Chapter 8

Linear Fractional Vector Optimization Problems

Linear Fractional Vector Optimization (LFVO) is an interesting area in the wider theory of vector optimization (see, for example, Choo and Atkins (1982, 1983), Malivert (1995), Malivert and Popovici (2000), and Steuer (1986)). LFVO problems have applications in finance and production management (see Steuer (1986)). In a LFVO problem, any point satisfying the first-oder necessary optimality condition is a solution. Therefore, solving a LFVO problem is to solve a monotone affine Vector Variational Inequality (VVI). The original concept of VVI was proposed by Giannessi (1980). In this chapter we will apply the results of the preceding two chapters to establish some facts about connectedness and stability of the solution sets in LFVO problems. In particular, we will prove that the efficient solution set of a LFVO problem with a bounded constraint set is connected.

8.1 LFVO Problems

Let $f_i : R^n \to R \quad (i = 1, 2, \cdots, m)$ be m *linear fractional functions*, that is

$$f_i(x) = \frac{a_i^T x + \alpha_i}{b_i^T x + \beta_i}$$

for some $a_i \in R^n, b_i \in R^n, \alpha_i \in R$, and $\beta_i \in R$. Let $\Delta = \{x \in R^n : Cx \leq d\}$, where $C \in R^{r \times n}$ and $d \in R^r$, be a nonempty polyhedral convex set. We assume that $b_i^T x + \beta_i > 0$ for all $i \in \{1, \cdots, m\}$ and for all $x \in \Delta$. Define

$$f(x) = (f_1(x), \dots, f_m(x)), \quad A = (a_1, \dots, a_m), \quad B = (b_1, \dots, b_m),$$
$$\alpha = (\alpha_1, \dots, \alpha_m), \quad \beta = (\beta_1, \dots, \beta_m), \quad \omega = (A, B, \alpha, \beta).$$

Thus A and B are $n \times m$-matrices, α and β are vectors of R^m, and ω is a parameter containing all the data related to the vector function f.

Consider the following vector optimization problem

(VP) Minimize $f(x)$ subject to $x \in \Delta$.

Definition 8.1. One says that $x \in \Delta$ is an *efficient solution* of (VP) if there exists no $y \in \Delta$ such that $f(y) \leq f(x)$ and $f(y) \neq f(x)$. If there exists no $y \in \Delta$ such that $f(y) < f(x)$, then one says that $x \in \Delta$ is a *weakly efficient solution* of (VP).

Let us denote the *efficient solution set* and the *weakly efficient solution set*) of (VP) by Sol(VP) and Sol^w(VP), respectively.

The following lemma will be useful for obtaining necessary and sufficient optimality conditions for (VP).

Lemma 8.1. (See Malivert (1995)) *Let $\varphi(x) = (a^T x + \alpha)/(b^T x + \beta)$ be a linear fractional function. Suppose that $b^T x + \beta \neq 0$ for every $x \in \Delta$. Then for any $x, y \in \Delta$, it holds*

$$\varphi(y) - \varphi(x) = \frac{b^T x + \beta}{b^T y + \beta} \langle \nabla \varphi(x), y - x \rangle, \tag{8.1}$$

where $\nabla \varphi(x)$ denotes the gradient of φ at x.

Proof. By the definition of gradient,

$$\begin{aligned}
&\langle \nabla \varphi(x), y - x \rangle \\
&= \lim_{t \downarrow 0} \frac{1}{t} \left[\varphi(x + t(y - x)) - \varphi(x) \right] \\
&= \lim_{t \downarrow 0} \frac{1}{t} \left[\frac{a^T (x + t(y - x)) + \alpha}{b^T (x + t(y - x)) + \beta} - \frac{a^T x + \alpha}{b^T x + \beta} \right] \\
&= \frac{a^T (y - x)(b^T x + \beta) - b^T (y - x)(a^T x + \alpha)}{(b^T x + \beta)^2}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\frac{b^T x + \beta}{b^T y + \beta} \langle \nabla \varphi(x), y - x \rangle \\
&= \frac{a^T (y - x)(b^T x + \beta) - b^T (y - x)(a^T x + \alpha)}{(b^T x + \alpha)(b^T x + \beta)} \\
&= \varphi(y) - \varphi(x),
\end{aligned}$$

which completes the proof. □

Given any $x, y \in \Delta$, $x \neq y$, we consider two points belonging to the line segment $[x, y]$:

$$z_t = x + t(y - x), \quad z_{t'} = x + t'(y - x) \quad (t \in [0, 1],\ t' \in [0, t)).$$

By (8.1),

$$\begin{aligned}
&\varphi(z_t) - \varphi(z_{t'}) \\
&= (\varphi(z_t) - \varphi(x)) - (\varphi(z_{t'} - \varphi(x)) \\
&= \frac{b^T + \beta}{b^T z_t + \beta} \langle \nabla\varphi(x), z_t - x \rangle - \frac{b^T x + \beta}{b^T z_{t'} + \beta} \langle \nabla\varphi(x), z_{t'} - x \rangle \\
&= \langle \nabla\varphi(x), y - x \rangle \frac{b^T x + \beta)^2}{(b^T z_t + \beta)(b^T z_{t'} + \beta)} (t - t').
\end{aligned}$$

From this we can conclude that: (i) If $\langle \nabla\varphi(x), y - x \rangle > 0$ then $\varphi(z_{t'}) < \varphi(z_t)$ for every $t' \in [0, t)$; (ii) If $\langle \nabla\varphi(x), y - x \rangle < 0$ then $\varphi(z_{t'}) > \varphi(z_t)$ for every $t' \in [0, t)$; (iii) If $\langle \nabla\varphi(x), y - x \rangle = 0$ then $\varphi(z_{t'}) = \varphi(z_t)$ for every $t' \in [0, t)$. Hence the function φ is monotonic on every line segment or ray contained in Δ.

By definition, a function $\varphi : \Delta \to R$ is *quasiconcave* on Δ if

$$\varphi((1 - t)x + ty) \geq \min\{\varphi(x), \varphi(y)\} \quad \forall x, y \in \Delta, \forall t \in (0, 1). \quad (8.2)$$

We say that φ is *semistrictly quasiconcave* on Δ if φ is quasiconcave on Δ and the inequality in (8.2) is strict whenever $\varphi(x) \neq \varphi(y)$. If φ is quasiconcave on Δ and the inequality in (8.2) is strict whenever $x \neq y$, then we say that φ is *strictly quasiconcave* on Δ. Function φ is said to be *quasiconvex* (resp., *semistrictly quasiconvex, strictly quasiconcave*)on Δ if the function $-\varphi$ is quasiconcave (resp., semistrictly quasiconcave, strictly quasiconcave) on Δ.

From the above-mentioned monotonicity of linear fractional functions it follows that if $\varphi : \Delta \to R$ is a linear fractional function then it is, at the same time, semistrictly quasiconcave and semistrictly quasiconvex on Δ. In general, linear fractional functions are not strictly quasiconcave on their effective domain. Indeed, for any $\mu \in R$, $t \in (0, 1)$, and x^1, x^2 from the set

$$\{x \in \Delta : \varphi(x) = \mu\} = \{x \in \Delta : a^T x + \alpha = b^T x + \beta\},$$

we have $\varphi((1 - t)x^1 + tx^2) = \min\{\varphi(x^1), \varphi(x^2)\}$.

Let

$$\Lambda = \{\lambda \in R^m_+ : \sum_{i=1}^m \lambda_i = 1\}.$$

Then

$$\mathrm{ri}\Lambda = \{\lambda \in R^m_+ : \sum_{i=1}^m \lambda_i = 1, \ \lambda_i > 0 \ \text{ for all } \ i\}.$$

Theorem 8.1. (See Malivert (1995)) *Let $x \in \Delta$. The following assertions hold:*

(i) $x \in \mathrm{Sol(VP)}$ *if and only if there exists* $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathrm{ri}\Lambda$ *such that*

$$\left\langle \sum_{i=1}^m \lambda_i \left[(b_i^T x + \beta_i) a_i - (a_i^T x + \alpha_i) b_i \right], y - x \right\rangle \geq 0, \quad \forall y \in \Delta. \tag{8.3}$$

(ii) $x \in \mathrm{Sol}^w(\mathrm{VP})$ *if and only if there exists* $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda$ *such that* (8.3) *holds.*

(iii) *Condition* (8.3) *is satisfied if and only if there exists* $\mu = (\mu_1, \ldots, \mu_r)$, $\mu_j \geq 0$ *for all* $j = 1, \ldots, r$, *such that*

$$\sum_{i=1}^m \lambda_i \left[(b_i^T x + \beta_i) a_i - (a_i^T x + \alpha_i) b_i \right] + \sum_{j \in I(x)} \mu_j C_j^T = 0, \tag{8.4}$$

where C_j *denotes the* j*-th row of the matrix* C *and*

$$I(x) = \{j : C_j x = d_j\}.$$

Proof. (i) We claim that $x \in\in \mathrm{Sol(VP)}$ if and only if

$$Q_x(\Delta - x) \cap (-R^m_+) = \{0\}, \tag{8.5}$$

where

$$Q_x = \begin{pmatrix} (b_1^T x + \beta_1) a_1^T - (a_1^T x + \alpha_1) b_1^T \\ \vdots \\ (b_m^T x + \beta_m) a_m^T - (a_m^T x + \alpha_m) b_m^T \end{pmatrix}$$

is an $(m \times n)$-matrix and $Q_x(\Delta - x) = \{Q_x(y - x) : y \in \Delta\}$. Indeed, $x \notin \mathrm{Sol(VP)}$ if and only if there exist $y \in \Delta$ and i_0 such that

$$f_i(y) \leq f_i(x) \ \ \forall i \in \{1, \ldots, m\}, \quad f_{i_0}(y) < f_{i_0}(x).$$

By Lemma 8.1, the last system of inequalities is equivalent to the following one:

$$\langle \nabla f_i(x), y-x\rangle \le 0 \ \ \forall i \in \{1,\ldots,m\}, \quad \langle \nabla f_{i_0}(x), y-x\rangle < 0. \tag{8.6}$$

Since

$$\langle \nabla f_i(x), y-x\rangle = \frac{a_i^T(y-x)(b_i^Tx+\beta_i) - b_i^T(y-x)(a_i^Tx+\alpha_i)}{(b_i^Tx+\beta_i)^2},$$

we can rewrite (8.6) as follows

$$\begin{cases} \left[(b_i^Tx+\beta_i)a_i^T - (a_i^Tx+\alpha_i)b_i^T\right](y-x) \le 0 \ \ \forall i \in \{1,\ldots,m\}, \\ \left[(b_{i_0}^Tx+\beta_{i_0})a_{i_0}^T - (a_{i_0}^Tx+\alpha_{i_0})b_{i_0}^T\right](y-x) < 0. \end{cases}$$

Therefore, $x \notin \mathrm{Sol(VP)}$ if and only if there exists $y \in \Delta$ such that

$$Q_x(y-x) \in -R_+^m \quad \text{and} \quad Q_x(y-x) \neq 0.$$

Our claim has been proved.

It is clear that $D := Q_x(\Delta - x)$ is a polyhedral convex set. Hence, by Corollary 19.7.1 from Rockafellar (1970), $K := \mathrm{cone}D$ is a polyhedral convex cone. In particular, K is a closed convex cone. It is easily seen that (8.5) is equivalent to the property $K \cap (-R_+^m) = \{0\}$. Setting

$$K^+ = \{z \in R^m : \langle z, v\rangle \ge 0 \quad \forall v \in K\},$$

we have $K^+ \cap \mathrm{int}R_+^k \neq \emptyset$. Indeed, on the contrary, suppose that $K^+ \cap \mathrm{int}R_+^k = \emptyset$. Then, by the separation theorem, there exists $\xi \in R^m \setminus \{0\}$ such that

$$\langle \xi, u\rangle \le 0 \le \langle \xi, z\rangle \quad \forall u \in \mathrm{int}R_+^m, \ \forall z \in K^+.$$

This implies that $\xi \in -R_+^m$ and $\xi \in (K^+)^+ = K$. So we get $\xi \in K \cap (-R_+^m) = \{0\}$, a contradiction.

Fix any $\widetilde{\lambda} \in K^+ \cap \mathrm{int}R_+^k$. For $\lambda := \widetilde{\lambda}/(\widetilde{\lambda}_1 + \ldots + \widetilde{\lambda}_m)$, we have $\lambda \in K^+ \cap \mathrm{ri}\Lambda$. Since $\langle \lambda, v\rangle \ge 0$ for every $v \in K$, we deduce that

$$\langle Q_x^T\lambda, (y-x)\rangle = \langle \lambda, Q_x(y-x)\rangle \ge 0$$

for every $x \in \Delta$. Hence (8.3) is valid.

(ii) It is easily seen that $x \in \mathrm{Sol}^w(\mathrm{VP})$ if and only if

$$Q_x(\Delta - x) \cap (-\mathrm{int}R_+^m) = \emptyset.$$

Using the separation theorem we find a multiplier $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda$ satisfying (8.3).

(iii) It suffices to apply the Farkas Lemma. □

Condition (8.3) can be rewritten in the form

$$(\mathrm{VI})_\lambda \qquad \langle M(\lambda)y + q(\lambda), y - x\rangle \geq 0 \quad \forall y \in \Delta,$$

where

$$M(\lambda) = (M_{kj}(\lambda)),$$

$$M_{kj}(\lambda) = \sum_{i=1}^{m} \lambda_i \left(b_{i,j}a_{i,k} - a_{i,j}b_{i,k}\right), \quad 1 \leq k \leq n, \;\; 1 \leq j \leq n,$$

$$q(\lambda) = (q_k(\lambda)), \quad q_k(\lambda) = \sum_{i=1}^{m} \lambda_i \left(\beta_i a_{i,k} - \alpha_i b_{i,k}\right), \quad 1 \leq k \leq n,$$

$a_{i,k}$ and $b_{i,k}$ are the $k-$th components of a_i and b_i, respectively.

It is clear that $(M(\lambda))^T = -M(\lambda)$. Therefore, $\langle M(\lambda)v, v\rangle = 0$ for every $v \in R^n$. In particular, $M(\lambda)$ is a positive semidefinite matrix.

8.2 Connectedness of the Solution Sets

Let X, Y be two topological spaces and $G : X \to 2^Y$ be a multifunction.

Definition 8.2. One says that G is *upper semicontinuous* (usc) at $a \in X$ if f for every open set $V \subset Y$ satisfying $G(a) \subset V$ there exists a neighborhood U of a, such that $G(a') \subset V$ for all $a' \in U$. One says that G is *lower semicontinuous* (lsc) at $a \in X$ if $G(a) \neq \emptyset$ and for every open set $V \subset Y$ satisfying $G(a) \cap V \neq \emptyset$ there exists a neighborhood U of a such that $G(a') \cap V \neq \emptyset$ for all $a' \in U$.

Multifunction G is said to be *continuous* at $a \in X$ if it is simultaneously upper semicontinuous and lower semicontinuous at that point.

Definition 8.3. A topological space Z is said to be *connected* if there exists no pair (Z_1, Z_2) of disjoint nonempty open subsets Z_1, Z_2 of Z, such that $Z = Z_1 \cup Z_2$. The space Z is *arcwise connected* if for any a, $b \in Z$ there exits a continuous mapping $\gamma : [0,1] \to Z$ such that $\gamma(0) = a$, $\gamma(1) = b$. One says that Z is *contractible* if there exist a point $z^0 \in Z$ and a continuous function $H : Z \times [0,1] \to Z$ such that $H(z,0) = z$ and $H(z,1) = z^0$ for every $z \in Z$.

It is clear that if Z is contractible then it is arcwise connected. It is also clear that if Z is arcwise connected then it is connected. The reverse implications are not true in general.

The simple proof of the following result is left after the reader.

Theorem 8.2. (See Warburton (1983), and Hirriart-Urruty (1985)) *Assume that X is connected. If*

(i) *for every $x \in X$ the set $G(x)$ is nonempty and connected, and*

(ii) *G is upper semicontinuous at every $a \in X$ (or G is lower semicontinuous at every $a \in X$),*

then the image set

$$G(X) := \bigcup_{x \in X} G(x),$$

which is equipped with the induced topology, is connected.

Remark 8.1. Let X and Y be two normed spaces, $G : X \to 2^Y$ a multifunction. If G is upper-Lipschitz at $a \in X$ and $G(a)$ is a compact set, then G is usc at a. So, according to Theorem 7.6, if M is a positive semidefinite matrix and if the solution set $\mathrm{Sol}(\mathrm{AVI}(M, q, \Delta))$ of (6.1) is nonempty and bounded, then the solution map $\mathrm{Sol}(\mathrm{AVI}(\cdot, \cdot, \Delta) : \mathcal{P}_n \times R^n \to 2^{R^n}$ is usc at (M, q). Here the symbol $\mathcal{P}_n$ stands for the set of all positive semidefinite $n \times n$-matrices.

We now turn our attention back to problem (VP). Denote by $F(\lambda)$ the solution set of the problem $(\mathrm{VI})_\lambda$ described in Section 8.1. By Proposition 5.4, $F(\lambda)$ is closed and convex. If Δ is compact then, by Theorem 5.1, $F(\lambda)$ is nonempty and bounded. Consider the set-valued map $F : \Lambda \to 2^{R^n}$, $\lambda \mapsto F(\lambda)$. According to Theorems 8.1 and 8.2,

$$\mathrm{Sol}(\mathrm{VP}) = \bigcup_{\lambda \in \mathrm{ri}\Lambda} F(\lambda) = F(\mathrm{ri}\Lambda), \tag{8.7}$$

$$\mathrm{Sol}^w(\mathrm{VP}) = \bigcup_{\lambda \in \Lambda} F(\lambda) = F(\Lambda). \tag{8.8}$$

Remark 8.2. Using the results and the terminology in Lee et al. (1998) and Lee and Yen (2001) we can say that solving problem (VP) is equivalent to solve the monotone affine VVI defined by Δ and the affine functions

$$g_i(x) = (b_i^T x + \beta_i)a_i - (a_i^T x + \alpha_i)b_i \quad (i = 1, 2, \ldots, m).$$

Thus the first-order optimality condition of a LFVO problem can be treated as a special vector variational inequality.

Theorem 8.3. (Benoist (1998), Yen and Phuong (2000)) *If* Δ *is compact, then* Sol(VP) *is a connected set.*

Proof. Since riΛ is a convex set (so it is connected), $F(\lambda)$ is nonempty and connected for every $\lambda \in \mathrm{ri}\Lambda$, and the map $F(\cdot)$ is upper semicontinuous at every $\bar{\lambda} \in \mathrm{ri}\Lambda$, Theorem 8.3 can be applied to the set-valued map $F : \mathrm{ri}\Lambda \to 2^{R^n}$. As a consequence, $F(\mathrm{ri}\Lambda)$ is connected. Hence, by (8.7), Sol(VP) is connected. □

Theorem 8.4. (Choo and Atkins (1983)) *If* Δ *is compact then* Sol^w(VP) *is a connected set.*

Proof. Apply Theorem 8.3 and formula (8.8) to the set-valued map $F : \Lambda \to 2^{R^n}$. □

In fact, Choo and Atkins (1983) established the following stronger result: *If* Δ *is compact then* Sol^w(VP) *is connected by line segments.* The latter means that for any points $x, \bar{y} \in \mathrm{Sol}^w(\mathrm{VP})$ there exists a finite sequence of points $x_0 = x, x_1, \ldots, x_k = \bar{y}$ such that each line segment $[x_j, x_{j+1}]$ $(j = 0, 1, \ldots, k-1)$ is a subset of Sol^w(VP).

If Δ is unbounded, then $\mathrm{Sol}(VP)$ and $\mathrm{Sol}(VP)^w$ may be disconnected.

Example 8.1. (Choo and Atkins (1983)) Consider problem (VP) with

$$\Delta = \{x = (x_1, x_2) \in R^2 \, : \, x_1 \geq 2, \; 0 \leq x_2 \leq 4\},$$
$$f_1(x) = \frac{-x_1}{x_1 + x_2 - 1}, \quad f_2(x) = \frac{-x_1}{x_1 - x_2 + 3}.$$

Using Theorem 8.1, one can verify that

$$\mathrm{Sol(VP)} = \mathrm{Sol}^w(\mathrm{VP}) = \{(x_1, 0) \, : \, x_1 \geq 2\} \cup \{(x_1, 4) \, : \, x_1 \geq 2\}.$$

Recall that a *component* of a topological space is a maximal connected subset (that is, a connected subset which is properly contained in no other connected subset).

Example 8.2. (Hoa et al. (2004)) Consider problem (VP) where $n = m$, $m \geq 2$,

$$\Delta = \{x \in R^m : x_1 \geq 0, x_2 \geq 0, \ldots, x_m \geq 0, \; \sum_{k=1}^{m} x_k \geq 1\},$$

and

$$f_i(x) = \frac{-x_i + \dfrac{1}{2}}{\sum_{k=1}^{m} x_k - \dfrac{3}{4}} \qquad (i = 1, \ldots, m).$$

Using Theorem 8.1 one can show that

$$\begin{aligned} \mathrm{Sol(VP)} = \mathrm{Sol}^w\mathrm{(VP)} = \ & \{(x_1, 0, \ldots, 0)^T : x_1 \geq 1\} \\ & \cup\{(0, x_2, \ldots, 0)^T : x_2 \geq 1\} \\ & \ldots \ \ldots \ \ldots \\ & \cup\{(0, \ldots, 0, x_m)^T : x_m \geq 1\}. \end{aligned}$$

Hence each of the sets Sol(VP) and Sol^w(VP) has exactly m components.

The following question (see Yen and Phuong (2000)) remains open: *Is it true that every component of Sol(VP) is connected by line segments?*

The following possible estimates have been mentioned in Hoa et al. (2004):

$$\chi(\mathrm{Sol(VP)}) \leq \min\{m, n\}, \quad \chi(\mathrm{Sol}^w\mathrm{(VP)}) \leq \min\{m, n\}.$$

Here $\chi(M)$ denotes the number of components of a subset $M \subset R^n$ in its induced topology. So far, the above estimates have not been proved even for bicriteria LFVO problems.

It is worthy to observe that the sets Sol(VP) and Sol^w(VP) may be not contractible, even in the case they are arcwise connected.

Example 8.3. (Huy and Yen (2004b)) Consider problem (VP), where

$$\begin{aligned} \Delta = \{x = (x_1, x_2, x_3) : \ & x_1 + x_2 + x_3 \geq 2, \ x_1 + x_2 - 2x_3 \leq 2, \\ & x_1 - 2x_2 + x_3 \leq 2, \ -2x_1 + x_2 + x_3 \leq 2\}, \end{aligned}$$

$$f_i(x) = \frac{-x_i}{x_1 + x_2 + x_3 - 1} \qquad (i = 1, 2, 3).$$

Using Theorem 8.1 one can prove that the sets Sol(VP) and Sol^w(VP) coincide with the surrounding surface of the parallelepiped Δ; that is

$$\mathrm{Sol}^w\mathrm{(VP)} = \mathrm{Sol(VP)} = F_1 \cup F_2 \cup F_3,$$

where

$$\begin{aligned} F_1 &= \{x : x_1 = 2 - x_2 + 2x_3, \ x_3 \geq 0, \ x_3 \leq x_2 \leq 2 + x_3\} \\ F_2 &= \{x : x_2 = 2 - x_3 + 2x_1, \ x_1 \geq 0, \ x_1 \leq x_3 \leq 2 + x_1\} \\ F_3 &= \{x : x_3 = 2 - x_1 + 2x_2, \ x_2 \geq 0, \ x_2 \leq x_1 \leq 2 + x_2\}. \end{aligned}$$

8.3 Stability of the Solution Sets

Since problem (VP) depends on the parameter $\omega = (A, B, \alpha, \beta)$, in this section it is convenient for us to rename the sets Sol(VP) and Sol^w(VP) into $E(\omega)$ and $E^w(\omega)$, respectively. The solution set of the problem $(\mathrm{VI})_\lambda$ is now denoted by $F(\omega, \lambda)$.

Theorem 8.5. *If Δ is compact, then the multifunction $\omega' \mapsto E^w(\omega')$ is upper semicontinuous at ω.*

Proof. Since Δ is compact and nonempty, $E^w(\omega)$ is a nonempty compact subset of Δ. Suppose that $\Omega \subset R^n$ is an open set containing $E^w(\omega)$. Choose $\delta > 0$ so that

$$E^w(\omega) + \delta \bar{B}_{R^n} \subset \Omega. \tag{8.9}$$

For each $y \in E^w(\omega)$, by Theorem 8.2 there exists $\lambda \in \Delta$ such that $y \in F(\omega, \lambda)$. By Theorem 7.6, there exist constants $\ell(\lambda) > 0$ and $\epsilon_1(\lambda) > 0$, such that $\ell(\lambda)\epsilon_1(\lambda) < 2^{-1/2}\delta$ and

$$F(\omega', \lambda') \subset F(\omega, \lambda) + \ell(\lambda)\|(\omega', \lambda') - (\omega, \lambda)\|\bar{B}_{R^n} \tag{8.10}$$

for all $\omega' = (A', B', \alpha', \beta')$ and $\lambda' \in \Lambda$ satisfying

$$\|\omega' - \omega\| < \epsilon_1(\lambda), \quad \|\lambda' - \lambda\| < \epsilon_1(\lambda).$$

By definition,

$$\|A\| = \max\{\|Av\| \,:\, v \in \bar{B}_{R^m}\}$$

and

$$\begin{aligned} \|(\omega', \lambda') - (\omega, \lambda)\| &= (\|\omega' - \omega\|^2 + \|\lambda' - \lambda\|^2)^{1/2} \\ &= \Big(\|A' - A\|^2 + \|B' - B\|^2 \\ &\quad + \|\alpha' - \alpha\|^2 + \|\beta' - \beta\|^2 + \|\lambda' - \lambda\|^2\Big)^{1/2}. \end{aligned}$$

Since Λ is compact and the family $\{U(\lambda)\}_{\lambda \in \Lambda}$, where

$$U(\lambda) := \{\lambda' \in \Lambda \,:\, \|\lambda' - \lambda\| < \epsilon_1(\lambda)\},$$

is an open covering of Λ, there exists a finite sequence $\lambda^{(1)}, \ldots, \lambda^{(k)} \in \Lambda$ such that

$$\Lambda \subset U(\lambda^{(1)}) \cup \ldots \cup U(\lambda^{(k)}). \tag{8.11}$$

Let

$$\varepsilon = \min\{\varepsilon_1(\lambda^{(i)}) \,:\, i = 1, \ldots, k\}.$$

Fix any $\omega' = (A', B', \alpha', \beta')$ satisfying $\|\omega' - \omega\| < \epsilon$. For every $y' \in E^w(\omega')$ there exists $\lambda' \in \Lambda$ such that $y' \in F(\omega', \lambda')$. By (8.11), there exists $i_0 \in \{1, \ldots, k\}$ such that $\lambda' \in U(\lambda^{(i_0)})$. By (8.10),

$$\begin{aligned} F(\omega', \lambda') &\subset F(\omega, \lambda^{(i_0)}) + \ell(\lambda^{(i_0)})\|(\omega', \lambda') - (\omega, \lambda^{(i_0)})\|\bar{B}_{R^n} \\ &\subset F(\omega, \lambda^{(i_0)}) + 2^{1/2}\ell(\lambda^{(i_0)})\epsilon_1(\lambda^{(i_0)})\bar{B}_{R^n} \\ &\subset F(\omega, \lambda^{(i_0)}) + \delta\bar{B}_{R^n}. \end{aligned}$$

Combining this with (8.9) we can deduce that $E(\omega') \subset \Omega$ for every ω' satisfying $\|\omega' - \omega\| < \delta$. The proof is complete. $\square$

If one can prove that there exists a finite upper bound for the family $\{\ell(\lambda)\}_{\lambda \in \Lambda}$ (see the preceding proof), then the multifunction $\omega' \mapsto E^w(\omega')$ is upper-Lipschitz at ω.

