, and AB, respectively, such that line segments AD, BE, and CF meet in a single point. Suppose that $ACDF$ and $BCEF$ are cyclic quadrilaterals. Prove that AD is perpendicular to BC, BE is perpendicular to AC, and CF is perpendicular to AB.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.



Let P be the point at which AD, BE, and CF meet.

Since $ACDF$ is cyclic, $\angle ACF = \angle ADF$; since $BCEF$ is cyclic, $\angle ECF = \angle EBF$. Therefore, $PDBF$ is cyclic. Analogously, $PEAF$ is cyclic.

Now, $\angle EFA = \angle EPA = \angle DPB = \angle DFB$. Also, $\angle PFE = \angle PAE = \angle PFD$ (the latter equality holds since $ACDF$ is cyclic). Thus, $\angle CFA = \angle CFB = 90^\circ$. It follows that $\angle BEA$ and $\angle ADB$ are each 90°.

7. Let a_0, a_1, a_2, $\ldots$ and b_0, b_1, b_2, $\ldots$ be two sequences of integers such that $a_0 = b_0 = 1$ and for each nonnegative integer k

(a) $a_{k+1} = b_0 + b_1 + b_2 + \cdots + b_k$, and

(b) $b_{k+1} = (0^2+0+1)a_0 + \big(1^2+1+1\big)a_1 + \cdots + \big(k^2+k+1\big)a_k$.

For each positive integer n show that

$$a_n = \frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n}.$$

Solved by Arkady Alt, San Jose, CA, USA; and Michel Bataille, Rouen, France. We use Alt's solution.

The recursions (a) and (b) can be rewritten as follows:

$$\begin{aligned} a_{n+1} &= a_n + b_n\,, \\ b_{n+1} &= \big(n^2+n+1\big)\,a_n + b_n\,; \quad n \geq 1\,. \end{aligned} \tag{1}$$

By making the substitutions $b_n = a_{n+1} - a_n$ and $b_{n+1} = a_{n+2} - a_{n+1}$ in $b_{n+1} = \left(n^2+n+1\right) a_n + b_n$ we obtain successively

$$\begin{aligned} a_{n+2} - a_{n+1} &= \left(n^2+n+1\right) a_n + a_{n+1} - a_n\,, \\ a_{n+2} &= 2a_{n+1} + n\,(n+1)\,a_n\,, \\ a_{n+2} &= 2a_{n+1} + n\,(n+1)\,a_n\,; \qquad n \geq 1\,, \end{aligned} \tag{2}$$

where $a_0 = 1$ and $a_1 = b_0 = 1$.

Using (2) we get $a_2 = 2$, $a_3 = 6$, $a_4 = 24$, and $a_5 = 120$, suggesting that $a_n = n!$, and we confirm this by using Mathematical Induction.

Indeed, supposing that $a_n = n!$ and $a_{n-1} = (n-1)!$ and using (2) we obtain, for any $n \geq 1$,

$$\begin{aligned} a_{n+1} &= 2a_n + (n-1)na_{n-1} = 2n! + (n-1)n(n-1)! \\ &= 2n! + (n-1)n! \qquad = (n-1+2)n! = (n+1)!\,. \end{aligned}$$

Since $a_n = n!$, then $b_n = a_{n+1} - a_n = (n+1)! - n! = n \cdot n! = na_n$, and therefore

$$\frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} = \frac{n! a_1 a_2 \cdots a_n}{a_1 a_2 \cdots a_n} = n! = a_n\,.$$

Next we look at solutions from our readers to problems of the 56th Belarusian Mathematical Olympiad 2006, Category C, Final Round, given at [2009 : 147–148].

1. (E. Barabanov) Is it possible to partition the set of all integers into three nonempty pairwise disjoint subsets so that for any two numbers a and b from different subsets

(a) there is a number c in the third subset such that $a + b = 2c$?

(b) there are numbers c_1 and c_2 in the third subset such that $a+b = c_1+c_2$?

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

(a) This is impossible. Suppose $\mathbb{Z} = A \cup B \cup C$ is a partition of $\mathbb{Z}$ satisfying the given condition. Without loss of generosity, assume $1 \in A$. If B contains any even integer b, then $1 + b$ is odd. Since $2c$ is even for all $c \in C$, we have a contradiction. Hence, B contains no even integers. Then $2 \in A$ or $2 \in C$. In either case, $2 + b$ is odd for any $b \in B$, again a contradiction.

(b) This is possible. Let $\mathbb{Z}$ be partitioned as $\mathbb{Z} = U \cup V \cup W$ where $U = \{3k \mid k \in \mathbb{Z}\}$, $V = \{3k+1 \mid k \in \mathbb{Z}\}$, and $W = \{3k+2 \mid k \in \mathbb{Z}\}$. Let a and b be two numbers from different subsets in the partition. There are three cases to consider:

If $a \in U$, $b \in V$, then write $a = 3k_1$ and $b = 3k_2 + 1$, and take $c_1 = 3k_1 + 2$ and $c_2 = 3(k_2 - 1) + 2$ as the required elements in W.

If $a \in U$, $b \in W$, then write $a = 3k_1$, and $b = 3k_2 + 2$, and take $c_1 = 3k_1 + 1$ and $c_2 = 3k_2 + 1$ as the required elements in V.

If $a \in V$, $b \in W$, then write $a = 3k_1 + 1$ and $b = 3k_2 + 2$, and take $c_1 = 3k_1$ and $c_2 = 3(k_2 + 1)$ as the required elements in U.

Therefore, U, V, and W satisfy the prescribed condition.

3. (V. Karamzin) Let a, b, and c be positive real numbers such that $abc = 1$. Prove that $2\left(a^2 + b^2 + c^2\right) + a + b + c \geq ab + bc + ca + 6$.

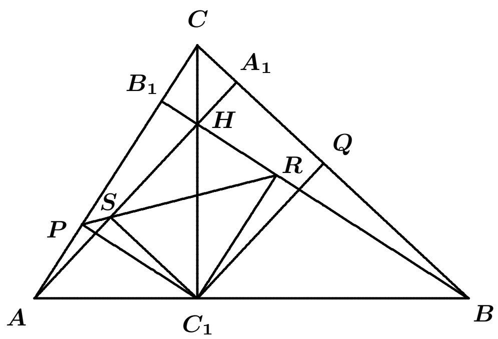
Solved by Arkady Alt, San Jose, CA, USA; George Apostolopoulos, Messolonghi, Greece; Michel Bataille, Rouen, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Alt's version.

Since $a + b + c \geq 3\sqrt[3]{abc} = 3$ and $ab + bc + ca \geq 3\sqrt[3]{a^2b^2c^2} = 3$ by the AM–GM Inequality, then we have

$$\begin{aligned}
&2\left(a^2 + b^2 + c^2\right) + a + b + c - (ab + bc + ca) - 6 \\
&\quad = 2\left(a^2 + b^2 + c^2 - ab - bc - ca\right) + a + b + c + ab + bc + ca - 6 \\
&\quad = (a-b)^2 + (b-c)^2 + (c-a)^2 \\
&\qquad\qquad + (a + b + c - 3) + (ab + bc + ca - 3) \geq 0 .
\end{aligned}$$

5. (I. Voronovich) Let AA_1, BB_1, and CC_1 be the altitudes of an acute triangle ABC. Prove that the feet of the perpendiculars from C_1 to the segments AC, BC, BB_1, and AA_1 are collinear.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; and Titu Zvonaru, Comănești, Romania. We give Kandall's version.



Let P, Q, R, S be the feet of the perpendiculars from C_1 to AC, BC, BB_1, AA_1, respectively, and let the orthocentre of ABC be H. Draw PS and SR.

