1^{st} United States of America Junior Mathematical Olympiad 2010

1. **Solution from Andy Niedermier:** Every integer in $[n]$ can be uniquely written in the form $x^2 \cdot q$, where q either 1 or *square free*, that is, a product of distinct primes. Let $\langle q \rangle$ denote the set $\{1^2 \cdot q, 2^2 \cdot q, 3^2 \cdot q, \ldots\} \subseteq [n]$.

   Note that for f to satisfy the square-free property, it must permute $\langle q \rangle$ for every $q = 1, 2, 3, \ldots$. To see this, notice that given an arbitrary square-free q, in order for $q \cdot f(q)$ to be a square, $f(q)$ needs to contribute one of every prime factor in q, after which it can take only even powers of primes. Thus, $f(q)$ is equal to the product of q and some perfect square.

   The number of f that permute the $\langle q \rangle$ is equal to

   $$\prod_{\substack{q \le n \\ q \text{ is square-free}}} \left\lfloor \sqrt{\frac{n}{q}} \right\rfloor !$$

   For $2010 = 2 \cdot 3 \cdot 5 \cdot 67$ to divide $P(n)$, we simply need $67!$ to appear in this product, which will first happen in $\langle 1 \rangle$ so long as $\sqrt{n/q} \ge 67$ for some n and q. The smallest such n is $67^2 = 4489$.

   This problem was proposed by Andy Niedermier.

2. **Solution from Răzvan Gelca:** There is a unique sequence $2, 4, 6, \ldots, 2n-2$ satisfying the conditions of the problem.

   Note that (b) implies $x_i < 2n$ for all i. We will examine the possible values of x_1.

   If $x_1 = 1$, then (c) implies that all numbers less than $2n$ should be terms of the sequence, which is impossible since the sequence has only $n-1$ terms.

If $x_1 = 2$, then by (c) the numbers $2, 4, 6, \ldots, 2n-2$ are terms of the sequence, and because the sequence has exactly $n-1$ terms we get $x_i = 2i$, $i = 1, 2, \ldots, n-1$. This sequence satisfies conditions (a) and (b) as well, so it is a solution to the problem.

For $x_1 \geq 3$, we will show that there is no sequences satisfying the conditions of the problem. Assume on the contrary that for some n there is such a sequence with $x_1 \geq 3$. If $n = 2$, the only possibility is $x_1 = 3$, which violates (b). If $n = 3$, then by (a) we have the possibilities $(x_1, x_2) = (3, 4)$, or $(3, 5)$, or $(4, 5)$, all three of which violate (b). Now we assume that $n > 3$. By (c), the numbers

$$x_1,\ 2x_1,\ \ldots,\ \left\lfloor \frac{2n}{x_i} \right\rfloor \cdot x \tag{1}$$

are terms of the sequence, and no other multiples of x_1 are. Because $x_1 \geq 3$, the above accounts for at most $\frac{2n}{3}$ terms of the sequence. For $n > 3$, we have $\frac{2n}{3} < n - 1$, and so there must be another term besides the terms in (1). Let x_j be the smallest term of the sequence that does not appear in (1). Then the first j terms of the sequence are

$$x_1,\ x_2 = 2x_1,\ \ldots,\ x_{j-1} = (j-1)x_1,\ x_j, \tag{2}$$

and we have $x_j < jx_1$. Condition (b) implies that the last j terms of the sequence must be

$$x_{n-j} = 2n - x_j,\ x_{n-j+1} = 2n - (j-1)x_1,\ \ldots,\\ x_{n-2} = 2n - 2x_1,\ x_{n-1} = 2n - x_1.$$

But then $x_1 + x_{n-j} < x_1 + x_{n-1} = 2n$, hence by condition (c) there exists k such that $x_1 + x_{n-j} = x_k$. On the one hand, we have

$$\begin{aligned} x_k &= x_1 + x_{n-j} = x_1 + 2n - x_j = 2n - (x_j - x_1) \\ &> 2n - (jx_1 - x_1) = 2n - (j-1)x_1 = x_{n-j+1}. \end{aligned}$$

One the other hand, we have

$$x_k = x_1 + x_{n-j} < x_1 + x_{n-j+1} = x_{n-j+2}.$$

This means that x_k is between two consecutive terms x_{n-j+1} and x_{n-j+2}, which is impossible by (a). (In the case $j = 2$, $x_k > x_{n-j+1} = x_{n-1}$, which is also impossible.) We conclude that there is no such sequence with $x_1 \geq 3$.

