# 20-th All-Russian Mathematical Olympiad 1994

## Final Round – Tver, April 19–25

### Grade 9

*First Day*

1. Prove that if $\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)=1$, then $x+y=0$. *(A. Galochkin)*

2. Two circles $S_1$ and $S_2$ touch externally at $F$. their external common tangent touches $S_1$ at $A$ and $S_2$ at $B$. A line, parallel to $AB$ and tangent to $S_2$ at $C$, intersects $S_1$ at $D$ and $E$. Prove that points $A, F, C$ are collinear. *(A. Kalinin)*

3. There are three piles of matches on the table: one with 100 matches, one with 200, and one with 300. Two players play the following game. They play alternatively, and a player on turn takes one of the piles and divides each of the remaining piles into two nonempty piles. The player who cannot make a legal move loses. Who has a winning strategy? *(K. Kokhas')*

4. On a line are given $n$ blue and $n$ red points. Prove that the sum of distances between pairs of points of the same color does not exceed the sum of distances between pairs of points of different colors. *(O. Musin)*

*Second Day*

5. Prove the equality
$$\frac{a_1}{a_2(a_1+a_2)}+\frac{a_2}{a_3(a_2+a_3)}+\cdots+\frac{a_n}{a_1(a_n+a_1)}$$
$$=\frac{a_2}{a_1(a_1+a_2)}+\frac{a_3}{a_2(a_2+a_3)}+\cdots+\frac{a_1}{a_n(a_n+a_1)}.$$
*(R. Zhenodarov)*

6. Cards numbered with numbers 1 to 1000 are to be placed on the cells of a $1\times 1994$ rectangular board one by one, according to the following rule: If the cell next to the cell containing the card $n$ is free, then the card $n+1$ must be put on it. Prove that the number of possible arrangements is not more than half a million. *(D. Karpov)*

7. A trapezoid $ABCD$ ($AB \parallel CD$) has the property that there are points $P$ and $Q$ on sides $AD$ and $BC$ respectively such that $\angle APB = \angle CPD$ and $\angle AQB = \angle CQD$. Show that the points $P$ and $Q$ are equidistant from the intersection point of the diagonals of the trapezoid. *(M. Smurov)*



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8. A plane is divided into unit squares by two collections of parallel lines. For any $n \times n$ square with sides on the division lines, we define its *frame* as the set of those unit squares which internally touch the boundary of the $n \times n$ square. Prove that there exists only one way of covering a given $100 \times 100$ square whose sides are on the division lines with frames of 50 squares (not necessarily contained in the $100 \times 100$ square). *(A. Perlin)*

## Grade 10

*First Day*

1. Let be given three quadratic polynomials:

$$P_1(x) = x^2 + p_1x + q_1, \quad P_2(x) = x^2 + p_2x + q_2, \quad P_3(x) = x^2 + p_3x + q_3.$$

Prove that the equation $|P_1(x)| + |P_2(x)| = |P_3(x)|$ has at most eight real roots. *(A. Golovanov)*

2. *Problem 3 for Grade 9.*

3. Let $a, b, c$ be the sides of a triangle, let $m_a, m_b, m_c$ be the corresponding medians, and let $D$ be the diameter of the circumcircle of the triangle. Prove that

$$\frac{a^2+b^2}{m_c} + \frac{b^2+c^2}{m_a} + \frac{c^2+a^2}{m_b} \le 6D.$$

*(D. Tereshin)*

4. In a regular $6n+1$-gon, $k$ vertices are painted in red and the others in blue. Prove that the number of isosceles triangles whose vertices are of the same color does not depend on the arrangement of the red vertices. *(D. Tamarkin)*

*Second Day*

5. Prove that, for any natural numbers $k, m, n$,

$$[k, m] \cdot [m, n] \cdot [n, k] \ge [k, m, n]^2.$$

6. Let functions $f$ and $g$ be defined on the set of integers not exceeding 1000 in absolute value. Let $m$ be the number of pairs $(x, y)$ with $f(x) = g(y)$, $n$ be the number of pairs with $f(x) = f(y)$, and $k$ be the number of pairs with $g(x) = g(y)$. Prove that $2m \le n + k$. *(A. Belov)*

7. Each of circles $S_1, S_2, S_3$ is tangent to two sides of a triangle $ABC$ and externally tangent to a circle $S$ a $A_1, B_1, C_1$ respectively. Prove that the lines $AA_1, BB_1, CC_1$ meet in a point. *(D. Tereshin)*

8. There are 30 pupils in a class, and each of them has the same number of friends among the classmates. What is the greatest possible number of pupils, each of which studies more than a majority of his/her friends? (For any two pupils it can be said who studies more). *(S. Tokarev)*



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## Grade 11

*First Day*

1. Natural numbers $a$ and $b$ are such that $\dfrac{a+1}{b}+\dfrac{b+1}{a}$ is an integer. If $d$ is the greatest common divisor of $a$ and $b$, prove that $d^2 \leq a+b$.
*(A. Golovanov, Ye. Malinnikova)*

2. Inside a convex 100-gon are selected $k$ points, $2 \leq k \leq 50$. Show that one can choose $2k$ vertices of the 100-gon so that the convex $2k$-gon determined by these vertices contains all the selected points. *(S. Berlov)*

3. Two circles $S_1$ and $S_2$ touch externally at $F$. their external common tangent touches $S_1$ at $A$ and $S_2$ at $B$. A line, parallel to $AB$ and tangent to $S_2$ at $C$, intersects $S_1$ at $D$ and $E$. Prove that the common chord of the circumcircles of triangles $ABC$ and $BDE$ passes through point $F$.
*(A. Kalinin)*

4. Real numbers are written on the squares of an infinite grid. Two figures consisting of finitely many squares are given. They may be translated anywhere on the grid as long as their squares coincide with those of the gid. It is known that wherever the first figure is translated, the sum of numbers it covers is positive. Prove that the second figure can be translated so that the sum of the numbers it covers is also positive.
*(B. Ginzburg, I. Solovyev)*

*Second Day*

5. Let $a_1$ be a natural number not divisible by 5. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_{n+1} = a_n + b_n$, where $b_n$ is the last digit of $a_n$. Prove that the sequence contains infinitely many powers of two. *(N. Agakhanov)*

6. *Problem 6 for Grade 10.*

7. The altitudes $AA_1, BB_1, CC_1, DD_1$ of a tetrahedron $ABCD$ intersect in the center $H$ of the sphere inscribed in the tetrahedron $A_1B_1C_1D_1$. Prove that the tetrahedron $ABCD$ is regular. *(D. Tereshin)*

8. Players $A$ and $B$ alternately move a knight on a $1994 \times 1994$ chessboard. Player $A$ makes only horizontal moves, i.e. such that the knight is moved to a neighboring row, while $B$ makes only vertical moves. Initially player $A$ places the knight to an arbitrary square and makes the first move. The knight cannot be moved to a square that was already visited during the game. A player who cannot make a move, loses. Prove that player $A$ has a winning strategy. *(A. Perlin)*



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