Real Analysis Exam: Part I (Spring 2002)

Do all five problems.

1. Let $f : \mathbf{R} \to \mathbf{R}$ be any function. Let E be the set of points x such that the limits

$$f'_+(x) = \lim_{h>0, h\to 0} \frac{f(x+h)-f(x)}{h} \quad , \quad f'_-(x) = \lim_{h<0, h\to 0} \frac{f(x+h)-f(x)}{h}$$

exist and are not equal. That is, $E = \{x : f'_+(x),\ f'_-(x) \text{ exist and } f'_+(x) \neq f'_-(x)\}$. Prove that E is countable.

Hint: for any fixed λ, consider the x for which $f'_+(x) > \lambda > f'_-(x)$.

2. Consider functions $f_n, g_n \in \mathcal{L}^2[0,1]$ such that $f_n \to f$ and $g_n \to g$ weakly in $\mathcal{L}^2$.
   (a) Show that the $\mathcal{L}^2$ norms of the f_n are uniformly bounded.
   (b) Show by example that $f_n g_n$ need not converge to fg in the weak star topology of $\mathcal{L}^1$ (where $\mathcal{L}^1$ is regarded as part of the dual space of $C[0,1]$.)
   (c) Suppose h_n ($n = 1, 2, \dots$) and h are in $\mathcal{L}^1[0,1]$, and that the $\mathcal{L}^1$ norms of the h_n are uniformly bounded. Show that $h_n \to h$ in the weak star topology of $\mathcal{L}^1$ if and only if each Fourier coefficient of h_n converges to the corresponding Fourier coefficient of h.
   (d) Suppose that f_n and g_n each have Fourier series of the form $\sum_{k\geq 0} c_k e^{2\pi i k x}$. Prove that $f_n g_n \to fg$ in the weak star topology of $\mathcal{L}^1$.

3. Consider a C^∞ function $f : \mathbf{R} \to \mathbf{R}$ with the following property: for every x, $f^{(k)}(x) = 0$ for some k. Let $U = \{x :\ f \text{ is equal to a polynomial in some neighborhood of } x\}$.
   (a) Prove that U is a dense open set.
   (b) Prove that the complement of U contains no isolated points.

Remark: one can prove that U must be all of $\mathbf{R}$, i.e., that f must be a polynomial.

4. Prove the inequality

$$\sum_m \left| \sum_{n\neq m} \frac{a_n}{n-m} \right|^2 \leq \pi^2 \sum_n |a_n|^2$$

for all sequences $a_n \in \ell^2(\mathbf{Z})$. Also, show that π^2 cannot be replaced by any smaller constant.

Hint: First show that $x - \frac{1}{2}$ has Fourier series $-\sum_{n\neq 0} \dfrac{e^{2\pi i n x}}{2\pi i n}$.

5. Let $K(x,y)$ be an $\mathcal{L}^1$ function on the unit square $[0,1]\times[0,1]$. Suppose for every continuous function f on $[0,1]$, we have

$$\int_x K(x,y) f(y)\, dy = 0 \text{ for almost every } x.$$

Prove that $K = 0$ almost everywhere.

Real Analysis Exam: Part II (Spring 2002)

Do all five problems.

1. Show that there do not exist measurable sets A and B in $\mathbf{R}$, each of positive measure, such that $A \cap (B - r) = \emptyset$ for all rational r.

2. Let $f_n : [0,1] \to \mathbf{R}$ be a sequence of Lebesgue measurable functions. Let E be the set of x such that $\sum_n f_n(x)$ converges. Show that for every $\epsilon > 0$, there is a set F and a $k < \infty$ such that

   (i) F is in the ring of sets generated by sets of the form $f_i^{-1}(A)$, where $i \le k$ and A is a Borel set, and

   (ii) $m(E \,\Delta\, F) < \epsilon$.

3. Let $0 < C < 1$. Show that there are numbers δ_N (depending on C) with the following properties:

   (i) If A_k $(1 \le k \le N)$ are measurable sets in $[0,1]$ each with measure C, then

$$m(A_i \cap A_j) \ge (1 - \delta_N) C^2$$

   for some pair i, j with $i \ne j$.

   (ii) $\delta_N \to 0$ as $N \to \infty$.

Hint: Let f_k be the characteristic function of A_k, let $F = f_1 + \cdots + f_k$, and consider F^2.

4. Suppose the Banach space X is uniformly convex. That is, suppose for every $\epsilon > 0$, there is an $\delta > 0$ with the following property:

$$\text{If } \|x\| = \|y\| = 1 \text{ and } \|x - y\| > \delta, \text{ then } \|(x + y)/2\| < 1 - \epsilon.$$

Let f be a linear functional on X with norm 1. Prove that there is a unique point $x \in X$ such that $\|x\| = 1$ and $f(x) = 1$.

5.

   (a) Let μ be a finite measure on $\mathbf{R}$, and let ν be the measure given by $d\nu(x) = e^{-x^2} d\mu(x)$. Show that the Fourier transform of ν is the restriction to $\mathbf{R}$ of an entire holomorphic function F.

   (b) Express the nth derivative $F^n(0)$ of F at 0 in terms of μ.

   (c) Show that the set

$$S = \{p(x) \exp(-x^2) : p \text{ a polynomial}\}$$

   is dense in $\mathcal{C}_0(\mathbf{R})$, the space of continuous functions $f : \mathbf{R} \to \mathbf{R}$ such that $\lim_{x \to \infty} f(x) = 0$ (with the sup norm).