By giving a counterexample, Kum, Lee and Yen (2004) have shown that, even in the case where Δ is a compact polyhedral convex set, the multifunction $\omega' \mapsto E(\omega')$ may be not upper semicontinuous at ω. Nevertheless, from Theorem 8.5 it is easy to derive the following sufficient condition for the usc property of this multifunction.

Theorem 8.6. *If Δ is compact and if $E(\omega) = E^w(\omega)$, then the multifunction $\omega' \mapsto E(\omega')$ is upper semicontinuous at ω.*

8.4 Commentaries

The material presented in this chapter is adapted from Yen and Phuong (2000). We have seen that the results on the solution existence and the stability of monotone AVI problems in Chapters 6 and 7 are very useful for studying LFVO problems on compact constraint sets.

Theorem 8.3 solves a question discussed in the final part of Choo and Atkins (1983). This result is a special corollary of the theorem of Benoist (1998) which asserts that the efficient solution set of the problem of maximizing a vector function with strictly quasiconcave components over a convex compact set is connected. The proof given here is due to Yen and Phuong (2000). Note that Warburton (1983) has extended the result in Theorem 8.4 by proving that, in a vector maximization problem with continuous quasiconvex functions and a compact convex constraint set, the weakly efficient solution set is connected. Connectedness and contractibility of the efficient sets of quasiconcave vector maximization problems have been discussed intensively in the literature (see Warburton (1983), Schaible (1983), Choo et al. (1985), Luc (1987), Hu and Sun (1993), Wantao and Kunping (1993), Benoist (2001), Huy and Yen (2004a, 2004b), and the references therein).

The reader is referred to Bednardczuck (1995) and Penot and Sterna-Karwat (1986) for some general results on stability of vector optimization problems. Stability of the solution maps of LFVO problems on noncompact sets deserves further investigations. It seems to us that some ideas and techniques related to asymptotic cones and asymptotic functions from Auslender and Teboulle (2003) can be useful for this purpose.

Chapter 9

The Traffic Equilibrium Problem

A traffic network can be modelled in a form of a variational inequality. Solutions of the variational inequality correspond to the equilibrium flows on the traffic network. Variational inequality can be also a suitable model for studying other kinds of economic equilibria. The aim of this chapter is to discuss the variational inequality model of the traffic equilibrium problem. Later on, in Chapter 17, by using some results on solution sensitivity of convex QP problems we will establish a fact about the Lipschitz continuity of the equilibrium flow in a traffic network where the travel costs and the demands are subject to change.

9.1 Traffic Networks Equilibria

Consider a traffic system with several cities and many roads connecting them. Suppose that the technical conditions (capacity and quality of roads, etc.) are established. Assume that we know the demands for transportation of some kind of materials or goods between each pair of two cities. The system is well functioning if all these demands are satisfied. The aim of the owner of the network is to keep the system well functioning. The users (drivers, passengers, etc.) do not behave blindly. To go from A to B they will choose one of the roads leading them from A to B with the minimum cost. This natural law is known as the user-optimizing principle or the Wardrop principle. The traffic flow satisfying demands and this law is said to be an *equilibrium flow* of the network. By using this principle, in most of the cases, the owner can compute or estimate the traffic flow on every road. The owner can affect on the network, for example, by requiring high fees from the users of the good roads to force them to use also some roads of lower quality. In this way, a new equilibrium flow, which is more suitable in the opinion of the owner, can be reached.

Traffic network is an example of networks acting in accordance with the Wardrop equilibrium principle. Other examples can be telephone networks or computer networks.

As it was proved by Smith (1979) and Dafermos (1980), a traffic network can be modelled in a variational inequality.

Consider a graph $\mathcal{G}$ consisting of a set $\mathcal{N}$ of nodes and a set $\mathcal{A}$ of arcs. Every *arc* is a pair of two nodes. The inclusion $a \in \mathcal{A}$ means that a is an arc. A *path* is an ordered family of arcs $a_1, \dots, a_m$, where the second node of a_s coincides with the first node of a_{s+1} for $s = 1, \dots, m-1$. We say that the path $\{a_1, \dots, a_m\}$ connects the first node of a_1 with the second node of a_m.

Let I be a given set of the *origin-destination pairs* ($OD-$pairs, for brevity). Each $OD-$pair consists of two nodes: the origin (the first node of the pair) and the destination (the second node of the pair). Denote by P_i the family of all paths connecting the origin with the destination of an $OD-$pair $i \in I$. Let $P = \bigcup_{i \in I} P_i$ and let $|P|$ denote the number of elements of P.

A vector $v = (v_a \,:\, a \in \mathcal{A})$, where $v_a \geq 0$ for all $a \in \mathcal{A}$, is said to be a *flow* (or *flow on arcs*) on the graph. Each v_a indicates the amount of material flow on arc a.

Let there be given a vector function

$$c(v) = (c_a(v) \,:\, a \in \mathcal{A}),$$

where $c_a(v) \geq 0$ for all $a \in \mathcal{A}$. This function $c(\cdot)$, which maps $R^{|\mathcal{A}|}$ to $R^{|\mathcal{A}|}$, is called the *travel cost function.* Each number $c_a(v)$ is interpreted as the travel cost for *one unit* of material flow to go through an arc a provided that the flow v exists on the network. There are many examples explaining why the travel cost on one arc should depend on the flows on other arcs.

The travel cost on a path $p \in P_i$ $(i \in I)$ is given by the formula

$$C_p(v) = \sum_{a \in p} c_a(v)$$

Let $C(v) = (C_p(v) : p \in P)$. For each $i \in I$, define the *minimum travel cost* $u_i(v)$ for the $OD-$pair i by setting

$$u_i(v) = \min\{C_p(v) : p \in P_i\}.$$

Obviously, $C_p(v) - u_i(v) \geq 0$ for each $i \in I$ and for each $p \in P_i$. Let $D = (\delta_{ap})$ be *the incidence matrix* of the relations "arcs-paths"; that is

$$\delta_{ap} = \begin{cases} 0 & \text{if} \quad p \notin P \\ 1 & \text{if} \quad p \in P \end{cases}$$

for all $a \in \mathcal{A}$ and $p \in P$.

It is natural to assume the fulfilment of the following *flow-invariant law*:

$$v_a = \sum_{p \in P} \delta_{ap} v_p. \tag{9.1}$$

Let there be given also a *vector of demands* $g = (g_i : i \in I)$. Every component g_i indicates the *demand* for an $OD-$pair i, that is the amount of the material flow going from the origin to the destination of the pair i. We say that a flow v on the network satisfies demands if

$$\sum_{p \in P_i} v_p = g_i \quad \forall i \in I. \tag{9.2}$$

Note that

$$\Delta := \left\{ v = (v_p : p \in P) \in R_+^{|P|} : \sum_{p \in P_i} v_p = g_i \quad \forall i \in I \right\} \tag{9.3}$$

(the set of flows satisfying demands) is a polyhedral convex set.

If there are given upper bounds $(\gamma_p : p \in P)$, $\gamma_p > 0$ for all $p \in P$, for the capacities of the arcs, then the set of flows satisfying demands is given by the formula

$$\Delta = \left\{ v \in R_+^{|P|} : \sum_{p \in P_i} v_p = g_i \ \ \forall i \in I, \ \ 0 \leq v_p \leq \gamma_p \ \ \forall \, p \in P \right\}. \tag{9.4}$$

In this case, Δ is a compact polyhedral convex set.

Definition 9.1. *A traffic network* $\{\mathcal{G}, I, c(v), g\}$ consists of a graph $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, a set I of $OD-$pairs, a travel cost function $c(v) = (c_a(v) : a \in \mathcal{A})$, and a vector of demands $g = (g_i : i \in I)$.

The following *user-optimizing principle* was introduced by Wardrop (1952). This equilibrium condition explains how the equilibrium flow must depend on the travel cost function.

Definition 9.2. (The Wardrop principle) A flow $\bar{v}$ on the network $\{\mathcal{G}, I, c(v), g\}$ is said to be *an equilibrium flow* if it satisfies demands and, for each $i \in I$ and for each $p \in P_i$, it holds

$$C_p(\bar{v}) - u_i(\bar{v}) = 0 \quad \text{if} \quad \bar{v}_p > 0.$$

The above principle can be stated equivalently as follows: If $C_p(\bar{v})$ (the travel cost on path $p \in P_i$) is greater than $u_i(\bar{v})$ (the minimum travel cost for the $OD-$pair i) then $\bar{v}_p = 0$ (the flow on p is zero). It is important to stress that *the fact that the flow on p is zero does not imply that the flows on all the arcs of p are zeros!*

The problem of finding an equilibrium flow $\bar{v}$ on the given network $\{\mathcal{G}, I, c(v), g\}$ is called *the network equilibrium problem.*

9.2 Reduction of the Network Equilibrium Problem to a Complementarity Problem

Let

$$S(v) = \{C_p(v) - u_i(v) \ : \ p \in P_i, \ \ i \in I\}.$$

We see that the number of components of vector $S(v)$ is equal to $|P|$. Since $C_p(v) - u_i(v) \geq 0$ for all $p \in P_i$ and $i \in I$, we have $S(v) \geq 0$. Note that $v = (v_p \ : \ p \in P)$ is also a nonnegative vector.

Proposition 9.1. *A flow $\bar{v} \in \Delta$ on a network $\{\mathcal{G}, I, c(v), g\}$ is an equilibrium flow if and only if*

$$\begin{cases} \bar{v} \geq 0, \quad S(\bar{v}) \geq 0, \\ \sum\limits_{p \in P_i} \bar{v}_p = g_i \quad \forall i \in I, \\ \langle S(\bar{v}), \bar{v} \rangle = 0. \end{cases} \tag{9.5}$$

Proof. Suppose that $\bar{v} = (\bar{v}_p \ : \ p \in P)$ is a flow satisfying (9.5). Since $\bar{v} \geq 0$ and $S(\bar{v}) \geq 0$, the equality $\langle S(\bar{v}), \bar{v} \rangle = 0$ is equivalent to

$$\bar{v}_p(C_p(\bar{v}) - u_i(\bar{v})) = 0 \quad \forall p \in P_i, \quad \forall i \in I.$$

Therefore, for each $i \in I$ and for each $p \in P_i$, if $\bar{v}_p > 0$ then $C_p(\bar{v}) - u_i(\bar{v}) = 0$. This means that the Wardrop principle is satisfied. Conversely, if $\bar{v} = (\bar{v}_p : p \in P)$ is an equilibrium flow then it is easy to show that (9.5) holds. □

Note that (9.5) is a (generalized) nonlinear complementarity problem under a polyhedral convex set constraint of the form

$$v \in \Delta, \quad f(v) \geq 0, \quad v^T f(v) = 0,$$

where $f(v) := S(v)$ and $\Delta \subset R_+^{|P|}$.

9.3 Reduction of the Network Equilibrium Problem to a Variational Inequality

Consider the incidence matrix $B = (\beta_{ip})$ of the relations "path-OD−pair", where

$$\beta_{ip} = \begin{cases} 1 & \text{if} \quad p \in P_i \\ 0 & \text{if} \quad p \notin P_i. \end{cases}$$

Note that a flow v satisfies demands if and only if

$$Bv = g. \tag{9.6}$$

Indeed, (9.6) means that $(Bv)_i = g_i$ for all $i \in I$. The latter is equivalent to (9.2). Let Δ be defined by (9.3) or (9.4).

The next proposition, which is due to Smith (1979) and Dafermos (1980), reduces the network equilibrium problem to a variational inequality. The proof of below is taken from De Luca and Maugeri (1989).

Proposition 9.2. *A flow $\bar{v} \in \Delta$ is an equilibrium flow of the network $\{\mathcal{G}, I, c(v), g\}$ if and only if*

$$\langle C(\bar{v}), v - \bar{v} \rangle \geq 0 \quad \forall v \in \Delta. \tag{9.7}$$

Proof. *Necessity:* Let $\bar{v} \in \Delta$ be an equilibrium flow. Let $v \in \Delta$. Define

$$P_i^{(1)} = \{p \in P_i : C_p(\bar{v}) = u_i(\bar{v})\},$$

$$P_i^{(2)} = \{p \in P_i \,:\, C_p(\bar{v}) > u_i(\bar{v})\}.$$

According to the Wardrop principle, we have

$$\begin{aligned}
&\langle C(v), v - \bar{v}\rangle \\
&= \sum_{p\in P} C_p(\bar{v})(v_p - \bar{v}_p) \\
&= \sum_{i\in I}\left(\sum_{p\in P_i} C_p(\bar{v})(v_p - \bar{v}_p)\right) \\
&= \sum_{i\in I}\left(\sum_{p\in P_i^{(1)}} C_p(\bar{v})(v_p - \bar{v}_p) + \sum_{p\in P_i^{(2)}} C_p(\bar{v})(v_p - \bar{v}_p)\right) \\
&\geq \sum_{i\in I}\left(\sum_{p\in P_i^{(1)}} u_i(\bar{v})(v_p - \bar{v}_p) + \sum_{p\in P_i^{(2)}} u_i(\bar{v})v_p\right) \\
&= \sum_{i\in I} u_i(\bar{v})\left(\sum_{p\in P_i^{(1)}} (v_p - \bar{v}_p) + \sum_{p\in P_i^{(2)}} v_p\right) \\
&= \sum_{i\in I} u_i(v)(g_i - g_i) \\
&= 0.
\end{aligned}$$

So, (9.7) is valid.

Sufficiency: Let $\bar{v} \in \Delta$ be such that (9.7) holds. It suffices to show that if for an $OD-$pair $i_0 \in I$ there exist two paths $\hat{p} \in P_{i_0}, \widetilde{p} \in P_{i_0}$ such that

$$C_{\hat{p}}(\bar{v}) > C_{\widetilde{p}}(\bar{v}),$$

then $\bar{v}_{\widetilde{p}} = 0$. Consider vector $v = (v_p \,:\, p \in P)$ defined by

$$v_p = \begin{cases} \bar{v}_p & \text{if} \quad p \notin \{\hat{p}, \widetilde{p}\}, \\ \bar{v}_{\hat{p}} + \bar{v}_{\widetilde{p}} & \text{if} \quad p = \widetilde{p}, \\ 0 & \text{if} \quad p = \hat{p}. \end{cases}$$

We have $v \geq 0$ and

$$\sum_{i\in P_i} v_p = g_i \quad \forall i \in I.$$

Indeed, if $i \neq i_0$ then the equality is obvious. If $i = i_0$ then we have

$$\begin{aligned}\sum_{p \in P_{i_0}} v_p &= \sum_{p \in P_{i_0} \setminus \{\hat{p}, \tilde{p}\}} + (\bar{v}_{\hat{p}} + \bar{v}_{\tilde{p}}) \\ &= \sum_{p \in P_{i_0}} \bar{v}_p \\ &= g_{i_0}.\end{aligned}$$

Hence $v \in \Delta$. From (9.7) it follows that

$$\begin{aligned}0 \leq \langle C(\bar{v}), v - \bar{v} \rangle &= \sum_{p \in P} C_p(\bar{v})(v_p - \bar{v}_p) \\ &= C_{\hat{p}}(\bar{v})(v_{\hat{p}} - \bar{v}_{\hat{p}}) + C_{\tilde{p}}(\bar{v})(v_{\tilde{p}} - \bar{v}_{\tilde{p}}) \\ &= -C_{\hat{p}}(\bar{v})\bar{v}_{\hat{p}} + C_{\tilde{p}}(\bar{v})\bar{v}_{\hat{p}} \\ &= -\bar{v}_{\hat{p}}(C_{\hat{p}}(\bar{v}) - C_{\tilde{p}}(\bar{v})).\end{aligned}$$

Since $C_{\hat{p}}(\bar{v}) - C_{\tilde{p}}(\bar{v}) > 0$, we have $\bar{v}_{\hat{p}} = 0$. □

We proceed to show that (9.7) can be expressed as a variational inequality on the set of flows on arcs.

According to (9.1), $v_{\mathcal{A}} = Dv$ for every $v \in \Delta$. Therefore, the set Z of flows on arcs can be defined as follows

$$\begin{aligned}Z &= \{z : z = Dv, \ \ v \in \Delta\} \\ &= \{z : z = Dv, \ \ Bv = g, \ \ v \geq 0\}.\end{aligned} \tag{9.8}$$

Proposition 9.3. *A flow $\bar{v}_A = (\bar{v}_a : a \in \mathcal{A}) \in Z$ is corresponding to an equilibrium flow of the network $\{\mathcal{G}, I, c(v), g\}$ if and only if*

$$\langle c(\bar{v}), v_{\mathcal{A}} - \bar{v}_{\mathcal{A}} \rangle \geq 0, \tag{9.9}$$

for all $v_{\mathcal{A}} = (v_a : a \in \mathcal{A}) \in Z$.

Proof. Recall that $c(v)$ is the vector of travel costs on arcs. By definition of the matrix $D = (\delta_{ap})$, we have

$$\begin{aligned}
\langle c(\bar{v}), v_{\mathcal{A}} - \bar{v}_{\mathcal{A}} \rangle &= \sum_{a\in\mathcal{A}} c_a(v)(v_a - \bar{v}_a) \\
&= \sum_{a\in\mathcal{A}} c_a(\bar{v}) \left(\sum_{p\in P} \delta_{ap} v_p - \sum_{p\in P} \delta_{ap} \bar{v}_p \right) \\
&= \sum_{a\in\mathcal{A}} \left(\sum_{p\in P} (c_a(\bar{v})\delta_{ap})(v_p - \bar{v}_p) \right) \\
&= \sum_{p\in P} \left(\sum_{a\in\mathcal{A}} (c_a(\bar{v})\delta_{ap})(v_p - \bar{v}_p) \right) \\
&= \sum_{p\in P} C_p(\bar{v})(v_p - \bar{v}_p) \\
&= \langle C(\bar{v}), v - \bar{v} \rangle.
\end{aligned}$$

The proof is completed by using (9.8), Proposition 9.2, and the equality

$$\langle c(\bar{v}), v_{\mathcal{A}} - \bar{v}_{\mathcal{A}} \rangle = \langle C(\bar{v}), v - \bar{v} \rangle$$

which holds for every $v \in \Delta$. $\square$

The variational inequality in (9.9), in some sense, is simpler than that one in (9.7). Both of them are variational inequalities on polyhedral convex sets, but the constraint set of (9.9) usually has a smaller dimension. Besides, in most of the cases we can assume that $c(v)$ is a *locally strongly monotone function* (see Chapter 17), while we cannot do so for $C(v)$.

9.4 Commentaries

The material of this chapter is adapted from the reports of Yen and Zullo (1992), Chen and Yen (1993).

Numerical methods for solving the traffic equilibrium problem can be seen in Patriksson (1999). Various network equilibrium problems leading to finite-dimensional variational inequalities are discussed in Nagurney (1993).

Chapter 10

Upper Semicontinuity of the KKT Point Set Mapping

We have studied QP problems in Chapters 1–4. Studying various stability aspects of QP programs is an interesting topic. Although the general stability theory in nonlinear mathematical programming is applicable to convex and nonconvex QP problems, the specific structure of the latter allows one to have more complete results.

In this chapter we obtain some conditions which ensure that a small perturbation in the data of a quadratic programming problem can yield only a small change in its Karush-Kuhn-Tucker point set. Convexity of the objective function and boundedness of the constraint set are not assumed. Obtaining *necessary* conditions for the upper semicontinuity of the KKT point set mapping will be our focus point. Sufficient conditions for the upper semicontinuity of the mapping will be developed on the framework of the obtained necessary conditions.

10.1 KKT Point Set of the Canonical QP Problems

Here we study QP problems of the canonical form:

$$\begin{cases} \text{Minimize} & f(x) := \dfrac{1}{2}x^T Dx + c^T x \\ \text{subject to} & x \in \Delta(A, b) := \{x \in R^n : Ax \geq b, \ x \geq 0\}, \end{cases} \tag{10.1}$$

where $D \in R_S^{n\times n}$, $A \in R^{m\times n}$, $c \in R^n$ and $b \in R^m$ are given data. In the sequel, sometime problem (10.1) will be referred to as $QP(D, A, c, b)$.

Recall that $\bar{x} \in R^n$ is a *Karush-Kuhn-Tucker point* of (10.1) if there exists a vector $\bar{\lambda} \in R^m$ such that

$$\begin{cases} D\bar{x} - A^T\bar{\lambda} + c \geq 0, \quad A\bar{x} - b \geq 0, \\ \bar{x} \geq 0, \quad \bar{\lambda} \geq 0, \\ \bar{x}^T(D\bar{x} - A^T\bar{\lambda} + c) + \bar{\lambda}^T(A\bar{x} - b) = 0. \end{cases} \tag{10.2}$$

The set of all the Karush-Kuhn-Tucker points of (10.1) is denoted by $S(D, A, c, b)$. In Chapter 3, we have seen that if $\bar{x}$ is a local solution of (10.1) then $\bar{x} \in S(D, A, c, b)$. This fact leads to the following standard way to solve (10.1): *Find first the set $S(D, A, c, b)$ then compare the values $f(x)$ among the points $x \in S(D, A, c, b)$.* Hence, one may wish to have some criteria for the (semi)continuity of the following multifunction

$$(D, A, c, b) \mapsto S(D, A, c, b). \tag{10.3}$$

In Section 10.2 we will obtain a necessary condition for the *upper semicontinuity* of the multifunction $S(\cdot, \cdot, c, b)$ at a given point $(D, A) \in R_S^{n\times n} \times R^{m\times n}$. In Section 10.3 we study a special class of QP problems for which the necessary condition obtained in this section is also a sufficient condition for the usc property of the multifunction in (10.3). This class contains some nonconvex QP problems. Sections 10.4 and 10.5 are devoted to sufficient conditions for the usc property of the multifunction in (10.3). In Section 10.5 we will investigate some questions concerning the usc property of the KKT point set mapping in a general QP problem.

Note that the upper Lipschitz property of the multifunction $S(D, A, \cdot, \cdot)$ with respect to the parameters (c, b) is a direct consequence of Theorem 7.3 in Chapter 7.

Since (10.2) can be rewritten as a linear complementarity problem, the study of continuity of the multifunction (10.3) is closely related to the study of continuity and stability of the solution map in linear complementarity theory (see Jansen and Tijs (1987), Cottle et al. (1992), Gowda (1992), Gowda and Pang (1992, 1994a)). However, when the data of (10.1) are perturbed, only some components of the matrix $M = M(D, A)$ (see formula (10.18) below) are perturbed. So, necessary conditions for (semi)continuity and stability of the Karush-Kuhn-Tucker point set cannot be derived from the corresponding results in linear complementarity theory (see, for example, Gowda and Pang (1992)) where all the components of M are perturbed.

10.2 A Necessary Condition for the usc Property of $S(\cdot)$

We now obtain a necessary condition for $S(\cdot, \cdot, c, b)$ to be upper semicontinuous at a given pair $(D, A) \in R_S^{n\times n} \times R^{m\times n}$.

Theorem 10.1. *Assume that the set $S(D, A, c, b)$ is bounded. If the multifunction $S(\cdot, \cdot, c, b)$ is upper semicontinuous at (D, A), then*

$$S(D, A, 0, 0) = \{0\}. \tag{10.4}$$

Proof. Arguing by contradiction, we assume that $S(D, A, c, b)$ is bounded, the multifunction $S(\cdot, \cdot, c, b)$ is usc at (D, A), but (10.4) is violated. The latter means that there is a nonzero vector $\hat{x} \in S(D, A, 0, 0)$. Hence there exists $\hat{\lambda} \in R^m$ such that

$$D\hat{x} - A^T\hat{\lambda} \geq 0, \quad A\hat{x} \geq 0, \tag{10.5}$$

$$\hat{x} \geq 0, \quad \hat{\lambda} \geq 0, \tag{10.6}$$

$$\hat{x}^T D\hat{x} = 0. \tag{10.7}$$

Setting

$$x_t = \frac{1}{t}\hat{x}, \quad \lambda_t = \frac{1}{t}\hat{\lambda}, \quad \text{for every } t \in (0, 1), \tag{10.8}$$

we claim that there exist matrices $D_t \in R_S^{n\times n}$ and $A_t \in R^{m\times n}$ such that $D_t \to D$, $A_t \to A$ as $t \to 0$, and

$$D_t x_t - A_t^T \lambda_t + c \geq 0, \quad A_t x_t - b \geq 0, \tag{10.9}$$

$$x_t \geq 0, \quad \lambda_t \geq 0, \tag{10.10}$$

$$x_t^T(D_t x_t - A_t^T\lambda_t + c) + \lambda_t^T(A_t x_t - b) = 0. \tag{10.11}$$

Matrices D_t and A_t will be of the form

$$D_t = D + tD_0, \quad A_t = A + tA_0, \tag{10.12}$$

where matrices D_0 and A_0 are to be constructed. Since

$$\begin{aligned} D_t x_t - A_t^T \lambda_t + c &= \frac{1}{t}(D + tD_0)\hat{x} - \frac{1}{t}(A^T + tA_0^T)\hat{\lambda} + c \\ &= \frac{1}{t}(D\hat{x} - A^T\hat{\lambda}) + D_0\hat{x} - A_0^T\hat{\lambda} + c, \end{aligned}$$

and

$$\begin{aligned} A_t x_t - b &= \frac{1}{t}(A + tA_0)\hat{x} - b \\ &= \frac{1}{t}A\hat{x} + A_0\hat{x} - b, \end{aligned}$$

the following conditions, due to (10.5), imply (10.9):

$$D_0\hat{x} - A_0^T\hat{\lambda} + c \geq 0, \quad A_0\hat{x} - b \geq 0. \tag{10.13}$$

As $x_t = \dfrac{1}{t}\hat{x}$ and $\lambda_t = \dfrac{1}{t}\hat{\lambda}$, (10.6) implies (10.10). Taking account of (10.7), we have

$$\begin{aligned} & x_t^T(D_t x_t - A_t^T\lambda_t + c) + \lambda_t^T(A_t x_t - b) \\ &= \frac{1}{t}\hat{x}^T\left[\frac{1}{t}(D\hat{x} - A^T\hat{\lambda}) + D_0\hat{x} - A_0^T\hat{\lambda} + c\right] \\ &\qquad + \frac{1}{t}\hat{\lambda}^T\left(\frac{1}{t}A\hat{x} + A_0\hat{x} - b\right) \\ &= \frac{1}{t^2}\left(\hat{x}^T D\hat{x} - \hat{x}^T A^T\hat{\lambda} + \hat{\lambda}^T A\hat{x}\right) \\ &\quad + \frac{1}{t}\hat{x}^T\left(D_0\hat{x} - A_0^T\hat{\lambda} + c\right) + \frac{1}{t}\hat{\lambda}^T(A_0\hat{x} - b) \\ &= \frac{1}{t}\left[\hat{x}^T(D_0\hat{x} - A_0^T\hat{\lambda} + c) + \hat{\lambda}^T(A_0\hat{x} - b)\right]. \end{aligned}$$

So the following equality implies (10.11):

$$\hat{x}^T(D_0\hat{x} - A_0^T\hat{\lambda} + c) + \hat{\lambda}^T(A_0\hat{x} - b) = 0. \tag{10.14}$$

Let $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$, where $\hat{x}_i > 0$ for $i \in I \subset \{1, \ldots, n\}$, and $\hat{x}_i = 0$ for $i \notin I$. Since $\hat{x} \neq 0$, I must be nonempty. Fixing an $i_0 \in I$, we define A_0 as the $m \times n$-matrix whose i_0–th column is $\hat{x}_0^{-1}b$, and whose other columns consist solely of zeros. For this A_0 we have $A_0\hat{x} - b = 0$, hence the second inequality in (10.13) is satisfied, and condition (10.14) becomes the following one:

$$\hat{x}^T(D_0\hat{x} - A_0^T\hat{\lambda} + c) = 0.$$

We have to find a matrix $D_0 \in R_S^{n\times n}$ such that this condition and the first inequality in (10.13) are valid. For this purpose it is enough to find a symmetric matrix D_0 such that

$$D_0\hat{x} - w = 0, \tag{10.15}$$

where $w := A_0^T\hat{\lambda} - c \in R^n$.