The quadrilaterals $APSC_1$ and $SHRC_1$ are cyclic, and so $\angle PSA = \angle PC_1A = 90^\circ - \angle CAB$ and $\angle HSR = \angle HC_1R = 90^\circ - \angle RC_1B = \angle RBA = 90^\circ - \angle CAB$. Thus, $\angle PSA = \angle HSR$, that is, the points P, S, and R are collinear. The proof that S, R, and Q are collinear is analogous. Therefore, P, S, R, and Q are collinear.

7. (I. Zhuk) Let x, y, and z be real numbers greater than 1 such that

$$\begin{aligned} xy^2 - y^2 + 4xy + 4x - 4y &= 4004\,, \\ xz^2 - z^2 + 6xz + 9x - 6z &= 1009\,. \end{aligned}$$

Determine all possible values of $xyz + 3xy + 2xz - yz + 6x - 3y - 2z$.

Solved by Arkady Alt, San Jose, CA, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give Zelator's solution.

The first equation is equivalent to $x(y^2 + 4y + 4) = 4004 + y^2 + 4y$, or $x(y+2)^2 = 4000 + (y+2)^2$, and we obtain

$$x = \frac{4000}{(y+2)^2} + 1\,. \tag{3}$$

By similar manipulations of the second equation we obtain

$$x = \frac{1000}{(z+3)^2} + 1\,. \tag{4}$$

Note that both (3) and (4) are consistent with the hypothesis that $x > 1$, $y > 1$, and $z > 1$.

By (3) and (4) we have

$$\frac{4000}{(y+2)^2} = \frac{1000}{(z+3)^2} \iff \left(\frac{y+2}{z+3}\right)^2 = 4\,,$$

and since $\frac{y+2}{z+3} > 0$ we have $\frac{y+2}{z+3} = 2$ and $y = 2z + 4$.

Next, we write

$$\begin{aligned} Q(x,y,z) &= xyz + 3xy + 2zx - yz + 6x - 3y - 2z \\ &= (xyz + 3xy + 2xz + 6x) + (-yz - 3y - 2z) \\ &= Q_1(x,y,z) + Q_2(x,y,z)\,. \end{aligned} \tag{5}$$

We have $Q_1(x,y,z) = x(yz + 3y + 2z + 6)$. Substituting $y = 2z + 4$ yields $Q_1(x,y,z) = 2x(z+3)^2$, and then by (4) we obtain

$$Q_1(x,y,z) = 2000 + 2(z+3)^2\,. \tag{6}$$

Next we substitute $y = 2z + 4$ into $Q_2(x,y,z) = -yz - 3y - 2z$ to obtain

$$Q_2(x,y,z) = 6 - 2(z+3)^2\,. \tag{7}$$

By virtue of (5), (6), and (7) we have $Q(x,y,z) = 2006$.

Thus, the expression $Q(x,y,z)$ has a fixed value, namely 2006, so the set of all possible values of $Q(x,y,z)$ is the singleton set $\{2006\}$.

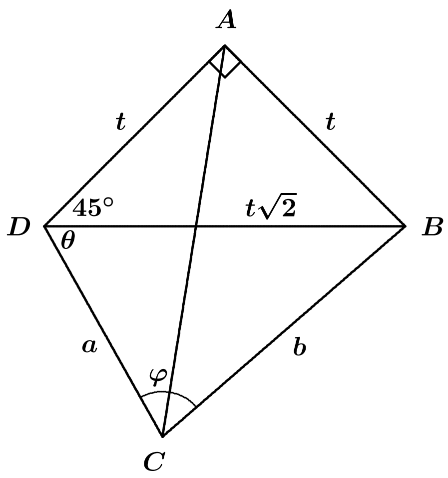
To finish the file of readers' solutions for the April 2009 number of the *Corner* we look at solutions to problems of the 56^{th} Belarusian Mathematical Olympiad 2006, Category B, Final Round, given at [2009 : 148–149].

1. (I.Voronovich) Given a convex quadrilateral $ABCD$ with $DC = a$, $BC = b$, $\angle DAB = 90^\circ$, $\angle DCB = \varphi$, and $AB = AD$, find the length of the diagonal AC.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give Kandall's solution.

Let $AB = AD = t$ and $\angle BDC = \theta$.
Then $DB = t\sqrt{2}$ and $\angle ADB = 45^\circ$.
By the Law of Cosines,

$$\begin{aligned} AC^2 &= a^2 + t^2 - 2at\cos(\theta + 45^\circ) \\ &= a^2 + t^2 - \sqrt{2}at(\cos\theta - \sin\theta)\,. \end{aligned}$$



In $\triangle BCD$ we have the relations

$$\begin{aligned} \cos\theta &= \frac{a^2 + 2t^2 - b^2}{2at\sqrt{2}}\,, \\ \sin\theta &= \frac{b\sin\varphi}{t\sqrt{2}}\,. \end{aligned}$$

Now we substitute these and simplify:

$$AC = \left(\frac{a^2 + b^2 + 2ab\sin\varphi}{2}\right)^{1/2}.$$

3. (I. Biznets) Let a, b, and c be positive real numbers. Prove that

$$\frac{a^3 - 2a + 2}{b + c} + \frac{b^3 - 2b + 2}{c + a} + \frac{c^3 - 2c + 2}{a + b} \geq \frac{3}{2}\,.$$

Solved by Mohammed Aassila, Strasbourg, France; Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; and by Titu Zvonaru, Cománeşti, Romania. We use Zvonaru's presentation.

Since $a^3 - 2a + 2 = a^3 - 3a + 2 + a = (a-1)^2(a+2) + a$, the given inequality is the same as

$$\left(\frac{(a-1)^2(a+2)}{b+c} + \frac{(b-1)^2(b+2)}{c+a} + \frac{(c-1)^2(c+2)}{a+b}\right) + \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \geq \frac{3}{2}\,.$$

But this inequality is true, as the first sum is obviously nonnegative and the second sum is greater than $\frac{3}{2}$ by Nesbitt's inequality.

6. (I. Voronovich) A sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$ of pairs of real numbers is such that $(a_{n+1}, b_{n+1}) = \left(a_n^2 - 2b_n, b_n^2 - 2a_n\right)$ for all $n \geq 1$. Find $2^{512}a_{10} - b_{10}$ if $4a_1 - 2b_1 = 7$.

Solution by Michel Bataille, Rouen, France.

Let $p(x) = x^3 - a_1x^2 + b_1x - 1$ and let α, β, γ be the complex roots of this polynomial. Then, $p(x) = (x-\alpha)(x-\beta)(x-\gamma)$ and

$$\begin{aligned} a_1 &= \alpha + \beta + \gamma\,, \\ b_1 &= \alpha\beta + \beta\gamma + \gamma\alpha\,, \\ 1 &= \alpha\beta\gamma\,. \end{aligned}$$

Now, easy calculations yield

$$-p(x)p(-x) \;=\; (x^2 - \alpha^2)(x^2 - \beta^2)(x^2 - \gamma^2)$$

as well as $-p(x)p(-x) = q(x^2)$, where

$$\begin{aligned} q(x) &= x^3 - (a_1^2 - 2b_1)x^2 + (b_1^2 - 2a_1)x - 1 \\ &= x^3 - a_2x^2 + b_2x - 1\,. \end{aligned}$$

Thus, the roots of $x^3 - a_2x^2 + b_2x - 1$ are α^2, β^2, γ^2 and so

$$\begin{aligned} a_2 &= \alpha^2 + \beta^2 + \gamma^2\,, \\ b_2 &= (\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2\,. \end{aligned}$$

Continuing this way, an easy induction argument yields

$$\begin{aligned} a_n &= \alpha^{2^{n-1}} + \beta^{2^{n-1}} + \gamma^{2^{n-1}}\,, \\ b_n &= (\alpha\beta)^{2^{n-1}} + (\beta\gamma)^{2^{n-1}} + (\gamma\alpha)^{2^{n-1}}\,, \end{aligned}$$

for all positive integers n.

