Mathematical Reflections

Junior problems

J67. Prove that among seven arbitrary perfect squares there are two whose difference is divisible by 20.

Proposed by Ivan Borsenco, University of Texas at Dallas, USA

First solution by Salem Malikic, Sarajevo, Bosnia and Herzegovina

It is easy to check that perfect squares can give one of the following residues:

$$1, 2, 4, 8, 16 \pmod{20}.$$

By the Pigeonhole principle we conclude that among seven perfect squares we must have at least two that have the same residue modulo 20. Hence their difference is divisible by 20 and our proof is complete.

Second solution by Vicente Vicario Garca, Huelva, Spain

Note that for all integer x we have $x^2 \equiv 1, 2, 4, 8, 16 \pmod{20}$ and we have six distinct possible residues. If we have seven arbitrary perfect squares x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, x_7^2, by the pigeonhole principle, there are two squares x_i^2 and x_j^2 with the same residue and they satisfy the requirement.

Third solution by Vishal Lama, Southern Utah University, USA

Observe that by the Pigeonhole Principle, there are at least four perfect squares which all have the same parity. Now, note that for any integer n, we have $n^2 \equiv -1, 0, 1 \pmod{5}$. Again by the Pigeonhole Principle, out of these four perfect squares, we have at least two perfect squares, say a^2 and b^2, such that $a^2 \equiv b^2 \pmod{5}$. This implies that $5 \mid a^2 - b^2$. Also, $2 \mid a - b$ and $2 \mid a + b$ since both a and b have the same parity. Hence, $4 \mid a^2 - b^2$, but $\gcd(5, 4) = 1$, thus we have $20 \mid a^2 - b^2$, and we are done.

Also solved by Andrea Munaro, Italy; Arkady Alt, San Jose, California, USA; Brian Bradie, VA, USA; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Ganesh Ajjanagadde Acharya Vidya Kula, Mysore, India; Jose Hernandez Santiago, UTM, Oaxaca, Mexico; Oleh Faynshteyn, Leipzig, Germany; G.R.A.20 Math Problems Group, Roma, Italy.

J68. Let ABC be a triangle with circumradius R. Prove that if the length of one of the medians is equal to R, then the triangle is not acute. Characterize all triangles for which the lengths of two medians are equal to R.

Proposed by Daniel Lasaosa, Universidad Publica de Navarra, Spain

First solution by Vicente Vicario Garca, Huelva, Spain

Let O be the circumcenter and M be the midpoint of the side BC. Without loss of generality we have that $a \geq b \geq c$, we have

$$m_A = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2},\ m_B = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2},\ m_C = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2},$$

and we deduce that $m_A \leq m_B \leq m_C$. On the other hand, if the triangle is acute angled, then its circumcenter lies int the interior of the triangle. Note that $m_A > R$, because $\angle AOM$ is obtuse, and the equality does not occur. Thus triangle ABC is not acute angled.

For the second part it is not difficult to see that if two medians in a triangle are equal, then the triangle is isosceles, because

$$m_B = m_C \Leftrightarrow \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2} = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2} \Leftrightarrow b = c.$$

Let the ABC be isosceles triangle with $b = c$. By the Law of Sines and the Law of Cosines we have

$$R = \frac{a}{2\sin A}, \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{2b^2 - a^2}{2b^2}$$

and if $m_B = m_C = R$, we have

$$m_B^2 = R^2, \Rightarrow\ \frac{1}{4}(2a^2 + 2b^2 - b^2) = R^2, \Rightarrow\ 2a^2 + b^2 = 4R^2 \quad (1)$$

and finally

$$R^2 = \frac{a^2}{4\sin^2 A} = \frac{a^2}{4(1 - \cos^2 A)} = \frac{a^2}{4\left[1 - \left(\frac{2b^2 - c^2}{2b^2}\right)^2\right]}, \Rightarrow\ 4R^2 = \frac{4b^4}{4b^2 - a^2}.$$

Finally, using (1) we get

$$2a^2 + b^2 = \frac{4b^4}{4b^2 - a^2}, \Rightarrow\ 7a^2b^2 - 2a^4 = 0 \Rightarrow a^2(7b^2 - 2a^2) = 0,$$

yielding $b = c = \sqrt{\frac{2}{7}}\, a$, and we are done.

Third solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Without loss of generality, let us assume that the length of the median from A equals R. The square of the length of this median is given by

$$\frac{b^2+c^2}{2}-\frac{a^2}{4}=\frac{a^2}{4}+bc\cos A=R^2\sin^2 A+4R^2\sin B\sin C\cos A.$$

Equating this result to R^2 and grouping terms in one side of the equality yields

$$\cos A\left(\cos A-4\sin B\sin C\right)=0.$$

One possible solution is that triangle ABC is right triangle at A, in which case the midpoint of BC is also the circumcenter, and the median from A is a radius of the circumcircle. Otherwise,

$$4\sin B\sin C=-\cos\left(B+C\right)=-\cos B\cos C+\sin B\sin C,$$

yielding

$$\tan B\tan C=-\frac{1}{3}.$$

Clearly, B and C cannot be simultaneously acute, and ABC is either rectangle or obtuse.

If the lengths of two medians are equal to R, say $m_a=m_b$, then

$$\frac{b^2+c^2}{2}-\frac{a^2}{4}=\frac{c^2+a^2}{2}-\frac{b^2}{4},$$

yielding $a=b$, or ABC is isosceles at C. Since $A=B$, C is obtuse, and using the well known identity $\tan A+\tan B+\tan C=\tan A\tan B\tan C$, we find $2\tan A-\frac{1}{3\tan A}=-\frac{\tan A}{3}$, and $\tan^2 A=\tan^2 B=\frac{1}{7}$, $\tan^2 C=\frac{7}{9}$. Using that $\sin^2\alpha=\frac{\tan^2\alpha}{1+\tan^2\alpha}$, we find that

$$\sin A=\sin B=\frac{\sqrt{2}}{4},$$

$$\sin C=\frac{\sqrt{7}}{4},$$

or the lengths of two medians in a triangle are equal to R if and only if it is similar to a triangle with sides $\sqrt{2},\sqrt{2},\sqrt{7}$.

J69. Consider a convex polygon $A_1A_2\dots A_n$ and a point P in its interior. Find the least number of triangles $A_iA_jA_k$ that contain P on their sides or in their interiors.

Proposed by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

We prove that a point P may be found such that it is not contained in the interior or on the sides of more than $n-2$ triangles. The result is true for $n=3$, since P will be in the interior of $A_1A_2A_3$ only. If $n \geq 4$, denote by Q the point where diagonals A_1A_3 and A_2A_4 intersect. Clearly, each triangle $A_iA_jA_k$ will have non void intersection with the interior of A_2A_3Q if and only if one of its sides is A_2A_3, in which case A_2A_3Q is contained in it. Since there are exactly $n-2$ different triangles $A_2A_3A_k$ that contain P in their interior, and no other triangle $A_iA_jA_k$ may contain P on its sides or in its interior, the least number is no larger than $n-2$.

The number cannot be less than $n-2$, we prove this by induction. The result is true for the case $n=3$. If the result is true for $n-1 \geq 3$, consider triangles $A_1A_2A_3$ and $A_3A_4A_5$ in an n-gon, $n \geq 4$ (if $n=4$, then $A_5 = A_1$).

If $n=4$, P is either on the common boundary A_1A_3 of these triangles, or completely outside one of them. If $n \geq 5$, P cannot be simultaneously on the sides or in the interior of both triangles, since they only have one common vertex A_3 which cannot be P. Therefore, either $(n-1)$-gon $A_1A_3A_4\dots A_n$, or $(n-1)$-gon $A_1A_2A_3A_5\dots A_n$, contains P in their interior. Assume without loss of generality, and by hypothesis of induction $n-3$ triangles $A_iA_jA_k$ may be found that contain P on their sides or in their interiors, where $i,j,k \neq 2$. Consider now the partition of the n-sided polygon on triangles by drawing all diagonals A_2A_k. Clearly, P is on the sides or in the interior of at least one of the triangles thus generated, and this triangle is different to the $n-3$ previously considered, or the number of triangles that contain P on their sides and in their interior is no less than $n-2$, and so this is the least number.

J70. Let l_a, l_b, l_c be the lengths of the angle bisectors of a triangle. Prove the following identity

$$\frac{\sin\frac{\alpha-\beta}{2}}{l_c}+\frac{\sin\frac{\beta-\gamma}{2}}{l_a}+\frac{\sin\frac{\gamma-\alpha}{2}}{l_b}=0,$$

where α, β, γ are the angles of the triangle.

Proposed by Oleh Faynshteyn, Leipzig, Germany

First solution by Courtis G. Chryssostomos, Larissa, Greece

Using the fact that $l_c = \frac{2R\sin a\sin b}{\cos\left(\frac{a-b}{2}\right)}$, we get

$$\begin{aligned}\frac{\sin\left(\frac{a-b}{2}\right)}{l_c}+\frac{\sin\left(\frac{b-\gamma}{2}\right)}{l_a}+\frac{\sin\left(\frac{\gamma-a}{2}\right)}{l_b} &= \frac{\sin\left(\frac{a-b}{2}\right)}{\frac{2R\sin a\sin b}{\cos\left(\frac{a-b}{2}\right)}}+\frac{\sin\left(\frac{b-\gamma}{2}\right)}{\frac{2R\sin b\sin\gamma}{\cos\left(\frac{b-\gamma}{2}\right)}}+\frac{\sin\left(\frac{\gamma-a}{2}\right)}{\frac{2R\sin a\sin\gamma}{\cos\left(\frac{\gamma-a}{2}\right)}}\\
&= \frac{\frac{1}{2}\sin(a-b)}{2R\sin a\sin b}+\frac{\frac{1}{2}\sin(b-\gamma)}{2R\sin b\sin\gamma}+\frac{\frac{1}{2}\sin(\gamma-a)}{2R\sin a\sin\gamma}\\
&= \frac{1}{4R}\left[\frac{\sin(a-b)}{\sin a\sin b}+\frac{\sin(b-\gamma)}{\sin b\sin\gamma}+\frac{\sin(\gamma-a)}{\sin\gamma\sin a}\right]\\
&= \frac{1}{4R}\cdot\frac{\sum_{cyc}(\sin a\cos b-\cos a\sin b)\sin\gamma}{\sin a\sin b\sin\gamma}\\
&= \frac{1}{4R}\cdot\frac{0}{\sin a\sin b\sin\gamma}=0.\end{aligned}$$

Second solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

By Mollweide's formula we have

$$\frac{a-b}{c}=\frac{\sin\alpha-\sin\beta}{\sin\gamma}=\frac{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}=\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\gamma}{2}}$$

and

$$l_c=\frac{2ab}{a+b}\cos\frac{\gamma}{2},$$

where a, b, c are the sides of the triangle; hence,

$$\frac{\sin\frac{\alpha-\beta}{2}}{l_c}=\frac{a^2-b^2}{2abc}.$$

With this and the two similar results, we obtain

$$\frac{\sin\frac{\alpha-\beta}{2}}{l_c}+\frac{\sin\frac{\beta-\gamma}{2}}{l_a}+\frac{\sin\frac{\gamma-\alpha}{2}}{l_b}=\frac{(a^2-b^2)+(b^2-c^2)+(c^2-a^2)}{2abc}=0,$$

as desired.

Also solved by Andrea Munaro, Italy; Arkady Alt, San Jose, California, USA; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Mihai Miculita, Oradea, Romania; Nguyen Manh Dung, Hanoi University of Science, Vietnam; Prithwijit De, ICFAI Business School, Calcutta, India; Vicente Vicario Garca, Huelva, Spain; Son Hong Ta, High School for Gifted Students, Hanoi University of Education, Hanoi, Vietnam.

J71. In the Cartesian plane call a line "good" if it contains infinitely many lattice points. Two lines intersect at a lattice point at an angle of 45° degrees. Prove that if one of the lines is good, then so is the other.

Proposed by Samin Riasat, Notre Dame College, Dhaka, Bangladesh

First solution by Brian Bradie, Newport University, VA, USA

Let ℓ_1 and ℓ_2 be lines that intersect at a lattice point at an angle of 45°. Further, suppose that ℓ_1 is good. As ℓ_1 contains two lattice points, its slope must either be undefined or rational. If the slope of ℓ_1 is undefined, then the slope of ℓ_2 is ± 1; in either situation, ℓ_2 contains one lattice point and has rational slope so must therefore contain infinitely many lattice points and is good. If, on the other hand, the slope of ℓ_1 is ± 1, then ℓ_2 is either a horizontal line or a vertical line; again, in either situation, because ℓ_2 is known to contain one lattice point it must therefore contain infinitely many lattice points and is good. Finally, suppose the slope of ℓ_1 is rational but neither ± 1, and let θ denote the angle of inclination of ℓ_1. Then $\tan\theta$ is rational and

$$\tan(\theta \pm 45^\circ) = \frac{\tan\theta \pm \tan 45^\circ}{1 \mp \tan\theta\ \tan 45^\circ} = \frac{\tan\theta \pm 1}{1 - \mp \tan\theta}$$

is also rational. Once again, ℓ_2 contains one lattice point and has rational slope so must therefore contain infinitely many lattice points and is good.

Second solution by Jose Hernandez Santiago, UTM, Oaxaca, Mexico

Let us suppose that l_1 and l_2 are two lines that satisfy the conditions stated in the hypothesis. Without loss of generality we may assume that l_1 is a "good" line, and that the coordinates of the lattice point at which those lines meet are (m, n).

The purported result clearly holds in any one of the following cases:

(a) The slope of line l_1 is 1.

(b) The slope of line is l_1 is -1.

(c) Line l_1 is vertical.

(d) Line l_1 is horizontal.