If $w = (w_1, \ldots, w_n)$, then we put $D_0 = (d_{ij})\,,\ 1 \le i, j \le n$, where

$$\begin{aligned} &d_{ii} := \hat{x}_i^{-1} w_i \qquad \text{for all } \ i \in I, \\ &d_{i_0 j} = d_{j i_0} := \hat{x}_{i_0}^{-1} w_j \quad \text{for all } \ j \in \{1, 2, \ldots, n\} \setminus I, \end{aligned}$$

and

$$d_{ij} := 0 \quad \text{for other pairs } \ (i,j),\ 1 \le i, j \le n.$$

A simple direct computation shows that this symmetric matrix D_0 satisfies (10.15).

We have thus constructed matrices A_0 and D_0 such that for x_t, λ_t, D_t and A_t defined by (10.8) and (10.12), the conditions (10.9)–(10.11) are satisfied. As a consequence, $x_t \in S(D_t, A_t, c, b)$. Since $S(D, A, c, b)$ is a bounded set, there exists a bounded open set Ω such that $S(D, A, c, b) \subset \Omega$. Since $D_t \to D$ and $A_t \to A$ as $t \to 0$, and the multifunction $S(\cdot, \cdot, c, b)$ is usc at (D, A), we have $x_t \in \Omega$ for all t sufficiently small. This is a contradiction, because $\|x_t\| = \dfrac{1}{t}\|\hat{x}\| \to \infty$ as $t \to 0$. The proof is complete. □

Observe also that, in general, (10.4) is not a sufficient condition for the upper semicontinuity of $S(\cdot)$ at (D, A, c, b).

Example 10.1. Consider the problem $QP(D, A, c, b)$ where

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = [0,\ -1], \quad b = (-1), \quad c = (0, 0).$$

For each $t \in (0,1)$, let $A_t = [-t,\ -1]$. By direct computation using (10.2) we obtain

$$\begin{aligned} &S(D, A, 0, 0) = \{0\}, \quad S(D, A, c, b) = \{(0,0),\ (0,1)\}, \\ &S(D, A_t, c, b) = \left\{(0,0),\ (0,1),\ \left(\frac{1}{t}, 0\right),\ \left(\frac{t}{t^2+1}, \frac{1}{t^2+1}\right)\right\}. \end{aligned}$$

Thus, for any bounded open set $\Omega \subset R^2$ containing $S(D, A, c, b)$, the inclusion

$$S(D, A_t, c, b) \subset \Omega$$

fails to hold for $t > 0$ small enough. Since $A_t \to A$ as $t \to 0$, $S(\cdot)$ cannot be usc at (D, A, c, b).

In the next section we will study a special class of quadratic programs for which (10.4) is not only a necessary but also a sufficient condition for the upper semicontinuity of $S(\cdot)$ at a given point (D, A, c, b).

10.3 A Special Case

We now study those canonical QP problems for which the following condition (H) holds:

(H) *There exists* $\bar{x} \in R^n$ *such that* $A\bar{x} > 0, \quad \bar{x} \geq 0$.

Denote by $\mathcal{H}$ the set of all the matrices $A \in R^{m \times n}$ satisfying (H).

The next statement can be proved easily by applying Lemma 3 from Robinson (1977) and the Farkas Lemma (Theorem 3.2).

Lemma 10.1. *Each one of the following two conditions is equivalent to* (H):

(i) *There exists* $\delta > 0$ *such that, for every matrix* A' *satisfying* $\|A' - A\| < \delta$ *and for every* $b \in R^n$, *the system* $A'x \geq b,\ x \geq 0$ *is solvable.*

(ii) *For any* $\lambda \in R^n$, *if*

$$-A^T\lambda \geq 0, \quad \lambda \geq 0, \tag{10.16}$$

then $\lambda = 0$.

Obviously, (H) implies the existence of an $\hat{x} \in R^n$ satisfying $A\hat{x} > 0,\ \hat{x} > 0$. Thus $\Delta(A, 0)$ has nonempty interior. Now suppose that (H) is fulfilled and $b \in R^n$ is an arbitrarily chosen vector. Since $\Delta(A, b) + \Delta(A, 0) \subset \Delta(A, b)$ and, by Lemma 10.1, $\Delta(A, b)$ is nonempty, we conclude that $\Delta(A, b)$ is an unbounded set with nonempty interior. Besides, it is clear that there exists $\widetilde{x} \in R^n$ satisfying

$$A\widetilde{x} > b, \quad \widetilde{x} > 0.$$

The latter property is a specialization of the Slater constraint qualification (Mangasarian (1969), p. 78), and the Mangasarian-Fromovitz constraint qualification (called by Mangasarian the modified Arrow-Hurwicz-Uzawa constraint qualification) (Mangasarian

(1969), pp. 172-173). These well-known constraint qualifications play an important role in the stability analysis of nonlinear optimization problems.

In the sequence, the inequality system $Ax \geq b$, where $A \in R^{m\times n}$ and $b \in R^m$, is said to be *regular* if there exists $x^0 \in R^n$ such that $Ax^0 > b$.

As it has been noted in Section 5.4, a pair $(\bar{x}, \bar{\lambda}) \in R^n \times R^m$ satisfies (10.2) if and only if $\bar{z} := \begin{pmatrix} \bar{x} \\ \bar{\lambda} \end{pmatrix}$ is a solution to the following linear complementarity problem

$$Mz + q \geq 0, \quad z \geq 0, \quad z^T(Mz + q) = 0, \tag{10.17}$$

where

$$M = M(D, A) := \begin{pmatrix} D & -A^T \\ A & 0 \end{pmatrix}, \; q := \begin{pmatrix} c \\ -b \end{pmatrix}, \; z = \begin{pmatrix} x \\ \lambda \end{pmatrix} \in R^{n+m}. \tag{10.18}$$

Denoting by $\mathrm{Sol}(M, q)$ the solution set of (10.17), we have

$$S(D, A, c, b) = \pi_1\left(\mathrm{Sol}(M, q)\right), \tag{10.19}$$

where $\pi_1 : R^{n+m} \to R^n$ is the linear operator defined by setting $\pi_1 \begin{pmatrix} x \\ \lambda \end{pmatrix} := x$ for every $\begin{pmatrix} x \\ \lambda \end{pmatrix} \in R^{m+n}$.

The notion of $\mathbf{R}_0$-matrix, which is originated to Garcia (1973), has proved to be useful in characterizing the upper semicontinuity property of the solution set of linear complementarity problems (see Jansen and Tijs (1987), Cottle et al. (1992), Gowda (1992), Gowda and Pang (1992), Oettli and Yen (1995, 1996a, 1996b)), and in studying other questions concerning these problems (see Cottle et al. (1992)). $\mathbf{R}_0$-matrices are called also pseudo-regular matrices (Gowda and Pang (1992), p. 78).

Definition 10.1. (See Cottle et al. (1992), Definition 3.8.7) A matrix $M \in R^{p\times p}$ is called an $\mathbf{R}_0$-matrix if the linear complementarity problem

$$Mz \geq 0, \quad z \geq 0, \quad z^T Mz = 0, \qquad (z \in R^p),$$

has the unique solution $z = 0$.

Theorem 10.2. *Assume that $A \in \mathcal{H}$ and that $S(D, A, c, b)$ is bounded. If the multifunction $S(\cdot, \cdot, c, b)$ is upper semicontinuous at (D, A), then $M(D, A)$ is an $\mathbf{R}_0$-matrix.*

Proof. Since $S(D, A, c, b)$ is bounded and $S(\cdot, \cdot, c, b)$ is usc at (D, A), by Theorem 10.1, (10.4) holds. Let $\hat{z} = \begin{pmatrix} \hat{x} \\ \hat{\lambda} \end{pmatrix}$ be such that

$$M\hat{z} \geq 0, \quad \hat{z} \geq 0, \quad \hat{z}^T M\hat{z} = 0, \tag{10.20}$$

where $M = M(D, A)$. This means that the system (10.5)–(10.7) is satisfied. Hence, $\hat{x} \in S(D, A, 0, 0)$. Then $\hat{x} = 0$ by (10.4), and the system (10.5)–(10.7) implies

$$-A^T\hat{\lambda} \geq 0, \quad \hat{\lambda} \geq 0.$$

Since $A \in \mathcal{H}$, $\hat{\lambda} = 0$. Thus any $\hat{z}$ satisfying (10.20) must be zero. So $M(D, A)$ is an $\mathbf{R}_0$-matrix. □

Corollary 10.1. *Let $A \in \mathcal{H}$. If for every $(c, b) \in R^n \times R^m$ the multifunction $S(\cdot, \cdot, c, b)$ is upper semicontinuous at (D, A), then $M(D, A)$ is an $\mathbf{R}_0$-matrix.*

Proof. Consider problem (10.17), where $M = M(D, A)$ and q are defined via D, A, c, b by (10.18). Lemma 1 from Oettli and Yen (1995) shows that there exists $\bar{q} \in R^{n+m}$ such that $\mathrm{Sol}(M, \bar{q})$ is bounded. If $(\bar{c}, \bar{b}) \in R^n \times R^m$ is the pair satisfying $\bar{q} = \begin{pmatrix} \bar{c} \\ -\bar{b} \end{pmatrix}$, then it follows from (10.19) that $S(D, A, \bar{c}, \bar{b})$ is bounded. Since $S(\cdot, \cdot, \bar{c}, \bar{b})$ is usc at (D, A), $M(D, A)$ is an $\mathbf{R}_0$-matrix by Theorem 10.2. □

The following statement gives a sufficient condition for the usc property of the multifunction $S(\cdot)$.

Theorem 10.3. *If $M(D, A)$ is an $\mathbf{R}_0$-matrix, then for any $(c, b) \in R^n \times R^m$ the set $S(D, A, c, b)$ is bounded, and the multifunction $S(\cdot)$ is upper semicontinuous at (D, A, c, b). If, in addition, $S(D, A, c, b)$ is nonempty, then there exist constants $\gamma > 0$ and $\delta > 0$ such that*

$$\begin{aligned} S(D', A', c', b') \subset\ & S(D, A, c, b) \\ & + \gamma(\|D' - D\| + \|A' - A\| + \|c' - c\| + \|b' - b\|)B_{R^n}, \end{aligned} \tag{10.21}$$

for all $(c', b') \in R^n \times R^m$, $D' \in R^{n \times n}$ and $A' \in R^{m \times n}$ satisfying $\|D' - D\| < \delta$, $\|A' - A\| < \delta$.

Proof. Since $M(D, A)$ is an $\mathbf{R}_0$-matrix, by Proposition 5.1 and Theorem 5.6 in Jansen and Tijs (1987) and the remarks before Theorem 2 of in Gowda (1992), $\mathrm{Sol}(M, q)$ is a bounded set, and the solution map $\mathrm{Sol}(\cdot)$ is usc at (M, q). It follows from (10.19) that

$S(D, A, c, b)$ is bounded. Let $\Omega \subset R^n$ be an arbitrary open set containing $S(D, A, b, c)$. By the upper semicontinuity of $\mathrm{Sol}(\cdot)$ at (M, q), we have

$$\mathrm{Sol}(M', q') \subset \Omega \times R^m, \tag{10.22}$$

for all (M', q') in a neighborhood of (M, q). Using (10.19) and (10.22) we get $S(D', A', c', b') \subset \Omega$, for all (D', A', c', b') in a neighborhood of (D, A, c, b).

The upper Lipschitz property described in (10.21) follows from a result of Gowda (1992). Indeed, since $S(D, A, c, b)$ is nonempty, $\mathrm{Sol}(M, q)$ is nonempty. Since M is an $\mathbf{R}_0$-matrix, by Theorem 9 of Gowda (1992) there exist γ_0 and δ_0 such that

$$\mathrm{Sol}(M', q') \subset \mathrm{Sol}(M, q) + \gamma_0(\|M' - M\| + \|q' - q\|)B_{R^{n+m}} \tag{10.23}$$

for all $q' \in R^{n+m}$ and for all $M' \in R^{(n+m)\times(n+m)}$ satisfying $\|M' - M\| < \delta_0$. The inclusion (10.21) follows easily from (10.23) and (10.19). □

Combining Theorem 10.3 with Corollary 10.1 we get the following result.

Corollary 10.2. *If $A \in \mathcal{H}$, then for every $(c, b) \in R^n \times R^m$ the multifunction $S(\cdot, \cdot, c, b)$ is upper semicontinuous at (D, A) if and only if $M(D, A)$ is an $\mathbf{R}_0$-matrix.*

We now find necessary and sufficient conditions for $M(D, A)$ to be an $\mathbf{R}_0$-matrix. By definition, $M = M(D, A)$ is an $\mathbf{R}_0$-matrix if and only if the system

$$D\hat{x} - A^T\hat{\lambda} \geq 0, \quad A\hat{x} \geq 0, \tag{10.24}$$

$$\hat{x} \geq 0, \quad \hat{\lambda} \geq 0, \tag{10.25}$$

$$\hat{x}^T D\hat{x} = 0 \tag{10.26}$$

has the unique solution $(\hat{x}, \hat{\lambda}) = (0, 0)$.

Proposition 10.1. *If $M = M(D, A)$ is an $\mathbf{R}_0$-matrix then $A \in \mathcal{H}$ and the following condition holds:*

$$\Big[D\hat{x} \geq 0,\ A\hat{x} \geq 0,\ \hat{x} \geq 0,\ \hat{x}^T D\hat{x} = 0\Big] \Longrightarrow \hat{x} = 0. \tag{10.27}$$

Proof. If $\hat{\lambda} \in R^m$ is such that $-A^T\hat{\lambda} \geq 0$, $\hat{\lambda} \geq 0$, then $(0, \hat{\lambda})$ is a solution of the system (10.24)–(10.26). If M is an $\mathbf{R}_0$-matrix then we must have $\hat{\lambda} = 0$. By Lemma 10.1, $A \in \mathcal{H}$. Furthermore, for any

$\hat{x} \in R^n$ satisfying $D\hat{x} \geq 0$, $A\hat{x} \geq 0$, $\hat{x} \geq 0$ and $\hat{x}^T D\hat{x} = 0$, it is clear that $(\hat{x}, 0)$ is a solution of (10.24)–(10.26). If M is an $\mathbf{R}_0$-matrix then $(\hat{x}, 0) = (0, 0)$. We have thus proved (10.27). □

The above proposition shows that the inclusion $A \in \mathcal{H}$ and the property (10.27) are necessary conditions for $M = M(D, A)$ to be an $\mathbf{R}_0$-matrix. Sufficient conditions for $M = M(D, A)$ to be an $\mathbf{R}_0$-matrix are given in the following proposition. Recall that a matrix is said to be *nonnegative* if each of its elements is a nonnegative real number.

Proposition 10.2. *Assume that $A \in \mathcal{H}$. The following properties hold:*

(i) *If A is a nonnegative matrix and D is an $\mathbf{R}_0$-matrix then $M(D, A)$ is an $\mathbf{R}_0$-matrix.*

(ii) *If D a positive definite or a negative definite matrix, then $M(D, A)$ is an $\mathbf{R}_0$-matrix.*

Proof. For proving (i), let D be an $\mathbf{R}_0$-matrix and let $(\hat{x}, \hat{\lambda})$ be a pair satisfying (10.24)–(10.26). Since A is a nonnegative matrix, the inequalities $D\hat{x} - A^T\hat{\lambda} \geq 0$ and $\hat{\lambda} \geq 0$ imply $D\hat{x} \geq A^T\hat{\lambda} \geq 0$. Hence (10.24)–(10.26) yield $D\hat{x} \geq 0$, $\hat{x} \geq 0$, $\hat{x}^T D\hat{x} = 0$. Since D is an $\mathbf{R}_0$-matrix, $\hat{x} = 0$. This fact and (10.24)–(10.26) imply $-A^T\hat{\lambda} \geq 0$, $\hat{\lambda} \geq 0$. Since $A \in \mathcal{H}$, $\hat{\lambda} = 0$ by Lemma 10.1. Thus $(\hat{x}, \hat{\lambda}) = (0, 0)$ is the unique solution of (10.24)–(10.26). Hence M is an $\mathbf{R}_0$-matrix. We omit the easy proof of (ii). □

Observe that in Proposition 10.2(i) the condition that A is a nonnegative matrix cannot be dropped.

Example 10.2. Let $n = 2$, $m = 1$, $D = \mathrm{diag}(1, -1)$, $A = (1, -1)$. It is clear that D is an $\mathbf{R}_0$-matrix and the condition $A \in \mathcal{H}$ is satisfied with $\bar{x} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$. Meanwhile, M is not an $\mathbf{R}_0$-matrix. Indeed, one can verify that the pair $(\hat{x}, \hat{\lambda})$, where $\hat{x} = (1, 1)$ and $\hat{\lambda} = 1$, is a solution of the system (10.24)–(10.26).

Definition 10.2 (Murty (1972), p. 67). We say that $D = (d_{ij}) \in R^{n \times n}$ is a *nondegenerate matrix* if, for any nonempty subset $\alpha \subset \{1, \ldots, n\}$, the determinant of the principal submatrix $D_{\alpha\alpha}$ consisting of the elements d_{ij} $(i \in \alpha,\ j \in \alpha)$ of D is nonzero.

Every nondegenerate matrix is an $\mathbf{R}_0$-matrix (see Cottle et al. (1992), p. 180). It can be proved that the set of nondegenerate

$n \times n$-matrices is open and dense in $R^{n\times n}$. From the following simple observation it follows that symmetric nondegenerate $\mathbf{R}_0$-matrices form a dense subset in the set of all symmetric matrices.

Proposition 10.3. *For any matrix $D \in R^{n\times n}$ and for any $\epsilon > 0$ there exists a nonnegative diagonal matrix U^ϵ such that $D + U^\epsilon$ is a nondegenerate matrix, and $\|U^\epsilon\| \le \epsilon$.*

Proof. The proposition is proved by induction on n. For $n = 1$, if $D = [d]$, $d \neq 0$, then we set $U^\epsilon = [0]$. If $D = [0]$ then we set $U^\epsilon = [\epsilon]$. Assume that the conclusion of the proposition is true for all indexes $n \le k - 1$. Let $D = (d_{ij})$ be a $k \times k$-matrix which is *not* nondegenerate. Denote by D_{k-1} the left-top submatrix of the order $(k-1) \times (k-1)$ of D. By induction, there is a diagonal matrix $U^\epsilon_{k-1} = \mathrm{diag}(\alpha_1, \dots, \alpha_{k-1})$ such that every principal minor of the matrix $D_{k-1} + U^\epsilon_{k-1}$ is nonzero, and $\|U^\epsilon_{k-1}\| \le \epsilon$. The required matrix U^ϵ is sought in the form

$$U^\epsilon = \mathrm{diag}(\alpha_1, \dots, \alpha_{k-1}, y),$$

where $y \in R$ is a parameter.

From the construction of U^ϵ it follows that all the determinants of the principal submatrices of $D + U^\epsilon$ which do not contain the element $d_{kk} + y$, are nonzero. Obviously, there are 2^{k-1} principal submatrices of $D + U^\epsilon$ containing the element $d_{kk} + y$. The determinant of each one of these submatrices has the form $\alpha_i y + \beta_i$, $1 \le i \le 2^{k-1}$, where α_i and β_i are certain real numbers. Moreover, α_i equals 1 or equals one of the principal minors of $D_{k-1} + U^\epsilon_{k-1}$. So $\alpha_i \neq 0$ for all i. Since the numbers $-\dfrac{\beta_i}{\alpha_i}$, $1 \le i \le 2^{k-1}$, cannot cover the segment $[0, \epsilon]$, there exists $\bar{y} \in [0, \epsilon]$ such that $\bar{y} \neq -\frac{\beta_i}{\alpha_i}$ for all i. From what has already been said, we conclude that for $U^\epsilon := \mathrm{diag}(\alpha_1, \dots, \alpha_{k-1}, \bar{y})$ the matrix $D + U^\epsilon$ is nondegenerate. In addition, it is clear that $\|U^\epsilon\| \le \epsilon$. The proof is complete. $\square$

Remark 10.1. The property of being a nondegenerate matrix is not invariant under the operation of matrix conjugation. This means that even if D is nondegenerate and P is nonsingular, the matrix $P^{-1}DP$ still may have zero principal minors. Examples can be found even in $R^{2\times 2}$. Consequently, a linear operator with a nondegenerate matrix in one basis may have a degenerate matrix in another basis.

It follows from Theorem 10.3 and Proposition 10.2 that the multifunction $S(\cdot)$ is usc at (D, A, c, b) if $A \in \mathcal{H}$, A is a nonnegative matrix and D is an $\mathbf{R}_0$-matrix. There are many nonconvex QP problems fulfilling these conditions. For example, in the quadratic programs whose objective functions are given by the formula

$$f(x) = c^T x + \sum_{i=1}^{s} \mu_i x_i^2 - \sum_{i=s+1}^{n} \mu_i x_i^2,$$

where $c \in R^n$, $1 \le s < n$ and $\mu_i > 0$ for all i, D is an $\mathbf{R}_0$-matrix. Proposition 10.3 shows that the set of symmetric $\mathbf{R}_0$-matrices is dense in $R_S^{n\times n}$.

10.4 Sufficient Conditions for the usc Property of $S(\cdot)$

Consider problem (10.1) whose Karush-Kuhn-Tucker point set is denoted by $S(D, A, c, b)$. A necessary condition for the usc property of $S(\cdot)$ was obtained in Section 10.2. Sufficient conditions for having that property were given in Section 10.3 only for a special class of QP problems. Our aim in this section is to find sufficient conditions for the usc property of the multifunction $S(\cdot)$ which are applicable for larger classes of QP problems.

For a matrix $A \in R^{m\times n}$, the dual of the cone

$$\Lambda[A] := \{\lambda \in R^m : -A^T\lambda \ge 0, \ \lambda \ge 0\}$$

is denoted by $(\Lambda[A])^*$. By definition, $(\Lambda[A])^* = \{\xi \in R^m : \lambda^T\xi \le 0 \ \ \forall \lambda \in \Lambda[A]\}$. The interior of $(\Lambda[A])^*$ is denoted by $\mathrm{int}\,(\Lambda[A])^*$. By Lemma 6.4,

$$\mathrm{int}\,(\Lambda[A])^* = \{\xi \in R^m : \lambda^T\xi < 0 \quad \forall \lambda \in \Lambda[A] \setminus \{0\}\}.$$

The proofs of Theorems 10.4–10.6 below are based on some observations concerning the structure of the Karush-Kuhn-Tucker system (10.2). It turns out that the desired stability property of the set $S(D, A, c, b)$ depends greatly on the behavior of the quadratic form $x^T Dx$ on the recession cone of $\Delta(A, b)$ and also on the position of b with respect to the set $\mathrm{int}\,(\Lambda[A])^*$.

One can note that in Example 10.1 the solution set $\mathrm{Sol}\,(D, A, 0, 0)$ is empty. In the following theorem, such "abnormal" situation will be excluded.

Theorem 10.4. *If* $\mathrm{Sol}(D, A, 0, 0) = \{0\}$ *and if* $b \in \mathrm{int}(\Lambda[A])^*$ *then, for any* $c \in R^n$*, the multifunction* $S(\cdot)$ *is upper semicontinuous at* (D, A, c, b).

Proof. Suppose the theorem were false. Then we could find an open set Ω containing $S(D, A, c, b)$, a sequence $\{(D^k, A^k, c^k, b^k)\}$ converging to (D, A, c, b) in $R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m$, a sequence $\{x^k\}$ with the property that $x^k \in S(D^k, A^k, c^k, b^k)$ and $x^k \notin \Omega$ for every k. By the definition of KKT point, there exists a sequence $\{\lambda^k\} \subset R^m$ such that

$$D^k x^k - (A^k)^T\lambda^k + c^k \geq 0, \quad A^k x^k - b^k \geq 0, \tag{10.28}$$

$$x^k \geq 0, \quad \lambda^k \geq 0, \tag{10.29}$$

$$(x^k)^T(D^k x^k - (A^k)^T\lambda^k + c^k) + (\lambda^k)^T(A^k x^k - b^k) = 0. \tag{10.30}$$

We first consider the case where the sequence of norms $\{\|(x^k, \lambda^k)\|\}$ is bounded. As the sequences $\{\|x^k\|\}$ and $\{\|\lambda^k\|\}$ are also bounded, from $\{x^k\}$ and $\{\lambda^k\}$, respectively, one can extract converging subsequences $\{x^{k_i}\}$ and $\{\lambda^{k_i}\}$. Assume that $x^{k_i} \to x^0 \in R^n$ and $\lambda^{k_i} \to \lambda^0 \in R^m$ as $i \to \infty$. From (10.28)–(10.30) it follows that

$$\begin{aligned}
&Dx^0 - A^T\lambda^0 + c \geq 0, \quad Ax^0 - b \geq 0,\\
&x^0 \geq 0, \quad \lambda^0 \geq 0,\\
&(x^0)^T(Dx^0 - A^T\lambda^0 + c) + (\lambda^0)^T(Ax^0 - b) = 0.
\end{aligned}$$

Hence $x^0 \in S(D, A, c, b) \subset \Omega$. On the other hand, since $x^{k_i} \notin \Omega$ for all i and Ω is open, we have $x^0 \notin \Omega$, a contradiction.

We now turn to the case where the sequence $\{\|(x^k, \lambda^k)\|\}$ is unbounded. In this case, there exists a subsequence, which is denoted again by $\{\|(x^k, \lambda^k)\|\}$, such that $\|(x^k, \lambda^k)\| \to \infty$ and $\|(x^k, \lambda^k)\| \neq 0$ for every k. Let

$$z^k := \frac{(x^k, \lambda^k)}{\|(x^k, \lambda^k)\|} = \left(\frac{x^k}{\|(x^k, \lambda^k)\|}, \frac{\lambda^k}{\|(x^k \lambda^k)\|}\right). \tag{10.31}$$

Since $\|z^k\| = 1$, there is a subsequence of $\{z^k\}$, which is denoted again by $\{z^k\}$, such that $z^k \to \bar{z} \in R^n \times R^m$, $\|\bar{z}\| = 1$. Let $\bar{z} = (\bar{x}, \bar{\lambda})$. By (10.31),

$$\frac{x^k}{\|(x^k, \lambda^k)\|} \to \bar{x}, \quad \frac{\lambda^k}{\|(x^k, \lambda^k)\|} \to \bar{\lambda}.$$

Dividing both sides of (10.28) and (10.29) by $\|(x^k, \lambda^k)\|$, both sides of (10.30) by $\|(x^k, \lambda^k)\|^2$, and taking limits as $k \to \infty$, we obtain

$$D\bar{x} - A^T\bar{\lambda} \geq 0, \quad A\bar{x} \geq 0, \tag{10.32}$$

$$\bar{x} \geq 0, \quad \bar{\lambda} \geq 0, \tag{10.33}$$

$$\bar{x}^T(D\bar{x} - A^T\bar{\lambda}) + \bar{\lambda}^T A\bar{x} = 0. \tag{10.34}$$

By (10.32) and (10.33), $\bar{x} \in \Delta(A, 0) = \{x \in R^n : Ax \geq 0, \ x \geq 0\}$. Let us suppose for the moment that $\bar{x} \neq 0$. It is obvious that (10.34) can be rewritten as $\bar{x}^T D\bar{x} = 0$. If $x^T Dx \geq 0$ for all $x \in \Delta(A, 0)$ then $\bar{x} \in \mathrm{Sol}\,(D, A, 0, 0)$, contrary to the assumption $\mathrm{Sol}\,(D, A, 0, 0) = \{0\}$. If there exists $\hat{x} \in \Delta(A, 0)$ such that $\hat{x}^T D\hat{x} < 0$ then

$$\inf\{x^T Dx : x \in \Delta(A, 0)\} = -\infty,$$

because $\Delta(A, 0)$ is a cone. Thus $\mathrm{Sol}\,(D, A, 0, 0) = \emptyset$, contrary to the condition $\mathrm{Sol}\,(D, A, 0, 0) = \{0\}$. Therefore $\bar{x} = 0$.