Since $p(2) = 7 - (4a_1 - 2b_1) = 0$, we have that 2 is a root of $p(x)$. Taking $\alpha = 2$, then $\beta\gamma = \frac{1}{2}$ and the above formulas give

$$\begin{aligned} a_{10} &= 2^{2^9} + \beta^{2^9} + \gamma^{2^9}\,, \\ b_{10} &= \frac{1}{2^{2^9}} + 2^{2^9}\left(\beta^{2^9} + \gamma^{2^9}\right)\,. \end{aligned}$$

It follows that

$$2^{512}a_{10} - b_{10} \;=\; 2^{2^9}a_{10} - b_{10} \;=\; 2^{2^9}\cdot 2^{2^9} - \frac{1}{2^{2^9}} \;=\; 2^{1024} - \frac{1}{2^{512}}\,.$$

That completes this number of the *Corner*. Send me your nice solutions, generalizations, and comments.

BOOK REVIEWS

Amar Sodhi

Origami Tessellations: Awe-Inspiring Geometric Designs
By Eric Gjerde, published by A K Peters Ltd., 2009
ISBN 978-1-56881-451-3, softcover, 121+vi pages, US$24.95

Ornamental Origami: Exploring 3D Geometric Designs
By Meenakshi Mukerji, published by A K Peters Ltd., 2009
ISBN 978-1-56881-445-2, softcover, 145+x pages, US$24.95

Combined review by **Georg Gunther**, *Sir Wilfred Grenfell College (MUN), Corner Brook, NL*

One of the never-ending appeals of mathematics is the way that simple initial ideas very quickly can lead to unexpected emergent concepts of astonishing complexity. Examples are myriad. Think of the natural numbers, marching on endlessly by increments of one, and giving rise to deep and profound questions that lie at the heart of number theory. Consider the evolution of cellular automata, whose complexities arise out of the simplest kinds of rules describing the birth, death, or survival of the individual cells.

Origami, the traditional Japanese art of paper folding, carries with it the same appeal. The starting components are very simple: a square piece of paper, and a number of simple folding rules. The end results are surprising, beautiful, and unexpected, and appeal to both the mathematician, who senses the underlying geometric regularities, and the non-mathematician, who responds to the artistic and aesthetic dimensions of the finished product. Origami is almost a paradox: rich in form and structure, austere in the purity with which it expresses underlying geometric law. In this, origami reminds one of two other forms of traditional Japanese artistic and intellectual expression: the poetic form of *Haiku*, and the game of *Go*.

The two books reviewed here demonstrate again that there is no clear dividing line between mathematics and the visual arts. The study of tessellations is at home as much in the mathematician's den as it is in the artist's studio. Correspondence between the Dutch graphic artist M.C. Escher and the Canadian geometer H.S.M. Coxeter makes it clear that both found inspiration from the other.

The book *Origami Tessellations* is a wonderful example of how the simple rules of origami can be applied to the mathematics of tessellations to create patterns beautiful enough to grace any wall. In an introductory chapter, the author, the paper-folding artist Eric Gjerde, provides clear and explicit instructions on how to perform the various creases that need to be mastered. The instructions are accompanied by a sequence of diagrams, showing each step and so even the most novice paper folder can learn to master techniques such as the rabbit-ear triangle sink, the rhombus twist, and the open-back hexagon twist. The rest of the book describes twenty-five origami tessellation projects. These are presented in three groupings. The first ten are beginner projects; this is followed by nine intermediate and six advanced projects. The designs are all beautiful and show a great deal of variation. For example, No. 11, called *Château-Chinon*, is an octagon-based design, while No. 25, called *Arms of Shiva*, shows a tessellation of stretched pentagons surrounding a central hexagon.

The second book, *Ornamental Origami*, is authored by Meenakshi Mukerji, who was awarded the 2005 Florence Temko Award by OrigamiUSA for her contributions to origami. This book develops and presents techniques for constructing 3-dimensional origami designs in which a number of origami modules are assembled in order to construct a complex 3-dimensional shape. Often the shapes created by modular origami are polyhedral, and so it comes as no surprise that many of the shapes presented in this book are based on either the Platonic or the Archimedean solids.

The book is beautifully organized. There is a brief introduction which provides useful folding tips and summarizes some of the basic facts about the underlying geometric solids. Following this, each chapter gives careful instructions for the construction of a number of models based upon a particular basic design feature. Thus, in Chapter 2, the models have a windmill base, while Chapter 3 builds models out of a Blintz base. This is followed by constructions based upon the icosahedron (Chapter 4), sonobe-type units (Chapter 5), floral balls (Chapter 6), finally concluding with a detailed chapter on planar models.

All constructions are clearly described, with detailed sequences of diagrams illustrating each step. Many of the models are stunning in their finished form, regardless of whether this is one of the floral models such as the lush 30-unit assembly of a zinnia, or the more austere star shapes arising out of the planar models.

Both books are lavishly illustrated and even though the two authors are non-mathematicians, these volumes will appeal to mathematicians for providing, in stunning visual form, so many models arising out of strict geometric laws. As for the many who have at one time or another folded paper to construct a boat, an airplane, or a delicate crane, the allure of these books will be hard to resist. They will feel a twitching in their fingers as they reach for a square piece of paper and start to fold, converting geometric regularities into aesthetically pleasing patterns.

Addendum to the November 2009 review of *Crocheting Adventures with Hyperbolic Planes* by Daina Taimiņa.

This book has won the coveted Diagram Prize for the Oddest Book Title. Details of the award can be found at `http://www.thebookseller.com/blogs//114968-non-euclidian-needlework.html`

PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er novembre 2010**. *Une étoile* ($\star$) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

A number of corrigenda have been pointed out by diligent readers.

In problem **3500** at [2009 : 517, 519] the expression

$$\text{“}\beta = -f(1) + \tfrac{1}{2}f\left(\tfrac{1}{4}\right) - \tfrac{1}{2}f\left(-\tfrac{1}{4}\right)\text{”}$$

should be replaced by

$$\text{“}\beta = -f(1) + \tfrac{1}{4}f\left(\tfrac{1}{2}\right) - \tfrac{1}{4}f\left(-\tfrac{1}{2}\right).\text{”}$$

The due date for solutions to the corrected version is 1st November, 2010.

In problem **3528** at [2009 : 171] the word "circles" should be replaced by "triangles". The French version of this problem is correct, and the due date for solutions to this problem remains the same.

In problem **3532** at [2009 : 172, 174] the "r" on the left of the displayed equation should be replaced by "$\sqrt{r}$". The due date for solutions to the corrected version of this problem remains the same.