Remark. This problem comes from the study of Weierstrass gaps in the theory of Riemann surfaces.

Alternate Solution from Richard Stong: Assume that $x_1, x_2, \ldots, x_{n-1}$ is a sequence satisfying the conditions of the problem. By condition (a), the following terms

$$x_1,\ 2x_1,\ x_1 + x_2,\ x_1 + x_3,\ x_1 + x_4, \ldots,\ x_1 + x_{n-2}$$

form an increasing sequence. By condition (c), this new sequence is a subsequence of the original sequence. Because both sequences have exactly $n-1$ terms, these two sequences are identical; that is, $2x_1 = x_2$ and $x_1 + x_j = x_{j+1}$ for $2 \leq j \leq n-2$. It follows that $x_j = jx_1$ for $1 \leq j \leq n-1$. By condition (b), we conclude that $(x_1, x_2, \ldots, x_{n-1}) = (2, 4, \ldots, 2n-2)$.

Remark. The core of the second solution is a result due to Freiman:

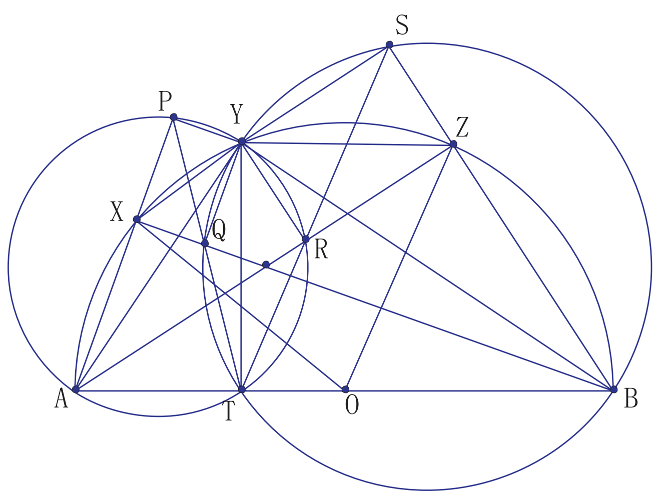
> Let A be a set of positive integers. Then the set $A+A = \{a_1 + a_2 \mid a_1, a_2 \in A\}$ has at least $2|A| - 1$ elements and equality holds if and only if A is a set of an arithmetic progression.

Freiman's theorem and its generalization below are very helpful in proofs of many contest problems, such as, USAMO 2009 problem 2, IMO 2000 problem 1, and IMO 2009 problem 5.

Let A and B be finite nonempty subsets of integers. Then the set $A+B = \{a+b \mid a \in A, b \in B\}$ has at least $|A|+|B|-1$ elements. Equality holds if and only if either A and B are arithmetic progressions with equal difference or $|A|$ or $|B|$ is equal to 1.

This problem was suggested by Răzvan Gelca.

3. **Solution by Titu Andreescu:** Let T be the foot of the perpendicular from Y to line AB. We note the P, Q, T are the feet of the perpendiculars from Y to the sides of triangle ABX. Because Y lies on the circumcircle of triangle ABX, points P, Q, T are collinear, by Simson's theorem. Likewise, points S, R, T are collinear.



We need to show that $\angle XOZ = 2\angle PTS$ or

$$\angle PTS = \frac{\angle XOZ}{2} = \frac{\widehat{XZ}}{2} = \frac{\widehat{XY}}{2} + \frac{\widehat{YZ}}{2}$$
$$= \angle XAY + \angle ZBY = \angle PAY + \angle SBY.$$

Because $\angle PTS = \angle PTY + \angle STY$, it suffices to prove that

$$\angle PTY = \angle PAY \quad \text{and} \quad \angle STY = \angle SBY;$$

that is, to show that quadrilaterals $APYT$ and $BSYT$ are cyclic, which is evident, because $\angle APY = \angle ATY = 90^\circ$ and $\angle BTY = \angle BSY = 90^\circ$.