As $\|(\bar{x}, \bar{\lambda})\| = 1$, from (10.32) and (10.33) it follows that $\bar{\lambda} \in \Lambda[A] \setminus \{0\}$. The assumption $b \in \mathrm{int}\,(\Lambda[A])^*$ implies

$$\bar{\lambda}^T b < 0. \tag{10.35}$$

Since $\|(x^k, \lambda^k)\| \to \infty$, $\dfrac{\lambda^k}{\|(x^k, \lambda^k)\|} \to \bar{\lambda}$ and $\|\bar{\lambda}\| = \|(\bar{x}, \bar{\lambda})\| = 1$, $\|\lambda^k\| \to \infty$. Using the obvious identity $(x^k)^T (A^k)^T \lambda^k = (\lambda^k)^T A^k x^k$ we can rewrite (10.30) as the following

$$(x^k)^T D^k x^k + (x^k)^T c^k = (\lambda^k)^T b^k. \tag{10.36}$$

If the sequence $\{x^k\}$ is bounded, then dividing both sides of (10.36) by $\|(x^k, \lambda^k)\|$ and letting $k \to \infty$ we obtain $\bar{\lambda}^T b = 0$, contrary to (10.35). So the sequence $\{x^k\}$ must be unbounded, and it has a subsequence, denoted again by $\{x^k\}$, such that $\|x^k\| \to \infty$, $\|x^k\| \neq 0$ for all k, and $\dfrac{x^k}{\|x^k\|} \to \hat{x}$ with $\|\hat{x}\| = 1$. For the sequence $\{(\lambda^k)^T b^k\}$ there are only two possibilities:

(α) There exists an integer i_0 such that

$$(\lambda^k)^T b^k \leq 0 \tag{10.37}$$

for all $k \geq i_0$, and

(β) For each i there exists an integer $k_i > i$ such that

$$(\lambda^{k_i})^T b^{k_i} > 0. \tag{10.38}$$

If case (α) arises, then (10.36) implies

$$(x^k)^T D^k x^k + (x^k)^T c^k \leq 0 \tag{10.39}$$

for all $k \geq i_0$. Dividing both sides of (10.39) by $\|x^k\|^2$ and letting $k \to \infty$ we get

$$\hat{x}^T D\hat{x} \leq 0. \tag{10.40}$$

By (10.28) and (10.29),

$$A^k x^k \geq b^k, \quad x^k \geq 0.$$

Dividing both sides of each of the last two inequalities by $\|x^k\|$ and letting $k \to \infty$ we obtain

$$A\hat{x} \geq 0, \quad \hat{x} \geq 0. \tag{10.41}$$

Since $0 \in \mathrm{Sol}\,(D, A, 0, 0)$, by (10.40) and (10.41) we have $\hat{x} \in \mathrm{Sol}\,(D, A, 0, 0)$, contrary to the condition $\mathrm{Sol}\,(D, A, 0, 0) = \{0\}$. Thus case (α) is impossible. If case (β) happens, then by dividing both sides of (10.38) by $\|(x^{k_i}, \lambda^{k_i})\|$ and letting $i \to \infty$ we obtain $\bar{\lambda}^T b \geq 0$, contrary to (10.35). The proof is complete, because neither (α) nor (β) can occur. $\square$

Theorem 10.5. *If* $\mathrm{Sol}\,(-D, A, 0, 0) = \{0\}$ *and* $b \in -\mathrm{int}\,(\Lambda[A])^*$ *then, for any* $c \in R^n$, *the multifunction* $S(\cdot)$ *is upper semicontinuous at* (D, A, c, b).

Proof. Except for several small changes, this proof is very similar to the proof of Theorem 10.4. Suppose, contrary to our claim, that there is an open set $\Omega \subset R^n$ containing $S(D, A, c, b)$, a sequence $\{(D^k, A^k, c^k, b^k)\}$ converging to (D, A, c, b) in $R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m$, a sequence $\{x^k\}$ with $x^k \in S(D^k, A^k, c^k, b^k)$ and $x^k \notin \Omega$ for every k. By the definition of KKT point, there is a sequence $\{\lambda^k\}$ satisfying (10.28)–(10.30). If the sequence of $\{\|(x^k, \lambda^k)\|\}$ is bounded then, arguing similarly as in the preceding proof, we will arrive at a contradiction. If the sequence $\{\|(x^k, \lambda^k)\|\}$ is unbounded then, without any loss of generality, we can assume that the sequence $\left\{\dfrac{(x^k, \lambda^k)}{\|(x^k, \lambda^k)\|}\right\}$ converges to a certain pair $(\bar{x}, \bar{\lambda})$ with $\|(\bar{x}, \bar{\lambda})\| = 1$. Dividing both sides of (10.28) and of (10.29) by $\|(x^k, \lambda^k)\|$, both sides of (10.30) by $\|(x^k, \lambda^k)\|^2$ and letting $k \to \infty$ we obtain (10.32)–(10.34). From (10.34) we have $\bar{x}^T(-D)\bar{x} = 0$. The assumption $\mathrm{Sol}\,(-D, A, 0, 0) = \{0\}$ forces $\bar{x} = 0$. Thus $\bar{\lambda} \in \Lambda[A] \setminus \{0\}$. Since $b \in -\mathrm{int}\,(\Lambda[A])^*$, we have

$$\bar{\lambda}^T b > 0. \tag{10.42}$$

Since $\|(x^k,\lambda^k)\| \to \infty$, $\dfrac{\lambda^k}{\|(x^k,\lambda^k)\|} \to \bar{\lambda}$, and $\|\bar{\lambda}\| = 1$, we must have $\|\lambda^k\| \to \infty$. Again, rewrite (10.30) in the form (10.36). If the sequence $\{x^k\}$ is bounded, we can divide both sides of (10.36) by $\|(x^k,\lambda^k)\|$ and let $k \to \infty$ to obtain $\bar{\lambda}^T b = 0$, which contradicts (10.42). Thus the sequence $\{x^k\}$ must be bounded, and it has a subsequence, denoted again by $\{x^k\}$, such that $\|x^k\| \to \infty$, $\|x^k\| \neq 0$ for all k, and $\dfrac{x^k}{\|x^k\|} \to \hat{x}$ with $\|\hat{x}\| = 1$.

If there exists an index i_0 such that (10.37) holds, then dividing both sides of (10.36) by $\|(x^k,\lambda^k)\|$ and taking limit as $k \to \infty$ we have $\bar{\lambda}^T b = 0$, contrary to (10.42).

Assume that for each i, there exists an integer $k_i > i$ such that (10.38) holds. From (10.36) and (10.38) it follows that

$$(x^{k_i})^T D_{k_i} x^{k_i} + (x^{k_i})^T c^{k_i} \geq 0 \tag{10.43}$$

for all i. Dividing both sides of (10.43) by $\|x^{k_i}\|^2$ and taking limit as $i \to \infty$ we get $\hat{x}^T D \hat{x} \geq 0$ or, equivalently,

$$\hat{x}^T(-D)\hat{x} \leq 0. \tag{10.44}$$

By (10.28) and (10.29), $A_{k_i} x^{k_i} \geq b^{k_i}$, $\quad x^{k_i} \geq 0$. Dividing both sides of each of the last two inequalities by $\|x^{k_i}\|$ and taking limits we obtain (10.41). Properties (10.41), (10.44), and the inclusion $0 \in \mathrm{Sol}\,(-D,A,0,0)$ yield $\hat{x} \in \mathrm{Sol}\,(-D,A,0,0)$, contrary to the condition $\mathrm{Sol}\,(-D,A,0,0) = \{0\}$. Thus, in all possible cases we have arrived at a contradiction. The proof is complete. □

Our third sufficient condition for the stability of the Karush-Kuhn-Tucker point set can be formulated as follows.

Theorem 10.6. *If $S(D,A,0,0) = \{0\}$ and $\Lambda[A] = \{0\}$ then, for any $(c,b) \in R^n \times R^m$, the multifunction $S(\cdot)$ is upper semicontinuous at (D,A,c,b).*

Proof. Repeat the arguments in the proof of Theorem 10.4 until reaching the system (10.32)–(10.34). Since $S(D,A,0,0) = \{0\}$, we have $\bar{x} = 0$, hence (10.32)–(10.34) imply $-A^T\bar{\lambda} \geq 0$, $\bar{\lambda} \geq 0$. By $\|\bar{\lambda}\| = \|(\bar{x},\bar{\lambda})\| = 1$, one has $\bar{\lambda} \in \Lambda[A] \setminus \{0\}$, contrary to the assumption that $\Lambda[A] = \{0\}$. □

10.5 Corollaries and Examples

We now consider some corollaries of the results established in the preceding section and give several illustrative examples.

Corollary 10.3. *If $\Lambda[A] = \{0\}$ and if the matrix D is a positive definite (or negative definite) then, for any pair $(c, b) \in R^n \times R^m$, the multifunction $S(\cdot)$ is upper semicontinuous at (D, A, c, b).*

Proof. If D is positive definite, then $S(D, A, 0, 0) = \mathrm{Sol}(D, A, 0, 0) = \{0\}$. So our assertion follows from Theorem 10.6.

If D is negative definite, then $S(D, A, 0, 0) = \mathrm{Sol}\,(-D, A, 0, 0) = \{0\}$, and again the assertion follows from Theorem 10.6. □

We proceed to show that the condition $b \in \mathrm{int}\,(\Lambda[A])^*$ in Theorem 10.4 is equivalent to the regularity of the following system of linear inequalities

$$Ax \geq b, \quad x \geq 0. \tag{10.45}$$

Lemma 10.2. *System* (10.45) *is regular if and only if* $b \in \mathrm{int}\,(\Lambda[A])^*$.

Proof. Assume (10.45) is regular, i.e. there exists x^0 such that $Ax^0 > b$, $x^0 > 0$. Let $q := Ax^0 - b > 0$ and let $\bar{\lambda}$ be any vector from $\Lambda[A] \setminus \{0\}$, that is $A^T\bar{\lambda} \leq 0, \bar{\lambda} \geq 0$, and $\bar{\lambda} \neq 0$. Then

$$\bar{\lambda}^T b = \bar{\lambda}^T(Ax^0 - q) = (x^0)^T A^T \bar{\lambda} - \bar{\lambda}^T q \leq -\bar{\lambda}^T q < 0.$$

Hence $b \in \mathrm{int}\,(\Lambda[A])^*$.

Conversely, assume that $b \in \mathrm{int}\,(\Lambda[A])^*$. Suppose for a moment that (10.45) is irregular. Since the system $Ax > b$, $x \geq 0$ has no solutions, for any sequence $b^k \to b$ with $b^k > b$ for all k, the systems

$$Ax \geq b^k, \quad x \geq 0$$

have no solutions. By Theorem 2.7.9 from Cottle et al. (1992), which is a corollary of the Farkas Lemma, there exists $\lambda^k \in R^m$ such that

$$-A^T\lambda^k \geq 0, \quad \lambda^k \geq 0, \quad (\lambda^k)^T b^k > 0. \tag{10.46}$$

Since $\lambda^k \neq 0$, without loss of generality, we can assume that $\|\lambda^k\| = 1$ for every k, and $\lambda^k \to \bar{\lambda}$ with $\|\bar{\lambda}\| = 1$. Taking limits in (10.46) as $k \to \infty$ we get

$$-A^T\bar{\lambda} \geq 0, \quad \bar{\lambda} \geq 0, \quad \bar{\lambda}^T b \geq 0.$$

Hence $\bar{\lambda} \in \Lambda[A] \setminus \{0\}$, and the inequality $\bar{\lambda}^T b \geq 0$ contradicts the assumption $b \in \mathrm{int}\,(\Lambda[A])^*$. We have thus proved that (10.45) is regular. □

Corollary 10.4. *If* (10.45) *is regular and if* $\Delta(A, b)$ *is bounded, then the multifunction* $S(\cdot)$ *is upper semicontinuous at* (D, A, c, b).

Proof. Since $\Delta(A, b)$ is nonempty, bounded, and $\Delta(A, b)+\Delta(A, 0) \subset \Delta(A, b)$, we have $\Delta(A, 0) = \{0\}$. Since (10.45) is regular, by Lemma 10.2 we have $b \in \operatorname{int}(\Lambda[A])^*$. Applying Theorem 10.4 we get the desired result. □

We have the following sufficient condition for stability of the KKT point set in QP problems with bounded constraint sets.

Corollary 10.5. *If* $\Delta(A, 0) = \{0\}$ *and* $\lambda^T b \neq 0$ *for all* $\lambda \in \Lambda[A] \setminus \{0\}$ *then, for any* $c \in R^n$, *the multifunction* $S(\cdot)$ *is upper semicontinuous at* (D, A, c, b).

Proof. Obviously, the condition $\Delta(A, 0) = \{0\}$ implies

$$S(D, A, 0, 0) = \operatorname{Sol}(D, A, 0, 0) = \operatorname{Sol}(-D, A, 0, 0) = \Delta(A, 0) = \{0\}. \tag{10.47}$$

Since $\Lambda[A]$ is a convex cone, the assumption $\lambda^T b \neq 0$ for all $\lambda \in \Lambda[A] \setminus \{0\}$ implies that one of the following two cases must occur:

(i) $\lambda^T b < 0$ for all $\lambda \in \Lambda[A] \setminus \{0\}$,

(ii) $\lambda^T b > 0$ for all $\lambda \in \Lambda[A] \setminus \{0\}$.

In the first case, the desired conclusion follows from (10.47) and Theorem 10.4. In the second case, the conclusion follows from (10.47) and Theorem 10.5. □

The following two examples show that the obtained sufficient conditions for stability can be applied to nonconvex QP problems.

Example 10.3. Consider problem (10.1) where $n = 2$, $m = 1$,

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ or } D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$A = \left[-\frac{1}{2}, -1\right], \ b = (-1), \ c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We have $\Delta(A, 0) = \{0\}$, $\operatorname{Sol}(D, A, 0, 0) = \{0\}$ and $b \in \operatorname{int}(\Lambda[A])^*$. By Theorem 10.4, $S(\cdot)$ is usc at (D, A, c, b).

Example 10.4. Consider problem (10.1) where $n = 2$, $m = 1$,

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ A = [-1, 0], \ b = (-1), \ c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

An easy computation shows that

$$S(D, A, 0, 0) = \{0\}, \ \operatorname{Sol}(-D, A, 0, 0) = \{0\}, \ \text{and } b \in -\operatorname{int}(\Lambda[A])^*.$$

The multifunction $S(\cdot)$ is usc at (D, A, c, b) by Theorem 10.5.

The next two examples show that when the condition $b \in \operatorname{int}(\Lambda[A])^*$ is violated the conclusion of Theorem 10.4 may hold or may not hold, as well.

Example 10.5. Let $D = [1]$, $A = [0]$, $b = (1)$, $c = (0)$, $A_t = [-t]$, where $t \in (0, 1)$. It is easily seen that

$$\begin{array}{l} S(D, A, 0, 0) = \{0\}, \quad \operatorname{Sol}(D, A, 0, 0) = \{0\}, \quad S(D, A, c, b) = \emptyset, \\ S(D, A_t, c, b) = \left\{\dfrac{1}{t}\right\}, \quad \Lambda[A] = R_+, \quad b \notin \operatorname{int}(\Lambda[A])^*. \end{array}$$

We have $S(D, A, c, b) \subset \Omega$, where $\Omega = \emptyset$. Since $A_t \to A$ and the inclusion $S(D, A_t, c, b) \subset \Omega$ cannot hold for sufficiently small $t > 0$, $S(\cdot)$ cannot be usc at (D, A, c, b).

Example 10.6. Let $D = [-1]$, $A = [-1]$, $b = (1)$, $c = (0)$. It is easy to verify that

$$\begin{array}{l} S(D, A, 0, 0) = \{0\}, \quad \operatorname{Sol}(D, A, 0, 0) = \{0\}, \quad S(D, A, c, b) = \emptyset, \\ \Lambda[A] = R_+, \quad b \notin \operatorname{int}(\Lambda[A])^*. \end{array}$$

The map $S(\cdot)$ is usc at (D, A, c, b). Indeed, since $S(-D, A, 0, 0) = \{0\}$ and $b \in -\operatorname{int}(\Lambda[A])^*$, Theorem 10.5 can be applied.

The following two examples show that if $b \notin -\operatorname{int}(\Lambda[A])^*$ then the conclusion of Theorem 10.5 may hold or may not hold, as well.

Example 10.7. Let D, A, c, b be defined as in Example 10.5. In this case we have

$$\begin{array}{l} S(D, A, 0, 0) = \{0\}, \quad \operatorname{Sol}(-D, A, 0, 0) = \{0\}, \\ \Lambda[A] = R_+, \quad b \notin -\operatorname{int}(\Lambda[A])^*. \end{array}$$

As it has been shown in Example 10.5, the map $S(\cdot)$ is not usc at (D, A, c, b).

Example 10.8. Let $D = [1]$, $A = [-1]$, $b = (-1)$, $c = (0)$. It is a simple matter to verify that

$$\begin{array}{l} S(D, A, 0, 0) = \{0\}, \quad \operatorname{Sol}(-D, A, 0, 0) = \{0\}, \\ \Lambda[A] = R_+, \quad b \notin -\operatorname{int}(\Lambda[A])^*. \end{array}$$

The fact that $S(\cdot)$ is usc at (D, A, c, b) follows from Theorem 10.4, because $\operatorname{Sol}(D, A, 0, 0) = \{0\}$ and $b \in \operatorname{int}(\Lambda[A])^*$.

10.6 USC Property of $S(\cdot)$: The General Case

In this section we obtain necessary and sufficient conditions for the stability of the Karush-Kuhn-Tucker point set in a general QP problem.

Given matrices $A \in R^{m\times n}$, $F \in R^{s\times n}$, $D \in R_S^{n\times n}$, and vectors $c \in R^n$, $b \in R^m$, $d \in R^s$, we consider the following general indefinite QP problem $QP(D, A, c, b, F, d)$:

$$\begin{cases} \text{Minimize } f(x) := \frac{1}{2}x^T Dx + c^T x \\ \text{subject to } x \in R^n, \ Ax \geq b, \ Fx \geq d \end{cases} \tag{10.48}$$

In what follows, *the pair* (F, d) *is not subject to change.* So the set $\Delta(F, d) := \{x \in R^n : Fx \geq d\}$ is fixed. Define $\Delta(A, b) = \{x \in R^n : Ax \geq b\}$ and recall (see Definition 3.1 and Corollary 3.2) that $\bar{x} \in \Delta(A, b) \cap \Delta(F, d)$ is said to be a Karush-Kuhn-Tucker point of $QP(D, A, c, b, F, d)$ if there exists a pair $(\bar{u}, \bar{v}) \in R^m \times R^s$ such that

$$\begin{aligned} &D\bar{x} - A^T\bar{u} - F^T\bar{v} + c = 0, \\ &A\bar{x} \geq b, \quad \bar{u} \geq 0, \\ &F\bar{x} \geq d, \quad \bar{v} \geq 0, \\ &\bar{u}^T(A\bar{x} - b) + \bar{v}^T(F\bar{x} - d) = 0. \end{aligned}$$

The KKT point set and the solution set of (10.48) are denoted, respectively, by $S(D, A, c, b, F, d)$ and $\mathrm{Sol}(D, A, c, b, F, d)$.

If $s = n$, $d = 0$, and F is the unit matrix in $R^{n\times n}$, then problem (10.48) has the following canonical form (10.1). In agreement with the notation of the preceding sections, we write $S(D, A, c, b)$ instead of $S(D, A, c, b, F, d)$, and $\mathrm{Sol}(D, A, c, b)$ instead of $\mathrm{Sol}(D, A, c, b, F, d)$ if (10.48) has the canonical form. The upper semicontinuity of the multifunction

$$p' \mapsto S(p'), \ p' = (D', A', c', b') \in R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m, \tag{10.49}$$

has been studied in Sections 10.3–10.5. This property can be interpreted as the *stability* of the KKT point set $S(D, A, c, b)$ with respect to the change in the problem parameters. In this section we are interested in finding out how the results proved in Sections 10.3–10.5 can be extended to the case of problem (10.48). We will obtain some necessary and sufficient conditions for the upper semicontinuity of the multifunction

$$p' \mapsto S(p', F, d), \; p' = (D', A', c', b') \in R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m. \tag{10.50}$$

As for the canonical problem, the obtained results can be interpreted as the necessary and sufficient *conditions for the stability* of the Karush-Kuhn-Tucker point set $S(D, A, c, b, F, d)$ with respect to the change in the problem parameters.

Our proofs are based on several observations concerning the system of equalities and inequalities defining the KKT point set. It is worthy to stress that the proofs in the preceding sections cannot be applied to the case of problem (10.48). This is because, unlike the case of the canonical problem (10.1), $\Delta(F, d)$ may fail to be a cone with nonempty interior and the vertex 0. So we have to use some new arguments. Fortunately, the proof schemes in the preceding sections will be useful also for the case of problem (10.48).

Theorem 10.7 below deals with the case where $\Delta(F, d)$ is a polyhedral cone with a vertex x^0, where $x^0 \in R^n$ is an arbitrarily given vector. Theorem 10.8 works for the case where $\Delta(F, d)$ is an arbitrary polyhedral set, but the conclusion is weaker than that of Theorem 10.7.

For any $M \in R^{r\times n}$ and $q \in R^r$, the set $\{x \in R^n : Mx \geq q\}$ is denoted by $\Delta(M, q)$. For $F \in R^{s\times n}$ and $A \in R^{m\times n}$, we abbreviate the set

$$\{(u, v) \in R^m \times R^s : A^T u + F^T v = 0, \; u \geq 0, \; v \geq 0\}$$

to $\Lambda[A, F]$. Note that

$$\begin{aligned}&\mathrm{int}(\Lambda[A, F])^* \\ &= \{(\xi, \eta) : \xi^T u + \eta^T v < 0 \;\; \forall\, (u, v) \in \Lambda[A, F] \setminus \{(0, 0)\}\}.\end{aligned}$$

The next two remarks clarify some points in the assumption and conclusion of Theorem 10.7 below.

Remark 10.2. If there is a point $x^0 \in R^n$ such that $F(x^0) = d$ then $\Delta(F, d) = x^0 + \Delta(F, 0)$. Conversely, for any $x^0 \in R^n$ and any polyhedral cone K, there exists a positive integer s and a matrix $F \in R^{s\times n}$ such that $x^0 + K = \Delta(F, d)$, where $d := F(x^0)$.

Remark 10.3. If $\Delta(F, d)$ and $\Delta(A, b)$ are nonempty, then $\Delta(F, 0)$ and $\Delta(A, 0)$, respectively, are the recession cones of $\Delta(F, d)$ and

$\Delta(A, b)$. By definition, $S(D, A, 0, 0, F, 0)$ is the Karush-Kuhn-Tucker point set of the following QP problem

$$\text{Minimize} \quad x^T Dx \quad \text{subject to} \quad x \in R^n, \ Ax \geq 0, \ Fx \geq 0,$$

whose constraint set is the intersection $\Delta(A, 0) \cap \Delta(F, 0)$.

Theorem 10.7. *Assume that the set $S(p, F, d)$, where $p = (D, A, c, b)$, is bounded and there exists $x^0 \in R^n$ such that $F(x^0) = d$. If the multifunction* (10.50) *is upper semicontinuous at p then*

$$S(D, A, 0, 0, F, 0) = \{0\}. \tag{10.51}$$

Proof. Suppose, contrary to our claim, that there is a nonzero vector $\bar{x} \in S(D, A, 0, 0, F, 0)$. By definition, there exists a pair $(\bar{u}, \bar{v}) \in R^m \times R^s$ such that

$$D\bar{x} - A^T\bar{u} - F^T\bar{v} = 0, \tag{10.52}$$

$$A\bar{x} \geq 0, \quad \bar{u} \geq 0, \tag{10.53}$$

$$F\bar{x} \geq 0, \quad \bar{v} \geq 0, \tag{10.54}$$

$$\bar{u}^T A\bar{x} + \bar{v}^T F\bar{x} = 0. \tag{10.55}$$

For every $t \in (0, 1)$, we set

$$x_t = x^0 + \frac{1}{t}\bar{x}, \quad u_t = \frac{1}{t}\bar{u}, \quad v_t = \frac{1}{t}\bar{v}, \tag{10.56}$$

where x^0 is given by our assumptions. We claim that there exist matrices $D_t \in R_S^{n\times n}$, $A_t \in R^{m\times n}$ and vectors $c_t \in R^n$, $b_t \in R^m$ such that

$$\max\{\|D_t - D\|, \|A_t - A\|, \|c_t - c\|, \|b_t - b\|\} \to 0 \quad \text{as } t \to 0,$$

and

$$D_t x_t - A_t^T u_t - F^T v_t + c_t = 0, \tag{10.57}$$

$$A_t x_t \geq b_t, \quad u_t \geq 0, \tag{10.58}$$

$$F x_t \geq d, \quad v_t \geq 0, \tag{10.59}$$

$$u_t^T(A_t x_t - b_t) + v_t^T(F x_t - d) = 0. \tag{10.60}$$

The matrices D_t, A_t and the vectors c_t, b_t will have the following representations

$$D_t = D + tD_0, \quad A_t = A + tA_0 \tag{10.61}$$

$$c_t = c + tc_0, \quad b_t = b + tb_0, \tag{10.62}$$

where the matrices D_0, A_0 and the vectors c_0, b_0 are to be constructed. First we observe that, due to (10.54) and (10.56), (10.59) holds automatically. Clearly,

$$\begin{aligned} A_t x_t - b_t &= (A + tA_0)\left(x^0 + \frac{\bar{x}}{t}\right) - (b + tb_0) \\ &= t(A_0 x^0 - b_0) + \frac{1}{t}A\bar{x} + A_0\bar{x} + Ax^0 - b \end{aligned}$$

and

$$\begin{aligned} & u_t^T(A_t x_t - b_t) + v_t^T(Fx_t - d) \\ &\quad = \frac{\bar{u}^T}{t}\left[t(A_0x^0 - b_0) + \frac{1}{t}A\bar{x} + A_0\bar{x} + Ax^0 - b\right] \\ &\quad\quad + \frac{\bar{v}^T}{t}\left[F\left(x^0 + \frac{\bar{x}}{t}\right) - d\right] \\ &\quad = \bar{u}^T(A_0x^0 - b_0) + \frac{1}{t^2}(\bar{u}^T A\bar{x} + \bar{v}^T F\bar{x}) + \frac{\bar{u}^T}{t}(A_0\bar{x} + Ax^0 - b). \end{aligned}$$

Therefore, by (10.53) and (10.55), if we have

$$A_0\bar{x} + Ax^0 - b = 0 \tag{10.63}$$

and

$$A_0 x^0 - b_0 = 0, \tag{10.64}$$

then (10.58) and (10.60) will be fulfilled. By (10.52),

$$\begin{aligned} D_t x_t \quad & -A_t^T u_t - F^T v_t + c_t \\ &= (D + tD_0)\left(x^0 + \frac{\bar{x}}{t}\right) - (A + tA_0)^T\frac{\bar{u}}{t} - F^T\frac{\bar{v}}{t} + c + tc_0 \\ &= \frac{1}{t}(D\bar{x} - A^T\bar{u} - F^T\bar{v}) + t(D_0x^0 + c_0) + Dx^0 \\ &\quad + D_0\bar{x} - A_0^T\bar{u} + c, \\ &= t(D_0x^0 + c_0) + Dx^0 + D_0\bar{x} - A_0^T\bar{u} + c. \end{aligned}$$

Hence, if we have

$$Dx^0 + D_0\bar{x} - A_0^T\bar{u} + c = 0 \tag{10.65}$$

and

$$D_0 x^0 + c_0 = 0, \tag{10.66}$$

then (10.57) will be fulfilled.

Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$, where $\bar{x}^i \neq 0$ for $i \in I$ and $\bar{x}_i = 0$ for $i \notin I$, $I \subset \{1, \ldots, n\}$. Since $\bar{x} \neq 0$, I is nonempty. Fixing an index

$i_0 \in I$, we define A_0 as the $m \times n$-matrix in which the i_0-th column is $\bar{x}_{i_0}^{-1}(b - Ax^0)$, and the other columns consist solely of zeros. Let $b_0 = A_0 x^0$. One can verify immediately that (10.63) and (10.64) are satisfied; hence conditions (10.58) and (10.60) are fulfilled. From what has been said it follows that our claim will be proved if we can construct a matrix $D_0 \in R_S^{n\times n}$ and a vector c_0 satisfying (10.65) and (10.66). Let $D_0 = (d_{ij})$, where d_{ij} $(1 \le i, j \le n)$ are defined by the following formulae:

$$\begin{aligned} d_{ii} &= \bar{x}_i^{-1}\left(A_0^T\bar{u} - Dx^0 - c\right)_i \quad \forall i \in I, \\ d_{i_0 j} &= d_{j i_0} = \bar{x}_{i_0}^{-1}\left(A_0^T\bar{u} - Dx^0 - c\right)_j \quad \forall j \in \{1, \dots, n\} \setminus I, \end{aligned}$$

and $d_{ij} = 0$ for other pairs (i, j), $1 \le i, j \le n$. Here $(A_0^T\bar{u} - Dx^0 - c)_k$ denotes the k-th component of the vector $A_0^T\bar{u} - Dx^0 - c$. Since D_0 is a symmetric matrix, $D_0 \in R_S^{n\times n}$. If we define $c_0 = -D_0 x^0$ then (10.66) is satisfied. A direct computation shows that (10.65) is also satisfied.

We have thus constructed matrices D_0, A_0 and vectors c_0, b_0 such that for x_t, u_t, v_t, D_t, A_t, c_t, b_t defined by (10.56), (10.61) and (10.62), conditions (10.57)–(10.60) are satisfied. Consequently, $x_t \in S(D_t, A_t, c_t, b_t, F, d)$. Since $S(p, F, d)$ is bounded, there is a bounded open set $\Omega \subset R^n$ such that $S(p, F, d) \subset \Omega$. Since

$$\max\{\|D_t - D\|, \|A_t - A\|, \|c_t - c\|, \|b_t - b\|\} \to 0$$

as $t \to 0$ and the multifunction $p' \mapsto S(p', F, d)$ is usc at $p = (D, A, c, b)$, $x_t \in \Omega$ for all sufficiently small t. This is impossible, because $\|x_t\| = \|x^0 + \dfrac{\bar{x}}{t}\| \to \infty$ as $t \to 0$. The proof is complete. □

Remark 10.4. If $d = 0$ then $\Delta(F, d)$ is a cone with the vertex 0. In order to verify the assumptions of Theorem 10.7, one can choose $x^0 = 0$. In particular, this is the case of the canonical problem (10.1). Applying Theorem 10.7 we obtain the following necessary condition for the upper semicontinuity of the multifunction (10.49): *If $S(p)$, $p = (D, A, c, b)$ is bounded and if the multifunction $p' \mapsto S(p')$, $p' = (D', A', c', b')$, is usc at p, then $S(D, A, 0, 0) = \{0\}$.* Thus Theorem 10.8 above extends Theorem 10.1 to the case where $\Delta(F, d)$ can be any polyhedral convex cone in R^n, merely the standard cone R^n_+.

In the sequel, $S(D, A)$ denotes the set of all $x \in R^n$ such that there exists $u = u(x) \in R^m$ satisfying the following system:

$$Dx - A^T u = 0, \quad Ax \ge 0, \quad u \ge 0, \quad u^T A x = 0.$$

Remark 10.5. From the definition it follows that $S(D,A) = S(D,A,0,0,F,0)$, where $s = n$ and $F = 0 \in R^{n\times n}$.

Theorem 10.8. *Assume that $\Delta(F,d)$ is nonempty and $S(p,F,d)$, where $p = (D,A,c,b)$, is bounded. If the multifunction* (10.50) *is upper semicontinuous at p then*

$$S(D,A) \cap \Delta(F,0) = \{0\}. \tag{10.67}$$

Remark 10.6. Observe that (10.51) implies (10.67). Indeed, suppose that (10.51) holds. The fact that $0 \in S(D,A) \cap \Delta(F,0)$ is obvious. So, if (10.67) does not hold then there exists $\bar{x} \in S(D,A) \cap \Delta(F,0)$, $\bar{x} \neq 0$. Taking $\bar{u} = u(\bar{x})$, $\bar{v} = 0 \in R^s$, we see at once that the system (10.52)–(10.55) is satisfied. This means that $\bar{x} \in S(D,A,0,0,F,0) \setminus \{0\}$, contrary to (10.51). Note that, in general, (10.67) does not imply (10.51).

Remark 10.7. If there exists x^0 such that $Fx^0 = d$ then, of course, $x^0 \in \Delta(F,d) = \{x \in R^n : Fx \geq d\}$. In particular, $\Delta(F,d) \neq \emptyset$. Thus Theorem 10.8 can be applied to a larger class of problems than Theorem 10.7. However, Remark 10.6 shows that the conclusion of Theorem 10.8 is weaker than that of Theorem 10.7. One question still unanswered is whether the assumptions of Theorem 10.8 always imply (10.51).

Proof of Theorem 10.8.

Assume that $\Delta(F,d)$ is nonempty, $S(D,A,c,b,F,d)$ is bounded and the multifunction $S(\cdot,F,d)$ is usc at p but (10.67) is violated. Then, there is a nonzero vector $\bar{x} \in S(D,A) \cap \Delta(F,0)$. Hence there exists $\bar{u} \in R^m$ such that

$$D\bar{x} - A^T\bar{u} = 0, \tag{10.68}$$

$$A\bar{x} \geq 0, \quad \bar{u} \geq 0, \tag{10.69}$$

$$\bar{u}^T A\bar{x} = 0, \tag{10.70}$$

$$F\bar{x} \geq 0. \tag{10.71}$$

Let x^0 be an arbitrary point of $\Delta(F,d)$. Setting

$$x_t = x^0 + \frac{1}{t}\bar{x}, \quad u_t = \frac{1}{t}\bar{u}$$

for every $t \in (0,1)$, we claim that there exist matrices

$$D_t \in R_S^{n\times n}, \quad A_t \in R^{m\times n}$$

and vectors $c_t \in R^n$, $b_t \in R^m$ such that

$$\max\{\|D_t - D\|, \|A_t - A\|, \|c_t - c\|, \|b_t - b\|\} \to 0$$

as $t \to 0$, and

$$\begin{aligned}
&D_t x_t - A_t^T u_t - F^T 0 + c_t = 0,\\
&A_t x_t \geq b_t, \quad u_t \geq 0,\\
&F x_t \geq d,\\
&u_t^T(A_t x_t - b_t) + 0^T(F x_t - d) = 0.
\end{aligned}$$

The matrices D_t, A_t and vectors c_t, b_t are defined by (10.61) and (10.62), where D_0, A_0, c_0, b_0 are constructed as in the proof of Theorem 10.7. Using (10.68)–(10.71) and arguing similarly as in the preceding proof we shall arrive at a contradiction. □

The following theorem gives three sufficient conditions for the upper semicontinuity of the multifunction (10.50). These conditions express some requirements on the behavior of the quadratic form $x^T D x$ on the cone $\Delta(A, 0) \cap \Delta(F, 0)$ and the position of the vector (b, d) relatively to the set $\operatorname{int}(\Lambda[A, F])^*$.

Theorem 10.9. *Suppose that one of the following three pairs of conditions*

$$\operatorname{Sol}(D, A, 0, 0, F, 0) = \{0\}, \quad (b, d) \in \operatorname{int}(\Lambda[A, F])^*, \tag{10.72}$$

$$\operatorname{Sol}(-D, A, 0, 0, F, 0) = \{0\}, \quad (b, d) \in -\operatorname{int}(\Lambda[A, F])^*, \tag{10.73}$$

and

$$S(D, A, 0, 0, F, 0) = \{0\}, \quad \operatorname{int}(\Lambda[A, F])^* = R^m \times R^s, \tag{10.74}$$

is satisfied. Then, for any $c \in R^n$ (and also for any $b \in R^m$ if (10.74) takes place), the multifunction $p' \mapsto S(p', F, d)$, where $p' = (D', A', c', b')$, is upper semicontinuous at $p = (D, A, c, b)$.

Proof. On the contrary, suppose that one of the three pairs of conditions (10.72)–(10.74) is satisfied but, for some $c \in R^n$ (and also for some $b \in R^m$ if (10.74) takes place), the multifunction $p' \mapsto S(p', F, d)$ is not usc at $p = (D, A, c, b)$. Then there exist an open subset $\Omega \subset R^n$ containing $S(p, F, d)$, a sequence $p^k = (D^k, A^k, c^k, b^k)$ converging to p in $R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m$, and a sequence $\{x^k\}$ such that, for each k, $x^k \in S(p^k, F, d)$ and $x^k \notin \Omega$. By the definition of KKT point, for each k there exists a pair $(u^k, v^k) \in R^m \times R^s$ such that

$$D^k x^k - (A^k)^T u^k - F^T v^k + c^k = 0, \tag{10.75}$$

$$A^k x^k \geq b^k, \quad u^k \geq 0, \tag{10.76}$$

$$F x^k \geq d, \quad v^k \geq 0, \tag{10.77}$$

$$(u^k)^T (A^k x^k - b^k) + (v^k)^T (F x^k - d) = 0. \tag{10.78}$$

If the sequence $\{(x^k, u^k, v^k)\}$ is bounded, then the sequences $\{x^k\}$, $\{u^k\}$ and $\{v^k\}$ are also bounded. Therefore, without loss of generality, we can assume that the sequences $\{x^k\}$, $\{u^k\}$ and $\{v^k\}$ converge, respectively, to some points $x^0 \in R^n$, $u^0 \in R^m$ and $v^0 \in R^s$, as $k \to \infty$. Letting $k \to \infty$, from (10.75)–(10.78) we get

$$\begin{aligned} & Dx^0 - A^T u - F^T v + c = 0, \\ & Ax^0 \geq b, \quad u^0 \geq 0, \\ & Fx^0 \geq d, \quad v^0 \geq 0, \\ & (u^0)^T (Ax^0 - b) + (v^0)^T (Fx^0 - d) = 0. \end{aligned}$$

Hence $x^0 \in S(p, F, d) \subset \Omega$. On the other hand, since $x^k \notin \Omega$ for each k, we must have $x^0 \notin \Omega$, a contradiction. We have thus shown that the sequence $\{(x^k, u^k, v^k)\}$ must be unbounded. By considering a subsequence, if necessary, we can assume that $\|(x^k, u^k, v^k)\| \to \infty$ and, in addition, $\|(x^k, u^k, v^k)\| \neq 0$ for all k. Since the sequence of vectors

$$\frac{(x^k, u^k, v^k)}{\|(x^k, u^k, v^k)\|} = \left(\frac{x^k}{\|(x^k, u^k, v^k)\|}, \frac{u^k}{\|(x^k, u^k, v^k)\|}, \frac{v^k}{\|(x^k, u^k, v^k)\|} \right)$$

is bounded, it has a convergent subsequence. Without loss of generality, we can assume that

$$\frac{(x^k, u^k, v^k)}{\|(x^k, u^k, v^k)\|} \to (\bar{x}, \bar{u}, \bar{v}) \in R^n \times R^m \times R^s, \quad \|(\bar{x}, \bar{u}, \bar{v})\| = 1. \tag{10.79}$$

Dividing both sides of (10.75), (10.76) and (10.77) by $\|(x^k, u^k, v^k)\|$, both sides of (10.78) by $\|(x^k, u^k, v^k)\|^2$, and letting $k \to \infty$, by (10.79) we obtain

$$D\bar{x} - A^T \bar{u} - F^T \bar{v} = 0, \tag{10.80}$$

$$A\bar{x} \geq 0, \ \bar{u} \geq 0, \tag{10.81}$$

$$F\bar{x} \geq 0, \ \bar{v} \geq 0, \tag{10.82}$$

$$\bar{u}^T A \bar{x} + \bar{v}^T F \bar{x} = 0. \tag{10.83}$$

We first consider the case where (10.72) is fulfilled. It is evident that (10.80)–(10.83) imply

$$\bar{x}^T D \bar{x} = 0, \quad A\bar{x} \geq 0, \quad F\bar{x} \geq 0. \tag{10.84}$$

If $\bar{x} \neq 0$ then, taking account of the fact that the constraint set $\Delta(A, 0) \cap \Delta(F, 0)$ of $QP(D, A, 0, 0, F, 0)$ is a cone, one can deduce from (10.84) that either $\mathrm{Sol}(D, A, 0, 0, F, 0) = \emptyset$ or

$$\bar{x} \in \mathrm{Sol}(D, A, 0, 0, F, 0).$$

This contradicts the first condition in (10.72). Thus $\bar{x} = 0$. Then it follows from (10.80)–(10.83) that $(\bar{u}, \bar{v}) \in \Lambda[A, F] \setminus \{(0, 0)\}$. Since $(b, d) \in \mathrm{int}\,(\Lambda[A, F])^*$ by (10.72),

$$\bar{u}^T b + \bar{v}^T d < 0. \tag{10.85}$$

Consider the sequence $\{(u^k)^T b^k + (v^k)^T d\}$. By (10.75) and (10.78),

$$(x^k)^T D^k x^k + (c^k)^T x^k = (u^k)^T b^k + (v^k)^T d. \tag{10.86}$$

If for each positive integer i there exists an integer k_i such that $k_i > i$ and

$$(u^{k_i})^T b^{k_i} + (v^{k_i})^T d > 0 \tag{10.87}$$

then, dividing both sides of (10.87) by $\|(x^{k_i}, u^{k_i}, v^{k_i})\|$ and letting $i \to \infty$, we have

$$\bar{u}^T b + \bar{v}^T d \geq 0,$$

contrary to (10.85). Consequently, there must exist a positive integer i_0 such that

$$(u^k)^T b^k + (v^k)^T d \leq 0 \quad \text{for every } k \geq i_0. \tag{10.88}$$

If the sequence $\{x^k\}$ is bounded then, dividing both sides of (10.86) by $\|(x^k, u^k, v^k)\|$ and letting $k \to \infty$, we get $\bar{u}^T b + v^T d = 0$, contrary to (10.85). Thus $\{x^k\}$ is unbounded. We can assume that $\|x^k\| \to \infty$ and $\|x^k\| \neq 0$ for each k. Then $\left\{ \frac{x^k}{\{x^k\|} \right\}$ is bounded. We can assume that

$$\frac{x^k}{\|x^k\|} \to \hat{x} \quad \text{with } \|\hat{x}\| = 1.$$

Combining (10.86) with (10.88) gives

$$(x^T)^k D^k x^k + (c^k)^T x^k \leq 0 \quad \text{for every } k \geq i_0. \tag{10.89}$$

Dividing both sides of (10.89) by $\|x^k\|^2$ and letting $k \to \infty$, we obtain

$$\hat{x}^T D \hat{x} \leq 0. \tag{10.90}$$

By (10.76) and (10.77),

$$A^k x^k \geq b^k, \quad F x^k \geq d.$$

Dividing both sides of each of the last inequalities by $\|x^k\|$ and letting $k \to \infty$, we have

$$A\hat{x} \geq 0, \quad F\hat{x} \geq 0. \tag{10.91}$$

Combining (10.90) with (10.91), we can assert that

$$\mathrm{Sol}(D, A, 0, 0, F, 0) \neq \{0\},$$

contrary to the first condition in (10.72). Thus we have proved the theorem for the case where (10.72) is fulfilled.

Now we turn to the case where condition (10.73) is fulfilled. We deduce (10.84) from (10.80)–(10.83). If $\bar{x} \neq 0$ then from (10.84) we get $\mathrm{Sol}(-D, A, 0, 0, F, 0) \neq \{0\}$, which contradicts the first condition in (10.73). Thus $\bar{x} = 0$. From (10.80)–(10.83) it follows that $(\bar{u}, \bar{v}) \in \Lambda[A, F] \setminus \{(0, 0)\}$. By the second condition in (10.73),

$$\bar{u}^T b + v^T d > 0. \tag{10.92}$$

Consider the sequence $\{(u^k)^T b^k + (v^k)^T d\}$. We have (10.86). If there exists a positive integer i_0 such that (10.88) is valid then, dividing both sides of (10.88) by $\|(x^k, u^k, v^k)\|$ and letting $k \to \infty$, we obtain $\bar{u}^T b + \bar{v}^T d \leq 0$, contrary to (10.92). Therefore, for each positive integer i, one can find an integer $k_i > i$ such that (10.87) holds. If the sequence $\{x^k\}$ is bounded then, dividing both sides of (10.86) by $\|(x^k, u^k, v^k)\|$ and letting $k \to \infty$, we have $\bar{u}^T b + \bar{v}^T d = 0$, contrary to (10.92). Thus the sequence $\{x^k\}$ is unbounded. We can assume that $\|x^k\| \to \infty$ and $\|x^k\| \neq 0$ for all k. Since the sequence $\left\{\dfrac{x^k}{\|x^k\|}\right\}$ is well defined and bounded, without loss of generality, we can assume that

$$\frac{x^k}{\|x^k\|} \to \hat{x} \quad \text{with } \|\hat{x}\| = 1.$$

Combining (10.86) with (10.87) gives

$$(x^{k_i})^T D^{k_i} x^{k_i} + (c^{k_i})^T x^{k_i} > 0 \quad \text{for all } i. \tag{10.93}$$

Dividing both sides of (10.93) by $\|x^{k_i}\|^2$ and letting $i \to \infty$, we obtain $\hat{x}^T D\hat{x} \geq 0$ or, equivalently,

$$\hat{x}^T(-D)\hat{x} \leq 0. \tag{10.94}$$

By (10.76) and (10.77),

$$A^{k_i}x^{k_i} \geq b^{k_i}, \quad Fx^{k_i} \geq d. \tag{10.95}$$

Dividing both sides of each of the inequalities in (10.95) by $\|x^{k_i}\|$ and letting $i \to \infty$, we have

$$A\hat{x} \geq 0, \quad F\hat{x} \geq 0. \tag{10.96}$$

Combining (10.94) with (10.96) yields $\mathrm{Sol}(-D, A, 0, 0, F, 0) \neq \{0\}$, contrary to the first condition in (10.73). This proves the theorem in the case where (10.73) is fulfilled.

Now let us consider the last case where (10.74) is assumed. From (10.80)–(10.83) we have $\bar{x} \in S(D, A, 0, 0, F, 0)$. By the first condition in (10.73), $\bar{x} = 0$. Then it follows from (10.80)–(10.83) that

$$A^T\bar{u} + F^T\bar{v} = 0, \quad \bar{u} \geq 0, \quad \bar{v} \geq 0, \quad \|(0, \bar{u}, \bar{v})\| = 1.$$

Therefore, $(\bar{u}, \bar{v}) \in \Lambda[A, F] \setminus \{(0,0)\}$. Since $\bar{u}^T\bar{u} + \bar{v}^T\bar{v} > 0$, then $(\bar{u}, \bar{v}) \notin \mathrm{int}(\Lambda[A, F])^*$. This contradicts the second condition in (10.74).

We have thus proved that if one of the pairs of conditions (10.72)–(10.74) is fulfilled, then the conclusion of the theorem must hold true. □

We now proceed to show how the sufficient conditions (10.72) and (10.73) look like in the case of the canonical problem (10.1). As in Section 10.4, for any $A \in R^{n\times n}$, $\Lambda[A] = \{\lambda \in R^m : -A^T\lambda \geq 0,\ \lambda \geq 0\}$. We have

$$\mathrm{int}(\Lambda[A])^* = \{\xi \in R^m : \lambda^T\xi < 0 \quad \forall \lambda \in \Lambda[A] \setminus \{0\}\}.$$

Lemma 10.3. *Suppose that, in problem* (10.48), $s = n$, $d = 0$, *and* F *is the unit matrix in* $R^{n\times n}$. *Then the following statements hold:*

(a_1) *If* $b \in \mathrm{int}\,(\Lambda[A])^*$ *then* $(b, 0) \in \mathrm{int}\,(\Lambda[A, F])^*$;

(a_2) *If* $\mathrm{Sol}\,(D, A, 0, 0) = \{0\}$ *then* $\mathrm{Sol}\,(D, A, 0, 0, F, 0) = \{0\}$;

(a_3) *If* $b \in -\mathrm{int}\,(\Lambda[A])^*$ *then* $(b, 0) \in -\mathrm{int}\,(\Lambda[A, F])^*$;

(a_4) *If* $\mathrm{Sol}\,(-D, A, 0, 0) = \{0\}$ *then* $\mathrm{Sol}\,(-D, A, 0, 0, F, 0) = \{0\}$.

Proof. If $b \in \mathrm{int}(\Lambda[A])^*$ then

$$\lambda^T b < 0 \quad \text{for all } \lambda \in \Lambda[A] \setminus \{0\}. \tag{10.97}$$

For any $(u, v) \in \Lambda[A, F] \setminus \{0\}$ we have

$$A^T u + F^T v = 0, \quad u \geq 0, \quad v \geq 0.$$

This yields

$$-A^T u = v \geq 0, \quad u \geq 0, \quad u \neq 0,$$

hence $u \in \Lambda[A] \setminus \{0\}$. By (10.97), $b^T u + 0^T v = b^T u = u^T b < 0$. This shows that $(b, 0) \in \mathrm{int}(\Lambda[A, F])^*$. Statement (a_1) has been proved. It is clear that (a_3) follows from (a_1).

For proving (a_2) and (a_4) it suffices to note that, under our assumptions,

$$\mathrm{Sol}(D, A, 0, 0) = \mathrm{Sol}(D, A, 0, 0, F, 0)$$

and

$$\mathrm{Sol}(-D, A, 0, 0) = \mathrm{Sol}(-D, A, 0, 0, F, 0).$$

□

We check at once that Theorems 10.4 and 10.5 follow from Theorem 10.10 and Lemma 10.3.

10.7 Commentaries

The material of this chapter is taken from Tam and Yen (1999, 2000), Tam (2001a).

Several authors have made efforts in studying stability properties of the QP problems. Daniel (1973) established some basic facts about the solution stability of a QP problem whose objective function is a positive definite quadratic form. Guddat (1976) studied continuity properties of the solution set of a convex QP problem. Robinson (1979) obtained a fundamental result (see Theorem 7.6 in Chapter 7) on the stable behavior of the solution set of a monotone affine generalized equation (an affine variational inequality in the terminology of Gowda and Pang (1994), which yields a fact on the Lipschitz continuity of the solution set of a convex QP problem. Best and Chakravarti (1990) obtained some results on the continuity and differentiability of the optimal value function in a perturbed convex QP problem. By using the linear complementarity theory, Cottle, Pang and Stone (1992), studied in detail the stability of convex QP problems. Best and Ding (1995) proved a result on the continuity of the optimal value function in a convex QP problem. Auslender and Coutat (1996) established some results on stability and differential stability of generalized linear-quadratic programs, which include convex QP problems as a special case. Several attempts have been made to study the stability of nonconvex

QP problems (see, for instance, Klatte (1985), Tam (1999), Tam (2001a, 2001b, 2002)).

The proof of Theorem 10.1 is based on a construction developed by Oettli and Yen (1995, 1996a) for linear complementarity problems, homogeneous equilibrium problems, and quasi-complementarity problems.

Chapter 11

Lower Semicontinuity of the KKT Point Set Mapping

Our aim in this chapter is to characterize the lower semicontinuity of the Karush-Kuhn-Tucker point set mapping in quadratic programming. Necessary and sufficient conditions for the lsc property of the KKT point set mapping in canonical QP problems are obtained in Section 11.1. The lsc property of the KKT point set mapping in standard QP problems under linear perturbations is studied in Section 11.2.

11.1 The Case of Canonical QP Problems

Consider the canonical QP problem of the form (10.1). The following statement gives a necessary condition for the lower semicontinuity of the multifunction (10.3).

Theorem 11.1. *Let $D \in R^{n\times n}_S$ and $A \in R^{m\times n}$ be given. If the multifunction $S(D, A, \cdot, \cdot)$ is lower semicontinuous at $(c, b) \in R^n \times R^m$, then the set $S(D, A, c, b)$ is finite.*

Proof. Setting

$$M = \begin{pmatrix} D & -A^T \\ A & 0 \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} c \\ -b \end{pmatrix},$$

and $s = n + m$, we consider the problem of finding a vector $z = \begin{pmatrix} x \\ \lambda \end{pmatrix} \in R^s$ satisfying

$$Mz + \bar{q} \geq 0, \quad z \geq 0, \quad z^T(Mz + \bar{q}) = 0. \tag{11.1}$$

For a nonempty subset $\alpha \subset \{1, 2, \ldots, s\}$, $M_{\alpha\alpha}$ denotes the corresponding principal submatrix of M. If $p \in R^s$, then the column-vector with the components $(p_i)_{i\in\alpha}$ is denoted by p_α.

Let $z = (z_1, z_2, \ldots, z_s)$ be a nonzero solution of (11.1), and let $J = \{j : z_j = 0\}$, $I = \{i : z_i > 0\}$. Since $z_J = 0$ and $(Mz + \bar{q})_I = 0$, we have $M_{II} z_I = -\bar{q}_I$. Therefore, if $\det M_{II} \neq 0$ then z is defined uniquely via $\bar{q}$ by the formulae

$$z_J = 0, \quad z_I = -M_{II}^{-1}(\bar{q}_I).$$

If $I \neq \emptyset$ and $\det M_{II} = 0$, then

$$\mathrm{Q}_I := \{q \in R^s : -q_I = M_{II} z_I \text{ for some } z \in R^s\}$$

is a proper subspace of R^s. By Baire's Lemma (Brezis (1987), p. 15), the union $\mathrm{Q} := \cup\{\mathrm{Q}_I : I \subset \{1, 2, \ldots, s\},\ I \neq \emptyset,\ \det M_{II} = 0\}$ is nowhere dense. So there exists a sequence $q^k = \begin{pmatrix} c^k \\ -b^k \end{pmatrix}$ converging to $\bar{q} = \begin{pmatrix} c \\ -b \end{pmatrix}$ such that $q^k \notin \mathrm{Q}$ for all k.

Fix any $x \in S(D, A, c, b)$ and let $\varepsilon > 0$ be given arbitrarily. Since the multifunction $S(D, A, \cdot, \cdot)$ is lsc at (c, b), there exists $\delta_\varepsilon > 0$ such that

$$x \in S(D, A, c', b') + \varepsilon B_{R^n}$$

for all (c', b') satisfying $\max\{\|c' - c\|, \|b' - b\|\} < \delta_\varepsilon$. Consequently, for each k sufficiently large, there exists $x^k \in S(D, A, c^k, b^k)$ such that

$$\|x - x^k\| \leq \varepsilon. \tag{11.2}$$

Since $x^k \in S(D, A, c^k, b^k)$, there exists λ^k such that $z^k := \begin{pmatrix} x^k \\ \lambda^k \end{pmatrix}$ is a solution of the LCP problem

$$Mz + q^k \geq 0, \quad z \geq 0, \quad z^T(Mz + q^k) = 0.$$

We put $J_k = \{j : z_j^k = 0\}$, $I_k = \{i : z_i^k > 0\}$. If $I_k = \emptyset$ then $z^k = 0$. If $I_k \neq \emptyset$ then $\det M_{I_k I_k} \neq 0$, because $q^k \notin \mathrm{Q}$. Hence

$$z_{J_k}^k = 0, \quad z_{I_k}^k = -M_{I_k I_k}^{-1}\left(q_{I_k}^k\right). \tag{11.3}$$

Obviously, there exists a subset $I \subset \{1, 2, \ldots, s\}$ and a subsequence $\{k_i\}$ of $\{k\}$ such that $I_{k_i} = I$ for all k_i. Let Z denote the set of all $z \in R^s$ such that there is a nonempty subset $I \subset \{1, \ldots, s\}$ with the property that $\det M_{II} \neq 0$, $z_I = -M_{II}^{-1}(\bar{q}_I)$ and $z_J = 0$, where $J := \{1, \ldots, s\} \setminus I$. It is clear that Z is finite. From (11.3) it follows that the sequence $z_{I_{k_i}}^{(k_i)}$ converges to a point from the finite set $\widetilde{Z} := Z \cup \{0\}$. For every $z = \begin{pmatrix} \xi \\ \lambda \end{pmatrix}$ let $\mathrm{pr}_1(z) := \xi$. Since $\mathrm{pr}_1(z^{(k_i)}) = x^{(k_i)}$ and $\mathrm{pr}_1(\cdot)$ is a continuous function, the sequence $\{x^{(k_i)}\}$ has a limit $\bar{\xi}$ in the finite set $\widetilde{X} := \{\mathrm{pr}_1(z) : z \in \widetilde{Z}\}$. By (11.2), $x \in \widetilde{X} + \varepsilon B_{R^n}$. As this inclusion holds for every $\varepsilon > 0$, we have $x \in \widetilde{X}$. Thus $S(D, A, c, b) \subset \widetilde{X}$. We have shown that $S(D, A, c, b)$ is a finite set. □

The following examples show that the finiteness of $S(D, A, c, b)$ may not be sufficient for the multifunction $S(\cdot)$ to be lower semicontinuous at (D, A, c, b).