If line l_1 falls into neither of those categories below, we infer that its slope is a rational number of the form $\frac{a}{b}$, where $a, b \in \mathbb{Z}\backslash\{0\}$, $a + b \neq 0$, and $a - b \neq 0$. Furthermore, the hypothesis that lines l_1 and l_2 intersect at an angle of 45° imply that one and only one of above relations holds

$$\alpha_2 = \alpha_1 + 45 \qquad (1)$$

$$\alpha_2 = \alpha_1 - 45, \qquad (2)$$

where α_1 and α_2 are the elevation angles of lines l_1 and l_2, respectively.

Now, assuming that (1) holds (similarly we can do for (2)), we get

$$\begin{aligned}\tan\alpha_2 &= \tan(\alpha_1+45)\\ &= \frac{\tan\alpha_1+\tan 45}{1-\tan\alpha_1\tan 45}\\ &= \frac{\frac{a}{b}+1}{1-\frac{a}{b}}\\ &= \frac{a+b}{b-a}.\end{aligned}$$

Hence, line l_2 is represented by the equation

$$\begin{aligned}y-n &= \tan\alpha_2(x-m)\\ &= \left(\frac{a+b}{b-a}\right)(x-m),\end{aligned}$$

or equivalently,

$$(b-a)y-(a+b)x = (b-a)n-(a+b)m. \qquad (3)$$

Since $\gcd(b-a,-(a+b)) \mid (b-a)n-(a+b)m$, the diophantine equation in (3) possesses an infinite number of solutions in integers. Each one of these solutions corresponds with a lattice point in l_2, and we are done.

Third solution by Vishal Lama, Southern Utah University, USA

Let the two lines, say, l_1 and l_2, intersect at a lattice point $P(a,b)$ at an angle of 45°. Without loss of generality we may assume that line l_1 is "good", i.e. l_1 contains an infinite number of lattice points.

Let $Q(c,d)$ be an arbitrary lattice point on l_1. Construct a perpendicular on l_1 passing through Q such that it intersects l_2 at $R(c',d')$. We show that R itself is a lattice point.

Note that triangle PQR is a right isosceles triangle, with $QP = QR$ and $\angle PQR = 90^\circ$. Now, consider the points on this plane as complex numbers. Recall that a complex number $a+ib$ when multiplied by $e^{i\theta}$ rotates it by an angle θ in the counterclockwise direction. We note that $\overrightarrow{QR}$ rotated by 90° in the counterclockwise direction coincides with $\overrightarrow{QP}$. Therefore we have $(c'-c+i(d'-d))e^{i\pi/2} = a-c+i(b-d)$, which implies $d-d'+i(c'-c) = a-c+i(b-d)$. Solving for c' and d', we obtain $c' = b+c-d$ and $d' = c+d-a$. Now, since a,b,c and d are all integers, so are c' and d', thus proving that $R(c',d')$ is a lattice point.

Hence, if an arbitrary point $Q(c,d)$ on line l_1 is a lattice point, then so is $R(c',d')$ on line l_2. This implies if l_1 is "good", then so is line l_2.

J72. Let a, b, c be real numbers such that $|a|^3 \leq bc$. Prove that $b^2 + c^2 \geq \frac{1}{3}$ whenever $a^6 + b^6 + c^6 \geq \frac{1}{27}$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Salem Malikic, Sarajevo, Bosnia and Herzegovina

Assume the contrary that $b^2 + c^2 < \frac{1}{3}$. Then note that $a^6 \leq (bc)^2$, thus

$$\begin{aligned}\frac{1}{27} &\leq a^6 + b^6 + c^6 \leq (bc)^2 + b^6 + c^6 \\ &= (b^2 + c^2)^3 + (bc)^2(1 - 3(b^2 + c^2)) \\ &\leq (b^2 + c^2)^3 + \left(\frac{b^2 + c^2}{2}\right)^2 (1 - 3(b^2 + c^2)) \\ &= \frac{(b^2 + c^2)^2}{4}(4(b^2 + c^2) + 1 - 3(b^2 + c^2)) \\ &= \frac{(b^2 + c^2)^2}{4}(1 + b^2 + c^2) < \frac{\left(\frac{1}{3}\right)^2}{4}\left(1 + \frac{1}{3}\right) \\ &= \frac{1}{27},\end{aligned}$$

a contradiction. Thus $b^2 + c^2 \geq \frac{1}{3}$, and we are done.

Also solved by Arkady Alt, San Jose, California, USA; Daniel Campos Salas, Costa Rica; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Oleh Faynshteyn, Leipzig, Germany; Vishal Lama, Southern Utah University, USA

Senior problems

S67. Let ABC be a triangle. Prove that

$$\cos^3 A + \cos^3 B + \cos^3 C + 5\cos A\cos B\cos C \le 1.$$

Proposed by Daniel Campos Salas, Costa Rica

First solution by Son Hong Ta, Hanoi, Vietnam

Using the equality

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$$

the initial inequality becomes equivalent to

$$\sum \cos^3 A + 3\prod \cos A \le \sum \cos^2 A,$$

or

$$3\prod \cos A \le \sum \cos^2 A\,(1-\cos A)$$

Now, by the AM-GM inequality, we have

$$\sum \cos^2 A\,(1-\cos A) \ge 3\sqrt[3]{\prod \cos^2 A \cdot \prod (1-\cos A)}$$

Thus, it suffices to prove that

$$\prod \cos A \le \prod (1-\cos A)\,.$$

When triangle ABC is obtuse, the above inequality is clearly true. So we will consider the case it is acute. We have

$$\begin{aligned}
\prod \cos A \le \prod (1-\cos A) & \\
&\Longleftrightarrow \prod \cos A\,(1+\cos A) \le \prod \left(1-\cos^2 A\right)\\
&\Longleftrightarrow 8\prod \cos A \cdot \prod \cos^2 \frac{A}{2} \le \prod \sin^2 A\\
&\Longleftrightarrow \frac{\prod \cos^2 \frac{A}{2}}{\prod \sin\frac{A}{2}\cos\frac{A}{2}} \le \frac{\prod \sin A}{\prod \cos A}\\
&\Longleftrightarrow \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \le \tan A\tan B\tan C\\
&\Longleftrightarrow \cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2} \le \tan A+\tan B+\tan C
\end{aligned}$$

Indeed, we have

$$\sum \tan A = \sum \frac{\tan B + \tan C}{2} \geq \sum \tan \frac{B+C}{2} = \sum \cot \frac{A}{2},$$

and the equality holds if and only if triangle ABC is equilateral.

Second solution by Arkady Alt, San Jose, California, USA
Since $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$ we will prove that

$$\sum_{cyc} \cos^2 A \,(1 - \cos A) \geq 3\cos A \cos B \cos C.$$

By the AM-GM Inequality we have

$$\sum_{cyc} \cos^2 A \,(1 - \cos A) \geq 3\sqrt[3]{\prod_{cyc} \cos^2 A \,(1 - \cos A)},$$

then it suffices to prove

$$3\sqrt[3]{\prod_{cyc} \cos^2 A \,(1 - \cos A)} \geq 3\cos A \cos B \cos C$$
$$\iff \prod_{cyc} (1 - \cos A) \geq \cos A \cos B \cos C$$
$$\iff \cos A \cos B \cos C \leq 8\sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{B}{2}.$$

Using that $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ and $2\sin^2 \dfrac{A}{2} = \dfrac{a^2 - (b-c)^2}{2bc}$ we get

$$\cos A \cos B \cos C \leq 8\sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2}$$
$$\iff \prod_{cyc} \frac{b^2 + c^2 - a^2}{2bc} \leq \prod_{cyc} \frac{a^2 - (b-c)^2}{2bc}$$
$$\iff \prod_{cyc} \left(b^2 + c^2 - a^2\right) \leq \prod_{cyc} (b + c - a)^2.$$

Without loss of generality we can assume that $\prod_{cyc} \left(b^2 + c^2 - a^2\right) > 0$.

Then $b^2 + c^2 > a^2$, $c^2 + a^2 > b^2, a^2 + b^2 > c^2$ and, therefore,

$$\prod_{cyc} \left(b^2 + c^2 - a^2\right) \leq \prod_{cyc} (b + c - a)^2 \iff \prod_{cyc} \left(b^2 + c^2 - a^2\right)^2 \leq \prod_{cyc} (b + c - a)^4.$$

Because
$$\prod_{cyc}\left(b^2+c^2-a^2\right)^2=\prod_{cyc}\left(b^4-\left(c^2-a^2\right)^2\right)$$
and
$$\prod_{cyc}(b+c-a)^4=\prod_{cyc}\left(b^2-(c-a)^2\right)^2,$$
it is enough to prove $b^4-\left(c^2-a^2\right)^2\leq\left(b^2-(c-a)^2\right)^2$. We have

$$\begin{aligned}\left(b^2-(c-a)^2\right)^2-b^4+\left(c^2-a^2\right)^2&=b^4-2b^2(c-a)^2+(c-a)^4-b^4+\left(c^2-a^2\right)^2\\&=(c-a)^2\left((c+a)^2-2b^2+(c-a)^2\right)\\&=(c-a)^2\left(2c^2+2a^2-2b^2\right)\\&=2(c-a)^2\left(c^2+a^2-b^2\right)\geq 0,\end{aligned}$$

and we are done.

Also solved by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

S68. Let ABC be an isosceles triangle with $AB = AC$. Let X ad Y be points on sides BC and CA such that $XY \parallel AB$. Denote by D the circumcenter of triangle CXY and by E be the midpoint of BY. Prove that $\angle AED = 90°$.

Proposed by Francisco Javier Garcia Capitan, Spain

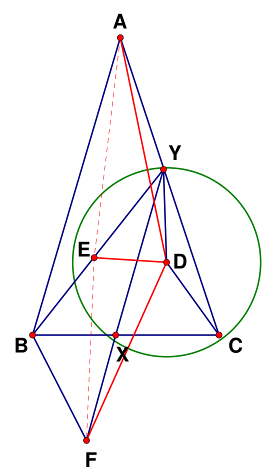
First solution by Andrea Munaro, Italy

Let M, N and H be the midpoints of CY, XY and BC, respectively. Then E, N and M are collinear. Note that $\angle EMD = 90° - \angle BCA = \angle DCA$. Since $AB \parallel XY$, triangles CHA and DMC are similar and so $\frac{BC}{2AC} = \frac{DM}{DC}$, or equivalently $\frac{DC}{AC} = \frac{DM}{\frac{BC}{2}}$. Thus triangles ADC and EDM are similar and

$$\begin{aligned} \angle AED &= \angle ABE + \angle BAE + \angle YBC + \angle DEM \\ &= \angle ABC + \angle BAE + \angle DAM \\ &= \angle ABC + \angle BAC - \angle EAD. \end{aligned}$$

Then $\angle AED + \angle EAD = \angle ABC + \angle BAC$ and so $\angle EDA = \angle ABC = \angle EMA$. Thus $AEDM$ is a cyclic quadrilateral and the result follows.

Second solution by Dinh Cao Phan, Pleiku, Vietnam

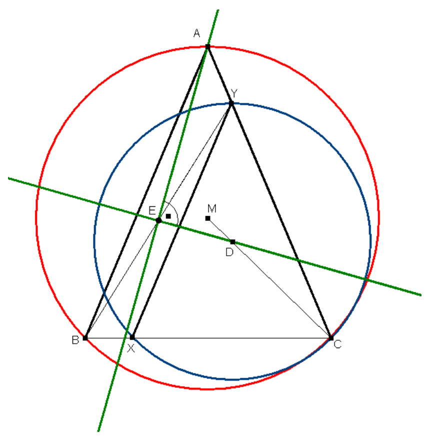


Draw $BF \parallel AY$. Let YX meet BF in F. We have $BF \parallel AY$ and $AB \parallel FY$, hence $ABFY$ is a parallelogram. Because E is the midpoint of BY, then E is the midpoint of AF, and since ΔABC is isosceles it follows that $\angle ABC = \angle ACB$. Thus $\angle YXC = \angle ABC$, yielding $\angle YXC = \angle YCX$ and we can conclude that ΔYXC is isosceles.

From point D we draw the circumcircle of the triangle ΔCXY, and D lies on the perpendicular bisector of XC. Consequently, YD is the angle bisector of $\angle XYC$, hence $\angle FYD = \angle CYD$. We have $AC = YF$, $DY = DC$, $\angle DCA = \angle DYF$, therefore $\Delta ADC = \Delta FDY$. Thus $DA = DF$ and $\triangle ADF$ is isosceles. Since DE is a median, it is also a perpendicular bisector of AF. Finally, $DE \perp AF$, hence $\angle AED = 90^\circ$, and we are done.