Alternate Solution from Lenny Ng and Richard Stong: Since YQ, YR are perpendicular to BX, AZ respectively, $\angle RYQ$ is equal to the acute angle between lines BX and AZ, which is $\frac{1}{2}(\widehat{AX} + \widehat{BZ}) = \frac{1}{2}(180^\circ - \widehat{XZ})$ since X, Z lie on the circle with diameter AB. Also, $\angle AXB = \angle AZB = 90^\circ$ and so $PXQY$ and $SZRY$ are rectangles, whence $\angle PQY = 90^\circ - \angle YXB = 90^\circ - \widehat{YB}/2$ and $\angle YRS = 90^\circ - \angle AZY = 90^\circ - \widehat{AY}/2$. Finally, the angle between PQ and RS is

$$\begin{aligned} \angle PQY + \angle YRS - \angle RYQ &= (90^\circ - \widehat{YB}/2) + (90^\circ - \widehat{AY}/2) - (90^\circ - \widehat{XZ}/2) \\ &= \widehat{XZ}/2 \\ &= (\angle XOZ)/2, \end{aligned}$$

as desired.

This problem was proposed by Titu Andreescu.

4. **Solution from Zuming Feng:**

Let $A = (a, a^2)$, $B = (b, b^2)$, and $C = (c, c^2)$, with $a < b < c$. We have $\overrightarrow{AB} = [b - a, b^2 - a^2]$ and $\overrightarrow{AC} = [c - a, c^2 - a^2]$. Hence the area of triangle ABC is equal to

$$\begin{aligned} [ABC] = (2^n m)^2 &= \frac{|(b-a)(c^2 - a^2) - (c-a)(b^2 - a^2)|}{2} \\ &= \frac{(b-a)(c-a)(c-b)}{2}. \end{aligned}$$

Setting $b-a=x$ and $c-b=y$ (where both x and y are positive integers), the above equation becomes

$$(2^n m)^2 = \frac{xy(x+y)}{2}. \tag{3}$$

If $n=0$, then $(m,x,y)=(1,1,1)$ is clearly a solution to (3). If $n \geq 1$, it is easy to check that,

$$(m,x,y) = \big((2^{4n-2}-1, 2^{2n+1}, (2^{2n-1}-1)^2)\big)$$

satisfies (3).

Alternate Solution from Jacek Fabrykowski:

The beginning is the same up to $(2^n m)^2 = \frac{xy(x+y)}{2}$. If $n=0$, we take $m=x=y=1$. If $n=1$, we take $m=3$, $x=1$, $y=8$. Assume that $n \geq 2$. Let a, b, c be a primitive Pythagorean triple with b even. Let $b=2^r d$ where d is odd and $r \geq 2$. Let $x=2^{2k}$, $y=2^{2k}b$ and $z=2^{2k}c$ where $k \geq 0$. We let $m=adc$ and $r=2$ if $n=3k+2$, $r=3$ if $n=3k+3$ and $r=4$ if $n=3k+4$.

Assuming that $x=a\cdot 2^s$, $y=b\cdot 2^2$, other triples are possible:

(a) If $n=3k$, then let $m=1$ and $x=y=2^{2k}$.

(b) If $n=3k+1$, then take $m=3$, $x=2^{2k}$, $y=2^{2k+3}$.

(c) If $n=3k+2$, then take $m=63$, $x=49\cdot 2^{2k}$, and $y=2^{2k+5}$.

This problem was suggested by Zuming Feng.

5. **Solution from Gregory Galperin:**

Let us create the following 1006 permutations $X_1,\ldots,X_{1006}$, the first 1006 positions of which are all possible cyclic rotations of the sequence

$1, 2, 3, 4, ..., 1005, 1006$, and the remaining 1004 positions are filled arbitrarily with the remaining numbers $1006, 1007, ..., 2009, 2010$:

$$X_1 = 1,\ 2,\ 3,\ 4,\ \dots,\ 1005,\ 1006,\ \ *,\ *,\ \dots,\ *\ ;$$
$$X_2 = 2,\ 3,\ 4,\ \dots,\ 1005,\ 1006,\ 1,\ \ *,\ *,\ \dots,\ *\ ;$$
$$X_3 = 3,\ 4,\ \dots,\ 1005,\ 1006,\ 1,\ 2,\ \ *,\ *,\ \dots,\ *\ ;$$
$$\dots\dots$$
$$X_{1006} = 1006,\ 1,\ 2,\ 3,\ 4,\ \dots,\ 1005,\ \ *,\ *,\ \dots,\ *\ .$$

We claim that at least one of these 1006 sequences has the same integer at the same position as the initial (unknown) permutation X.