Example 11.1. Consider the problem (P_ε) of minimizing the function

$$f_\varepsilon(x) = -\frac{1}{2}x_1^2 - x_2^2 + x_1 - \varepsilon x_2$$

on the set $\Delta = \{x \in R^2 : x \geq 0, -x_1 - x_2 \geq -2\}$. Note that Δ is a compact set with nonempty interior. Denote by $S(\varepsilon)$ the KKT point set of (P_ε). A direct computation using (10.2) gives $S(0) = \left\{(0,0), (1,0), (2,0), \left(\frac{5}{3}, \frac{1}{3}\right), (0,2)\right\}$, and

$$S(\varepsilon) = \left\{(2,0), \left(\frac{5+\varepsilon}{3}, \frac{1-\varepsilon}{3}\right), (0,2)\right\}$$

for $\varepsilon > 0$ small enough. For $U := \{x \in R^2 : \frac{1}{2} < x_1 < \frac{3}{2}, -1 < x_2 < 1\}$ we have $S(\varepsilon) \cap U = \emptyset$ for every $\varepsilon > 0$ small enough. Meanwhile, $S(0) \cap U = \{(1,0)\}$. Hence the multifunction $\varepsilon \mapsto S(\varepsilon)$ is not lsc at $\varepsilon = 0$.

Example 11.2. Consider the problem $(\widetilde{P}_\varepsilon)$ of minimizing the function

$$\widetilde{f}_\varepsilon(x) = \frac{1}{2}x_1^2 - x_2^2 - x_1 - \varepsilon x_2$$

on the set $\Delta = \{x \in R^2 : x \geq 0, -x_1 - x_2 \geq -2\}$. Denote by $\widetilde{S}(\varepsilon)$ the KKT point set of $(\widetilde{P}_\varepsilon)$. Using (10.2) we can show that $\widetilde{S}(0) = \{(1,0), (0,2)\}$, and $\widetilde{S}(\varepsilon) = \{(0,2)\}$ for every $\varepsilon > 0$. For $U := \{x \in R^2 : \frac{1}{2} < x_1 < \frac{3}{2}, -1 < x_2 < 1\}$ we have $\widetilde{S}(0) \cap U =$

$\{(1,0)\}$, but $\widetilde{S}(\varepsilon) \cap U = \emptyset$ for every $\varepsilon > 0$. Hence the multifunction $\varepsilon \mapsto \widetilde{S}(\varepsilon)$ is not lsc at $\varepsilon = 0$.

In the KKT point set $S(D, A, c, b)$ of (10.1) we distinguish three types of elements: (1) Local solutions of $QP(D, A, c, b)$; (2) Local solutions of $QP(-D, A, c, b)$ which are not local solutions of $QP(D, A, c, b)$; (3) Points of $S(D, A, c, b)$ which do not belong to the first two classes. Elements of the first type (of the second type, of the third type) are called, respectively, the *local minima*, the *local maxima*, and the *saddle points* of (10.1).

In Example 11.1, $(1,0) \in S(0)$ is a local maximum of (P_0) which lies on the boundary of Δ. Similarly, in Example 11.2, $(1,0) \in \widetilde{S}(0)$ is a saddle point of $\widetilde{P}_0$ which lies on the boundary of Δ. If such situations do not happen, then the set of the KKT points is lower semicontinuous at the given parameter.

Theorem 11.2. *Assume that the inequality system $Ax \geq b$, $x \geq 0$ is regular. If the set $S(D, A, c, b)$ is nonempty, finite, and in $S(D, A, c, b)$ there exist no local maxima and no saddle points of (10.1) which are on the boundary of $\Delta(A, b)$, then the multifunction $S(\cdot)$ is lower semicontinuous at (D, A, c, b).*

Proof. For proving the lower semicontinuity of $S(\cdot)$ at (D, A, c, b) it suffices to show that: For any $\bar{x} \in S(D, A, c, b)$ and for any neighborhood U of $\bar{x}$ there exists $\delta > 0$ such that $S(D', A', c', b') \cap U \neq \emptyset$ for every (D', A', c', b') satisfying

$$\max\{\|D' - D\|, \|A' - A\|, \|c' - c\|, \|b' - b\|\} < \delta.$$

First, suppose that $\bar{x}$ is a local minimum of (10.1). As $S(D, A, c, b)$ is a finite set, $\bar{x}$ is an isolated local minimum. Using Theorem 3.7 we can verify that, for any Lagrange multiplier $\bar{\lambda}$ of $\bar{x}$, the second-order sufficient condition in the sense of Robinson (1982) is satisfied at $(\bar{x}, \bar{\lambda})$. According to Theorem 3.1 from Robinson (1982), for each neighborhood U of $\bar{x}$ there exists $\delta > 0$ such that for every (D', A', c', b') satisfying

$$\max\{\|D' - D\|, \|A' - A\|, \|c' - c\|, \|b' - b\|\} < \delta$$

there is a local minimum x' of the problem $QP(D', A', c', b')$ belonging to U. Since $x' \in S(D', A', c', b')$, we have $S(D', A', c', b') \cap U \neq \emptyset$, as desired. Now, suppose that $\bar{x}$ is a local maximum or a saddle point of (10.1). By our assumption, $\bar{x}$ belongs to the interior of $\Delta(A, b)$. Hence $\nabla f(\bar{x}) = D\bar{x} + c = 0$, or equivalently,

$$D\bar{x} = -c. \tag{11.4}$$

As $S(D, A, c, b)$ is finite, $\bar{x}$ is an isolated KKT point of (10.1). Then $\bar{x}$ must be the unique solution of the linear system (11.4). Therefore, the matrix D is nonsingular, and

$$\bar{x} = -D^{-1}c. \tag{11.5}$$

Since the system $Ax \geq b,\ x \geq 0$ is regular, using Lemma 3 from Robinson (1977) we can prove that there exist $\delta_0 > 0$ and an open neighborhood U_0 of $\bar{x}$ such that $U_0 \subset \Delta(A', b')$ for every (A', b') satisfying $\max\{\|A' - A\|, \|b' - b\|\} < \delta_0$. For any neighborhood U of $\bar{x}$, by (11.5) there exists $\delta \in (0, \delta_0)$ such that, for every (D', A', c', b') satisfying $\max\{\|D' - D\|, \|A' - A\|, \|c' - c\|, \|b' - b\|\} < \delta$, the matrix D' is nonsingular and $x' := -(D')^{-1}c'$ belongs to $U \cap U_0$. Since x' is an interior point $\Delta(A', b')$, this implies that $x' \in S(D', A', c', b')$. (It is easily seen that $\lambda' := 0$ is a Lagrange multiplier corresponding to x'.) We have thus shown that, for every (D', A', c', b') satisfying $\max\{\|D' - D\|, \|A' - A\|, \|c' - c\|, \|b' - b\|\} < \delta$, $S(D', A', c', b') \cap U \neq \emptyset$. The proof is complete. $\square$

11.2 The Case of Standard QP Problems

In this section we consider the following QP problem

$$\begin{cases} \text{Minimize} & \frac{1}{2}x^T Dx + c^T x \\ \text{subject to} & x \in \Delta(A, b) \end{cases} \tag{11.6}$$

where $A \in R^{m \times n}$ and $D \in R^{n \times n}_S$ are given matrices, $b \in R^m$ and $c \in R^n$ are given vectors,

$$\Delta(A, b) = \{x \in R^n \ : \ Ax \geq b\}.$$

Recall that $x \in R^n$ is a Karush-Kuhn-Tucker point of (11.6) if there exists $\lambda \in R^m$ such that

$$\begin{cases} Dx - A^T\lambda + c = 0, \\ Ax \geq b, \quad \lambda \geq 0, \\ \lambda^T(Ax - b) = 0. \end{cases}$$

The KKT point set (resp., the local solution set, the solution set) of (11.6) are denoted by $S(D, A, c, b)$, (resp., $\mathrm{loc}(D, A, c, b)$, $\mathrm{Sol}(D, A, c, b)$).

We will study the lower semicontinuity of the multifunctions

$$(D', A', c', b') \mapsto S(D', A', c', b') \tag{11.7}$$

and

$$(c', b') \mapsto S(D, A, c', b'), \tag{11.8}$$

which will be denoted by $S(\cdot)$ and $S(D, A, \cdot, \cdot)$, respectively. It is obvious that if (11.7) is lsc at $(D, A, c, b) \in R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m$ then (11.8) is lsc at $(c, b) \in R^n \times R^m$.

Necessary conditions for the lsc property of the multifunction (11.8) can be stated as follows.

Theorem 11.3. *Let $(D, A, c, b) \in R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m$. If the multifunction $S(D, A, \cdot, \cdot)$ is lower semicontinuous at (c, b), then*

(a) *the set $S(D, A, c, b)$ is finite, nonempty, and*

(b) *the system $Ax \geq b$ is regular.*

Proof. (a) For each index set $I \subset \{1, \cdots, m\}$, we define a matrix $M_I \in R^{(n+|I|)\times(n+|I|)}$, where $|I|$ is the number of elements of I, by setting

$$M_I = \begin{bmatrix} D & -A_I^T \\ A_I & O \end{bmatrix}.$$

(If $I = \emptyset$ then we set $M_I = D$). Let

$$\begin{aligned} Q_I \;&= \Big\{(u, v) \in R^n \times R^m \,:\, \tbinom{u}{v_I} = M_I \tbinom{x}{\lambda_I} \\ &\qquad \text{for some } (x, \lambda) \in R^n \times R^m\Big\}, \end{aligned}$$

and

$$Q = \bigcup \{Q_I \,:\, I \subset \{1, \cdots, m\},\ \det M_I = 0\}.$$

If $\det M_I = 0$ then it is clear that Q_I is a proper linear subspace of $R^n \times R^m$. Since the number of the index sets $I \subset \{1, \ldots, m\}$ is finite, the set Q is nowhere dense in $R^n \times R^m$ according to the Baire Lemma (see Brezis (1987), p. 15). So there exists a sequence $\{(c^k, b^k)\}$ converging to the given point $(c, b) \in R^n \times R^m$ such that $(-c^k, b^k) \notin Q$ for all k.

Fix any $\bar{x} \in S(D, A, c, b)$. Since $S(D, A, \cdot, \cdot)$ is lower semicontinuous at (c, b), one can find a subsequence $\{(c^{k_l}, b^{k_l})\}$ of $\{(c^k, b^k)\}$ and a sequence $\{x^{k_l}\}$ converging to $\bar{x}$ in R^n such that

$$x^{k_l} \in S(D, A, c^{k_l}, b^{k_l})$$

for all k_l. As $x^{k_l} \in S(D, A, c^{k_l}, b^{k_l})$, there exists $\lambda^{k_l} \in R^m$ such that

$$\begin{cases} Dx^{k_l} - A^T\lambda^{k_l} + c^{k_l} = 0, \\ Ax^{k_l} \geq b^{k_l}, \quad \lambda^{k_l} \geq 0, \\ (\lambda^{k_l})^T(Ax^{k_l} - b^{k_l}) = 0. \end{cases} \tag{11.9}$$

For every k_l, let $I_{k_l} := \{i \in \{1, \ldots, m\} : \lambda_i^{k_l} > 0\}$. (It may happen that $I_{k_l} = \emptyset$.) Since the number of the index sets $I \subset \{1, \ldots, m\}$ is finite, there must exist an index set $I \subset \{1, \cdots, m\}$ such that $I_{k_l} = I$ for infinitely many k_l. Without loss of generality we can assume that $I_{k_l} = I$ for all k_l. From (11.9) we deduce that

$$Dx^{k_l} - A_I^T\lambda_I^{k_l} + c^{k_l} = 0, \quad A_I x^{k_l} = b_I^{k_l}.$$

or, equivalently,

$$M_I \begin{pmatrix} x^{k_l} \\ \lambda_I^{k_l} \end{pmatrix} = \begin{pmatrix} -c^{k_l} \\ b_I^{k_l} \end{pmatrix}. \tag{11.10}$$

We claim that $\det M_I \neq 0$. Indeed, if $\det M_I = 0$ then, by (11.10) and by the definitions of Q_I and Q, we have

$$(-c^{k_l}, b^{k_l}) \in \mathrm{Q}_I \subset \mathrm{Q},$$

contrary to the fact that $(-c^k, b^k) \notin \mathrm{Q}$ for all k. We have proved that $\det M_I \neq 0$. By (11.10), we have

$$\begin{pmatrix} x^{k_l} \\ \lambda_I^{k_l} \end{pmatrix} = M_I^{-1} \begin{pmatrix} -c^{k_l} \\ b_I^{k_l} \end{pmatrix}.$$

Therefore

$$\lim_{l \to \infty} \begin{pmatrix} x^{k_l} \\ \lambda_I^{k_l} \end{pmatrix} = M_I^{-1} \begin{pmatrix} -c \\ b_I \end{pmatrix}. \tag{11.11}$$

If $I = \emptyset$ then formula (11.11) has the form

$$\lim_{l \to \infty} x^{k_l} = D^{-1}(-c). \tag{11.12}$$

From (11.11) it follows that the sequence $\{\lambda_I^{k_l}\}$ converges to some $\lambda_I \geq 0$ in $R^{|I|}$. Since the sequence $\{x^{k_l}\}$ converges to $\bar{x}$, from (11.11) and (11.12) it follows that

$$\begin{pmatrix} \bar{x} \\ \lambda_I \end{pmatrix} = M_I^{-1} \begin{pmatrix} -c \\ b_I \end{pmatrix}. \tag{11.13}$$

(Recall that $M_I = D$ if $I = \emptyset$). We set

$$\begin{aligned} \mathrm{Z} = \ & \{(x,\lambda) \in R^n \times R^m \ : \ \text{there exists } J \subset \{1,\cdots,m\} \\ & \quad \text{such that } \det M_J \neq 0 \text{ and } \tbinom{x}{\lambda_J} = M_J^{-1}\tbinom{-c}{b_J}\Big\}, \end{aligned}$$

and

$$X = \{x \in R^n \ : \ \text{there exists } \lambda \in R^m \text{ such that } (x,\lambda) \in \mathrm{Z}\}.$$

From the definitions of Z and X, we can deduce that X is a finite set (although Z may have infinitely many elements). We observe also that Z and X do not depend on the choice of $\bar{x}$. Actually, these sets depend only on the parameters (D, A, c, b). From (11.13) we have $\bar{x} \in X$. Since $\bar{x} \in S(D,A,c,b)$ can be chosen arbitrarily and since X is finite, we conclude that $S(D,A,c,b)$ is a finite set.

(b) If $Ax \geq b$ is irregular then there exists a sequence $\{b^k\}$ converging in R^n to b such that $\Delta(A, b^k)$ is empty for all k (Robinson (1977), Lemma 3). Clearly, $S(D,A,c,b^k) = \emptyset$ for all k. As $\{b^k\}$ converges to b, this shows that $S(D,A,\cdot,\cdot)$ cannot be lower semicontinuous at (c,b). The proof is complete. □

Examples 11.1 and 11.2 show that finiteness and nonemptiness of $S(D,A,c,b)$ together with the regularity of the system $Ax \geq b$, in general, does not imply that $S(D,A,\cdot,\cdot)$ is lower semicontinuous at (c,b).

Let $(D,A,c,b) \in R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m$. Let $x \in S(D,A,c,b)$ and let $\lambda \in R^m$ be a Lagrange multiplier corresponding to x. We define $I = \{1,2,\ldots,m\}$,

$$K = \{i \in I \ : \ A_i x = b_i, \ \lambda_i > 0\} \tag{11.14}$$

and

$$J = \{i \in I \ : \ A_i x = b_i, \ \lambda_i = 0\}. \tag{11.15}$$

It is clear that K and J are two disjoint sets (possibly empty).

We now obtain a sufficient condition for the lsc property of the multifunction $S(D,A,\cdot,\cdot)$ at a given point $(c,b) \in R^n \times R^m$.

Theorem 11.4. *Let $(D,A,c,b) \in R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m$. Suppose that*

(i) *the set $S(D,A,c,b)$ is finite, nonempty,*

(ii) *the system $Ax \geq b$ is regular,* *and suppose that for every $x \in S(D, A, c, b)$ there exists a Lagrange multiplier λ corresponding to x such that at least one of the following conditions holds:*

(c1) $x \in \mathrm{loc}(D, A, c, b)$,

(c2) $J = K = \emptyset$,

(c3) $J = \emptyset$, $K \neq \emptyset$, *and the system* $\{A_i : i \in K\}$ *is linearly independent,*

(c4) $J \neq \emptyset$, $K = \emptyset$, *D is nonsingular and* $A_J D^{-1} A_J^T$ *is a positive definite matrix,*

where K and J are defined via (x, λ) by (11.14) *and* (11.15). *Then, the multifunction $S(D, A, \cdot, \cdot)$ is lower semicontinuous at (c, b).*

Proof. Since $S(D, A, c, b)$ is nonempty, in order to prove that $S(D, A, \cdot, \cdot)$ is lower semicontinuous at (c, b) we only need to show that, for any $x \in S(D, A, c, b)$ and for any open neighborhood V_x of x, there exists $\delta > 0$ such that

$$S(D, A, c', b') \cap V_x \neq \emptyset \tag{11.16}$$

for every $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta$.

Let $x \in S(D, A, c, b)$ and let V_x be an open neighborhood of x. By our assumptions, there exists a Lagrange multiplier λ corresponding to x such that at least one of the four conditions (c1)-(c4) holds.

We first examine the case where (c1) holds, that is

$$x \in \mathrm{loc}(D, A, c, b).$$

Since $S(D, A, c, b)$ is finite by (i), $\mathrm{loc}(D, A, c, b)$ is finite. So x is an isolated local solution of (11.1). Using Theorem 3.7 we can verify that, for any Lagrange multiplier $\bar{\lambda}$ of $\bar{x}$, the second-order sufficient condition in the sense of Robinson (1982), Definition 2.1, is satisfied at $(\bar{x}, \bar{\lambda})$. By assumption (ii), we can apply Theorem 3.1 from Robinson (1982) to find an $\delta > 0$ such that

$$\mathrm{loc}(D, A, c', b') \cap V_x \neq \emptyset$$

for every $(c', b') \in R^n \times R^n$ with $\|(c', b') - (c, b)\| < \delta$. Since $\mathrm{loc}(D, A, c, b) \subset S(D, A, c', b')$, we conclude that (11.16) is valid for every (c', b') satisfying $\|(c', b') - (c, b)\| < \delta$.

Consider the case where (c2) holds, that is $A_i x > b_i$ for every $i \in I$. Since λ is a Lagrange multiplier corresponding to x, the system

$$Dx - A^T\lambda + c = 0, \quad Ax \geq b, \quad \lambda \geq 0, \quad \lambda^T(Ax - b) = 0$$

is satisfied. As $Ax > b$, from this we deduce that $\lambda = 0$. Hence the first equality in the above system implies that $Dx = -c$. Thus x is a solution of the linear system

$$Dz = -c \quad (z \in R^n). \tag{11.17}$$

Since $S(D, A, c, b)$ is finite, x is a locally unique KKT point of (11.6). Combining this with the fact that x is an interior point of $\Delta(A, b)$, we can assert that x is a unique solution of (11.17). Hence matrix D is nonsingular and we have

$$x = -D^{-1}c. \tag{11.18}$$

Since $Ax > b$, there exist $\delta_1 > 0$ and an open neighborhood $U_x \subset V_x$ of x such that $U_x \subset \Delta(A, b')$ for all $b' \in R^m$ satisfying $\|b' - b\| < \delta_1$. By (11.18), there exists $\delta_2 > 0$ such that if $\|c' - c\| < \delta_2$ and $x' = -D^{-1}c'$ then $x' \in U_x$. Set $\delta = \min\{\delta_1, \ \delta_2\}$. Let (c', b') be such that $\|(c', b') - (c, b)\| < \delta$. Since $x' := -D^{-1}c'$ belongs to the open set $U_x \subset \Delta(A, b')$, we deduce that

$$Dx' + c' = 0, \quad Ax' > b'.$$

From this it follows that $x' \in S(D, A, c', b')$. (Observe that $\lambda' = 0$ is a Lagrange multiplier corresponding to x'.) We have thus shown that (11.16) is valid for every $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta$.

We now suppose that (c3) holds. First, we establish that the matrix $M_K \in R^{(n+|K|)\times(n+|K|)}$ defined by setting

$$M_K = \begin{bmatrix} D & -A_K^T \\ A_K & 0 \end{bmatrix},$$

where $|K|$ denotes the number of elements in K, is nonsingular. To obtain a contradiction, suppose that M_K is singular. Then there exists a nonzero vector $(v, w) \in R^n \times R^{|K|}$ such that

$$M_K \begin{pmatrix} v \\ w \end{pmatrix} = \begin{bmatrix} D & -A_K^T \\ A_K & 0 \end{bmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = 0.$$

This implies that

$$Dv - A_K^T w = 0, \quad A_K v = 0. \tag{11.19}$$

Since the system $\{A_i : i \in K\}$ is linearly independent by (c3), from (11.19) it follows that $v \neq 0$. As $A_{I\setminus K}x > b_{I\setminus K}$ and $\lambda_K > 0$, there exists $\delta_3 > 0$ such that $A_{I\setminus K}(x + tv) \geq b_{I\setminus K}$ and $\lambda_K + tw \geq 0$ for every $t \in [0, \delta_3]$. By (11.19), we have

$$\begin{cases} D(x+tv) - A_K^T(\lambda_K + tw) + c = 0, \\ A_K(x+tv) = b_K, \quad \lambda_K + tw \geq 0, \\ A_{I\setminus K}(x+tv) \geq b_{I\setminus K}, \quad \lambda_{I\setminus K} = 0 \end{cases} \tag{11.20}$$

for every $t \in [0, \delta_3]$. From (11.20) we deduce that $x+tv \in S(D, A, c, b)$ for all $t \in [0, \delta_3]$. This contradicts the assumption that $S(D, A, c, b)$ is finite. We have thus proved that M_K is nonsingular. From the definition of K it follows that

$$\begin{cases} Dx - A_K^T\lambda_K + c = 0, \\ A_K x = b_K, \quad \lambda_K > 0, \\ A_{I\setminus K}x > b_{I\setminus K}, \quad \lambda_{I\setminus K} = 0. \end{cases}$$

The last system can be rewritten equivalently as follows

$$M_K\begin{pmatrix} x \\ \lambda_K \end{pmatrix} = \begin{pmatrix} -c \\ b_K \end{pmatrix}, \quad \lambda_K > 0, \quad \lambda_{I\setminus K} = 0, \quad A_{I\setminus K}x > b_{I\setminus K}. \tag{11.21}$$

As M_K is nonsingular, (11.21) yields

$$\begin{pmatrix} x \\ \lambda_K \end{pmatrix} = M_K^{-1}\begin{pmatrix} -c \\ b_K \end{pmatrix}, \quad \lambda_K > 0, \quad \lambda_{I\setminus K} = 0, \quad A_{I\setminus K}x > b_{I\setminus K}.$$

So there exists $\delta > 0$ such that if $(c', b') \in R^n \times R^m$ is such that $\|(c', b') - (c, b)\| < \delta$, then the formula

$$\begin{pmatrix} x' \\ \lambda'_K \end{pmatrix} = M_K^{-1}\begin{pmatrix} c' \\ b'_K \end{pmatrix}$$

defines a vector $(x', \lambda'_K) \in R^n \times R^{|K|}$ satisfying the conditions

$$x' \in V_x, \quad \lambda'_K > 0, \quad A_{I\setminus K}x' > b'_{I\setminus K}.$$

We see at once that vector x' defined in this way belongs to the set

$$S(D, A, c', b') \cap V_x$$

and $\lambda' := (\lambda'_K, \lambda'_{I\setminus K})$, where $\lambda'_{I\setminus K} = 0$, is a Lagrange multiplier corresponding to x'. We have shown that (11.16) is valid for every $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta$.

Finally, suppose that (c4) holds. In this case, we have

$$Dx + c = 0, \quad A_J x = b_J, \quad \lambda_J = 0, \quad A_{I\setminus J}x > b_{I\setminus J}, \quad \lambda_{I\setminus J} = 0. \tag{11.22}$$

To prove that there exists $\delta > 0$ such that (11.16) is valid for every $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta$, we consider the following system of equations and inequalities of variables $(z, \mu) \in R^n \times R^m$:

$$\begin{cases} Dz - A_J^T \mu_J + c' = 0, \quad A_J z \geq b'_J, \quad \mu_J \geq 0, \\ A_{I\setminus J} z \geq b'_{I\setminus J}, \quad \mu_{I\setminus J} = 0, \quad \mu_J^T(A_J z - b'_J) = 0. \end{cases} \tag{11.23}$$

Since D is nonsingular, (11.23) is equivalent to the system

$$\begin{cases} z = D^{-1}(-c' + A_J^T \mu_J), \quad A_J z \geq b'_J, \quad \mu_J \geq 0, \\ A_{I\setminus J} z \geq b'_{I\setminus J}, \quad \mu_{I\setminus J} = 0, \quad \mu_J^T(A_J z - b'_J) = 0. \end{cases} \tag{11.24}$$

By (11.22), $A_{I\setminus J}x > b_{I\setminus J}$. Hence there exist $\delta_4 > 0$ and an open neighborhood $U_x \subset V_x$ of x such that $A_{I\setminus J} z \geq b'_{I\setminus J}$ for any $z \in U_x$ and $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta_4$. Consequently, for every (c', b') satisfying $\|(c', b') - (c, b)\| < \delta_4$, the verification of (11.16) is reduced to the problem of finding $z \in U_x$ and $\mu_J \in R^{|J|}$ such that (11.24) holds. Here $|J|$ denotes the number of elements in J. We substitute z from the first equation of (11.24) into the first inequality and the last equation of that system to get

$$\begin{cases} A_J D^{-1} A_J^T \mu_J \geq b'_J + A_J D^{-1} c', \quad \mu_J \geq 0, \\ \mu_J^T(A_J D^{-1} A_J^T \mu_J - b'_J - A_J D^{-1} c') = 0. \end{cases} \tag{11.25}$$

Let $S := A_J D^{-1} A_J^T$ and $q' := -b'_J - A_J D^{-1} c'$. We can rewrite (11.25) as follows

$$S\mu_J + q' \geq 0, \quad \mu_J \geq 0, \quad (\mu_J)^T(S\mu_J + q') = 0. \tag{11.26}$$

Problem of finding $\mu_J \in R^{|J|}$ satisfying (11.26) is the linear complementarity problem defined by the matrix $S \in R^{|J|\times|J|}$ and the vector $q' \in R^{|J|}$. By assumption (c4), S is a positive definite matrix, that is $y^T S y > 0$ for every $y \in R^{|J|} \setminus \{0\}$. Then S is a *P-matrix*. The latter means that every principal minor of S is positive (see Cottle et al. (1992), Definition 3.3.1). According to Cottle et al. (1992), Theorem 3.3.7, for each $q' \in R^{|J|}$, problem (11.26) has a unique solution $\mu_J \in R^{|J|}$. Since D is nonsingular, from (11.22) it follows that

$$A_J D^{-1}(-c) - b_J = 0.$$

Setting $q = -b_J - A_J D^{-1}c$ we have $q = 0$. Substituting $q' = q = 0$ into (11.26) we find the unique solution $\bar{\mu}_J = 0 = \lambda_J$. By Theorem 7.2.1 from Cottle et al. (1992), there exist $\ell > 0$ and $\varepsilon > 0$ such that for every $q' \in R^{|J|}$ satisfying $\|q' - q\| < \varepsilon$ we have

$$\|\mu_J - \lambda_J\| \le \ell\|q' - q\|.$$

Therefore

$$\|\mu_J\| = \|\mu_J - \lambda_J\| \le \ell\|b'_J - b_J + A_J D^{-1}(c' - c)\|.$$

From this we conclude that there exists $\delta \in (0, \delta_4]$ such that if (c', b') satisfies the condition $\|(c', b') - (c, b)\| < \delta$, then the vector

$$z = D^{-1}(-c' + A_J^T \mu_J),$$

where μ_J is the unique solution of (11.26), belongs to U_x. From the definition of μ_J and z we see that system (11.24), where $\mu_{I\setminus J} := 0$, is satisfied. Then $z \in S(D, A, c', b')$. We have thus shown that, for any (c', b') satisfying $\|(c', b') - (c, b)\| < \delta$, property (11.16) is valid.