3539. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne et Pantelimon George Popescu, Bucarest, Roumanie.*

Soit A et B deux matrices réelles carrées 2×2. Montrer que les équations $\det(xA \pm B) = 0$ ont toutes leurs racines réelles si et seulement si

$$[\text{trace}(AB) - \text{trace}(A)\text{trace}(B)]^2 \;\geq\; 4\det(A)\det(B)\,.$$

3540. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Dans un triangle ABC de demi-périmètre s et de surface F, on inscrit un carré $PQRS$ de côté x, avec P et Q sur BC, R sur AC et S sur AB. De manière analogue, soit y et z les côtés des carrés dont deux sommets sont respectivement sur AC et AB. Montrer que

$$x^{-1} + y^{-1} + z^{-1} \;\leq\; \frac{s(2+\sqrt{3})}{2F}\,.$$

3541. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Dans un triangle ABC, soit O le centre du cercle circonscrit, de rayon R, H son orthocentre, a, b et c les longueurs des côtés, les hauteurs AD, BE et CF, où les points D, E et F sont respectivement sur les côtés BC, AC et AB. La droite d'Euler du triangle ABC coupe BC en P et HC en Q et le quadrilatère $ABPQ$ possède un cercle inscrit. Montrer que $a^2 + b^2 = 6R^2$ et exprimer la longueur de PQ en fonction de a, b et c.

3542★. *Proposé par Cosmin Pohoaţă, Collège National Tudor Vianu, Bucarest, Roumanie.*

Les cercles inscrits mixtilineaires d'un triangle ABC sont les trois cercles chacun étant tangent à deux côtés et intérieurement au cercle inscrit. Soit Γ le cercle tangent intérieurement à ces trois cercles. Montrer que Γ est orthogonal au cercle passant par le centre du cercle inscrit et par les points isodynamiques du triangle ABC.

[*Ed. : Soit* Γ_A *le cercle passant par* A *et par les points d'intersection des bissectrices interne et externe en* A *avec la droite* BC. *Les points isodynamiques sont les deux points communs aux cercles* Γ_A, Γ_B *et* Γ_C.]

3543. *Proposé par Mehmet Şahin, Ankara, Turquie.*

Dans un triangle ABC, soit r le rayon du cercle inscrit, R celui du cercle circonscrit, et $[AD]$, $[BE]$ et $[CF]$ les bissectrices joignant les sommets aux points D, E et F sur les côtés BC, AC et AB respectivement. Soit R' le rayon du cercle circonscrit au triangle DEF. Montrer que

$$R' \leq \frac{R^4}{16r^3}\,.$$

3544. *Proposé par Mehmet Şahin, Ankara, Turquie.*

Soit I_a, I_b et I_c les excentres (les centres des cercles exinscrits) d'un triangle ABC et H_a, H_b et H_c les orthocentres respectifs des triangles I_aBC, I_bCA et I_cAB. Montrer que

$$\text{Aire}(H_aCH_bAH_cB) = 2\text{Aire}(ABC)\,.$$

3545. *Proposé par Michel Bataille, Rouen, France.*

On donne une droite ℓ et les points A et B avec $A \notin \ell$ et $B \in \ell$. Dans le plan qu'ils déterminent, trouver le lieu des points P tels que $PA+QB = PQ$ pour un unique point Q sur ℓ.

3546. *Proposé par Michel Bataille, Rouen, France.*

Soit n un entier positif. Montrer que

$$0 < \sum_{k=0}^{\binom{n}{2}} \frac{(-1)^k}{n+k}\binom{\binom{n}{2}}{k} \leq \frac{1}{n^n}\,.$$

3547. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

On donne un triangle ABC de périmètre 1 et soit r le rayon de son cercle inscrit, R celui de son cercle circonscrit et a, b et c les longueurs de ses côtés. Montrer que

$$\frac{a}{\sqrt{1-a}}+\frac{b}{\sqrt{1-b}}+\frac{c}{\sqrt{1-c}} \;\geq\; \sqrt{\frac{2}{1+4r(r+4R)}}\,.$$

3548. *Proposé par Pham Van Thuan, Université de Science de Hanoï, Hanoï, Vietnam.*

Soit x, y et z trois nombres réels non négatifs. Montrer que

$$\sum_{\text{cyclique}} \sqrt{x^2-xy+y^2} \;\leq\; x+y+z+\sqrt{x^2+y^2+z^2-xy-yz-zx}\,.$$

3549. *Proposé par Pham Kim Hung, étudiant, Université de Stanford, Palo Alto, CA, É-U.*

Soit x, y et z trois nombres réels non négatifs tels que $a+b+c=3$. Montrer que $\left(1+a^2b\right)\left(1+b^2c\right)\left(1+c^2a\right)\leq 5+3abc$.

3550. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Trouver la somme

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m}\left(\ln 2-\sum_{i=1}^{n+m}\frac{1}{n+m+i}\right)\,.$$

. . . .

3539. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania.*

Let A and B be 2×2 square matrices with real entries. Prove that the equations $\det(xA\pm B)=0$ have all of their roots real if and only if

$$[\text{trace}(AB)-\text{trace}(A)\text{trace}(B)]^2 \;\geq\; 4\det(A)\det(B)\,.$$

3540. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Triangle ABC has semiperimeter s and area F. A square $PQRS$ with side length x is inscribed in ABC with P and Q on BC, R on AC, and S on AB. Similarly y and z are the sides of squares two vertices of which lie on AC and AB, respectively. Prove that

$$x^{-1}+y^{-1}+z^{-1} \;\leq\; \frac{s(2+\sqrt{3})}{2F}\,.$$

3541. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Triangle ABC has circumcentre O, circumradius R, orthocentre H, side lengths a, b, c, and altitudes AD, BE, CF, where points D, E, F lie on the sides BC, AC, AB, respectively. The Euler line of triangle ABC intersects BC in P and HC in Q, and the quadrilateral $ABPQ$ has an inscribed circle.

Show that $a^2 + b^2 = 6R^2$, and express the length of PQ in terms of a, b, c.

3542★. *Proposed by Cosmin Pohoaţă, Tudor Vianu National College, Bucharest, Romania.*

The mixtilinear incircles of a triangle ABC are the three circles each tangent to two sides and to the circumcircle internally. Let Γ be the circle tangent to each of these three circles internally. Prove that Γ is orthogonal to the circle passing through the incentre and the isodynamic points of the triangle ABC.

[*Ed.: Let* Γ_A *be the circle passing through* A *and the intersection points of the internal and external angle bisectors at* A *with the line* BC. *The isodynamic points are the two points that* Γ_A, Γ_B, *and* Γ_C *have in common.*]

3543. *Proposed by Mehmet Mehmet Şahin, Ankara, Turkey.*

Triangle ABC has inradius r, circumradius R, and angle bisectors $[AD]$, $[BE]$, $[CF]$, where points D, E, F lie on the sides BC, AC, AB, respectively. Let R' be the circumradius of triangle DEF. Prove that

$$R' \leq \frac{R^4}{16r^3}.$$

3544. *Proposed by Mehmet Şahin, Ankara, Turkey.*

Triangle ABC has excentres I_a, I_b, I_c and H_a, H_b, H_c are the orthocentres of triangles I_aBC, I_bCA, I_cAB, respectively. Prove that

$$\text{Area}(H_aCH_bAH_cB) = 2\text{Area}(ABC).$$

3545. *Proposed by Michel Bataille, Rouen, France.*

Given a line ℓ and points A and B with $A \notin \ell$ and $B \in \ell$, find the locus of points P in their plane such that $PA + QB = PQ$ for a unique point Q of ℓ.