Third solution by Oleh Faynshteyn, Leipzig, Germany

Let the vertices of a triangle ABC correspond to the complex numbers $A(a), B(b), C(c)$. Assume that the circumcircle of triangle ABC is the unit circle, and denote by M its center, which will also be the origin of the complex plane.



$$a \cdot \overline{a} = 1, \quad b \cdot \overline{b} = 1, \quad c \cdot \overline{c} = 1.$$

As $AB \parallel XY$, the triangles ABC and XYZ are similar, yielding

$$\lambda = \frac{BX}{XC} = \frac{AY}{YC}.$$

In addition, since ABC is isosceles, we have $a^2 = bc$. Knowing the complex coordinates of A, B, C we can calculate the corresponding complex coordinates of points $X(x), Y(y), E(e), D(d)$. We obtain

$$x = \frac{b+\lambda c}{1+\lambda},\ y = \frac{a+\lambda c}{1+\lambda},\ e = \frac{b+y}{2} = \frac{(a+b)+\lambda(b+c)}{2(1+\lambda)},\ d = \frac{0+\lambda c}{1+\lambda}, = \frac{\lambda c}{1+\lambda}.$$

Further, we get

$$\overline{e} = \frac{(a+c)+\lambda(b+c)}{2a^2(1+\lambda)},\ \overline{d} = \frac{\lambda}{(1+\lambda)c}.$$

For the triangle XYZ, the necessary and sufficient condition for it to be isosceles is $y^2 = xc$, or using the above identities we can rewrite it as

$$\left(\frac{a+\lambda c}{1+\lambda}\right)^2 = \frac{b+\lambda c}{1+\lambda} \cdot c.$$

Wince $\lambda \neq 0$, we obtain $c + b - 2a = 0$. Using the above relations we calculate the slope between the lines AE and DE. We have

$$k_{AE} = \frac{e-a}{\overline{e}-\overline{a}} = \frac{a^2(a-b)}{a-c}, \quad k_{DE} = \frac{e-d}{\overline{e}-\overline{d}} = \frac{a^2(a+b+\lambda(b-c))}{a+c+\lambda(b-c)}.$$

We have that

$$\begin{aligned} k_{AE} + k_{DE} = \frac{e-a}{\overline{e}-\overline{a}} &= \frac{a^2(a-b)}{a-c} + \frac{e-d}{\overline{e}-\overline{d}} = \frac{a^2(a+b+\lambda(b-c))}{a+c+\lambda(b-c)} \\ &= \frac{a^2(a-b)(a+c+\lambda(b-c)) + a^2(a-c)(a+b+\lambda(b-c))}{(a-c)(a+c+\lambda(b-c))} = 0. \end{aligned}$$

Therefore, $AE \perp DE$, and we are done.

Also solved by Ricardo Barroso Campos, Universidad de Sevilla, Spain; Andrei Iliasenco, Chisinau, Moldova; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Daniel Campos Salas, Costa Rica; Miguel Amengual Covas, Mallorca, Spain; Mihai Miculita, Oradea, Romania; Courtis G. Chryssostomos, Larissa, Greece; Son Hong Ta, Ha Noi University, Vietnam; Vicente Vicario Garca, Huelva, Spain; Vishal Lama, Southern Utah University, USA.

S69. Circles ω_1 and ω_2 intersect at X and Y. Let AB be a common tangent with $A \in \omega_1$, $B \in \omega_2$. Point Y lies inside triangle ABX. Let C and D be the intersections of an arbitrary line, parallel to AB, with ω_1 and ω_2, such that $C \in \omega_1$, $D \in \omega_2$, C is not inside ω_2, and D is not inside ω_1. Denote by Z the intersection of lines AC and BD. Prove that XZ is the bisector of angle CXD.

Proposed by Son Hong Ta, Ha Noi University, Vietnam

Solution by Andrei Iliasenco, Chisinau, Moldova

Denote $\alpha = \angle ZCD$, by $\beta = \angle ZDC$, by $\gamma = \angle XCZ$, and by $\delta = \angle XDZ$.

$$\frac{2R_1 \sin\gamma}{2R_2 \sin\delta} = \frac{AX}{BX} = \frac{\sin\angle ABX}{\sin\angle BAX} = \frac{\sin\delta}{\sin\gamma}, \Rightarrow \ \frac{R_1}{R_2} = \frac{\sin^2\delta}{\sin^2\gamma}.$$

Denote by E and F the second intersections of line CD with circles ω_1 and ω_2 respectively. Note that CD is parallel to AD, thus $\angle CEA \equiv \angle ECA \equiv \alpha$ and by analogy $\angle DFB \equiv \angle FDB \equiv \beta$. Let d be the distance between CD and AB. We have

$$1 = \frac{d}{d} = \frac{CA\sin\alpha}{BD\sin\beta} = \frac{2R_1\sin^2\alpha}{2R_2\sin^2\beta}, \Rightarrow \ \frac{\sin^2\delta}{\sin^2\gamma} = \frac{R_1}{R_2} = \frac{\sin^2\beta}{\sin^2\alpha}.$$

Because $\sin\alpha, \sin\beta, \sin\gamma$, and $\sin\delta$ have positive values, we get that $\frac{\sin\alpha}{\sin\beta} = \frac{\sin\gamma}{\sin\delta}$. Now we will use the Law of Sines for the triangles CXZ and DXZ:

$$\frac{CZ}{\sin CXZ} = \frac{XZ}{\sin XCZ} \text{ and } \frac{DZ}{\sin DXZ} = \frac{XZ}{\sin XDZ},$$

$$\frac{\sin\alpha}{\sin\beta} = \frac{DZ}{CZ} = \frac{\sin\gamma}{\sin\delta}\frac{\sin DXZ}{\sin CXZ}.$$

Hence $\sin CXZ = \sin DXZ$. It follows that either $\angle CXZ = \angle DXZ$ or $\angle CXZ + \angle DXZ = \pi$. If CD does not pass through X, then $\angle CXZ + \angle DXZ \neq \pi$ and therefore $\angle CXZ = \angle DXZ$. If CD passes through X and $X = E = F$, then let T be the point on the line perpendicular to CD, passing through the point X, such that $XT = 2d$. Let A_1 and B_1 be the projections of the points A and B on the line CD. Because of the similarity of triangles $\triangle CAA_1$ and $\triangle CTX$ we get that the points C, A, and T are collinear. Analogously, points D, B, and T are collinear. This means that $T = Z$ and $\angle CXZ = \angle DXZ = 90°$.

S70. Find the least odd positive integer n such that for each prime p, $\frac{n^2-1}{4}+np^4+p^8$ is divisible by at least four primes.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

First solution by Andrea Munaro, Italy

Let $n = 2k+1$ with k nonnegative integer. For $k = 0, 1, 2, 3$ it is easy to see that when $p = 2$ there are less than four prime divisors.

$$\begin{aligned} M &= p^8 + np^4 + \frac{n^2-1}{4} \\ &= \left(p^4 + \frac{n}{2}\right)^2 - \frac{1}{4} \\ &= \left(p^4 + \frac{n-1}{2}\right)\left(p^4 + \frac{n+1}{2}\right) = (p^4+k)(p^4+k+1). \end{aligned}$$

Let $k = 4$, then $M = (p^4+4)(p^4+5) = (p^2+2p+2)(p^2-2p+2)(p^4+5)$.

If $p = 2$, then M is divisible by $2, 3, 5, 7$. If p is odd we have

$$(p^2+2p+2, p^2-2p+2) = (p^2+2p+2, 4p) = 1,$$

$$\begin{aligned} (p^2+2p+2, p^4+5) &= (p^2+2p+2, p^4+5-p^4-8p^2-4-4p^3-4p) \\ &= (p^2+2p+2, 4p^3+8p^2+4p+1) \\ &= (p^2+2p+2, 4p^3+8p^2+4p+1-4p^3-8p^2-4p) \\ &= (p^2+2p+2, 1) = 1, \end{aligned}$$

and

$$(p^2-2p+2, p^4+5) = (p^2-2p+2, 4p^3-8p^2+4p+1) = (p^2-2p+2, 1) = 1.$$

Thus p^2+2p+2, p^2-2p+2 and p^4+5 are pairwise coprime. As $p^4+5 \equiv 2 \pmod 4$ for all odd p, then 2^1 is the greatest power of 2 dividing p^4+5. Since both p^2+2p+2 and p^2-2p+2 are odd, there is another prime different from 2 and from all the divisors of p^2+2p+2 and p^2-2p+2 which divides p^4+5, and so $n = 9$ is the least desired number.

Second solution by Daniel Campos Salas, Costa Rica

Let $n = 2k+1$, then

$$\tfrac{n^2-1}{4} + np^4 + p^8 = k(k+1) + (2k+1)p^4 + p^8 = (p^4+k)(p^4+k+1).$$

Note that for $k = 0, 1, 2, 3$ the result does not hold for $p = 2$. We prove that $k = 4$ is the least integer that satisfies the condition. For $k = 4$ we have

$$(p^4 + 4)(p^4 + 5) = (p^2 + 2p + 2)(p^2 - 2p + 2)(p^4 + 5).$$

Since $(p^2 + 2p + 2)(p^2 - 2p + 2) = (p^4 + 5) - 1$ we have that

$$(p^2 + 2p + 2, p^4 + 5) = (p^2 - 2p + 2, p^4 + 5) = 1.$$

This implies that any prime that divides $(p^2 + 2p + 2)(p^2 - 2p + 2)$ does not divide $p^4 + 5$ and viceversa. Then, it is enough to prove that two primes divide $(p^2 + 2p + 2)(p^2 - 2p + 2)$ and another two divide $p^4 + 5$.

For $p = 2$ the result holds. Assume that p is an odd prime. Note that $2|p^4 + 5$. To prove that another prime divides $p^4 + 5$ it is enough to prove that $4 \nmid p^4 + 5$. This results follows from the fact that $4|p^4 + 3$.

In order to prove that two primes divide $(p^2 + 2p + 2)(p^2 - 2p + 2)$ it is enough to prove that $(p^2 + 2p + 2, p^2 - 2p + 2) = 1$. Let $(p^2 + 2p + 2, p^2 - 2p + 2) = d$. Note that d is odd and that $d|4p$. This implies that $d|p$. If $d = p$ then $p|p^2 + 2p + 2$, which is a contradiction. Therefore, $d = 1$, as we wanted to prove. This implies that $k = 4$ is the least integer value, from where we conclude that $n = 9$ is the least odd positive integer that satisfies the condition.

Also solved by Andrei Iliasenco, Chisinau, Moldova; G.R.A.20 Math Problems Group,Roma, Italy; Salem Malikic, Sarajevo, Bosnia and Herzegovina.

S71. Let ABC be a triangle and let P be a point inside the triangle. Denote by $\alpha = \frac{\angle BPC}{2}, \beta = \frac{\angle CPA}{2}, \gamma = \frac{\angle APB}{2}$. Prove that if I is the incenter of ABC, then

$$\frac{\sin\alpha\sin\beta\sin\gamma}{\sin A\sin B\sin C} \geq \frac{R}{2(r+PI)},$$

where R and r are the circumcenter and incenter, respectively.

Proposed by Khoa Lu Nguyen, Massachusetts Institute of Technology, USA

Solution by Khoa Lu Nguyen, Massachusetts Institute of Technology, USA

First of all we prove the following lemma:

Lemma. Given a triangle ABC with sidelengths a, b, c and a fixed point P. Then for all the point Q in the same plane,

$$a \cdot PA \cdot QA + b \cdot PB \cdot QB + c \cdot PC \cdot QC \geq abc.$$

Moreover, if P lies inside ABC, the equality happens only at the isogonal conjugate P^* of P with respect to ABC.

Proof. Consider two cases

1^{st} case: P is inside the triangle ABC.

Without loss of generality, we may assume P is not A. It is easy to easy that $f(Q) > abc$ for every point Q lying outside the closed disk centered at A with radius $\frac{bc}{PA}$. Since the disk is compact and f is continuous, we obtain that f must have a minimum value.

Suppose Q lies inside triangle ABC. Denote by A', B', C' the projections of P onto BC, CA, AB. Because Q is inside triangle ABC, we must have

$$a \cdot PA \cdot QA = 2R\sin A \cdot PA \cdot QA = 2R \cdot B'C' \cdot QA \geq 4R \cdot S_{AB'QC'},$$

where $S_{AB'QC'}$ denotes the area of the quadrilateral $AB'QC'$. Similarly, we obtain

$$b \cdot PB \cdot QB \geq 4R. \cdot S_{BC'QA'}$$

$$c \cdot PC \cdot QC \geq 4R \cdot S_{CA'QB'}.$$

Summing up, we obtain

$$a \cdot PA \cdot QA + b \cdot PB \cdot QB + c \cdot PC \cdot QC \geq 4R \cdot S_{ABC} = abc.$$

The equality holds if and only if $(QA, B'C'), (QB, C'A'), (QC, A'B')$ are pairs of perpendicular lines. This means that Q is the isogonal conjugate of P.

Suppose now that there is a point Q_0 lying outside triangle ABC and Q_0 is a critical point of f. Since Q_0 is outside ABC, we can set up a Cartesian coordinate O_{xy} such that $x_{Q_0} > \max\{x_A, x_B, x_C\}$. Then we have

$$\begin{aligned} f(Q) = m\sqrt{(x-x_A)^2+(y-y_A)^2} + n\sqrt{(x-x_B)^2+(y-y_B)^2} \\ + p\sqrt{(x-x_C)^2+(y-y_C)^2} \end{aligned}$$

where $Q = (x, y), m = a \cdot PA, n = b \cdot PB, p = c \cdot PC$. Since

$$x_{Q_0} - x_A > 0, x_{Q_0} - x_B > 0, x_{Q_O} - x_C > 0,$$

we must have

$$\frac{\partial f}{\partial x}(x_{Q_0}, y_{Q_0}) > 0.$$

Hence Q_0 cannot be a critical point of f. Thus $f(Q) \geq abc$ and the equality occurs when $Q = P^*$.

2^{nd} case: P is outside the triangle ABC.

By a similar argument, one can show that $f(Q)$ has a minimal value when $Q = Q_0$ and Q_0 cannot be outside ABC. Thus by interchanging the role of Q_0 and P and applying the result in case 1, we obtain

$$f(Q) \geq f(Q_0) \geq abc,$$

and the lemma is proved.

Returning to the problem we know that P is inside a triangle ABC. Denote by A', B', C' the projections of P onto BC, CA, AB, respectively. Clearly, P is inside the triangle $A'B'C'$. Now by applying the lemma to triangle $A'B'C'$ and P, we obtain $f(I) \geq B'C' \cdot C'A' \cdot A'B'$. By replacing

$$B'C' = PA \cdot \sin A,\ C'A' = PB \cdot \sin B,\ C'A' = PC \cdot \sin C$$

and $PA' = \dfrac{PB \cdot PC \sin \angle BPC}{a}$, $PB' = \dfrac{PC \cdot PA \cdot \sin \angle CPA}{b}$,
$PC' = \dfrac{PA \cdot PB \cdot \sin \angle APB}{c}$, we obtain

$$\frac{1}{2R}(IA' \sin 2\alpha + IB' \sin 2\beta + IC' \sin 2\gamma) \geq \sin A \sin B \sin C,$$

where $\alpha = \frac{\angle BPC}{2}, \beta = \frac{\angle CPA}{2}, \gamma = \frac{\angle APB}{2}$.

