Fall 2001

Ph.D. Qualifying Examination

Algebra

Part I

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. Determine the number of isomorphism classes of groups of order $1705 = 5 \cdot 11 \cdot 31$.

2. If A is a commutative Noetherian ring with 1 prove that $(0) = \mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_k$ for some finite collection of (not necessarily distinct) prime ideals $\mathfrak{p}_i \subset A$.

[Hint: Consider the set of all ideals of A which do not contain a finite product of prime ideals.]

3. Determine the number of similarity classes of matrices over $\mathbb{C}$ and which have characteristic polynomial $(X^4 - 1)(X^8 - 1)$. Do the same thing over $\mathbb{Q}$.

4. Let F be a field. Consider the polynomial $f(X) = X^4 - a$ where $a \in F$. Determine (with explanation) all possible Galois groups of $f(X)$ as the field F and the element $a \in F$ vary. Give an example for every possible Galois group.

5. Let G be a finite group and let $z \in G$. Suppose for every irreducible complex character χ of G we have $|\chi(z)| = |\chi(1)|$. Prove that z is in the center of G.

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Part II

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. If A is a finite abelian group and m is a positive integer show that every automorphism of the subgroup mA of A can be extended to an automorphism of A.

[Hint: Reduce to the case where m is prime. Then use the structure theorem for finite abelian groups.]

2. Let k be a field and let A and B be k-algebras with unit having centers $Z(A)$ and $Z(B)$. Prove that the center of the k-algebra $A \otimes_k B$ is $Z(A) \otimes_k Z(B)$.

[Hint: First express $z \in A \otimes B$ as $\sum_{i=1}^{n} a_i \otimes b_i$ where a_i are linearly independent over k. Show all $b_i \in Z(B)$.]

3. Suppose V is a finite dimensional vector space over a field k and $T : V \to V$ is a linear tranformation. Let $\wedge^2 T : \wedge^2 V \to \wedge^2 V$ be the induced endomorphism of the second exterior power of V. Explain why the characteristic polynomial of $\wedge^2 T$ depends only on the characteristic polynomial of T, and express the characteristic polynomial of $\wedge^2 T$ in terms of the eigenvalues of T.

4. Let G be the group of matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

in $GL(3, \mathbb{F}_3)$. Find the conjugacy classes in G and compute it character table.

5. Suppose F is a field and K and E are finite extensions of F in some algebraic closure of F. Suppose that E is Galois over F (normal and separable). Show that $L = KE$ is Galois over K with $[L : K] = [E : E \cap K]$.