3546. *Proposed by Michel Bataille, Rouen, France.*

Let n be a positive integer. Prove that

$$0 < \sum_{k=0}^{\binom{n}{2}} \frac{(-1)^k}{n+k} \binom{\binom{n}{2}}{k} \leq \frac{1}{n^n}.$$

3547. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Triangle ABC has perimeter equal to 1, inradius r, circumradius R, and side lengths a, b, c. Prove that

$$\frac{a}{\sqrt{1-a}}+\frac{b}{\sqrt{1-b}}+\frac{c}{\sqrt{1-c}} \;\geq\; \sqrt{\frac{2}{1+4r(r+4R)}}\,.$$

3548. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let x, y, and z be nonnegative real numbers. Prove that

$$\sum_{\text{cyclic}}\sqrt{x^2-xy+y^2} \;\leq\; x+y+z+\sqrt{x^2+y^2+z^2-xy-yz-zx}\,.$$

3549. *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c=3$. Prove that $\left(1+a^2b\right)\left(1+b^2c\right)\left(1+c^2a\right)\leq 5+3abc$.

3550. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Find the sum

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m}\left(\ln 2-\sum_{i=1}^{n+m}\frac{1}{n+m+i}\right)\,.$$

A brief word here on the current situation regarding articles in ***CRUX with MAYHEM***.

For various reasons, no articles have appeared in the first four issues of this year, and there has been a backlog of articles for a while now.

One reason is that there is not much space for articles in ***CRUX with MAYHEM*** to begin with. For instance, only nine articles appeared in all of 2008, for a total of 46 out of 512 pages, which is less than 9% of the total page count. Another reason is the quantum nature of the page count, which is either 64 or 96 pages per issue, and producing a 96 page issue (which is naturally richer in articles) requires a larger "energy packet" to achieve.

We will be aiming to clear the backlog in the last four issues of 2010 and early in 2011, and thank our contributors for their patience and their continued interest and enthusiasm for articles in ***CRUX with MAYHEM***.

Václav (Vazz) Linek

SOLUTIONS

Aucun problème n'est immuable. L'éditeur est toujours heureux d'envisager la publication de nouvelles solutions ou de nouvelles perspectives portant sur des problèmes antérieurs.

3440. [2009 : 233, 236] *Proposed by Hidetoshi Fukagawa, Kani, Gifu, Japan.*

There are N coins on a table all of the same size. These N coins can be arranged in a square and they can also be arranged into an equilateral triangle. Find N.

Solution by John Hawkins and David R. Stone, Georgia Southern University, Statesboro, GA, USA.

We are told that the number of coins satisfies $N = s^2$ for some $s \geq 1$, while at the same time it is a triangular number so that $N = \frac{t(t+1)}{2}$ for some $t \geq 1$. After some algebra we find these conditions to be equivalent to the existence of positive integers $x = 2t + 1$ and $y = 2s$ for which

$$x^2 - 2y^2 \;=\; 1\,.$$

We recognize this to be a Pell equation; since the time of Brahmagupta in the seventh century, it has been known that if such an equation has any solution, then it has infinitely many solutions. [*Ed.: The solution to this Pell equation was obtained 1100 years before Brahmagupta by the Pythagoreans in Greece and independently around that time in India.*] According to the theory, the pairs (x, y) that satisfy the equation can be calculated recursively, based upon the initial solution $(x_1, y_1) = (3, 2)$ and the two recursive equations $x_{k+1} = 3x_k + 4y_k$, $y_{k+1} = 2x_k + 3y_k$ for $k \geq 1$. Therefore, for our problem, the pairs (s, t) can also be calculated recursively:

$$\begin{aligned} s_{k+1} &= 2t_k + 3s_k + 1\,, \\ t_{k+1} &= 3t_k + 4s_k + 1\,. \end{aligned}$$

We list the first few solutions of the Pell equation, also giving s, t, and N.

x	y	$s = \frac{y}{2}$	$t = \frac{x-1}{2}$	$N = s^2 = \frac{t(t+1)}{2}$
3	2	1	1	1
17	12	6	8	36
99	70	35	49	1225
577	408	204	288	41616
3363	2378	1189	1681	1413721
19601	13860	6930	9800	48024900

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUNEDOARA PROBLEM SOLVING GROUP, Hunedoara, Romania; PETER HURTHIG, Columbia College, Vancouver, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; GEORGES MELKI, Fanar, Lebanon; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; DANIEL REISZ, Auxerre, France; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

The proposer found the problem in a small manuscript with the title Fukyu Sanpou*, or* Masterpiece of Mathematics*, written by Ajima Naonobu (1732-1798) and edited by one of his students in 1799. At that same time in Europe (and independently, because Japan was then in the midst of its long period of isolation) Euler answered this question and more in a 1778 paper. There is now a vast literature on these* square triangular numbers*; the two web pages listed below contain further formulas and references. For example, the formula for the n^{th} square triangle number is*

$$N_n = \left(\frac{(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}}{4\sqrt{2}} \right)^2 .$$

Almost all submissions assumed the theory of Pell equations to be well known. Bataille, however, used the recursive formula for N_n that is established in [3]:

$$N_{n+1} = \left(6\sqrt{N_n} - \sqrt{N_{n-1}} \right)^2 .$$

Also, Hurthig's solution displayed noteworthy ingenuity; obtaining the solution by manipulating diagrams.

Schlosberg addressed the question of what quantity of coins could, in fact, be arranged to fit on a table. The smallest North American coin has a diameter of about 1.8 cm (the US dime measures 1.791 cm across while the Canadian dime measures 1.803 cm). An equilateral triangle consisting of 1225 dimes, 49 along a side, would fit on a table 88 cm $\times$ 76 cm, which is a reasonable size for a table, but who could afford that many dimes? If Scrooge McDuck, the world's richest duck, wanted to arrange a square of 41616 dimes (with 204 per side), he would need a table whose width is about 3.7 m. This computation suggests that the practical answer to the question is that N would have to be 1, 36, or 1225. Konečný went a step further and sent us a picture of a Christmas tree whose trunk consists of a square of 36 pennies, topped by an equilateral triangle of 36 pennies; we decided that it would be rushing the Christmas season a bit to reproduce his picture in our May issue.

References

[1] http://mathworld.wolfram.com/SquareTriangularNumber.html

[2] http://en.wikipedia.org/wiki/Square_triangular_number

[3] D. Keedwell, Square-triangular numbers. *Math. Gazette* **84** (July 2000), 292-294.

3441★. [2009 : 233, 236] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $ABCD$ be a convex quadrilateral and let P be a point in the interior of $ABCD$ such that $PA = \frac{AB}{\sqrt{2}}$, $PB = \frac{BC}{\sqrt{2}}$, $PC = \frac{CD}{\sqrt{2}}$, and $PD = \frac{DA}{\sqrt{2}}$. Prove or disprove that $ABCD$ is a square.

Solution by Missouri State University Problem Solving Group, Springfield, MO, USA and Jan Verster, Kwantlen University College, BC.

We shall show that $ABCD$ need not be a square. For a counterexample define P to be the midpoint of a segment AC of length 2, and let B be any point of the circle with centre A and radius $\sqrt{2}$ that is not on the line AC. The median from B in triangle ABC satisfies

$$4PB^2 \;=\; 2AB^2 + 2BC^2 - AC^2 \;=\; 4 + 2BC^2 - 4 \;=\; 2BC^2\,.$$

Thus, we already have both $PA = \dfrac{AB}{\sqrt{2}}$ and $PB = \dfrac{BC}{\sqrt{2}}$. Similarly, if D is a point on the circle with centre C and radius $\sqrt{2}$, we have $PC = \dfrac{CD}{\sqrt{2}}$ and $PD = \dfrac{DA}{\sqrt{2}}$. To satisfy the condition that $ABCD$ be convex, we must restrict D to that portion of its circle in the interior of $\angle ABC$ and in the exterior of $\triangle ABC$. For a specific example, choose D to lie on the line BP; then, since P is the midpoint of both AC and BD, $ABCD$ is a parallelogram and, therefore, convex. It will not be a square for any B that avoids the perpendicular bisector of AC.