The proof is complete. □

To verify condition (c1), we can use Theorem 3.5.

We now consider three examples to see how the conditions (c1)–(c4) can be verified for concrete QP problems.

Example 11.3. (See Robinson (1980), p. 56) Let

$$f(x) = \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - x_1 \quad \text{for all} \;\; x = (x_1, x_2) \in R^2. \tag{11.27}$$

Consider the QP problem

$$\min\{f(x) : x = (x_1, x_2) \in R^2, x_1 - 2x_2 \ge 0, x_1 + 2x_2 \ge 0\}. \tag{11.28}$$

For this problem, we have

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}, \quad c = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$S(D, A, c, b) = \left\{(1, 0), \left(\frac{4}{3}, \frac{2}{3}\right), \left(\frac{4}{3}, -\frac{2}{3}\right)\right\},$$

$$\operatorname{loc}(D, A, c, b) = \left\{ \left(43, \frac{2}{3}\right), \left(\frac{4}{3}, -\frac{2}{3}\right) \right\}.$$

For any feasible vector $x = (x_1, x_2)$ of (11.28), we have $x_1 \geq 2|x_2|$. Therefore

$$f(x) + \frac{2}{3} = \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - x_1 + \frac{2}{3} \geq \frac{3}{8}x_1^2 - x_1 + \frac{2}{3} \geq 0. \tag{11.29}$$

For $\bar{x} := \left(\frac{4}{3}, \frac{2}{3}\right)$ and $\hat{x} := \left(\frac{4}{3}, -\frac{2}{3}\right)$, we have $f(\bar{x}) = f(\hat{x}) = -\frac{2}{3}$. Hence from (11.29) it follows that $\bar{x}$ and $\hat{x}$ are the solutions of (11.28). Actually,

$$\operatorname{Sol}(D, A, c, b) = \operatorname{loc}(D, A, c, b) = \{\bar{x},\ \hat{x}\}.$$

Setting $\widetilde{x} = (1, 0)$ we have $\tilde{x} \in S(D, A, c, b) \setminus \operatorname{loc}(D, A, c, b)$. Note that $\widetilde{\lambda} := (0, 0)$ is a Lagrange multiplier corresponding to $\tilde{x}$. We check at once that conditions (i) and (ii) in Theorem 11.4 are satisfied and, for each KKT point $x \in S(D, A, c, b)$, either (c1) or (c2) is satisfied. Theorem 11.4 shows that the multifunction $S(D, A, \cdot, \cdot)$ is lower semicontinuous at (c, b).

Example 11.4. Let $f(\cdot)$ be defined by (11.27). Consider the QP problem

$$\min\{f(x) \,:\, x = (x_1, x_2) \in R^2, x_1 - 2x_2 \geq 0, x_1 + 2x_2 \geq 0, x_1 \geq 1\}.$$

For this problem, we have

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad c = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let $\bar{x}$, $\hat{x}$, $\widetilde{x}$ be the same as in the preceding example. Note that $\widetilde{\lambda} := (0, 0, 0)$ is a Lagrange multiplier corresponding to $\widetilde{x}$. We have

$$S(D, A, c, b) = \{\widetilde{x},\ \bar{x},\ \hat{x}\}, \quad \operatorname{Sol}(D, A, c, b) = \operatorname{loc}(D, A, c, b) = \{\bar{x},\ \hat{x}\}.$$

Clearly, for $x = \bar{x}$ and $x = \hat{x}$, assumption (c1) is satisfied. It is easily seen that, for the pair $(\widetilde{x}, \widetilde{\lambda})$, we have $K = \emptyset$, $J = \{3\}$. Since $A_J = (1\ 0)$ and $D^{-1} = D$, we get $A_J D^{-1} A_J^T = 1$. Thus (c4) is satisfied. By Theorem 11.4, $S(D, A, \cdot, \cdot)$ is lower semicontinuous at (c, b).

Example 11.5. Let $f(x)$ be as in (11.27). Consider the QP problem

$$\min\{f(x) : x = (x_1, x_2) \in R^2, x_1 - 2x_2 \geq 0, x_1 + 2x_2 \geq 0, x_1 \geq 2\}.$$

For this problem, we have

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad c = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix},$$

$$S(D, A, c, b) = \{(2,0), (2,1), (2,-1)\},$$
$$\mathrm{Sol}(D, A, c, b) = \mathrm{loc}(D, A, c, b) = \{(2,1), (2,-1)\}.$$

Let $\bar{x} = (2,-1)$, $\hat{x} = (2,1)$, $\widetilde{x} = (2,0)$. Note that $\widetilde{\lambda} := (0,0,1)$ is a Lagrange multiplier corresponding to $\widetilde{x}$. For $x = \bar{x}$ and $x = \hat{x}$, we see at once that (c1) is satisfied. For the pair $(\widetilde{x}, \widetilde{\lambda})$, we have $K = \{3\}$, $J = \emptyset$. Since

$$\{A_i : i \in K\} = \{A_3\} = \{(1\ 0)\},$$

assumption (c3) is satisfied. According to Theorem 11.4, $S(D, A, \cdot, \cdot)$ is lower semicontinuous at (c, b).

The idea of the proof of Theorem 11.4 is adapted from Robinson (1980), Theorem 4.1, and the proof of Theorem 11.2. In Robinson (1980), some results involving Schur complements were obtained.

Let $(D, A, c, b) \in R_S^{n \times n} \times R^{m \times n} \times R^n \times R^m$. Let $x \in S(D, A, c, b)$ and let $\lambda \in R^m$ be a Lagrange multiplier corresponding to x. We define K and J by (11.14) and (11.15), respectively. Consider the case where both the sets K and J are nonempty. If the matrix

$$M_K = \begin{bmatrix} D & -A_K^T \\ A_K & 0 \end{bmatrix} \in R^{(n+|K|) \times (n+|K|)}$$

is nonsingular, then we denote by S_J the *Schur complement* (see Cottle et al. (1992), p. 75) of M_K in the following matrix

$$\begin{bmatrix} D & -A_K^T & -A_J^T \\ A_K & 0 & 0 \\ A_J & 0 & 0 \end{bmatrix} \in R^{(n+|K|+|J|) \times (n+|K|+|J|)}.$$

This means that

$$S_J = [A_J\ 0] M_K^{-1} [A_J\ 0]^T.$$

Note that S_J is a symmetric matrix (see Robinson (1980), p. 56). Consider the following condition:

(c5) *$J \neq \emptyset$, $K \neq \emptyset$, the system $\{A_i : i \in K\}$ is linearly independent, $v^T Dv \neq 0$ for every nonzero vector v satisfying $A_K v = 0$, and S_J is positive definite.*

Modifying some arguments of the proof of Theorem 11.4 we can show that if $J \neq \emptyset$, $K \neq \emptyset$, the system $\{A_i : i \in K\}$ is linearly independent, and $v^T Dv \neq 0$ for every nonzero vector v satisfying $A_K v = 0$, then M_K is nonsingular.

It can be proved that the assertion of Theorem 11.4 remains valid if instead of (c1)–(c4) we use (c1)–(c3) and (c5). The method of dealing with (c5) is similar to that of dealing with (c4) in the proof of Theorem 11.4. Up to now we have not found any example of QP problems of the form (11.1) for which there exists a pair (x, λ), $x \in S(D, A, c, b)$ and λ is a Langrange multiplier corresponding to x, such that (c1)–(c4) are not satisfied, but (c5) is satisfied. Thus the usefulness of (c5) in characterizing the lsc property of the multifunction $S(D, A, \cdot, \cdot)$ is to be investigated furthermore. This is the reason why we omit (c5) in the formulation of Theorem 11.4.

We observe that the sufficient condition in Theorem 11.2 for the lsc property of the following multifunction

$$(D', A', c', b') \to S(D', A', c', b'), \tag{11.30}$$

where $(D', A', c', b') \in R_S^{n \times n} \times R^{m \times n} \times R^n \times R^m$, can be reformulated equivalently as follows.

Theorem 11.5. *Let $(D, A, c, b) \in R_S^{n \times n} \times R^{m \times n} \times R^n \times R^m$. Suppose that*

(i) *the set $S(D, A, c, b)$ is finite, nonempty,*

(ii) *the system $Ax \geq b$ is regular,*

and suppose that for every $x \in S(D, A, c, b)$ at least one of the following conditions holds:

(c1) $x \in \mathrm{loc}(D, A, c, b)$,

(c2) $Ax > b$.

Then, multifunction (11.30) is lower semicontinuous at (D, A, c, b).

It is easy to check that (c2) in the above theorem is equivalent to (c2) in Theorem 11.4.

11.3 Commentaries

The material of this chapter is taken from Tam and Yen (1999) and Lee et al. (2002b, 2002c).

Chapter 12

Continuity of the Solution Map in Quadratic Programming

In this chapter we study the lower semicontinuity and the upper semicontinuity properties of the multifunction $(D, A, c, b) \mapsto \mathrm{Sol}(D, A, c, b)$, where $\mathrm{Sol}(D, A, c, b)$ denotes the solution set of the canonical quadratic programming problem.

12.1 USC Property of the Solution Map

Let $D \in R^{n\times n}_S$, $A \in R^{m\times n}$, $c \in R^n$, and $b \in R^m$. Consider the following QP problem of the canonical form:

$$(P) \qquad \begin{cases} \text{Minimize} & f(x) := \frac{1}{2}x^T Dx + c^T x \\ \text{subject to} & Ax \geq b, \quad x \geq 0. \end{cases}$$

Let $\Delta(A, b)$, $\mathrm{Sol}(D, A, c, b)$, and $S(D, A, c, b)$ denote, respectively, the constraint set, the solution set, and the Karush-Kuhn-Tucker point set of (P).

In Chapter 10 we have studied the upper semicontinuity of the set-valued map

$$(D, A, c, b) \mapsto S(D, A, c, b).$$

In this section we will examine in detail the usc property of the solution map

$$(D, A, c, b) \mapsto \mathrm{Sol}(D, A, c, b). \tag{12.1}$$

A complete characterization of the lsc property of the map will be given in the next section.

Recall that the inequality system

$$Ax \geq b, \quad x \geq 0 \tag{12.2}$$

is called regular if there exists $x^0 \in R^n$ such that $Ax^0 > b,\ x^0 > 0$.

The next result is due to Nhan (1995), Theorem 3.4.

Theorem 12.1. *Assume that*

(a_1) $\mathrm{Sol}(D, A, 0, 0) = \{0\}$,

(a_2) *the system* (12.2) *is regular.*

Then, for any $c \in R^n$, *the multifunction* $\mathrm{Sol}(\cdot)$ *is upper semicontinuous at* (D, A, c, b).

Proof. To obtain a contradiction, suppose that there is a pair $(c, b) \in R^n \times R^m$ such that $\mathrm{Sol}(D, A, c, b) \neq \emptyset$ and there exist an open set Ω containing $\mathrm{Sol}(D, A, c, b)$, a sequence $\{(D^k, A^k, c^k, b^k)\}$ converging to (D, A, c, b), a sequence $\{x^k\}$ such that

$$x^k \in \mathrm{Sol}(D^k, A^k, c^k, b^k) \setminus \Omega \quad \text{for every} \quad k \in N.$$

If the sequence $\{x^k\}$ is bounded, then there is no loss of generality in assuming that $x^k \to x^0$ for some $x^0 \in R^n$. It is clear that $x^0 \in \Delta(A, b)$. Fix any $x \in \Delta(A, b)$. By assumption (b_2), there exists a sequence $\{\xi^k\}$, $\xi^k \in \Delta(A^k, b^k)$ for all $k \in N$, such that $\lim_{k\to\infty} x^k = x$ (see Lemma 13.1 in Chapter 13). Since $x^k \in \mathrm{Sol}(D^k, A^k, c^k, b^k)$, we have $f(x^k) \leq f(\xi^k)$. Letting $k \to \infty$ we get $f(x^0) \leq f(x)$. This shows that $x^0 \in \mathrm{Sol}(D, A, c, b) \subset \Omega$. We have arrived at a contradiction, because $x^k \notin \Omega$ for all k, and Ω is open.

Now suppose that the sequence $\{x^k\}$ is bounded. Without loss of generality we can assume that $\|x^k\|^{-1}x^k \to \bar{v}$, $\bar{v} \in \Delta(A, 0)$. Fix any $x \in \Delta(A, b)$. By (a_2) there exists a sequence $\{\xi^k\}$, $\xi^k \in \Delta(A^k, b^k)$ for all k and $\xi^k \to x$. Dividing the inequality

$$\frac{1}{2}(x^k)^T D^k x^k + (c^k)^T x^k \leq \frac{1}{2}(\xi^k)^T D^k \xi^k + (c^k)^T \xi^k$$

by $\|x^k\|^2$ and letting $k \to \infty$ we get $\bar{v}^T D\bar{v} \leq 0$. If $\bar{v}^T D\bar{v} < 0$, then $\mathrm{Sol}(D, A, 0, 0) = \emptyset$, contrary to ($a_1$). If $\bar{v}^T D\bar{v} = 0$, then we have $\bar{v} \in \mathrm{Sol}(D, A, 0, 0)$, which is also impossible.

The proof is complete. □

Corollary 12.1. *If the system* (12.2) *is regular and if the set* $\Delta(A,b)$ *is bounded, then* $\mathrm{Sol}(\cdot)$ *is upper semicontinuous at* (D,A,c,b).

Proof. Since the system (12.2) is regular, $\Delta(A,b)$ is nonempty. The boundedness of $\Delta(A,b)$ implies that $\Delta(A,0) = \{0\}$. Hence $\mathrm{Sol}(D,A,0,0) = \{0\}$, and the desired property follows from Theorem 12.1. □

Condition (a_1) amounts to saying that $x^TDx > 0$ for every $x \in \Delta(A,0)\setminus\{0\}$, i.e., the quadratic form x^TDx is strictly copositive on the cone $\Delta(A,0)$.

The next statement is a complement to Theorem 12.1.

Theorem 12.2. *Assume that:*

(b_1) $S(D,A,0,0) = \{0\}$,

(b_2) *the system* $Ax \geq 0,\ x \geq 0$ *is regular.*

Then, for any $(c,b) \in R^n \times R^m$, *the multifunction* $\mathrm{Sol}(\cdot)$ *is upper semicontinuous at* (D,A,c,b).

Proof. Suppose that the assertion of the theorem is false. Then there is a pair $(c,b) \in R^n \times R^m$ such that $\mathrm{Sol}(D,A,c,b) \neq \emptyset$ and there exist an open set Ω containing $\mathrm{Sol}(D,A,c,b)$, a sequence $\{(D^k,A^k,c^k,b^k)\}$ converging to (D,A,c,b), a sequence $\{x^k\}$ such that

$$x^k \in \mathrm{Sol}(D^k,A^k,c^k,b^k)\setminus\Omega \quad \text{for every} \quad k \in N.$$

If the sequence $\{x^k\}$ is bounded, then we can assume that $x^k \to x^0$ for some $x^0 \in R^n$. We have $x^0 \in \Delta(A,b)$. Fix any $x \in \Delta(A,b)$. Assumption (b_2) implies that system (12.2) is regular. Then there exists a sequence $\{\xi^k\}$, $\xi^k \in \Delta(A^k,b^k)$ for all $k \in N$, such that $\lim_{k\to\infty} x^k = x$. Since $f(x^k) \leq f(\xi^k)$, letting $k \to \infty$ we obtain $f(x^0) \leq f(x)$. Thus $x^0 \in \mathrm{Sol}(D,A,c,b) \subset \Omega$. This contradicts the fact that $x^k \notin \Omega$ for all k.

Now assume that $\{x^k\}$ is unbounded. By taking a subsequence if necessary, we may assume that $||x^k|| \to \infty$. Since $x^k \in \mathrm{Sol}(D^k,A^k,c^k,b^k)$, for each k there exists $\lambda^k \in R^m$ such that

$$D^kx^k - (A^k)^T\lambda^k + c^k \geq 0, \quad A^kx^k - b^k \geq 0, \tag{12.3}$$

$$x^k \geq 0, \quad \lambda^k \geq 0, \tag{12.4}$$

$$(x^k)^T(D^kx^k - (A^k)^T\lambda^k + c^k) + (\lambda^k)^T(A^kx^k - b^k) = 0. \tag{12.5}$$

Since $||(x^k, \lambda^k)|| \to \infty$, without loss of generality we can assume that $||(x^k, \lambda^k)|| \neq 0$ for all k, and the sequence of vectors

$$\frac{(x^k, \lambda^k)}{||(x^k, \lambda^k)||} = \left(\frac{x^k}{||(x^k, \lambda^k)||}, \frac{\lambda^k}{||(x^k, \lambda^k)||} \right)$$

converges to some $(\bar{x}, \bar{\lambda}) \in R^n \times R^m$ with $||(\bar{x}, \bar{\lambda})|| = 1$. Dividing both sides of (12.3) and of (12.4) by $||(x^k, \lambda^k)||$, dividing both sides of (12.5) by $||(x^k, \lambda^k)||^2$, and taking the limits as $k \to \infty$, we obtain

$$D\bar{x} - A^T\bar{\lambda} \geq 0, \quad A\bar{x} \geq 0, \tag{12.6}$$

$$\bar{x} \geq 0, \quad \bar{\lambda} \geq 0, \tag{12.7}$$

$$\bar{x}^T(D\bar{x} - A^T\bar{\lambda}) + \bar{\lambda}^T A\bar{x} = 0. \tag{12.8}$$

The system (12.6)–(12.8) proves that $\bar{x} \in S(D, A, 0, 0)$. By (b_1), $\bar{x} = 0$. Hence

$$-A^T\bar{\lambda} \geq 0, \quad \bar{\lambda} \geq 0. \tag{12.9}$$

Combining (12.9) and (b_2) yields $\bar{\lambda} = 0$ (see Lemma 10.1), hence $||(\bar{x}, \bar{\lambda})|| = 0$, a contradiction. The proof is complete. □

Remark 12.1. Since $\Delta(A, b) + \Delta(A, 0) \subset \Delta(A, b)$, (b_2) implies (a_2). However, (b_1) does not imply (a_1).

Observe that neither (a_1) nor (a_2) is a necessary condition for the upper semicontinuity of the solution map $\mathrm{Sol}(\cdot)$ at a given point (D, A, c, b).

Example 12.1. Let $n = m = 1$, $D = [0]$, $A = [1]$, $c = 1$, $b = 1$. It is easily verified that $\mathrm{Sol}(D, A, c, b) = \{1\}$ and the multifunction $\mathrm{Sol}(\cdot)$ is usc at (D, A, c, b). Meanwhile, $\mathrm{Sol}(D, A, 0, 0) = \{x \in R : x \geq 0\}$, so (a_1) fails to hold.

Example 12.2. Let $n = m = 1$, $A = [-1]$, $b = 0$. If $A' = [-1 + \alpha]$, $b' = \beta$, where α and β are sufficiently small, then $\Delta(A', b') = \left\{ x \in R \,:\, 0 \leq x \leq \dfrac{-\beta}{1 - \alpha} \right\}$. It is easily seen that, for arbitrarily chosen D and c, the multifunction $\mathrm{Sol}(\cdot)$ is usc at (D, A, c, b), while condition (a_2) does not hold.

12.2 LSC Property the Solution Map

By definition, multifunction $\mathrm{Sol}(\cdot)$ is continuous at (D, A, c, b) if it is simultaneously upper semicontinuous and lower semicontinuous at that point.

Our main result in this section can be stated as follows.

Theorem 12.3. *The solution map* Sol$(\cdot)$ *of* (P) *is lower semicontinuous at* (D, A, c, b) *if and only if the following three conditions are satisfied:*

(a) *the system* $Ax \geq b,\ x \geq 0$ *is regular,*

(b) $\mathrm{Sol}(D, A, 0, 0) = \{0\}$,

(c) $|\mathrm{Sol}(D, A, c, b)| = 1$.

For proving Theorem 12.3 we need some lemmas.

Lemma 12.1. *If* Sol$(\cdot)$ *is lower semicontinuous at* (D, A, c, b)*, then the system* $Ax \geq b,\ x \geq 0$ *is regular.*

Proof. If the system $Ax \geq b,\ x \geq 0$ is irregular then, according to Lemma 3 in Robinson (1977), there exists a sequence $(A^k, b^k) \in R^{m\times n} \times R^m$ tending to (A, b) such that $\Delta(A^k, b^k) = \emptyset$ for each k. Therefore, $\mathrm{Sol}(D, A^k, c, b^k) = \emptyset$ for each k, contrary to the assumed lower semicontinuity of the solution map. $\square$

Lemma 12.2. *If the multifunction* Sol$(\cdot)$ *is lower semicontinuous at* (D, A, c, b)*, then* $\mathrm{Sol}(D, A, 0, 0) = \{0\}$.

Proof. On the contrary, suppose that $\mathrm{Sol}(D, A, 0, 0) \neq \{0\}$. Then there is a nonzero vector $\bar{x} \in R^n$ such that

$$A\bar{x} \geq 0, \quad \bar{x} \geq 0, \quad \bar{x}^T D\bar{x} \leq 0. \tag{12.10}$$

Since $\Delta(A, b) \neq \emptyset$, from (12.10) it follows that $\Delta(A, b)$ is unbounded. For every $\varepsilon > 0$, we get from (12.10) that $\bar{x}^T(D - \varepsilon E)\bar{x} < 0$. Hence, for any $x \in \Delta(A, b)$,

$$f(x + t\bar{x}) = \frac{1}{2}(x + t\bar{x})^T(D - \varepsilon E)(x + t\bar{x}) + c^T(x + t\bar{x}) \to -\infty$$

as $t \to \infty$. Thus, $\mathrm{Sol}(D - \varepsilon E, A, c, b) = \emptyset$. This contradicts our assumption that Sol$(\cdot)$ is lower semicontinuous at (D, A, c, b). $\square$

Lemma 12.3. (i) *If* $\mathrm{Sol}(D, A, 0, 0) = \{0\}$ *then, for any* $(c, b) \in R^n \times R^m$, $\mathrm{Sol}(D, A, c, b)$ *is a compact set.*

(ii) *If* $\mathrm{Sol}(D, A, 0, 0) = \{0\}$ *and if* $\Delta(A, b)$ *is nonempty, then* $\mathrm{Sol}(D, A, c, b)$ *is nonempty for every* $c \in R^n$.

Proof. (i) Suppose that $\mathrm{Sol}(D, A, 0, 0) = \{0\}$, but $\mathrm{Sol}(D, A, c, b)$ is unbounded for some (c, b). Then there is a sequence $\{x^k\} \subset$

$\mathrm{Sol}(D, A, c, b)$ such that $||x^k|| \to \infty$ as $k \to \infty$. Fixing any $x \in \Delta(A, b)$, one has

$$\frac{1}{2}(x^k)^T D x^k + c^T x^k \leq \frac{1}{2} x^T D x + c^T x, \tag{12.11}$$

$$A x^k \geq b, \quad x^k \geq 0. \tag{12.12}$$

We can assume that the sequence $||x^k||^{-1} x^k$ converges to some $\bar{x}$ with $||\bar{x}|| = 1$. Using (12.11) and (12.12) it is easy to show that $\bar{x}^T D \bar{x} \leq 0$, $A\bar{x} \geq 0$, $\bar{x} \geq 0$. This contradicts the fact that $\mathrm{Sol}(D, A, 0, 0) = \{0\}$. We have thus proved that $\mathrm{Sol}(D, A, c, b)$ is a bounded set. Then $\mathrm{Sol}(D, A, c, b)$, being closed, is a compact set.

(ii) Let $\mathrm{Sol}(D, A, 0, 0) = \{0\}$, $\Delta(A, b) \neq \emptyset$, and let $c \in R^n$ be given arbitrarily. If the quadratic form $f(x) = \dfrac{1}{2} x^T D x + c^T x$ is bounded below on the polyhedron $\Delta(A, b)$ then, by the Frank-Wolfe Theorem (see Theorem 2.1), the solution set $\mathrm{Sol}(D, A, c, b)$ is nonempty. Now assume that there exists a sequence $x^k \in \Delta(A, b)$ such that $f(x^k) \to -\infty$ as $k \to \infty$. By taking a subsequence, if necessary, we can assume that

$$\frac{1}{2}(x^k)^T D x^k + (c^k)^T x^k \leq 0 \tag{12.13}$$

for all k, $||x^k|| \to \infty$, and $||x^k||^{-1} x^k$ converges to some $\bar{x}$ as $k \to \infty$. It is a simple matter to show that $\bar{x} \in \Delta(A, 0)$. Dividing both sides of (12.13) by $||x^k||^2$ and letting $k \to \infty$ one has $\bar{x}^T D \bar{x} \leq 0$. As $||\bar{x}|| = 1$, one gets $\mathrm{Sol}(D, A, 0, 0) \neq \{0\}$, which is impossible. $\square$

We omit the proof of the next lemma, because it is similar to the proof of Theorem 11.1.

Lemma 12.4. *If the multifunction* $\mathrm{Sol}(\cdot)$ *is lower semicontinuous at* (D, A, c, b)*, then the set* $\mathrm{Sol}(D, A, c, b)$ *is finite.*

Lemma 12.5. *The set* $\mathrm{G} := \{(D, A) : \mathrm{Sol}(D, A, 0, 0) = \{0\}\}$ *is open in* $R_S^{n \times n} \times R^{m \times n}$.

Proof. On the contrary, suppose that there is a sequence $\{(D^k, A^k)\}$ converging to $(D, A) \in \mathrm{G}$ such that $\mathrm{Sol}(D^k, A^k, 0, 0) \neq \{0\}$ for all k. Then for each k there exists a vector x^k such that $||x^k|| = 1$ and

$$A^k x^k \geq 0, \quad x^k \geq 0, \quad (x^k)^T D^k x^k \leq 0. \tag{12.14}$$

Without loss of generality, we can assume that $\{x^k\}$ converges to some x^0 with $||x^0|| = 1$. Taking the limits in (12.14) as $k \to \infty$, we obtain

$$A x^0 \geq 0, \quad x^0 \geq 0, \quad (x^0)^T D x^0 \leq 0.$$

This contradicts the assumption that $\mathrm{Sol}(D, A, 0, 0) = \{0\}$. The proof of is complete. $\square$

For each subset $\alpha \subset \{1, \ldots, m\}$ with the complement $\bar{\alpha}$, and for each subset $\beta \subset \{1, \ldots, n\}$ with the complement $\bar{\beta}$, let

$$F(\alpha, \beta) := \{x \in R^n : (Ax)_\alpha > b_\alpha,\ (Ax)_{\bar{\alpha}} = b_{\bar{\alpha}},\ x_\beta > 0,\ x_{\bar{\beta}} = 0\}.$$

Note that $F(\alpha, \beta)$ is a pseudo-face of $\Delta(A, b)$. Obviously,

$$\Delta(A, b) = \bigcup_{(\alpha,\beta)} F(\alpha, \beta).$$

Besides, for every $x \in \Delta(A, b)$ there exists a unique pair (α, β) such that $x \in F(\alpha, \beta)$. In addition, if $(\alpha, \beta) \neq (\alpha', \beta')$ then $F(\alpha, \beta) \cap F(\alpha', \beta') = \emptyset$.