By triangle inequality, we have $\max\{IA', IB', IC'\} \leq r + PI$. Hence

$$\frac{r+PI}{2R}(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) \geq \sin A \sin B \sin C.$$

To obtain the inequality in the problem, it is now sufficient to show that $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4 \sin \alpha \sin \beta \sin \gamma$. This is true because $\alpha + \beta + \gamma = \pi$.

The equality of the inequality occurs only if P is I.

Undergraduate problems

U67. Let $(a_n)_{n\geq 0}$ be a decreasing sequence of positive real numbers. Prove that if the series $\sum_{k=1}^{\infty} a_k$ diverges, then so does the series $\sum_{k=1}^{\infty}\left(\frac{a_0}{a_1}+\cdots+\frac{a_{k-1}}{a_k}\right)^{-1}$.

Proposed by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

Solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

As $(a_n)_{n\geq 0}$ is a decreasing sequence, we have

$$\left(\frac{a_0}{a_1}+\cdots+\frac{a_{k-1}}{a_k}\right)^{-1} \geq \left(\frac{a_0}{a_k}+\cdots+\frac{a_{k-1}}{a_k}\right)^{-1} = \frac{a_k}{S_{k-1}} \geq \frac{a_k}{S_k},$$

where $S_k = \sum_{i=1}^{k} a_i$. Further, using Abel-Dini Theorem, if $a_k \geq 0$ and $\sum a_k$ diverges, then $\sum \frac{a_k}{S_k}$ also diverges. This can proved writing

$$\frac{a_k}{S_k}+\ldots+\frac{a_n}{S_n} \geq \frac{S_n - S_{k-1}}{S_n} = 1-\frac{S_{k-1}}{S_n},$$

the last quantity that can is close to one as n goes to infinity. This assures the divergence of the series $\sum \frac{a_k}{S_k}$, and we are done.

U68. In the plane consider two lines d_1 and d_2 and let $B, C \in d_1$ and $A \in d_2$. Denote by M the midpoint of BC and by A' the orthogonal projection of A onto d_1. Let P be a point on d_2 such that $T = PM \cap AA'$ lies in the halfplane bounded by d_1 and containing A. Prove that there is a point Q on segment AP such that the angle bisector of the angle BQC passes through T.

Proposed by Nicolae Nica and Cristina Nica, Romania

Solution by Nicolae Nica and Cristina Nica, Romania

Recall the well known fact - the angle bisector always lies between the altitude and the median with respect to the same vertex. Now let us consider the function $f : [AP] \to \mathbb{R}$, $f(x) = d(T, XC) - d(T, XB)$, where $d(U, VW)$ is the distance from U to the line VW. Clearly, this function is continuous and has Darboux property. We have

$$f(A) = d(T, AC) - d(T, AB), \;\; f(P) = d(T, PC) - d(T, PB)$$

and using the above observation $f(A) \cdot f(P) < 0$. From the Darboux property it follows that there is a point Q such that $f(Q) = 0$, or $d(T, QC) = d(T, QB)$. Thus point Q lies on the angle bisector of triangle QBC.

U69. Evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\arctan\frac{k}{n}\right)\sin\frac{1}{n+k}.$$

Proposed by Cezar Lupu, University of Bucharest, Romania

First solution by Brian Bradie, VA, USA

Using Taylor series,

$$\sin\frac{1}{n+k}=\sum_{j=0}^{\infty}(-1)^j\frac{1}{(n+k)^{2j+1}}.$$

For $j>0$,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\arctan\frac{k}{n}\right)\frac{1}{(n+k)^{2j+1}}=$$

$$\begin{aligned}
&= \lim_{n\to\infty}\frac{1}{n^{2j}}\sum_{k=1}^{n}\frac{1}{n}\left(1+\arctan\frac{k}{n}\right)\frac{1}{(1+\frac{k}{n})^{2j+1}}\\
&= \lim_{n\to\infty}\frac{1}{n^{2j}}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(1+\arctan\frac{k}{n}\right)\frac{1}{(1+\frac{k}{n})^{2j+1}}\\
&= \lim_{n\to\infty}\frac{1}{n^{j-1}}\cdot\int_0^1\frac{1+\arctan x}{(1+x)^{2j+1}}\,dx\\
&= 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\arctan\frac{k}{n}\right)\sin\frac{1}{n+k} &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1+\arctan\frac{k}{n}}{1+\frac{k}{n}}\cdot\frac{1}{n}\\
&= \int_0^1\frac{1+\arctan x}{1+x}\,dx\\
&= \ln 2+\int_0^1\frac{\arctan x}{1+x}\,dx.
\end{aligned}$$

By integration by parts,

$$\begin{aligned}
\int_0^1\frac{\arctan x}{1+x}\,dx &= \ln(1+x)\arctan x\Big|_0^1-\int_0^1\frac{\ln(1+x)}{1+x^2}\,dx\\
&= \frac{\pi}{4}\ln 2-\int_0^1\frac{\ln(1+x)}{1+x^2}\,dx
\end{aligned}$$

Now, making the substitution $x = \tan\theta$ into the above integral on the right-hand side, we find

$$\begin{aligned}
\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx &= \int_0^{\pi/4} \ln(1+\tan\theta)\,d\theta = \int_0^{\pi/4} \ln\left(\frac{\sin\theta+\cos\theta}{\cos\theta}\right) d\theta \\
&= \int_0^{\pi/4} \ln\frac{\sqrt{2}\cos(\theta-\pi/4)}{\cos\theta}\,d\theta \\
&= \frac{\pi}{8}\ln 2 + \int_0^{\pi/4} \ln\cos(\theta-\pi/4)\,d\theta - \int_0^{\pi/4} \ln\cos\theta\,d\theta. \quad (4)
\end{aligned}$$

Finally, with the substitution $w = \pi/4 - \theta$, we find

$$\int_0^{\pi/4} \ln\cos(\theta-\pi/4)\,d\theta = -\int_{\pi/4}^{0} \ln\cos w\,dw = \int_0^{\pi/4} \ln\cos w\,dw. \qquad (5)$$

Combining all the results above we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(1+\arctan\frac{k}{n}\right)\sin\frac{1}{n+k} = \ln 2 + \frac{\pi}{4}\ln 2 - \frac{\pi}{8}\ln 2 = \left(1+\frac{\pi}{8}\right)\ln 2.$$

Second solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

Note that we may substitute $\sin\frac{1}{n+k}$ by $\frac{1}{n+k}$ without altering the value of the limit, since

$$\sin\frac{1}{n+k} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!\,(n+k)^{2m+1}},$$

or the difference between the proposed limit and the result of the proposed substitution is, in absolute value,

$$0 \le \left|\lim_{n\to\infty} \sum_{k=1}^{n} \left(1+\arctan\frac{k}{n}\right) \sum_{m=1}^{\infty} (-1)^m \frac{1}{(2m+1)!\,(n+k)^{2m+1}}\right|$$

$$\le \lim_{n\to\infty} \sum_{m=1}^{\infty}\sum_{k=1}^{n} \left|\frac{1+\arctan 1}{(2m+1)!n^{2m+1}}\right| \le \lim_{n\to\infty} \sum_{m=1}^{\infty} \frac{2}{n^{2m}} = \lim_{n\to\infty} \frac{2}{n^2-1} = 0.$$

Let us write now the limit obtained after performing the substitution in the following way:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1+\arctan\frac{k}{n}}{1+\frac{k}{n}} \cdot \frac{1}{n}.$$

This is clearly the Riemann sum of the function $\frac{1+\arctan x}{1+x}$, evaluating the function at all upper extrema of intervals $\left(\frac{k-1}{n}, \frac{k}{n}\right)$, each interval with length $\frac{1}{n}$, for

$k = 1, 2, ..., n$, i.e, when x goes from 0 to 1. Therefore, the proposed limit is equal to

$$\int_0^1 \frac{1+\arctan x}{1+x} dx = \ln 2 + \int_0^1 \frac{\arctan x}{1+x} dx.$$

In order to calculate this second integral, we perform first the substitution $x = \tan \frac{\alpha}{2}$, yielding $\arctan x = \frac{\alpha}{2}$, $dx = \frac{d\alpha}{2\cos^2 \frac{\alpha}{2}}$, and lower and upper integration limits over α equal to 0 and $\frac{\pi}{2}$. Therefore,

$$\int_0^1 \frac{\arctan x}{1+x} dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\alpha d\alpha}{2\cos^2 \frac{\alpha}{2} + 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\alpha d\alpha}{1+\cos\alpha+\sin\alpha}$$

$$\frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{\alpha d\alpha}{1+\cos\alpha+\sin\alpha} - \frac{1}{2} \int_{\frac{\pi}{4}}^0 \frac{\left(\frac{\pi}{2} - \beta\right) d\beta}{1+\sin\beta+\cos\beta} = \frac{\pi}{4} \int_0^{\frac{\pi}{4}} \frac{d\alpha}{1+\cos\alpha+\sin\alpha},$$

where we have substituted $\alpha = \frac{\pi}{2} - \beta$ in the interval $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ first, and we have then substituted $\beta = \alpha$.

This final integral is east to find after performing the standard substitution $\tan \frac{\alpha}{2} = y$, resulting in $\cos\alpha = \frac{1-y^2}{1+y^2}$, $\sin\alpha = \frac{2y}{1+y^2}$, $d\alpha = \frac{2dy}{1+y^2}$, and lower and upper integration limits of 0 and $\tan \frac{\pi}{8}$:

$$\int_0^1 \frac{\arctan x}{1+x} dx = \frac{\pi}{4} \int_0^{\tan \frac{\pi}{8}} \frac{2dy}{(1+y^2)+(1-y^2)+2y} = \frac{\pi}{4} \int_0^{\tan \frac{\pi}{8}} \frac{dy}{1+y}$$

$$= \frac{\pi}{4} \ln\left(1 + \tan \frac{\pi}{8}\right) = \frac{\pi}{8} \ln 2,$$

since $\tan \frac{\pi}{8} = \sqrt{2} - 1$. This last result may be found by setting $2\gamma = \frac{\pi}{4}$ (and therefore $\tan(2\gamma) = 1$ in the well-known relation $\tan(2\gamma) = \frac{2\tan\gamma}{1-\tan^2\gamma}$, and solving for $\tan\gamma = \tan \frac{\pi}{8}$, keeping the positive root since $\frac{\pi}{8}$ is in the first quadrant. We thus finally arrive to

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(1 + \arctan \frac{k}{n}\right) \sin \frac{1}{n+k} = \left(1 + \frac{\pi}{8}\right) \ln 2.$$

Third solution by G.R.A.20 Math Problems Group, Roma, Italy

First we note that

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} \left(1 + \arctan \frac{k}{n}\right) \sin \frac{1}{n+k} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(1 + \arctan \frac{k}{n}\right) \frac{1}{n+k}$$

because $|\sin x - x| \leq x^2$ and

$$\begin{aligned}\left|\sum_{k=1}^{n}\left(1+\arctan\frac{k}{n}\right)\left(\sin\frac{1}{n+k}-\frac{1}{n+k}\right)\right| &\leq \sum_{k=1}^{n}\left(1+\frac{\pi}{4}\right)\frac{1}{(n+k)^2}\\ &\leq n\left(1+\frac{\pi}{4}\right)\frac{1}{n^2}\to 0.\end{aligned}$$

Moreover, the new limit is a limit of Riemann sums

$$\begin{aligned}L &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1+\arctan\frac{k}{n}}{1+\frac{k}{n}}\cdot\frac{1}{n}=\int_0^1\frac{1+\arctan x}{1+x}\,dx\\ &= \ln 2+\int_0^1\frac{\arctan x}{1+x}\,dx\\ &= \ln 2+[\arctan x\log(1+x)]_0^1-\int_0^1\frac{\ln(1+x)}{1+x^2}\,dx\\ &= \left(1+\frac{\pi}{4}\right)\ln 2-\int_0^1\frac{\ln(1+x)}{1+x^2}\,dx.\end{aligned}$$

Now, letting $x=\tan(\theta)$ we have that

$$\begin{aligned}\int_0^1\frac{\ln(1+x)}{1+x^2}\,dx &= \int_0^{\pi/4}\ln(1+\tan\theta)\,d\theta\\ &= \int_0^{\pi/4}\ln(\cos\theta+\sin\theta)\,d\theta-\int_0^{\pi/4}\ln(\cos\theta)\,d\theta\\ &= \int_0^{\pi/4}\ln\left(\sqrt{2}\cos(\frac{\pi}{4}-\theta)\right)\,d\theta-\int_0^{\pi/4}\ln(\cos\theta)\,d\theta\\ &= \int_0^{\frac{\pi}{4}}\ln\sqrt{2}\,d\theta+\int_0^{\frac{\pi}{4}}\ln\cos(\frac{\pi}{4}-\theta)\,d\theta-\int_0^{\frac{\pi}{4}}\ln\cos\theta\,d\theta\\ &= \int_0^{\pi/4}\ln\sqrt{2}\,d\theta=\frac{\pi}{8}\ln 2.\end{aligned}$$

Finally,

$$L=\left(1+\frac{\pi}{4}\right)\ln 2-\frac{\pi}{8}\ln 2=\left(1+\frac{\pi}{8}\right)\ln 2.$$

Fourth solution by Vishal Lama, Southern Utah University, USA

Let $L=\lim_{n\to\infty}\sum_{k=1}^{n}\big(1+\arctan\frac{k}{n}\big)\sin\frac{1}{n+k}$.