Also solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; HUNEDOARA PROBLEM SOLVING GROUP, Hunedoara, Romania; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

3442. [2009 : 234, 236] *Proposed by Iyoung Michelle Jung, student, Hanyoung Foreign Language High School, Seoul, South Korea and Sung Soo Kim, Hanyang University, Seoul, South Korea.*

Let C be a right circular cone and let D be a disk of fixed radius lying within the base of the cone C. Prove that if A is the area of that part of the cone lying directly above D, then A is independent of the position of the disk D.

Solution by Albert Stadler, Herrliberg, Switzerland.

Without loss of generality we can assume that the base of the cone is the unit circle and that the equation of the cone is

$$z \;=\; f(u,v) \;=\; a\left(1 - \sqrt{u^2+v^2}\right)\,.$$

Then

$$\begin{aligned} f_u &= \frac{-au}{\sqrt{u^2+v^2}}\,, \\ f_v &= \frac{-av}{\sqrt{u^2+v^2}}\,, \end{aligned}$$

and the area of that part of the cone lying above a region D in plane and within the unit circle is

$$\begin{aligned} A &= \iint_D \sqrt{1+(f_u)^2+(f_v)^2}\,du\,dv \\ &= \iint_D \sqrt{1+\frac{a^2u^2}{u^2+v^2}+\frac{a^2v^2}{u^2+v^2}}\,du\,dv \\ &= \iint_D \sqrt{1+a^2}\,du\,dv \;=\; \text{Area}(D)\sqrt{a^2+1}\,, \end{aligned}$$

which yields the desired conclusion.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; and the proposers.

3443. [2009 : 234, 236] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a, b, and c be positive real numbers such that $a+b+c=3$. Prove that

$$\sum_{\text{cyclic}} \frac{a^2(b+1)}{a+b+ab} \;\geq\; 2\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

We have

$$\begin{aligned} \sum_{\text{cyclic}} \frac{a^2(b+1)}{a+b+ab} &= \sum_{\text{cyclic}} \left(\frac{a^2(b+1)}{a+b+ab} - a + 1\right) \\ &= \sum_{\text{cyclic}} \frac{a+b}{a+b+ab} \;\geq\; \sum_{\text{cyclic}} \frac{a+b}{a+b+\dfrac{(a+b)^2}{4}} \\ &= \sum_{\text{cyclic}} \frac{4}{4+a+b} \;=\; \frac{4}{18}\cdot\sum_{\text{cyclic}}(4+a+b)\sum_{\text{cyclic}}\frac{1}{4+a+b} \\ &\geq\; \frac{4}{18}\cdot 9 \;=\; 2\,, \end{aligned}$$

where we used the fact that $(x+y+z)(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}) \geq 9$ for positive real numbers x, y, z and that $\sum\limits_{\text{cyclic}}(4+a+b) = 18$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; HUNEDOARA PROBLEM SOLVING GROUP, Hunedoara, *Romania; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; DUNG NGUYEN MANH, Student, Hanoi University of Technology, Hanoi, Vietnam; DRAGOLJUB MILOSEVIĆ, Gornji Milanovac, Serbia; ALBERT STADLER, Herrliberg, Switzerland; PANOS E. TSAOUSSOGLOU, Athens, Greece; STAN WAGON, Macalester College, St. Paul, MN, USA; TITU ZVONARU, Comănești, Romania; and the proposer.*

Michel Bataille, Rouen, France, pointed out that this problem appeared as Problem No. 322 in Math. Excalibur, Vol. 14, No. 1, by the same proposer, with a solution appearing in Vol. 14, No. 2. The solution presented here is different from that one.

3444. [2009 : 234, 236] *Proposed by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Let a, b, and c be positive real numbers such that $a+b+c=1$. Prove that

$$\sum_{\text{cyclic}} \frac{ab}{3a^2+2b+3} \;\le\; \frac{1}{12}\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

The function $f(x) = \dfrac{x(1-x)}{3x+2}$ is concave for $0 \le x \le 1$, because its second derivative, $f''(x) = -\dfrac{20}{(3x+2)^2}$, is negative in this range. Hence, by Jensen's inequality,

$$f(a)+f(b)+f(c) \;\le\; 3f\left(\frac{1}{3}\right) \;=\; \frac{2}{9}\,.$$

We have

$$\begin{aligned}
\sum_{\text{cyclic}} \frac{ab}{3a^2+2b+3} &= 3\sum_{\text{cyclic}} \frac{ab}{(3a-1)^2+6a+(6b+8)} \\
&\le 3\sum_{\text{cyclic}} \frac{ab}{6a+(6b+8)} \\
&= \frac{3}{2}\sum_{\text{cyclic}} \frac{ab}{(3a+2)+(3b+2)} \\
&\le \frac{3}{2}\sum_{\text{cyclic}} \frac{1}{4}\left(\frac{ab}{3a+2}+\frac{ab}{3b+2}\right) \\
&= \frac{3}{8}\sum_{\text{cyclic}} \frac{a(b+c)}{3a+2} \\
&= \frac{3}{8}\sum_{\text{cyclic}} f(a) \;\le\; \frac{3}{8}\cdot\frac{2}{9} \;=\; \frac{1}{12}\,.
\end{aligned}$$

The proof is complete.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; JOE HOWARD, Portales, NM, USA; HUNEDOARA PROBLEM SOLVING GROUP, Hunedoara, Romania; PETER HURTHIG, Columbia College, Vancouver, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; DUNG NGUYEN MANH, Student, Hanoi University of Technology, Hanoi, Vietnam; ALBERT STADLER, Herrliberg, Switzerland; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Stan Wagon, Macalester College, St. Paul, MN, USA, used Mathematica *to determine that the inequality is true and that equality holds for $a = b = c = 1/3$.*

3445. [2009 : 234, 236] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina, in memory of Murray S. Klamkin.*

Let a, b, and c be nonnegative real numbers such that $ab+bc+ac = 1$. Prove that

$$\text{(a)} \sum_{\text{cyclic}} \frac{a}{1+bc} \ \geq\ \frac{3\sqrt{3}}{4}\,; \qquad \text{(b)} \sum_{\text{cyclic}} \frac{a^2}{1+a} \ \geq\ \frac{\sqrt{3}}{\sqrt{3}+1}\,.$$

Solution by Peter Hurthig, Columbia College, Vancouver, BC.

(a) By the AM–GM Inequality,

$$ab + bc + ca + bc \ \geq\ 4\sqrt[4]{a^2b^3c^3}$$

and

$$2a + b + c \ \geq\ 4\sqrt[4]{a^2bc}\,.$$

Using these inequalities, we have

$$\begin{aligned}\frac{a}{1+bc} &= a - \frac{abc}{1+bc} = a - \frac{abc}{ab+bc+ca+bc} \\ &\geq a - \frac{abc}{4\sqrt[4]{a^2b^3c^3}} = a - \frac{\sqrt[4]{a^2bc}}{4} \\ &\geq a - \frac{2a+b+c}{16} = \frac{7}{8}a - \frac{1}{16}b - \frac{1}{16}c\,.\end{aligned}$$

Similarly,

$$\frac{b}{1+ca} \ \geq\ \frac{7}{8}b - \frac{1}{16}c - \frac{1}{16}a$$

and

$$\frac{c}{1+ab} \ \geq\ \frac{7}{8}c - \frac{1}{16}a - \frac{1}{16}b\,.$$

Using the well-known and easy to prove inequality

$$(a+b+c)^2 \ \geq\ 3(ab+bc+ca)$$

and the condition $ab + bc + ca = 1$, we obtain $a + b + c \geq \sqrt{3}$, and then

$$\sum_{\text{cyclic}} \frac{a}{1+bc} \;\geq\; \frac{3}{4}(a+b+c) \;\geq\; \frac{3\sqrt{3}}{4}\,,$$

as claimed.