Suppose not. Then the set of the first (leftmost) integers in the permutation X contains no integers from 1 to 1006. Hence it consists of the 1004 integers in the range from 1007 to 2010 only. By the pigeon-hole principle, some two of the integers from the permutation X must be equal, which is a contradiction: there are not two identical integers in the permutation X.

Consequently, the permutation X has at last one common element with some sequence X_i, $i = 1, \dots 1006$ and we are done.

This problem was proposed by Gregory Galperin.

6. **Solution from Zuming Feng:** The answer is *no*, it is not possible for segments AB, BC, BI, ID, CI, IE to all have integer lengths.

   Assume on the contrary that these segments do have integer side lengths. We set $\alpha = \angle ABD = \angle DBC$ and $\beta = \angle ACE = \angle ECB$. Note that I is the incenter of triangle ABC, and so $\angle BAI = \angle CAI = 45^\circ$. Applying the Law of Sines to triangle ABI yields

$$\frac{AB}{BI} = \frac{\sin(45^\circ + \alpha)}{\sin 45^\circ} = \sin\alpha + \cos\alpha,$$

by the addition formula (for the sine function). In particular, we conclude that $s = \sin\alpha + \cos\alpha$ is rational. It is clear that $\alpha + \beta = 45^\circ$. By the subtraction formulas, we have

$$s = \sin(45^\circ - \beta) + \cos(45^\circ - \beta) = \sqrt{2}\cos\beta,$$

from which it follows that $\cos\beta$ is not rational. On the other hand, from right triangle ACE, we have $\cos\beta = AC/EC$, which is rational by assumption. Because $\cos\beta$ cannot not be both rational and irrational, our assumption was wrong and not all the segments AB, BC, BI, ID, CI, IE can have integer lengths.

Alternate Solution from Jacek Fabrykowski: Using notations as introduced in the problem, let $BD = m$, $AD = x$, $DC = y$, $AB = c$, $BC = a$ and $AC = b$. The angle bisector theorem implies

$$\frac{x}{b-x} = \frac{c}{a}$$

and the Pythagorean Theorem yields $m^2 = x^2 + c^2$. Both equations imply that

$$2ac = \frac{(bc)^2}{m^2 - c^2} - a^2 - c^2$$

and since $a^2 = b^2 + c^2$ is rational, a is rational too (observe that to reach this conclusion, we only need to assume that b, c, and m are integers). Therefore, $x = \frac{bc}{a+c}$ is also rational, and so is y. Let now (similarly to the notations above from the solution by Zuming Feng) $\angle ABD = \alpha$ and $\angle ACE = \beta$ where $\alpha + \beta = \pi/4$. It is obvious that $\cos\alpha$ and $\cos\beta$are both rational and the above shows that also $\sin\alpha = x/m$ is rational. On the other hand, $\cos\beta = \cos(\pi/4 - \alpha) = (\sqrt{2}/2)(\sin\alpha + \sin\beta)$, which is a contradiction. The solution shows that a stronger statement holds true: There is no right triangle with both legs and bisectors of acute angles all having integer lengths.

Alternate Solution from Zuming Feng: Prove an even stronger result: there is no such right triangle with AB, AC, IB, IC having rational side lengths. Assume on the contrary, that AB, AC, IB, IC have rational side lengths. Then $BC^2 = AB^2 + AC^2$ is rational. On the other hand, in triangle BIC, $\angle BIC = 135°$. Applying the law of cosines to triangle BIC yields

$$BC^2 = BI^2 + CI^2 - \sqrt{2}BI \cdot CI$$

which is irrational. Because BC^2 cannot be both rational and irrational, we conclude that our assumption was wrong and that not all of the segments AB, AC, IB, IC can have rational lengths.

This problem was proposed by Zuming Feng.