Now, let $E=\{1,2,\ldots,n\}$, and let $\{f_n\}, n=1,2,\ldots$, be a sequence of functions defined by

$$f_n(k)=(n+k)\sin\left(\frac{1}{n+k}\right),\ \forall k\in E.$$

We prove that $\{f_n\}$ uniformly converges to 1. Indeed, using the Taylor series expansion for $\sin x$, for any $k \in E$, we have

$$\left|(n+k)\sin\left(\frac{1}{n+k}\right)-1\right| = \left|(n+k)\left(\frac{1}{n+k}-\frac{1}{3!(n+k)^3}+\dots\right)-1\right|$$
$$= \left|\frac{1}{3!(n+k)^2}-\frac{1}{5!(n+k)^4}+\dots\right|$$
$$< \frac{1}{6(n+k)^2} < \frac{1}{n}.$$

Now, for $\epsilon > 0$, we choose the least integer $N > 1/\epsilon$. Note that our choice for N is independent of k. So, for any $\epsilon > 0, |f_n(k) - 1| < \epsilon \forall n \geq N$, for all $k \in E$. Hence, $\{f_n\}$ uniformly converges to 1.

Thus

$$L = \lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\arctan\frac{k}{n}\right)\frac{1}{n+k}.$$

Now, setting $x = k/n$ and considering the above summation as a Riemann sum, we have

$$L = \int_0^1 \frac{1+\tan^{-1}x}{1+x}\,dx = \int_0^1 (1+\tan^{-1}x)\,d(\ln(1+x))$$
$$= (1+\tan^{-1}x)\ln(1+x)\big|_0^1 - \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$$
$$= (1+\frac{\pi}{4})\ln 2 - I$$

To evaluate I, we use the substitution $x = \tan\theta$, to get

$$I = \int_0^{\pi/4} \ln(1+\tan\theta)\,d\theta$$
$$= \int_0^{\frac{\pi}{4}} \ln(1+\tan(\pi/4-\theta))d\theta \text{ using the identity } \int_0^a f(x)dx = \int_0^a f(a-x)dx$$
$$= \int_0^{\pi/4} \ln(\frac{2}{1+\tan\theta})\,d\theta$$
$$= \frac{\pi}{4}\ln 2 - I.$$

Thus $I = \frac{\pi}{8}\ln 2$, and therefore $L = (1+\frac{\pi}{4})\ln 2 - \frac{\pi}{8}\ln 2 = (1+\frac{\pi}{8})\ln 2.$

Remark. The evaluation of I above was Problem $A5$ in the Putnam Competition 2005.

Also solved by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

U70. For all integers $k, n \geq 2$ prove that

$$\sqrt[n]{1+\frac{n}{k}} \leq \frac{1}{n}\log\left(1+\frac{n}{k-1}\right)+1.$$

Proposed by Oleg Golberg, Massachusetts Institute of Technology, USA

First solution by Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

Let $n/k = x$, $0 < x \leq n/2$. Consider the function

$$f(x) = \frac{1}{n}\ln\Big(1+\frac{nx}{n-x}\Big)+1-(1+x)^{1/n}$$

and we study it for all $n \geq 2$. We have

$$1)\ \lim_{x\to 0} f(x) = 0, \qquad 2)\ f'(x) = \frac{1}{n}\Big(\frac{n^2}{(nx+n-x)(n-x)} - \frac{\sqrt[n]{1+x}}{1+x}\Big).$$

Recall Bernoulli's inequality: $(1+x)^a \leq 1+ax$ for $0 \leq a \leq 1$. Hence

$$\sqrt[n]{1+x} \leq 1+\frac{x}{n}.$$

To prove that the derivative of f is positive for any $0 < x < n$ and for any fixed $n \geq 2$, it is enough to prove that

$$\frac{n^2}{(nx+n-x)(n-x)} \geq \frac{1+\frac{x}{n}}{1+x},$$

$$n^3(x+1) \geq (nx+n-x)(n^2-x^2),$$

$$n^3x+n^3 \geq n^3x+n^3-n^2x-nx^3-nx^2+x^3,$$

$$n^2x+nx^2+nx^3 \geq x^3,$$

which is clearly true. Thus the derivative is positive and therefore $f(x) \geq 0$. The proof is completed.

Second solution by Oleg Golberg, Massachusetts Institute of Technology, USA

We will use the following simple result.

Lemma. For all positive a

$$\frac{1}{a+1} < \log(a+1) - \log a < \frac{1}{a}.$$

Let $f(x) = \log x$. Then $f'(x) = \frac{1}{x}$ and due to the Mean-Value theorem there exists $c \in (a, a+1)$ such that

$$\frac{1}{c} = f'(c) = \frac{\log(a+1) - \log a}{(a+1) - a} = \log(a+1) - \log a.$$

Since $c \in (a, a+1)$, it must be that

$$\frac{1}{a+1} < \frac{1}{c} < \frac{1}{a}.$$

Combining these two results yields the desired inequalities.

Returning to the problem, by the AM-GM inequality we have

$$\frac{k+1}{k} + \frac{k+2}{k+1} + \cdots + \frac{k+n}{k+n-1} \geq n\sqrt[n]{\frac{k+1}{k} \cdot \frac{k+2}{k+1} \cdots \frac{k+n}{k+n-1}} = n\sqrt[n]{\frac{k+n}{k}}.$$

We also have

$$\begin{aligned}\frac{k+1}{k} + \frac{k+2}{k+1} + \cdots + \frac{k+n}{k+n-1} &= 1 + \frac{1}{k} + 1 + \frac{1}{k+2} + \cdots + 1 + \frac{1}{k+n-1} \\ &= n + \left(\frac{1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{k+n-1}\right).\end{aligned}$$

Due to the result of the lemma, we have

$$\begin{aligned}\frac{1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{k+n-1} &\leq \log k - \log(k-1) + \log(k+1) - \log k + \cdots \\ &\quad + \log(k+n-1) - \log(k+n-2) \\ &= \log(k+n-1) - \log(k-1) = \log\left(1 + \frac{n}{k-1}\right).\end{aligned}$$

Combining the obtained results, we finally obtain

$$\begin{aligned}\sqrt[n]{1 + \frac{n}{k}} &\leq \frac{1}{n}\left(\frac{k+1}{k} + \frac{k+2}{k+1} + \cdots + \frac{k+n}{k+n-1}\right) \leq \frac{1}{n}\left(n + \log\left(1 + \frac{n}{k-1}\right)\right) \\ &= 1 + \frac{1}{n}\log\left(1 + \frac{n}{k-1}\right),\end{aligned}$$

and the inequality is proved.

U71. A polynomial $p \in \mathbb{R}[X]$ is called a "mirror" if $|p(x)| = |p(-x)|$. Let $f \in \mathbb{R}[X]$ and consider polynomials $p, q \in \mathbb{R}[X]$ such that $p(x) - p'(x) = f(x)$, and $q(x) + q'(x) = f(x)$. Prove that $p + q$ is a mirror polynomial if and only if f is a mirror polynomial.

Proposed by Iurie Boreico, Harvard University, USA

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

It is well known that a polynomial $p \in \mathbb{R}[X]$ has nonzero coefficients only for terms with even degree of x if and only if $p(x) = p(-x)$ for all x; we call such a polynomial an "even" polynomial (or polynomial with even symmetry). Similarly, a polynomial $p \in \mathbb{R}[X]$ has nonzero coefficients only for terms with odd degree of x if and only if $p(x) = -p(-x)$ for all x; we call such a polynomial an "odd"polynomial (or polynomial with odd symmetry).

Lemma. p is a mirror polynomial if and only if it is either odd or even.

Proof. if p is either odd or even, it is clearly a mirror. If p is a mirror, then either $p(x) = p(-x)$ for an infinitude of values of x, or $p(x) = -p(-x)$ for an infinite of values of x. In either case, either finite-degree polynomial $p(x) - p(-x)$ or finite-degree polynomial $p(x) + p(-x)$ has an infinite number of real roots, and needs to be thus identically zero, ie, either $p(x) - p(-x) = 0$ for all x (and p is even), or $p(x) + p(-x) = 0$ for all x (and p is odd), or both (and p is identically zero).

We prove our statement using induction on the degree n of f, which is by definition equal to the degree of p and q, since the degree of p' and q' is less than the degree of p and q, unless p and q, and therefore also f, are constant. By the previous argument, it is also clearly true that the highest degree of x has the same coefficient in f, p, q. When $n = 0$, $f(x) = p(x) = q(x) = \frac{p(x)+q(x)}{2}$ are constant, thus even, thus mirrors. When $n = 1$, write without loss of generality $f(x) = a_1x + a_0$ with $a_1 \neq 0$. Then $p(x) = a_1x + a_0 + a_1$ and $q(x) = a_1x + a_0 - a_1$, and $p(x) + q(x) = 2f(x) = 2a_1x + 2a_0$. Since neither f nor $p + q$ may be even, and f is odd if and only if $a_0 = 0$ or $p + q$ is odd, then f is a mirror polynomial if and only if $p + q$ is a mirror polynomial.

Assume now that the proposed result is true for all polynomials of degree less than $n \geq 2$, and write without loss of generality $f(x) = \sum_{k=0}^{n} a_k x^k$, where $a_n \neq 0$. Define now polynomials $r, s, \Delta f, \Delta p, \Delta q$ as follows: $\Delta f(x) = f(x) - a_n x^n$, $\Delta p(x) = p(x) - r(x)$, $\Delta q(x) = q(x) - s(x)$, where

$$r(x) = a_n \sum_{k=0}^{n} \binom{n}{k} k! x^{n-k},$$

$$s(x) = a_n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \, k! x^{n-k}.$$

Note that

$$r'(x) = a_n \sum_{k=0}^{n-1} \binom{n}{k} (n-k) \, k! x^{n-(k+1)} = a_n \sum_{l=1}^{n} \binom{n}{l} l! x^{n-l} = r(x) - a_n x^n,$$

$$s'(x) = -a_n \sum_{l=1}^{n} \binom{n}{l} (-1)^l \, l! x^{n-l} = a_n x^n - s(x),$$

where we have performed the substitution $k = l - 1$. With this definitions, it is clear that the degree of $\Delta f, \Delta p, \Delta q$ is less than the degree of f, p, q, while

$$\Delta p(x) - \Delta p'(x) = p(x) - p'(x) - r(x) + r'(x) = f(x) - a_n x^n = \Delta f(x),$$

$$\Delta q(x) + \Delta q'(x) = q(x) + q'(x) - s(x) - s'(x) = f(x) - a_n x^n = \Delta f(x),$$

$$\Delta p(x) + \Delta q(x) = p(x) + q(x) - a_n \sum_{k=0}^{n} \binom{n}{k} \left(1 - (-1)^k\right) k! x^{n-k}.$$

From the first two of the last relations, we find that $\Delta f, \Delta p, \Delta q$ satisfy the conditions given in the problem, and from the third that the only terms that are different in $\Delta p + \Delta q$ with respect to $p + q$ are those whose degree has the same parity as n. Therefore, f is a mirror if and only if f is even or odd, if and only if Δf is even or odd (we obtain Δf by making 0 one coefficient in f which has the same parity of all other nonzero coefficients in f), if and only if $\Delta p + \Delta q$ is even or odd (by hypothesis of induction), if and only if $p + q$ is even or odd (because $\Delta p + \Delta q$ is obtained by modifying only coefficients of $p + q$ that have the same parity as n), if and only if $p + q$ is a mirror, and we are done.

Second solution by Iurie Boreico, Harvard University, USA

The condition $|p(x)| = |p(-x)|$ is equivalent to $p^2(x) = p^2(-x)$ i.e. $(p(x) - p(-x))(p(x) + p(-x)) = 0$. This can happen only when one of the two factors is identically 0, so either $p(x) = p(-x)$, or $p(x) = -p(-x)$. By comparing the coefficients of the two polynomials, this can happen if and only if all monomials appearing if f are either of even degree, or of odd degree. Thus f is a mirror polynomial if and only if $f(x) = g(x^2)$ or $f(x) = xg(x^2)$ for some polynomial g.

The next idea is that p and q can be exhibited in a rather explicit form. For example, p must be unique, because if $p_1 - p_1' = p_2 - p_2'$, then $(p_1 - p_2) = (p_1 - p_2)'$ and a polynomial equals its derivative if and only if it is identically zero. Then, clearly the sum $p = f(x) + f'(x) + f''(x) + \ldots$ is finite, as the higher-order derivatives of a polynomials are all eventually zero. Moreover, $p - p' = (f + f' + \ldots) - (f' + f" + \ldots) = f$. Thus we have found p, and analogously $q = f - f' + f'' - \ldots$ and hence $p + q = 2(f + f'' + \ldots)$ (the sum of all derivatives of even order).

We are left to prove that f is a mirror polynomial if and only if $f + f'' + \ldots$ is a mirror polynomial.

If f is a mirror polynomial, then all monomials appearing in f have either even degree, or odd. Since differentiating twice preserves the parity of the degree, all monomials appearing in $f'', \ldots$ are also all even or odd (according to whether f is even or odd), and so the sum of all these polynomials has all monomials of even degree, or odd degree, thus $p + q$ is a mirror polynomial.

For the converse, observe that $2f = (p + q) - (p + q)''$. If $p + q$ is mirror polynomial, then $(p + q)''$ is a mirror polynomial of the same type, and hence so is their difference $2f$.

Remark. It can be proved directly that $2f = (p+q)-(p+q)''$, by differentiating the initial relations and manipulating, and by the method exposed above of computing p and q, we can conclude that $p + q = 2(f + f'' + \ldots)$.