(b) By the AM–HM Inequality,

$$\sum_{\text{cyclic}} \frac{1}{a+1} \;\geq\; \frac{9}{a+b+c+3}\,,$$

so that

$$\begin{aligned}\sum_{\text{cyclic}} \frac{a^2}{1+a} &= \sum_{\text{cyclic}} \left(a - 1 + \frac{1}{1+a}\right) \\ &= a+b+c-3+\sum_{\text{cyclic}} \frac{1}{1+a} \\ &\geq a+b+c-3+\frac{9}{a+b+c-3}\,.\end{aligned}$$

We have shown in part (a) that $a + b + c \geq \sqrt{3}$; also, it is easy to check that the function $f(x) = x - 3 + \dfrac{9}{x+3}$ is increasing on the interval $[\sqrt{3}, \infty)$. Hence,

$$\begin{aligned}\sum_{\text{cyclic}} \frac{a^2}{1+a} &\geq a+b+c-3+\frac{9}{a+b+c-3} \\ &\geq \sqrt{3}-3+\frac{9}{\sqrt{3}+3} \\ &= \frac{\sqrt{3}}{\sqrt{3}+1}\,,\end{aligned}$$

which completes the proof.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; HUNEDOARA PROBLEM SOLVING GROUP, Hunedoara, Romania; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece (part (a) only); DUNG NGUYEN MANH, Student, Hanoi University of Technology, Hanoi, Vietnam; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; ALBERT STADLER, Herrliberg, Switzerland; PANOS E. TSAOUSSOGLOU, Athens, Greece; STAN WAGON, Macalester College, St. Paul, MN, USA (part (b) only); PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3446. [2009 : 234, 237] *Proposed by Mihály Bencze, Brasov, Romania.*

For any positive integer n prove that

$$\left\lfloor \sqrt{n^2-n+1} + \sqrt{n^2+n+1} \right\rfloor + \left\lfloor \sqrt{n^2+n} + \sqrt{n^2+3n+2} \right\rfloor$$
$$= \left\lfloor \sqrt{4n^2+3} \right\rfloor + \left\lfloor \sqrt{4n^2+8n+3} \right\rfloor ,$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.

Solution by Michel Bataille, Rouen, France.

We will show the following two chains of inequalities:

$$2n < \sqrt{4n^2+3} < \sqrt{n^2-n+1} + \sqrt{n^2+n+1} < 2n+1 \qquad (1)$$
$$2n+1 < \sqrt{n^2+n} + \sqrt{n^2+3n+2} < \sqrt{4n^2+8n+3} < 2n+2 \qquad (2)$$

Then from (1),

$$\left\lfloor \sqrt{4n^2+3} \right\rfloor = \left\lfloor \sqrt{n^2-n+1} + \sqrt{n^2+n+1} \right\rfloor = 2n ,$$

and from (2),

$$\left\lfloor \sqrt{n^2+n} + \sqrt{n^2+3n+2} \right\rfloor = \left\lfloor \sqrt{4n^2+8n+3} \right\rfloor = 2n+1 ,$$

so that both sides of the required equality equal $4n+1$.

To prove (1) we first observe that

$$2n = \sqrt{4n^2} < \sqrt{4n^2+3}$$

and

$$\sqrt{n^2-n+1} + \sqrt{n^2+n+1} < \sqrt{n^2} + \sqrt{n^2+2n+1} = 2n+1 .$$

By squaring, the middle inequality of (1) becomes equivalent to

$$2n^2+1 < 2\sqrt{n^2-n+1}\sqrt{n^2+n+1} ,$$

which holds since, squaring again, it becomes equivalent to

$$4n^4+4n^2+1 < 4n^4+4n^2+4 .$$

Now to prove (2), we first observe that

$$2n+1 = \sqrt{n^2} + \sqrt{n^2+2n+1} < \sqrt{n^2+n} + \sqrt{n^2+3n+2}$$

and

$$\sqrt{4n^2+8n+3} < \sqrt{4n^2+8n+4} = 2n+2 .$$

By squaring, the middle inequality of (2) becomes equivalent to

$$2\sqrt{n^2+n}\sqrt{n^2+3n+2} < 2n^2+4n+1 ,$$

which holds since, squaring again, it becomes equivalent to

$$4n^4 + 16n^3 + 20n^2 + 8n \; < \; 4n^4 + 16n^3 + 20n^2 + 8n + 1\,.$$

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. There was one incomplete solution submitted.

3447. [2009 : 234, 237] *Proposed by Mihály Bencze, Brasov, Romania.*

Let n be a positive integer. Prove that

$$\frac{2}{n!(n+2)!} \; < \; \prod_{k=1}^{n}\left(\sqrt[k+1]{\frac{k+1}{k}} - 1\right) \; < \; \frac{1}{(n+1)(n!)^2}\,.$$

Solution by Hunedoara Problem Solving Group, Hunedoara, Romania.

Let $H_k = \sqrt[k+1]{\dfrac{k+1}{k}} - 1$. By the AM–GM Inequality, we have

$$H_k \; = \; \sqrt[k+1]{\frac{k+1}{k}\cdot 1^k} - 1 \; < \; \frac{\left(\dfrac{k+1}{k} + k\right)}{k+1} - 1 \; = \; \frac{1}{k(k+1)}\,.$$

Hence, $\displaystyle\prod_{k=1}^{n} H_k < \prod_{k=1}^{n}\frac{1}{k(k+1)} = \frac{1}{n!(n+1)!} = \frac{1}{(n+1)(n!)^2}$.

On the other hand, we have, by the GM–HM Inequality,

$$\begin{aligned} H_k \; &= \; \sqrt[k+1]{\frac{k+1}{k}\cdot 1^k} \; > \; \frac{k+1}{\left(\dfrac{k}{k+1} + k\right)} - 1 \\ &= \; \frac{(k+1)^2}{k^2+2k} - 1 \; = \; \frac{1}{k(k+2)}\,. \end{aligned}$$

Hence, $\displaystyle\prod_{k=1}^{n} H_k > \prod_{k=1}^{n}\frac{1}{k(k+2)} = \frac{2}{n!(n+2)!}$.

This completes the proof.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ARKADY ALT, San Jose, CA, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3448. [2009 : 235, 237] *Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let F_n be the n^{th} Fibonacci number, that is, $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that

$$a^2F_n + b^2F_{n+1} + c^2F_{n+2} \ \geq \ 4S\left(\sum_{k=1}^{n+2} F_k^2 - F_{n+1}^2\right)^{1/2}$$

holds for any triangle ABC, where a, b, c, and S are the side lengths and area of the triangle, respectively.

Similar solutions by Thanos Magkos, 3rd High School of Kozani, Kozani, Greece and Dung Nguyen Manh, Student, Hanoi University of Technology, Hanoi, Vietnam.

We make use of an inequality of Oppenheim. Namely, if x, y, z are positive real numbers and ABC is a triangle with side lengths a, b, c and area S, then

$$xa^2 + yb^2 + zc^2 \ \geq \ 4S\sqrt{xy + yz + zx}\,.$$

If we set $x = F_n$, $y = F_{n+1}$, $z = F_{n+2}$, then we obtain

$$a^2F_n + b^2F_{n+1} + c^2F_{n+2} \ \geq \ 4S\sqrt{F_nF_{n+1} + F_{n+1}F_{n+2} + F_{n+2}F_n}\,.$$

We complete the proof by showing that

$$F_nF_{n+1} + F_{n+1}F_{n+2} + F_{n+2}F_n \ = \ \left(\sum_{k=1}^{n+2} F_k^2\right) - F_{n+1}^2\,.$$

We have

$$\begin{aligned}
&F_nF_{n+1} + F_{n+1}F_{n+2} + F_{n+2}F_n + F_{n+1}^2 \\
&\qquad = \ F_{n+1}(F_n + F_{n+1} + F_{n+2}) + F_nF_{n+2} \\
&\qquad = \ 2F_{n+1}F_{n+2} + F_nF_{n+2} \ = \ F_{n+2}(F_n + 2F_{n+1}) \\
&\qquad = \ F_{n+1}(F_n + F_{n+1} + F_{n+1}) \ = \ F_{n+2}(F_{n+2} + F_{n+1}) \\
&\qquad = \ F_{n+2}^2 + F_{n+1}F_{n+2}\,.
\end{aligned}$$

Now, it remains to prove that $\sum_{k=1}^{n+1} F_k^2 = F_{n+1}F_{n+2}$, which is easily verified by induction.

Also solved by MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

Janous located this particular inequality and more in an online paper by the proposer at `http://rgmia.org/papers/v7n2/Triangle.pdf`.

3449. [2009 : 235, 237] *Proposed by an anonymous proposer.*

Let $ABCD$ be a unit square, M the midpoint of AB, and N the midpoint of CD. Is there a point P on MN such that the lengths of AP and PC are both rational numbers?

Solution by the proposer.

The answer is negative. We first establish a lemma in which $\mathbb{Q}$ denotes the set of rational numbers.

Lemma Let α, $\beta \in \mathbb{Q}$ be such that $2 - 2\alpha + \beta = 0$ and $\beta \neq 2$. Then $\alpha^2 - 2\beta = \gamma^2$ for some $\gamma \in \mathbb{Q} - \{0\}$.

Proof: Let $\gamma = 1 - \frac{\beta}{2}$. Then $\gamma \neq 0$. From $\alpha = 1 + \frac{\beta}{2}$ we obtain

$$\alpha^2 - 2\beta = \left(1 + \frac{\beta}{2}\right)^2 - 2\beta = \left(1 - \frac{\beta}{2}\right)^2 = \gamma^2 . \qquad \blacksquare$$

Now suppose P is a point on MN such that $a = AP$ and $b = PC$ are positive rational numbers. Let $x = MP$. Then $x \in [0, 1]$, and $NP = 1 - x$. If $a = b$, then $x = 1 - x$ or $x = \frac{1}{2}$, which implies that $a = b = \frac{\sqrt{2}}{2}$, a contradiction. Hence, $a \neq b$. Note that

$$a^2 = x^2 + \frac{1}{4}, \tag{1}$$

$$b^2 = (1 - x)^2 + \frac{1}{4} = x^2 - 2x + \frac{5}{4}. \tag{2}$$

From (1) and (2) we obtain $a^2 - b^2 + 1 = 2x$.

Hence, $4a^2 = (2x)^2 + 1 = (a^2 - b^2 + 1)^2 + 1$, which is equivalent to the equation $2 - 2(a^2 + b^2) + (a^2 - b^2)^2 = 0$. Let $a^2 + b^2 = \alpha$ and $(a^2 - b^2)^2 = \beta$. Then α, $\beta \in \mathbb{Q}$ and $2 - 2\alpha + \beta = 0$. If $\beta = 2$, then we have $a^2 - b^2 = \pm\sqrt{2}$, a contradiction, and thus $\beta \neq 2$. Using the Lemma, we obtain $\alpha^2 - 2\beta^2 = \gamma^2$ for some $\gamma \in \mathbb{Q} - \{0\}$.

That is, $(a^2 + b^2)^2 - 2(a^2 - b^2)^2 = \gamma^2$, or $6a^2b^2 - a^4 - b^4 = \gamma^2$.

By straightforward computations we find that

$$\begin{aligned}(a^2 + b^2)^4 - \gamma^4 &= (a^2 + b^2)^4 - (6a^2b^2 - a^4 - b^4)^2 \\ &= a^8 + 4a^6b^2 + 6a^4b^4 + 4a^2b^6 + b^8 \\ &\quad - 36a^4b^4 - a^8 - b^8 + 12a^6b^2 + 12a^2b^6 - 2a^4b^4 \\ &= 16a^6b^2 - 32a^4b^4 + 16a^2b^6 \\ &= 16a^2b^2(a^4 - 2a^2b^2 + b^4) = \left[4ab(a^2 - b^2)\right]^2 .\end{aligned}$$

That is, a^2+b^2, γ, and $4ab(a^2-b^2)$ are nonzero rational numbers which satisfy the Diophantine equation $X^4 - Y^4 = Z^2$. It then follows easily that this equation has nonzero integer solutions.

This contradicts the known results of Fermat, and our proof is complete.

It is well known that the equation $X^4 + Y^4 = Z^2$ has no nonzero integer solutions. The proof of this result by Fermat is based on his method of infinite descent. Using exactly the same argument, it can be shown that the equation $X^4 - Y^4 = Z^2$ has no nonzero integer solutions. For example, see Theorem 13.3 on pages 520-522 and Exercise No. 4 on p. 525 of the book Elementary Number Theory and its Applications, *5th edition, by Kenneth Rosen.*

3450. [2009 : 235, 237] *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let $\triangle ABC$ have inradius r, exradii r_a, r_b, r_c, and altitudes h_a, h_b, h_c. Prove that

$$\frac{h_a + 2r_a}{r + r_a} + \frac{h_b + 2r_b}{r + r_b} + \frac{h_c + 2r_c}{r + r_c} \geq \frac{27}{4}.$$

Solution by Arkady Alt, San Jose, CA, USA; Dung Nguyen Manh, Student, Hanoi University of Technology, Hanoi, Vietnam; Thanos Magkos, 3rd High School of Kozani, Kozani, Greece; and Panos E. Tsaoussoglou, Athens, Greece, independently.

Let a, b, c be the sides, A the area, and s the semiperimeter of the triangle ABC. We have

$$\begin{aligned}
\sum_{\text{cyclic}} \frac{h_a + 2r_a}{r + r_a} &= \sum_{\text{cyclic}} \frac{\left(\frac{2A}{a} + \frac{2A}{s-a}\right)}{\left(\frac{A}{s} + \frac{A}{s-a}\right)} \\
&= \sum_{\text{cyclic}} \frac{2s^2}{a(2s-a)} = \sum_{\text{cyclic}} \frac{(a+b+c)^2}{2a(b+c)}.
\end{aligned}$$

Using the well-known and easy to prove inequality

$$(a+b+c)^2 \geq 3(ab+bc+ca)$$

and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\sum_{\text{cyclic}} \frac{(a+b+c)^2}{2a(b+c)} &\geq \sum_{\text{cyclic}} \frac{3(ab+bc+ca)}{2a(b+c)} \\
&= \frac{3}{2}(ab+bc+ca) \sum_{\text{cyclic}} \frac{1}{a(b+c)} \\
&= \frac{3}{4}\left(\sum_{\text{cyclic}} a(b+c)\right)\left(\sum_{\text{cyclic}} \frac{1}{a(b+c)}\right) \\
&\geq \frac{3}{4}(1+1+1)^2 = \frac{27}{4},
\end{aligned}$$

as claimed.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; HUNEDOARA PROBLEM SOLVING GROUP, Hunedoara, Romania; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; WEI-DONG, Weihai Vocational College, Weihai, Shandong Province, China; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.