U72. Let n be an even integer. Evaluate

$$\lim_{x\to-1}\left[\frac{n(x^n+1)}{(x^2-1)(x^n-1)}-\frac{1}{(x+1)^2}\right].$$

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

First solution by Daniel Campos Salas, Costa Rica

Since n is even we have that

$$\lim_{x\to-1}\left[\frac{n(x^n+1)}{(x^2-1)(x^n-1)}-\frac{1}{(x+1)^2}\right]=\lim_{x\to1}\left[\frac{n(x^n+1)}{(x^2-1)(x^n-1)}-\frac{1}{(x-1)^2}\right].$$

Note that for $x\neq 1$ we have that

$$\begin{aligned}\frac{n(x^n+1)}{(x^2-1)(x^n-1)}-\frac{1}{(x-1)^2} &= \frac{n(x^n+1)-(x+1)\sum_{i=0}^{n-1}x^i}{(x-1)^2(x+1)\sum_{i=0}^{n-1}x^i}\\ &= \frac{\sum_{i=0}^{n-1}x^n+1-x^i-x^{n-i}}{(x-1)^2(x+1)\sum_{i=0}^{n-1}x^i}\\ &= \frac{\sum_{i=0}^{n-1}(x^i-1)(x^{n-i}-1)}{(x-1)^2(x+1)\sum_{i=0}^{n-1}x^i}\\ &= \frac{\sum_{i=1}^{n-1}(x^i-1)(x^{n-i}-1)}{(x-1)^2(x+1)\sum_{i=0}^{n-1}x^i}\end{aligned}$$

$$= \frac{(x-1)^2 \sum_{i=1}^{n-1} \left(\sum_{j=0}^{i-1} x^j\right)\left(\sum_{k=0}^{n-i-1} x^k\right)}{(x-1)^2(x+1)\sum_{i=0}^{n-1} x^i}$$

$$= \frac{\sum_{i=1}^{n-1} \left(\sum_{j=0}^{i-1} x^j\right)\left(\sum_{k=0}^{n-i-1} x^k\right)}{(x+1)\sum_{i=0}^{n-1} x^i}.$$

It follows that

$$\begin{aligned}
\lim_{x\to 1}\left[\frac{n(x^n+1)}{(x^2-1)(x^n-1)} - \frac{1}{(x-1)^2}\right] &= \lim_{x\to 1}\left[\frac{\sum_{i=1}^{n-1} \left(\sum_{j=0}^{i-1} x^j\right)\left(\sum_{k=0}^{n-i-1} x^k\right)}{(x+1)\sum_{i=0}^{n-1} x^i}\right] \\
&= \frac{\sum_{i=1}^{n-1} i\cdot(n-i)}{2n} \\
&= \frac{n\cdot\frac{n(n-1)}{2} - \frac{n(n-1)(2n-1)}{6}}{2n} \\
&= \frac{n^2-1}{12},
\end{aligned}$$

and we are done.

Second solution by G.R.A.20 Math Problems Group, Roma, Italy
Let

$$f_n(x) = \frac{n(x^n+1)}{(x^2-1)(x^n-1)} - \frac{1}{(x+1)^2}.$$

Let $y = -1 - x$ then, since n is even,

$$\begin{aligned}
f_n(x) &= \frac{n((1+y)^n+1)}{((1+y)^2-1)((1+y)^n-1)} - \frac{1}{y^2} \\
&= \frac{n(2+y\sum_{k=1}^{n}\binom{n}{k}y^{k-1}))}{y^2(y+2)(\sum_{k=1}^{n}\binom{n}{k}y^{k-1})} - \frac{1}{y^2} \\
&= \frac{n(2+y\sum_{k=1}^{n}\binom{n}{k}y^{k-1})) - (y+2)(\sum_{k=1}^{n}\binom{n}{k}y^{k-1})}{y^2(y+2)(\sum_{k=1}^{n}\binom{n}{k}y^{k-1})} \\
&= \frac{2n + ((n-1)y-2)(n+\binom{n}{2}y+\binom{n}{3}y^2+o(y^2))}{2ny^2+o(y^2)} \\
&= \frac{2n+n(n-1)y+\binom{n}{2}(n-1)y^2-2n-n(n-1)y-2\binom{n}{3}y^2+o(y^2))}{2ny^2+o(y^2)} \\
&= \frac{\binom{n}{2}(n-1)y^2-2\binom{n}{3}y^2+o(y^2))}{2ny^2+o(y^2)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{x\to -1} f_n(x) &= \lim_{y\to 0}\left(\frac{\binom{n}{2}(n-1)y^2-2\binom{n}{3}y^2+o(y^2))}{2ny^2+o(y^2)}\right) \\
&= \frac{\binom{n}{2}(n-1)-2\binom{n}{3}}{2n} = \frac{n^2-1}{12}.
\end{aligned}$$

Also solved by Arkady Alt, San Jose, California, USA; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Courtis G. Chryssostomos, Larissa, Greece; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Vicente Vicario Garca, Huelva, Spain; Brian Bradie, VA, USA.

Olympiad problems

O67. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \ldots + a_n = 0$. Prove that for $a \geq 0$, $\;a + a_1^2 + a_2^2 + \ldots + a_n^2 \geq m(|a_1| + |a_2| + \ldots + |a_n|)$, where $m = 2\sqrt{\dfrac{a}{n}}$, if n is even, and $m = 2\sqrt{\dfrac{an}{n^2-1}}$, if n is odd.

Proposed by Pham Kim Hung, Stanford University, USA

Solution by Pham Kim Hung, Stanford University, USA

The first step is to dismiss the absolute value sign. We separate the sequence $a_1, a_2, \ldots, a_n$ into a sequence of non-negative real numbers $x_1, x_2, ..., x_k$ and a sequence of negative real numbers $y_1, y_2, \ldots, y_{n-k}$. Denote $z_j = -y_j$, $j \in \{1, 2, \ldots, n-k\}$, we have to prove that

$$\sum_{i=1}^{k} x_i^2 + \sum_{j=1}^{n-k} z_j^2 + a \geq m \sum_{i=1}^{k} x_i + m \sum_{j=1}^{n-k} z_j.$$

Denote $x = \frac{1}{k} \sum\limits_{i=1}^{k} z_i$ and $z = \frac{1}{n-k} \sum\limits_{j=1}^{n-k} z_j$. Clearly,

$$\sum_{i=1}^{k} x_i^2 \geq k \left(\frac{x_1 + x_2 + ... + x_k}{k} \right)^2 = kx^2,$$

$$\sum_{j=1}^{n-k} z_i^2 \geq (n-k) \left(\frac{z_1 + z_2 + ... + z_{n-k}}{n-k} \right)^2 = (n-k)z^2.$$

After all, we would like to prove that

$$kx^2 + (n-k)z^2 + a \geq m(kx + (n-k)z).$$

From the the condition $kx = (n-k)z$, as $a_1 + a_2 + ... + a_n = 0$, the above inequality becomes

$$kx^2 \left(1 + \frac{k}{n-k} \right) + a \geq 2mkx.$$

Using the AM-GM inequality we get

$$\text{LHS} \geq 2\sqrt{a}x \sqrt{k \left(1 + \frac{k}{n-k} \right)} = \frac{2k\sqrt{an}x}{\sqrt{k(n-k)}}.$$

If n is even, the maximum of $k(n-k)$ is $\frac{n^2}{4}$. If n is odd, the maximum of $k(n-k)$ is $\frac{n^2-1}{4}$. The conclusion follows.

Also solved by Kee Wai Lau, Hong Kong, China; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy

O68. Let $ABCD$ be a quadrilateral and let P be a point in its interior. Denote by K, L, M, N the orthogonal projections of P onto lines AB, BC, CD, DA, and by H_a, H_b, H_c, H_d the orthocenters of triangles AKN, BKL, CLM, DMN, respectively. Prove that H_a, H_b, H_c, H_d are the vertices of a parallelogram.

Proposed by Mihai Miculita, Oradea, Romania

First solution by Son Hong Ta, Hanoi University of Education, Vietnam

We have $PK \perp AB$ and $NH_a \perp AK$, so $PK \parallel NH_a$. We also have $PN \perp AD$, $KH_a \perp AN$, so $PN \parallel KH_a$. Hence, we deduce that the quadrilateral PKH_aN is a parallelogram. Similarly, the quadrilateral PLH_bK is also a parallelogram. It implies that NH_a is parallel and equal to LH_b (both are parallel and equal to PK). Thus means NH_aH_bL is a parallelogram. Similarly, NH_dH_cL is a parallelogram. Therefore we conclude that H_aH_b and H_cH_d are parallel and equal. Hence, H_a, H_b, H_c, H_d are the vertices of a parallelogram, as desired.

Second solution by Daniel Campos Salas, Costa Rica

Note that $H_aN||KP$ and $H_aK||NP$, hence H_aKPN is a parallelogram. This implies that $H_aK = PN$ and $H_aK||PN$. Analogously, it follows that $H_dM = PN$ and $H_dM||PN$. Thus t $H_aK = H_dM$ and $H_aK||H_dM$.

Similarly, $H_bK = H_cM$ and $H_bK||H_cM$. Therefore, triangles H_aH_bK and H_dH_cM are congruent and with all of its correspondent sides parallel. In particular, H_aH_b is parallel to H_dH_c. Analogously, we prove that H_aH_d is parallel to H_bH_c and this completes the proof.

Third solution by Andrei Iliasenco, Chisinau, Moldova

Let us prove that $H_aH_b \parallel NL$. Denote by O_a, O_b, O_c, O_d the circumcircles of triangles AKN, BKL, CLM, and DMN, respectively, and by G_a, G_b, G_c, G_d the gravity gravity of these triangles, respectively.

Using following properties:

- G is between H and O and $OH = 3OG$
- O_a is midpoint of AP and O_b is midpoint of BP

we get

$$
\begin{aligned}
\overrightarrow{H_aH_b} &= \overrightarrow{H_aO_a} + \overrightarrow{O_aA} + \overrightarrow{AB} + \overrightarrow{BO_b} + \overrightarrow{O_bH_b} \\
&3\overrightarrow{G_aO_a} + \overrightarrow{O_aA} + \overrightarrow{AB} + \overrightarrow{BO_b} + 3\overrightarrow{O_bG_b} \\
&= 3\frac{\overrightarrow{AO_a} + \overrightarrow{KO_a} + \overrightarrow{NO_a}}{3} + \overrightarrow{O_aA} + \overrightarrow{AB} + \overrightarrow{BO_b} + 3\frac{\overrightarrow{O_bB} + \overrightarrow{O_bK} + \overrightarrow{O_bL}}{3} \\
&= \overrightarrow{AO_a} + \overrightarrow{KO_a} + \overrightarrow{NO_a} + \overrightarrow{O_aA} + \overrightarrow{AB} + \overrightarrow{BO_b} + \overrightarrow{O_bB} + \overrightarrow{O_bK} + \overrightarrow{O_bL} \\
&= \frac{\overrightarrow{AO_a} + \overrightarrow{OA_a}}{2} + \frac{\overrightarrow{KP} + \overrightarrow{KA}}{2} + \frac{\overrightarrow{NP} + \overrightarrow{NA}}{2} \\
&\quad + \frac{\overrightarrow{AB} + \overrightarrow{AB}}{2} + \frac{\overrightarrow{BO_b} + \overrightarrow{O_bB}}{2} + \frac{\overrightarrow{PK} + \overrightarrow{BK}}{2} + \frac{\overrightarrow{PL} + \overrightarrow{BL}}{2} \\
&= \frac{\overrightarrow{KP} + \overrightarrow{PK}}{2} + \frac{\overrightarrow{KA} + \overrightarrow{AB} + \overrightarrow{BK}}{2} + \frac{\overrightarrow{NP} + \overrightarrow{PL}}{2} + \frac{\overrightarrow{NA} + \overrightarrow{AB} + \overrightarrow{BL}}{2} \\
&= \frac{\overrightarrow{NL}}{2} + \frac{\overrightarrow{NL}}{2} = \overrightarrow{NL}.
\end{aligned}
$$

Analogously, $H_cH_d \parallel NL \parallel H_aH_b$ and $H_aH_c \parallel KM \parallel H_bH_d$, hence $H_aH_bH_cH_d$ is a parallelogram.

Also solved by Salem Malikic, Sarajevo, Bosnia and Herzegovina; Daniel Lasaosa, Universidad Publica de Navarra, Spain

O69. Find all integers a, b, c for which there is a positive integer n such that

$$\left(\frac{a+bi\sqrt{3}}{2}\right)^n = c + i\sqrt{3}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Romania

Solution by Titu Andreescu, University of Texas at Dallas, USA and Dorin Andrica, Babes-Bolyai University, Romania

If $n = 1$, then we get $a = 2c$, $b = 2$, where c is any integer.

If $n \geq 2$, then taking the absolute values in both sides we obtain

$$\left(\frac{a^2+3b^2}{2}\right)^n = c^2 + 3$$

which is a Diophantine equation of the form

$$x^2 + 3 = y^n \tag{1}$$

For $n = 2$ the solutions (x, y) are $(1, 2)$, $(1, -2)$, $(-1, 2)$ and $(-1, -2)$. In this case we get $(a, b, c) = (\pm 1, \pm 1, 1)$, $(\pm 1, \pm 1, -1)$, $(\pm 2, 0, 1)$, $(\pm 2, 0, -1)$.

For n even, $n \geq 4$, the equation is not solvable, since no other squares differ by 3. For n odd, $n \geq 3$, we may assume that n is a prime p. Indeed, if $n = qk$, where q is an odd prime, we obtain an equation of the same type:

$$x^2 + 3 = (y^k)^q.$$

We will use the uniqueness of prime factorization in the ring R of integers of $\mathbb{Q}[\sqrt{-3}]$. It is known that the integers in $\mathbb{Q}[\sqrt{-3}]$ are $\dfrac{\alpha+\beta\sqrt{-3}}{2}$, where α and β are integers of the same parity. Write the equation as

$$(x + \sqrt{-3})(x - \sqrt{-3}) = y^p,$$

where $u = \dfrac{\alpha+3\beta^2}{4}$.

Clearly, x must be even, otherwise $x^2 + 3 \equiv 4 \pmod 8$, while $y^p \equiv 0 \pmod 8$.

The equation $x^2 - x + 1 = y^3$ is equivalent to

$$(2x-1)^2 + 3 = 4y^3,$$

that is

$$\frac{(2x-1)+\sqrt{-3}}{2}\cdot\frac{(2x-1)-\sqrt{-3}}{2}=y^3. \tag{2}$$

Let

$$d=\gcd\left(\frac{2x-1+\sqrt{-3}}{2},\frac{2x-1-\sqrt{-3}}{2}\right).$$

Then

$$d|\left(\frac{2x-1+\sqrt{-3}}{2}-\frac{2x-1-\sqrt{-3}}{2}\right)=\sqrt{-3}.$$

Hence $N(d)|N(\sqrt{-3})$, that is $d^2|3$. It follows that $d=1$, i.e. the integers $\frac{2x-1+\sqrt{-3}}{2}$ and $\frac{2x-1-\sqrt{-3}}{2}$ are relatively prime in R, the ring of integers of $\mathbb{Q}[\sqrt{-3}]$.

Using the uniqueness of prime factorization in R, we get

$$\frac{2x-1+\sqrt{-3}}{2}=w^k\left(\frac{\alpha+\beta\sqrt{-3}}{2}\right)^3 \tag{3}$$

and

$$\frac{2x-1-\sqrt{-3}}{2}=w^{6-k}\left(\frac{\alpha-\beta\sqrt{-3}}{2}\right)^3,$$

where $w=\frac{-1+\sqrt{-3}}{2}$ and $\frac{\alpha^2+3\beta^2}{4}=y$.

Then $\gcd(x+\sqrt{-3},x-\sqrt{-3})=1$ and

$$x+\sqrt{-3}=w^k\left(\frac{\alpha+\beta\sqrt{-3}}{2}\right)^p,\quad x-\sqrt{-3}=w^{6-k}\left(\frac{\alpha-\beta\sqrt{-3}}{2}\right)^p,$$

where $w=\frac{-1+\sqrt{-3}}{2}$. The first relation can be written as

$$x+\sqrt{-3}=\left(\frac{m+n\sqrt{-3}}{2}\right)^p, \tag{4}$$

for some integers m and n of the same parity.

Indeed, for each $k\in\{0,1,\ldots,5\}$, there is a positive integer s such that $w^k=w^{sp}$. The choice of s depends upon the residue of p modulo 6. If $p\equiv 1 \pmod 6$ we take $s=k$, while for $p\equiv 5 \pmod 6$ we take $s=6-k$.

Taking the conjugate in (4) we obtain

$$x-\sqrt{-3}=\left(\frac{m-n\sqrt{-3}}{2}\right)^p,$$

hence

$$2\sqrt{-3} = \left(\frac{m+n\sqrt{-3}}{2}\right)^p - \left(\frac{m-n\sqrt{-3}}{2}\right)^p.$$

Factoring the expression in the right-hand side as

$$A^p - B^p = (A-B)(A^{p-1} + A^{p-2}B + \cdots + AB^{p-2} + B^{p-1})$$

we get $2\sqrt{-3} = n\sqrt{-3}\cdot u$, where u is an integer in $\mathbb{Q}[\sqrt{-3}]$. It follows that $2 = n\cdot u$, and so $N(2) = N(n\cdot u) = N(n)\cdot N(u)$, i.e. $4 = n^2 N(u)$. Hence $n|2$.

For $n = \pm 1$, from (1) we obtain

$$\pm 2^p = \binom{p}{1} m^{p-1} - 3\binom{p}{3} m^{p-3} + \cdots + (-3)^{\frac{p-1}{2}}. \qquad (5)$$

Looking modulo p, from Fermat's Little Theorem we get

$$\pm 2 \equiv (-3)^{\frac{p-1}{2}} \pmod p,$$

hence $4 \equiv (-3)^{p-1} \equiv 1 \pmod p$, so $p = 3$.

The equation becomes $x^2 + 3 = y^3$. This equation is not solvable for $y \equiv 1 \pmod 4$. Hence $y \equiv 3 \pmod 4$ and $x^2 + 4 = y^3 + 1 = (y+1)(y^2 - y + 1)$, which is again impossible, since $y^2 - y + 1$ is of the form $4m + 3$ and it cannot divide the sum of squares $x^2 + 4$.

For $n = \pm 2$, $m = 2a$ and (4) becomes

$$x + \sqrt{-3} = (a + \sqrt{-3})^p,$$

so

$$1 = \binom{p}{1} a^{p-1} - 3\binom{p}{3} a^{p-3} + 9\binom{p}{5} a^{p-5} - \cdots + (-3)^{\frac{p-1}{2}}. \qquad (6)$$

Clearly, $3 \nmid a$, so $a^2 \equiv 1 \pmod 3$. From (6), we get $1 \equiv pa^{p-1} \pmod 3$, hence $p \equiv 1 \pmod 3$. Let $p = 3^u \cdot 2q + 1$, where $3 \nmid q$. Looking at (6) modulo $3^{\mu+2}$ we get

$$1 \equiv pa^{p-1} + \frac{p-1}{2} a^{p-3} \pmod{3^{\mu+2}}. \qquad (7)$$

Indeed, $3^{\mu+2} | 9\binom{p}{5}$ and

$$3\binom{p}{3} = \frac{p-1}{2} p(p-2) = \frac{p-1}{2}[(p-1)^2 - 1] \equiv -\frac{p-1}{2} \pmod{3^{\mu+2}}.$$

We have

$$a^{p-1} = (a^2)^{\frac{p-1}{2}} = (1+3k)^{3^\mu q} \equiv 1 + 3^{\mu+1} kq \pmod{3^{\mu+2}}.$$

Multiplying (7) by $a^2 = 1 + 3k$ and looking mod $3^{\mu+2}$, we obtain

$$\frac{p-1}{2}a^{p-1} \equiv 3^\mu q(1 + 3^{\mu+1}kq) \equiv 3^\mu q \pmod{3^{\mu+2}}. \tag{8}$$

On the other hand,

$$a^2(pa^{p-1} - 1) \equiv -\frac{p-1}{2}a^{p-1} \pmod{3^{\mu+2}},$$

and

$$a^2(pa^{p-1} - 1) = (1 + 3k)[p(1 + 3k)^{\frac{p-1}{2}} - 1] = (1 + 3k)[p(1 + 3k)^{3^\mu q} - 1]$$

$$\equiv (1 + 3k)p + (1 + 3k)p \cdot 3^{\mu+1}kq - (1 + 3k) \pmod{3^{\mu+2}}$$

$$\equiv (1 + 3k)(p - 1) + pkq \cdot 3^{\mu+1} \pmod{3^{\mu+2}}$$

$$\equiv 3^\mu \cdot 2q + 3^{\mu+1} \cdot 2kq + (3^\mu \cdot 2q + 1)kq3^{\mu+1} \pmod{3^{\mu+2}}$$

$$\equiv 3^\mu \cdot 2q + 3^{\mu+1}(2kq + kq) \pmod{3^{\mu+2}} \equiv 3^\mu \cdot 2q \pmod{3^{\mu+2}}.$$

Using (8) we obtain

$$-3^\mu q \equiv 3^\mu \cdot 2q \pmod{3^{\mu+2}},$$

hence $3^{\mu+2}|3^{\mu+1}q$, i.e. $3|q$, a contradiction.

In conclusion, the equation is not solvable for $n \geq 3$.

O70. In triangle ABC let M_a, M_b, M_c be the midpoints of BC, CA, AB, respectively. The incircle (I) of triangle ABC touches the sides BC, AC, AB at points A', B', C'. The line r_1 is the reflection of line BC in AI, and line r_2 is the perpendicular from A' to IM_a. Denote by X_a the intersection of r_1 and r_2, and define X_b and X_c analogously. Prove that X_a, X_b, X_c lie on a line that is tangent to the incircle of triangle ABC.

Proposed by Jan Vonk, Ghent University, Belgium

Solution by Daniel Lasaosa, Universidad Publica de Navarra, Spain

The theorem of Thales ensures that $M_aM_b \parallel AB$, $M_bM_c \parallel BC$ and $M_cM_a \parallel CA$, or triangles $M_aM_bM_c$ and ABC are homothetic. Denote by A'', B'', C'' the respective reflections of A', B', C' in AI, BI, CI. By symmetry with respect to AI, it is clear that A'' is the point where r_1 touches the incircle. The lines r_1, AB, AC determine a triangle equal to ABC, where the side on r_1 corresponds to BC, the side on AB corresponds to AC and the side on AC corresponds to AB. Furthermore, the length of the segment on r_1 between its intersection with AC and A'' has the same length as BA' by the symmetry with respect to AI. Since r_1 and AC form an angle equal to $\angle B$ also by symmetry around AI, then the distance from A'' to AC is $BA' \sin B$. Similarly, we may show that the distance from C'' to AC is $BC' \sin B$. Using the fact that $BA' = BC' = \frac{c+a-b}{2}$, we conclude that $C''A'' \parallel CA$. Analogously, $A''B'' \parallel AB$ and $B''C'' \parallel BC$, or triangles $A''B''C''$ and ABC are homothetic. Therefore triangle $A''B''C''$ and $M_aM_bM_c$ are also homothetic. Lines M_aA'', M_bB'' and M_cC'' meet at a point U; with respect to P, the circumcircle of $M_aM_bM_c$ (i.e., the nine-point circle) is the result of scaling the circumcircle of $A''B''C''$ (i.e., the incircle). Since the nine-point circle and the incircle touch at the Feuerbach point, then U is the Feuerbach point, and it is clearly on the incircle.

Denote by P_a, P_b, P_c the respective midpoints of IM_a, IM_b, IM_c. By Thales' theorem, $P_bP_c \parallel M_bM_c \parallel BC$, and similarly $P_cP_a \parallel M_cM_a \parallel CA$ and $P_aP_b \parallel M_aM_b \parallel AB$. Denote by Q_a, Q_b, Q_c the respective midpoints of IA'', IB'', IC''. Again by Thales' theorem, $Q_aQ_b \parallel A''B'' \parallel AB$, $Q_bQ_c \parallel B''C'' \parallel BC$, and $Q_cQ_a \parallel C''A'' \parallel CA$, or triangles ABC, $M_aM_bM_c$, $A''B''C''$, $P_aP_bP_c$ and $Q_aQ_bQ_c$ are pairwise homothetic. Furthermore, since triangles $P_aP_bP_c$, $Q_aQ_bQ_c$ are the result of scaling triangles $M_aM_bM_c$ and $A''B''C''$ with respect to the incenter with scale factor $\frac{1}{2}$, then the point V, where lines P_aQ_a, P_bQ_b and P_cQ_c meet, is the midpoint of IU, i.e., the midpoint between the incenter and the Feuerbach point.

Consider now the inversion of r_1 and r_2 with respect to the incircle. Since r_1 is tangent to the incircle at A'', the result of performing the inversion is a circle through I and touching the incircle at A'', or the inversion of r_1 yields the circle with diameter IA'', ie, the circle with center Q_a and radius $\frac{r}{2}$, r being the inradius of ABC. Since $\angle IA'M_a = \frac{\pi}{2}$, A' is on the circle with diameter IM_a. Thus, r_2 is the line that contains a chord of this circle, and by symmetry around IM_a, the reflection of A' in IM_a is on r_2, on the circle diameter IM_a, and on the incircle. Therefore, the inversion of r_2 with respect to the incircle yields the circle with diameter IM_a, or the circle through I with center P_a. Now, the circles through I with centers P_a and Q_a, meet at I, and at a second point that we will call Y_a, which is the result of performing the inversion of X_a with respect to the incircle. Note that P_aQ_a is the perpendicular bisector of IY_a. The circles through I with centers P_b and Q_b meet at I and at Y_b, and the circles through I with centers P_c and Q_c meet at I and at Y_c, where Y_b, Y_c are the results of inverting X_b, X_c with respect to the incircle; furthermore, P_bQ_b and P_cQ_c are the respective perpendicular bisectors of IY_b, IY_c. Therefore, since the perpendicular bisectors P_aQ_a, P_bQ_b and P_cQ_c of IY_a, IY_b and IY_c meet at V, then V is the circumcenter of $IY_aY_bY_C$, or the inverse of the circumcircle of $IY_aY_bY_C$ with respect to the incircle is a line through X_a, X_b and X_c. Furthermore, the circumcircle of $IY_aY_bY_C$ has center V at a distance $\frac{r}{2}$ of I, and radius $IV = \frac{r}{2}$, where r is the inradius of ABC, or the circumcircle of $IY_aY_bY_C$ touches the incircle at U, and its inverse, the line through X_a, X_b and X_c, touches the incircle at the Feuerbach point U.

O71. Let n be a positive integer. Prove that $\displaystyle\sum_{k=1}^{n-1} \frac{1}{\cos^2 \frac{k\pi}{2n}} = \frac{2}{3}(n^2-1).$

Proposed by Dorin Andrica, Babes-Bolyai University, Romania

First solution by John Mangual, New York, USA

From De Moivre's fomula and calculus one can show:

$$\frac{\sin 2n\theta}{\sin\theta} = (-1)^{n+1}\left[2n\cos\theta - \frac{2^3 n(n^2-1)}{3!}\cos^3\theta + \dots\right]$$

Then we can define a function

$$f(x) = (-1)^{n+1}2nx\left[1 - \frac{2^2(n^2-1)}{3!}x^2 + \frac{2^4(n^2-1)(n^2-2^2)}{5!}x^4 - \dots\right]$$

In fact, $f(x)$ is a polynomial of degree $2n-1$ with the roots $x = \cos\frac{k\pi}{2n}$, where $1 \le k \le 2n-1$.

Consider $\frac{1}{x}f(\frac{1}{x})$ and substitute $y = \frac{1}{x^2}$, then

$$xf(x) = g(y) = y^{n-1} - \frac{2^2(n^2-1)}{3!}y^{n-3} + \frac{2^4(n^2-1)(n^2-2^2)}{5!}x^{2n-5} - \dots$$

This is a polynomial of degree $n-1$ in y whose roots are $y = \sec^2\frac{k\pi}{2n}$ with $1 \le k \le n-1$. Using Vieté's theorem we find that the sum of coefficients is

$$\frac{2}{3}(n^2-1) = \frac{2^2(n^2-1)}{3!} = \sum_{i=1}^{n-1}\sec^2\frac{k\pi}{2n},$$

and we are done.

Second solution by Arkady Alt, California, USA

Note that for any polynomial $P(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$, $a_0 \neq 0$ with non-zero roots $x_1, x_2, \dots, x_n$ we have

$$\sum_{i=1}^{n}\frac{1}{x_i} = -\frac{P'(0)}{P(0)}. \quad (1)$$

Let $P(x) = a_0(x-x_1)(x-x_2)\dots(x-x_n)$, then

$$\sum_{i=1}^{n}\frac{1}{x-x_i} = \sum_{i=1}^{n}(\ln(x-x_i))' = \left(\sum_{i=1}^{n}\ln(x-x_i)\right)' = \left(\ln\frac{P(x)}{a_0}\right)' = \frac{P'(x)}{P(x)},$$

and plugging $x = 0$ the conclusion follows.

Let $U_n(x) := \dfrac{T'_{n+1}(x)}{n+1} = \dfrac{\sin(n+1)\varphi}{\sin\varphi}$ be the Chebishev Polynomial of the Second Kind. Then $U_n(x)$ satisfies to recurrence

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), n \in \mathbb{N} \text{ and } U_0(x) = 1, U_1(x) = 2x.$$

Because $\dfrac{\sin n\varphi}{\sin\varphi} = 0$ if and only if $\varphi = \dfrac{k\pi}{n}, n \in \mathbb{Z}$, we get $U_{n-1}(x) = 0$ if and only if $x = \cos\dfrac{k\pi}{n}$, $k = 1, 2, \ldots, n-1$ and

$$U_{n-1}(x) = 2^{n-1}\left(x - \cos\frac{\pi}{n}\right)\left(x - \cos\frac{2\pi}{n}\right)\ldots\left(x - \cos\frac{(n-1)\pi}{n}\right),$$

as the coefficient of x^n in $U_n(x)$ is 2^{n-1}.

In particular,

$$\begin{aligned} U_{2n-1}(x) &= 2^{2n-1}\prod_{k=1}^{2n-1}\left(x - \cos\frac{k\pi}{2n}\right) \\ &= 2^{2n-1}\left(x - \cos\frac{n\pi}{2n}\right)\prod_{k=1}^{n-1}\left(x - \cos\frac{k\pi}{2n}\right)\prod_{k=1}^{n-1}\left(x - \cos\frac{(2n-k)\pi}{2n}\right) \\ &= 2^{2n-1}x\prod_{k=1}^{n-1}\left(x^2 - \cos^2\frac{k\pi}{2n}\right). \end{aligned}$$

Let $P_n(x) := \dfrac{U_{2n-1}(\sqrt{x})}{2\sqrt{x}}$, then $P_n(x) = 4^{n-1}\prod_{k=1}^{n-1}\left(x - \cos^2\dfrac{k\pi}{2n}\right)$.

Note that $U_{2n-1}(x)$ can be defined by the recurrence

$$U_{2n+1}(x) = 2\left(2x^2 - 1\right)U_{2n-1}(x) - U_{2n-3}(x), \text{ with } U_{-1}(x) = 0, U_1(x) = 2x.$$

Since $U_{2n-1}(x)$ is divisible by $2x$, then polynomial $P_n(x)$ satisfy the recurrence

$$P_{n+1}(x) = 2(2x-1)P_n(x) - P_{n-1}(x), n \in \mathbb{N} \text{ with } P_0(x) = 0, P_1(x) = 1. \quad (2)$$

Thus applying (1) to the polynomial $P_n(x)$ we obtain

$$\sum_{k=1}^{n-1}\frac{1}{\cos^2\frac{k\pi}{2n}} = -\frac{P'_n(0)}{P_n(0)}.$$

In particularly, from (2) follows recurrence

$$P_{n+1}(0) + 2P_n(0) + P_{n-1}(0) = 0, n \in \mathbb{N} \text{ with } P_0(0) = 0, P_1(0) = 1. \quad (3)$$

Let $b_n := \dfrac{P_n(0)}{(-1)^n}$, then (3) can be rewritten as

$$b_{n+1} - 2b_n + b_{n-1} = 0, n \in \mathbb{N} \quad b_0 = 0, b_1 = -1.$$

Since $b_{n+1} - b_n = b_n - b_{n-1}$ we have $b_n - b_{n-1} = -1$ and $\sum\limits_{k=1}^{n} (b_k - b_{k-1}) = -n$. Therefore $b_n - b_0 = -n$, implying $b_n = -n$.

From the other hand,

$$P'_{n+1}(x) = 2(2x-1)P'_n(x) + 4P_n(x) - P'_{n-1}(x), \text{with } P'_0(x) = 0, P'_1(x) = 0,$$

then

$$P'_{n+1}(0) + 2P'_n(0) + P'_{n-1}(0) = 4P_n(0), \text{ with } P'_0(0) = 0, P'_1(0) = 0. \quad (4)$$

Let $a_n := \dfrac{P'_n(0)}{(-1)^n}$, then $\dfrac{P_n(0)}{(-1)^{n+1}} = -b_n = n$ and (4) can be rewritten as

$$a_{n+1} - 2a_n + a_{n-1} = 4n, n \in \mathbb{N} \text{ with } a_0 = a_1 = 0. \quad (5)$$

Since sequence $\dfrac{2n\left(n^2-1\right)}{3}$ is particular solution of nonhomogeneous recurrence (5), then $a_n = \dfrac{2n\left(n^2-1\right)}{3} + \alpha n + \beta$, where $\alpha = \beta = 0$, because $a_0 = a_1 = 0$.

Thus $a_n = \dfrac{2n\left(n^2-1\right)}{3}$ and

$$\sum_{k=1}^{n-1} \frac{1}{\cos^2 \dfrac{k\pi}{2n}} = -\frac{P'_n(0)}{P_n(0)} = \frac{\dfrac{P'_n(0)}{(-1)^n}}{\dfrac{P_n(0)}{(-1)^{n+1}}} = \frac{a_n}{n} = \frac{2}{3}(n^2-1).$$

Third solution by Brian Bradie, Christopher Newport University, USA

Using the double angle formula

$$\cos^2 \frac{k\pi}{2n} = \frac{1 + \cos \frac{k\pi}{n}}{2},$$

we can rewrite the indicated sum as

$$\begin{aligned}\sum_{k=1}^{n-1} \frac{1}{\cos^2 \frac{k\pi}{2n}} &= \sum_{k=1}^{n-1} \frac{2}{1 + \cos \frac{k\pi}{n}} = \sum_{k=1}^{n-1} \frac{2(1 - \cos \frac{k\pi}{n})}{1 - \cos^2 \frac{k\pi}{n}} \\ &= 2\sum_{k=1}^{n-1} \csc^2 \frac{k\pi}{n} - 2\sum_{k=1}^{n-1} \csc \frac{k\pi}{n} \cot \frac{k\pi}{n}.\end{aligned}$$

From K.R. Stromberg, *Introduction to Classical Real Analysis*, 1981 we know that

$$\sum_{k=1}^{n-1} \csc^2 \frac{k\pi}{n} = \frac{1}{3}(n^2 - 1).$$

Now, if n is even, then

$$\begin{aligned}\sum_{k=1}^{n-1} \csc \frac{k\pi}{n} \cot \frac{k\pi}{n} &= \sum_{k=1}^{n/2-1} \csc \frac{k\pi}{n} \cot \frac{k\pi}{n} + \csc \frac{\pi}{2} \cot \frac{\pi}{2} + \sum_{k=n/2+1}^{n-1} \csc \frac{k\pi}{n} \cot \frac{k\pi}{n} \\ &= \sum_{k=1}^{n/2-1} \csc \frac{k\pi}{n} \cot \frac{k\pi}{n} + 0 + \sum_{k=1}^{n/2-1} \csc \frac{(n-k)\pi}{n} \cot \frac{(n-k)\pi}{n} \\ &= \sum_{k=1}^{n/2-1} \csc \frac{k\pi}{n} \cot \frac{k\pi}{n} - \sum_{k=1}^{n/2-1} \csc \frac{k\pi}{n} \cot \frac{k\pi}{n} \\ &= 0.\end{aligned}$$

On the other hand, if n is odd, then

$$\begin{aligned}\sum_{k=1}^{n-1} \csc \frac{k\pi}{n} \cot \frac{k\pi}{n} &= \sum_{k=1}^{(n-1)/2} \csc \frac{k\pi}{n} \cot \frac{k\pi}{n} + \sum_{k=(n+1)/2}^{n-1} \csc \frac{k\pi}{n} \cot \frac{k\pi}{n} \\ &= \sum_{k=1}^{(n-1)/2} \csc \frac{k\pi}{n} \cot \frac{k\pi}{n} + \sum_{k=1}^{(n-1)/2} \csc \frac{(n-k)\pi}{n} \cot \frac{(n-k)\pi}{n} \\ &= \sum_{k=1}^{(n-1)/2} \csc \frac{k\pi}{n} \cot \frac{k\pi}{n} - \sum_{k=1}^{(n-1)/2} \csc \frac{k\pi}{n} \cot \frac{k\pi}{n} \\ &= 0.\end{aligned}$$

Thus,

$$\sum_{k=1}^{n-1} \frac{1}{\cos^2 \frac{k\pi}{2n}} = \frac{2}{3}(n^2 - 1),$$

and we are done.

Remark. Kunihiko Chikaya point out that the similar problem was proposed in the Tokyo Institute of Technology entrance exam in 1990.

Also solved by Jingjun Han, Shanghai, China; Paolo Perfetti, Universita degli studi di Tor Vergata, Italy; Daniel Lasaosa, Universidad Publica de Navarra, Spain; Oleh Faynstein, Leipzig, Germany; G.R.A.20 Math Problems Group, Roma, Italy

O72. For $n \geq 2$, let S_n be the set of divisors of all polynomials of degree n with coefficients in $\{-1, 0, 1\}$. Let $C(n)$ be the greatest coefficient of a polynomial with integer coefficients that belongs to S_n. Prove that there is a positive integer k such that for all $n > k$,

$$n^{2007} < C(n) < 2^n.$$

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For a polynomial $f(x)$ with coefficients in $\{0, 1\}$ and degree at most n we define a function $\phi(f(x)) = (f(1), f'(1), \ldots, f^{N-1}(1))$. Because all the coefficients of $f(x)$ are 0 or 1, using mathematical induction we can deduce that $f^{(j)}(1) \leq (1 + n)^{j+1}$, for all j. Thus the image of $\phi(f(x))$ has at most

$$(1 + n)^{1+2+\ldots+N} < (1 + n)^{N^2}$$

elements. On the other hand, $f(x)$ is defined on a set of 2^{n+1} elements. Therefore, if $2^{n+1} > (1 + n)^{N^2}$, by the Pigeonhole Principle there exist two polynomials $f_1(x), f_2(x)$ that have the same image. Clearly, their difference has all coefficients in $\{-1, 0, 1\}$ and its degree is at most n. Also from the construction we get $f_1(x) - f_2(x)$ is divisible by $(x - 1)^N$. Thus $C(n) \geq \binom{2N}{N}$, because the largest coefficient of $(x-1)^N$ is $\binom{2N}{N}$. It is not difficult to prove that $\binom{2N}{N} > 2^N$, for $N \geq 2$. Thus taking $N = \left\lfloor \sqrt{\frac{n}{\log_2(n+1)}} \right\rfloor$, we assure that $(1 + n)^{N^2} < 2^{n+1}$, and therefore $C(n) > 2^N$. Thus for a sufficiently large n we have $2^N > n^{2007}$.

The right part is much more subtle. For a polynomial $f(x) = a_n x^n + \ldots + a_1 x + a_0$ define its Mahler measure by $M(f(x)) = |a_n| \prod_{i=1}^{n} \max(1, |x_i|)$, where x_i are the roots of the polynomial $f(x)$. The following inequality is true and due to Landau

$$M(f(x)) \leq \sqrt{|a_0|^2 + |a_1|^2 + \ldots + |a_n|^2}.$$

Thus polynomials with all coefficients of absolute value at most 1 have Mahler measure at most $\sqrt{n+1}$. Take now any divisor $g(x)$ of a polynomial $f(x)$ with all coefficients in $\{-1, 0, 1\}$ and write $f(x) = g(x)h(x)$. Suppose that $g(x)$ has integer coefficients. It is not difficult to see that

$$M(f(x)) = M(g(x))M(h(x)) \geq M(g(x)).$$

Therefore $M(g(x)) \leq \sqrt{n+1}$. Now, observe that by Viete's formula, the triangular inequality, and the fact that $|x_{i_1}x_{i_2}\cdots x_{i_s}| \leq M(f(x))$ for all distinct $i_1, \ldots, i_s$ and all s, we get that all coefficients of f are bounded in absolute value by $\binom{n}{\lfloor \frac{n}{2} \rfloor} M(f(x))$. Thus $C(n) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} M(f(x)) \leq \sqrt{n+1} \cdot \binom{n}{\lfloor \frac{n}{2} \rfloor} < 2^n$, for n sufficiently large, and we are done.