The following lemma is immediate from Theorem 4.5.

Lemma 12.6. *If the solution set* $\mathrm{Sol}(D, A, c, b)$ *is finite then, for any* $\alpha \subset \{1, \ldots, m\}$ *and for any* $\beta \subset \{1, \ldots, n\}$, *we have*

$$|\mathrm{Sol}(D, A, c, b) \cap F(\alpha, \beta)| \leq 1.$$

Lemma 12.7. *If the multifunction* $\mathrm{Sol}(\cdot)$ *is lower semicontinuous at* (D, A, c, b), *then*

$$|\mathrm{Sol}(D, A, c, b)| = 1.$$

Proof. On the contrary, suppose that in $\mathrm{Sol}(D, A, c, b)$ we can find two distinct vectors $\bar{x}$, $\bar{y}$. Let $J(\bar{x}) = \{j : \bar{x}_j = 0\}$, $J(\bar{y}) = \{j : \bar{y}_j = 0\}$.

If $J(\bar{x}) \neq J(\bar{y})$, then there exists j_0 such that $\bar{x}_{j_0} = 0$ and $\bar{y}_{j_0} > 0$, or there exists j_1 such that $\bar{x}_{j_1} > 0$ and $\bar{y}_{j_1} = 0$. By symmetry, it is enough to consider the first case. As $\bar{y} \in \mathrm{Sol}(D, A, c, b)$ and $y_{j_0} > 0$, there is an open neighborhood U of $\bar{y}$ such that $f(y) \geq f(\bar{y})$ and $y_{j_0} > 0$ for every $y \in U$. Fix any $\varepsilon > 0$ and put $c(\varepsilon) = (c_i(\varepsilon))$ where

$$c_i(\varepsilon) = \begin{cases} c_i & \text{if } i \neq j_0 \\ c_i + \varepsilon & \text{if } i = j_0. \end{cases}$$

Let $f_\varepsilon(x) = f(x) + \varepsilon x_{j_0}$, where, as before, $f(x) = \dfrac{1}{2} x^T D x + c^T x$. Consider the quadratic program

$$\text{Minimize } f_\varepsilon(x) \quad \text{subject to } x \in \Delta(A, b),$$

whose solution set is $\mathrm{Sol}(D, A, c(\varepsilon), b)$. For every $y \in U$, we have

$$\begin{aligned} f_\varepsilon(y) = f(y) + \varepsilon y_{j_0} > f(y) &\geq f(\bar{y}) \\ &= f(\bar{x}) = f_\varepsilon(\bar{x}). \end{aligned}$$

Hence $y \notin \mathrm{Sol}(D, A, c(\varepsilon), b)$. So

$$\mathrm{Sol}(D, A, c(\varepsilon), b) \cap U = \emptyset. \tag{12.15}$$

Since $\varepsilon > 0$ can be arbitrarily small, (12.15) contradicts our assumption that $\mathrm{Sol}(\cdot)$ is lower semicontinuous at (D, A, c, b).

We now suppose that $J(\bar{x}) = J(\bar{y})$. Let α and α' be the index sets such that

$$\bar{x} \in F(\alpha, \beta), \quad \bar{y} \in F(\alpha', \beta),$$

where β is the complement of $J(\bar{x}) = J(\bar{y})$ in $\{1, \ldots, n\}$. By Lemma 12.4, $\mathrm{Sol}(D, A, c, b)$ is a finite set. Then, by Lemma 12.6, $\alpha \neq \alpha'$. Hence at least one of the sets $\alpha \setminus \alpha'$ and $\alpha' \setminus \alpha$ must be nonempty. By symmetry, it suffices to consider the first case. Let $i_0 \in \alpha \setminus \alpha'$. Then we have

$$(A\bar{x})_{i_0} > b_{i_0}, \quad (A\bar{y})_{i_0} = b_{i_0}. \tag{12.16}$$

As $\mathrm{Sol}(D, A, c, b)$ is finite, one can find a neighborhood W of $\bar{y}$ such that

$$\mathrm{Sol}(D, A, c, b) \cap W = \{\bar{y}\}. \tag{12.17}$$

Fix any $\varepsilon > 0$ and put $b(\varepsilon) = (b_i(\varepsilon))$, where

$$b_i(\varepsilon) = \begin{cases} b_i & \text{if } i \neq i_0 \\ b_i + \varepsilon & \text{if } i = i_0. \end{cases}$$

By (12.16), there exists $\delta > 0$ such that $\bar{x} \in \Delta(A, b(\varepsilon))$ for every $\varepsilon \in (0, \delta)$. Since $\Delta(A, b(\varepsilon)) \subset \Delta(A, b)$, we have

$$\inf_{x \in \Delta(A, b(\varepsilon))} f(x) \geq \inf_{x \in \Delta(A, b)} f(x) = f(\bar{x}).$$

Therefore, for every $\varepsilon \in (0, \delta)$, $\bar{x} \in \mathrm{Sol}(D, A, c, b(\varepsilon))$. Moreover,

$$\mathrm{Sol}(D, A, c, b(\varepsilon)) \subset \mathrm{Sol}(D, A, c, b).$$

It is clear that $\bar{y} \notin \Delta(A, b(\varepsilon))$. Then we have $\mathrm{Sol}(D, A, c, b(\varepsilon)) \subset \mathrm{Sol}(D, A, c, b) \setminus \{\bar{y}\}$. Hence, by (12.17), $\mathrm{Sol}(D, A, c, b(\varepsilon)) \cap W = \emptyset$ for every $\varepsilon \in (0, \delta)$. This contradicts the lower semicontinuity of $\mathrm{Sol}(\cdot)$ at (D, A, c, b). Lemma 12.7 is proved. $\square$

Proof of Theorem 12.3.

If $\mathrm{Sol}(\cdot)$ is lower semicontinuous at (D, A, c, b) then from Lemmas 12.1, 12.2, and 12.7, we get (a), (b), and (c).

Conversely, assume that the conditions (a), (b) and (c) are fulfilled. Let Ω be an open set containing the unique solution $\bar{x} \in \mathrm{Sol}(D, A, c, b)$. By (a), there exists $\delta_1 > 0$ such that $\Delta(A', b') \neq \emptyset$ for every pair (A', b') satisfying $\max\{||A' - A||, ||b' - b||\} < \delta_1$ (see Lemma 13.1 in Chapter 13). By (b) and by Lemma 12.5, there exists $\delta_2 > 0$ such that $\mathrm{Sol}(D', A', 0, 0) = \{0\}$ for every pair (D', A') satisfying $\max\{||D' - D||, ||A' - A||\} \leq \delta_2$. For $\delta := \min\{\delta_1, \delta_2\}$, by the second assertion of Lemma 12.3 we have $\mathrm{Sol}(D', A', c', b') \neq \emptyset$ for every (D', A', c', b') satisfying

$$\max\{||D' - D||, ||A' - A||, ||c' - c||, ||b' - b||\} < \delta. \tag{12.18}$$

By (a) and (b), it follows from Theorem 2.1 that $\mathrm{Sol}(\cdot)$ is upper semicontinuous at (D, A, c, b). Hence $\mathrm{Sol}(D', A', c', b') \subset \Omega$ for every (D', A', c', b') satisfying (12.18), if $\delta > 0$ is small enough. For such a δ, from what has been said it follows that $\mathrm{Sol}(D', b', c', b') \cap \Omega \neq \emptyset$ for every (D', A', c', b') satisfying (12.18). This shows that $\mathrm{Sol}(\cdot)$ is lower semicontinuous at (D, A, c, b). The proof is complete. □

The following fact follows directly from Theorems 12.3 and 12.1.

Corollary 12.2. *If the multifunction* $\mathrm{Sol}(\cdot)$ *is lower semicontinuous at* (D, A, c, b) *then it is upper semicontinuous at* (D, A, c, b)*, hence it is continuous at the point.*

Let us mention two other interesting consequences of Theorem 12.3.

Corollary 12.3. *If* D *is a negative semidefinite matrix, then the multifunction* $\mathrm{Sol}(\cdot)$ *is continuous at* (D, A, c, b) *if and only if the following conditions are satisfied*

(i) *the system* $Ax \geq b,\ x \geq 0$ *is regular,*

(ii) $\Delta(A, b)$ *is a compact set, and*

(iii) $|\mathrm{Sol}(D, A, c, b)| = 1$.

Proof. Assume that $\mathrm{Sol}(\cdot)$ is lower semicontinuous at (D, A, c, b). By Theorem 12.3, conditions (i) and (iii) are satisfied. Moreover,

$$\mathrm{Sol}(D, A, 0, 0) = \{0\}. \tag{12.19}$$

We claim that $\Delta(A, 0) = \{0\}$. Indeed, by assumption, $x^T Dx \leq 0$ for every $x \in \Delta(A, 0)$. If there exists no $\bar{x} \in \Delta(A, 0)$ with the property that $\bar{x}^T D\bar{x} < 0$ then $\mathrm{Sol}(D, A, 0, 0) = \Delta(A, 0)$, and (12.19) forces $\Delta(A, 0) = \{0\}$. If $\bar{x}^T D\bar{x} < 0$ for some $\bar{x} \in \Delta(A, 0)$ then it is obvious that $\mathrm{Sol}(D, A, 0, 0) = \emptyset$, which is impossible. Our claim is proved. Property (ii) follows directly from the equality $\Delta(A, 0) = \{0\}$.

Conversely, suppose that (i), (ii) and (iii) are satisfied. As $\Delta(A, b) \neq \emptyset$ by (i), assumption (ii) implies that $\Delta(A, 0) = \{0\}$. Therefore, $\mathrm{Sol}(D, A, 0, 0) = \{0\}$. Since the conditions (a), (b) and (c) in Theorem 12.3 are satisfied, $\mathrm{Sol}(\cdot)$ is lower semicontinuous at (D, A, c, b). The proof is complete. $\square$

Corollary 12.4. *If D is a positive definite matrix, then the multifunction* $\mathrm{Sol}(\cdot)$ *is continuous at (D, A, c, b) if and only if the system $Ax \geq b,\ x \geq 0$ is regular.*

The proof of this corollary is simple, so it is omitted.

12.3 Commentaries

The material of this chapter is adapted from Tam (1999). The proof of the 'necessity' part of Theorem 12.3 can be shortened greatly by using an argument in Phu and Yen (2001).

Chapter 13

Continuity of the Optimal Value Function in Quadratic Programming

In this chapter we will characterize the continuity property of the optimal value function in a general parametric QP problem. The lower semicontinuity and upper semicontinuity properties of the optimal value function are studied as well. Directional differentiability of the optimal value function in QP problems will be addressed in the next chapter.

13.1 Continuity of the Optimal Value Function

Consider the following general quadratic programming problem with linear constraints, which will be denoted by $QP(D, A, c, b)$,

$$\begin{cases} \text{Minimize } \ f(x, c, D) := \dfrac{1}{2}x^T Dx + c^T x \\ \text{subject to } \ x \in \Delta(A, b) := \{x \in R^n \,:\, Ax \geq b\} \end{cases} \tag{13.1}$$

depending on the parameter $\omega = (D, A, c, b) \in \Omega$, where

$$\Omega := R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m.$$

The solution set of (13.1) will be denoted by $\mathrm{Sol}(D, A, c, b)$. The function $\varphi : \Omega \longrightarrow \overline{R}$ defined by

$$\varphi(\omega) = \inf\{f(x, c, D) \,:\, x \in \Delta(A, b)\}.$$

is the optimal value function of the parametric problem (13.1).

If $v^T Dv \geq 0$ (resp., $v^T Dv \leq 0$) for all $v \in R^n$ then $f(\cdot, c, D)$ is a convex (resp., concave) function and (13.1) is a convex (resp., concave) QP problem. If such conditions are not required then (13.1) is an *indefinite* QP problem (see Section 1.5).

In this section, complete characterizations of the continuity of the function φ at a given point are obtained. In Section 13.2, sufficient conditions for the upper and lower semicontinuity of φ at a given point will be established. For proving the results, we rely on some results due to Robinson (1975, 1977) on stability of the feasible region $\Delta(A, b)$ and the Frank-Wolfe Theorem.

Before obtaining the desired characterizations, we state some lemmas.

Lemma 13.1. *Let $A \in R^{m\times n}$, $b \in R^m$. The system $Ax \geq b$ is regular if and only if the multifunction $\Delta(\cdot) : R^{m\times n} \times R^m \longrightarrow 2^{R^n}$, defined by $\Delta(A', b') = \{x \in R^n : A'x \geq b'\}$, is lower semicontinuous at (A, b).*

Proof. Suppose that $Ax \geq b$ is a regular system and $x^0 \in R^n$ is such that $Ax^0 > b$. Obviously, $\Delta(A, b)$ is nonempty. Let V be an open subset in R^n satisfying $\Delta(A, b) \cap V \neq \emptyset$. Take $x \in \Delta(A, b) \cap V$. For every $t \in [0, 1]$, we set

$$x_t := (1 - t)x + tx^0.$$

Since $x_t \to x$ as $t \to 0$, there is $t_0 > 0$ such that $x_{t_0} \in V$. Since

$$Ax_{t_0} = (1 - t_0)Ax + t_0 Ax^0 > (1 - t_0)b + t_0 b = b,$$

there exists $\delta_{t_0} > 0$ such that

$$A'x_{t_0} > b'$$

for all $(A', b') \in R^{m\times n} \times R^m$ satisfying

$$\|(A', b') - (A, b)\| < \delta_{t_0}. \tag{13.2}$$

Thus $x_t \in \Delta(A', b')$ for every (A', b') fulfilling (13.2). Therefore $\Delta(\cdot)$ is lower semicontinuous at (A, b).

Conversely, if $\Delta(\cdot)$ is lower semicontinuous at (A, b) then there exists $\delta > 0$ such that $Ax \geq b'$ is solvable for every $b' \in R^m$ satisfying $b' > b$ and $\|b' - b\| < \delta$. This implies that $Ax > b$ is solvable. Thus $Ax \geq b$ is a regular system. □

Remark 13.1. If the inequality system $Ax \geq b$ is irregular then there exists a sequence $\{(A^k, b^k)\}$ in $R^{m\times n} \times R^m$ converging to (A, b) such that, for every k, the system $A^k x \geq b^k$ has no solutions. This fact follows from the results of Robinson (1977).

Lemma 13.2 (cf. Robinson (1977), Lemma 3). *Let $A \in R^{m\times n}$. If the system $Ax \geq 0$ is regular then, for every $b \in R^m$, the system $Ax \geq b$ is regular.*

Proof. Assume that $Ax \geq 0$ is a regular and $\bar{x} \in R^n$ is such that $A\bar{x} > 0$. Setting $\bar{b} = A\bar{x}$, we have $\bar{b} > 0$. Let $b \in R^m$ be given arbitrarily. Then there exists $t > 0$ such that $t\bar{b} > b$. We have $A(t\bar{x}) = tA\bar{x} = t\bar{b}$. Therefore $A(t\bar{x}) > b$, hence the system $Ax \geq b$ is regular. $\square$

The set

$$G := \{(D, A) \in R_S^{n\times n} \times R^{m\times n} \,:\, \mathrm{Sol}(D, A, 0, 0) = \{0\}\} \tag{13.3}$$

is open in $R_S^{n\times n} \times R^{m\times n}$. This fact can be proved similarly as Lemma 12.5. It is worthy to stress that Lemma 12.5 is applicable only to canonical QP problems while, in this chapter, the standard QP problems are considered.

Lemma 13.3. *If $\Delta(A, b)$ is nonempty and if* $\mathrm{Sol}(D, A, 0, 0) = \{0\}$ *then, for every $c \in R^n$,* $\mathrm{Sol}(D, A, c, b)$ *is a nonempty compact set.*

Proof. Let $\Delta(A, b)$ be nonempty and $\mathrm{Sol}(D, A, 0, 0) = \{0\}$. Suppose that $\mathrm{Sol}(D, A, c, b) = \emptyset$ for some $c \in R^n$. By the Frank-Wolfe Theorem, there exists a sequence $\{x^k\}$ such that $Ax^k \geq b$ for every k and

$$f(x^k, c, D) = \frac{1}{2}(x^k)^T Dx^k + c^T x^k \to -\infty \quad \text{as} \quad k \to \infty.$$

It is clear that $\|x^k\| \to +\infty$ as $k \to \infty$. By taking a subsequence if necessary, we can assume that $\|x^k\|^{-1}x^k \to \bar{x} \in R^n$ and

$$f(x^k, c, D) = \frac{1}{2}(x^k)^T Dx^k + c^T x^k < 0 \quad \text{for every} \ \ k. \tag{13.4}$$

We have

$$A\frac{x^k}{\|x^k\|} \geq \frac{b}{\|x^k\|}.$$

Letting $k \to \infty$, we obtain $\bar{x} \in \Delta(A, 0)$. Dividing both sides of the inequality in (13.4) by $\|x^k\|^2$ and letting $k \to \infty$, we get $\bar{x}^T D\bar{x} \leq 0$. Since $\|\bar{x}\| = 1$, we have $\mathrm{Sol}(D, A, 0, 0) \neq \{0\}$. This contradicts the assumption $\mathrm{Sol}(D, A, 0, 0) = \{0\}$. Thus $\mathrm{Sol}(D, A, c, b)$ is nonempty for each $c \in R^n$.

Suppose, contrary to our claim, that $\mathrm{Sol}(D, A, c, b)$ is unbounded for some $c \in R^n$. Then there exists a sequence $\{x^k\} \subset \mathrm{Sol}(D, A, c, b)$ such that $\|x^k\| \to \infty$ as $k \to \infty$ and $\{\|x^k\|^{-1}x^k\}$ converges to a certain $\bar{x} \in R^n$. Taking any $x \in \Delta(A, b)$, we have

$$\frac{1}{2}(x^k)^T Dx^k + c^T x^k \le \frac{1}{2}x^T Dx + c^T x, \tag{13.5}$$

$$Ax^k \ge b. \tag{13.6}$$

Dividing both sides of (13.5) by $\|x^k\|^2$, both sides of (13.6) by $\|x^k\|$, and letting $k \to \infty$, we obtain

$$\bar{x}^T D\bar{x} \le 0, \quad A\bar{x} \ge 0.$$

Thus $\mathrm{Sol}(D, A, 0, 0) \ne \{0\}$, a contradiction. We have proved that, for every $c \in R^n$, the solution set $\mathrm{Sol}(D, A, c, b)$ is bounded. Fixing any $\bar{x} \in \mathrm{Sol}(D, A, c, b)$ one has

$$\mathrm{Sol}(D, A, c, b) = \{x \in \Delta(A, b) \, : \, f(x, c, D) = f(\bar{x}, c, D)\}.$$

Hence $\mathrm{Sol}(D, A, c, b)$ is a closed set and, therefore, $\mathrm{Sol}(D, A, c, b)$ is a compact set. □

We are now in a position to state our first theorem on the continuity of the optimal value function φ. This theorem gives a set of conditions which is necessary and sufficient for the continuity of φ at a point $\omega = (D, A, c, b)$ where φ has a finite value.

Theorem 13.1. *Let $(D, A, c, b) \in \Omega$. Assume that $\varphi(D, A, c, b) \ne \pm\infty$. Then, the optimal value function $\varphi(\cdot)$ is continuous at (D, A, c, b) if and only if the following two conditions are satisfied:*

(a) *the system $Ax \ge b$ is regular,*

(b) $\mathrm{Sol}(D, A, 0, 0) = \{0\}$.

Proof. *Necessity:* First, suppose that $\varphi(\cdot)$ is continuous at $\omega := (D, A, c, b)$ and $\varphi(\omega) \ne \pm\infty$. If (a) is violated then, by Remark 13.1, there exists a sequence $\{(A^k, b^k)\}$ in $R^{m \times n} \times R^m$ converging to (A, b) such that, for every k, the system $A^k x \ge b^k$ has no solutions. Consider the sequence $\{(D, A^k, c, b^k)\}$. Since $\Delta(A^k, b^k) = \emptyset$,

$\varphi(D, A^k, c, b^k) = +\infty$ for every k. As $\varphi(\cdot)$ is continuous at ω and $\{(D, A^k, c, b^k)\}$ converges to ω, we have

$$\lim_{k\to\infty} \varphi(D, A^k, c, b^k) = \varphi(D, A, c, b) \neq \pm\infty.$$

We have arrived at a contradiction. Thus (a) is fulfilled.

Now we suppose that (b) fails to hold. Then there is a nonzero vector $\bar{x} \in R^n$ such that

$$A\bar{x} \geq 0, \quad \bar{x}^T D\bar{x} \leq 0. \tag{13.7}$$

Consider the sequence $\{(D^k, A, c, b)\}$, where $D^k := D - \frac{1}{k}E$, E is the unit matrix in $R^{n\times n}$. From the assumption $\varphi(\omega) \neq \pm\infty$ it follows that $\Delta(A, b)$ is nonempty. Then from (13.7) we can deduce that $\Delta(A, b)$ is unbounded. For every k, by (13.7) we have

$$\bar{x}^T D^k \bar{x} = \bar{x}^T (D - \frac{1}{k}E)\bar{x} < 0.$$

Hence, for any x belonging to $\Delta(A, b)$ and for any $t > 0$, we have $x + t\bar{x} \in \Delta(A, b)$ and

$$f(x + t\bar{x}, c, D^k) = \frac{1}{2}(x + t\bar{x})^T D^k (x + t\bar{x}) + c^T (x + t\bar{x}) \to -\infty$$

as $t \to \infty$. This implies that, for all k, $\mathrm{Sol}(D^k, A, c, b) = \emptyset$ and $\varphi(D^k, A, c, b) = -\infty$. We have arrived at a contradiction, because $\varphi(\cdot)$ is continuous at ω and $\varphi(\omega) \neq \pm\infty$. We have proved that (b) holds true.

Sufficiency: Suppose that (a), (b) are satisfied and

$$\{(D^k, A^k, c^k, b^k)\} \subset \Omega$$

is a sequence converging to ω. By Lemma 13.1, assumption (a) implies the existence of a positive integer k_0 such that $\Delta(A^k, b^k) \neq \emptyset$ for every $k \geq k_0$. From assumption (b) it follows that the set G defined by (13.3) is open. Hence there exists a positive integer $k_1 \geq k_0$ such that $\mathrm{Sol}(D^k, A^k, 0, 0) = \{0\}$ for every $k \geq k_1$. By Lemma 13.3, $\mathrm{Sol}(D^k, A^k, c^k, b^k) \neq \emptyset$ for every $k \geq k_1$. Therefore, for every $k \geq k_1$ there exists $x^k \in R^n$ satisfying

$$\varphi(D^k, A^k, c^k, b^k) = \frac{1}{2}(x^k)^T D x^k + (c^k)^T x^k, \tag{13.8}$$

$$A^k x^k \geq b^k. \tag{13.9}$$

Since $\varphi(\omega) \neq \pm\infty$, the Frank-Wolfe Theorem shows that

$$\mathrm{Sol}(D, A, c, b) \neq \emptyset.$$

Taking any $x^0 \in \mathrm{Sol}(D, A, c, b)$, we have

$$\varphi(D, A, c, b) = \frac{1}{2}(x^0)^T D x^0 + c^T x^0, \tag{13.10}$$

$$A x^0 \geq b. \tag{13.11}$$

By Lemma 13.1, there exists a sequence $\{y^k\}$ in R^n converging to x^0 and

$$A^k y^k \geq b^k \quad \text{for every } k \geq k_1. \tag{13.12}$$

From (13.12) it follows that $y^k \in \Delta(A^k, b^k)$ for $k \geq k_1$. So

$$\varphi(D^k, A^k, c^k, b^k) \leq \frac{1}{2}(y^k)^T D^k y^k + (c^k)^T y^k. \tag{13.13}$$

From (13.13) it follows that

$$\begin{aligned}
&\limsup_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) \\
&\quad \leq \limsup_{k\to\infty} \left[\frac{1}{2}(y^k)^T D^k y^k + (c^k)^T y^k\right] \\
&\quad = \lim_{k\to\infty} \left[\frac{1}{2}(y^k)^T D^k y^k + (c^k)^T y^k\right].
\end{aligned}$$

Therefore, taking account of (13.10) and (13.11), we get

$$\limsup_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) \leq \varphi(D, A, c, b). \tag{13.14}$$

We now claim that the sequence $\{x^k\}$, $k \geq k_1$, is bounded. Indeed, if it is unbounded then, by taking a subsequence if necessary, we can assume that $\|x^k\| \to \infty$ as $k \to \infty$ and $\|x^k\| \neq 0$ for all $k \geq k_1$. Then the sequence $\{\|x^k\|^{-1} x^k\}$, $k \geq k_1$, has a convergent subsequence. Without loss of generality we can assume that $\|x^k\|^{-1} x^k \to \hat{x}$, $\|\hat{x}\| = 1$. From (13.9) we have

$$A^k \frac{x^k}{\|x^k\|} \geq \frac{b^k}{\|x^k\|}.$$

Letting $k \to \infty$, we obtain

$$A\hat{x} \geq 0. \tag{13.15}$$

By (13.8) and (13.13),

$$\frac{1}{2}(x^k)^T D^k x^k + (c^k)^T x^k \leq \frac{1}{2}(y^k)^T D^k y^k + (c^k)^T y^k. \tag{13.16}$$

Dividing both sides of (13.16) by $\|x^k\|^2$ and taking limits as $k \to \infty$, we get

$$\hat{x}^T D \hat{x} \leq 0. \tag{13.17}$$

By (13.15) and (13.17), we have $\mathrm{Sol}(D, A, 0, 0) \neq \{0\}$. This contradicts (b). We have thus shown that the sequence $\{x^k\}, \quad k \geq k_1$, is bounded; hence it has a convergent sequence. There is no loss of generality in assuming that $x^k \to \tilde{x} \in R^n$. By (13.8) and (13.9),

$$\lim_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) = \frac{1}{2}\tilde{x}^T D\tilde{x} + c^T\tilde{x} = f(\tilde{x}, c, D), \tag{13.18}$$

$$A\tilde{x} \geq b. \tag{13.19}$$

From (13.19) it follows that $\tilde{x} \in \Delta(A, b)$. Hence

$$f(\tilde{x}, c, D) \geq \varphi(D, A, c, b).$$

Therefore, by (13.18),

$$\lim_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) \geq \varphi(D, A, c, b). \tag{13.20}$$

Combining (13.14) with (13.20) gives

$$\lim_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) = \varphi(D, A, c, b).$$

This shows that φ is continuous at (D, A, c, b). The proof is complete. $\square$

Example 13.1. Consider the problem $QP(D, A, c, b)$ where $m = 3, \ n = 2$,

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -2 \end{bmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It can be verified that $\varphi(D, A, c, b) = 0, \ \mathrm{Sol}(D, A, 0, 0) = \{0\}$, and the system $Ax \geq b$ is regular. By Theorem 13.1, φ is continuous at (D, A, c, b).

Example 13.2. Consider the problem $QP(D, A, c, b)$ where $m = n = 1, \ D = [1], \ A = [0], \ c = (1), \ b = (0)$. It can be shown that

$\varphi(D, A, c, b) = 0$, and the system $Ax \geq b$ is irregular. By Theorem 13.1, φ is not continuous at (D, A, c, b).

Remark 13.2. If $\Delta(A, b)$ is nonempty then $\Delta(A, 0)$ is the recession cone of $\Delta(A, b)$. By definition, $\mathrm{Sol}(D, A, 0, 0)$ is the solution set of the problem $QP(D, A, 0, 0)$. So, verifying the assumption $\mathrm{Sol}(D, A, 0, 0) = \{0\}$ is equivalent to solving one special QP problem.

Now we study the continuity of the optimal value function $\varphi(\cdot)$ at a point where its value is infinity. Let $\alpha \in \{+\infty, -\infty\}$ and $\varphi(D, A, c, b) = \alpha$. We say that $\varphi(\cdot)$ is continuous at (D, A, c, b) if, for every sequence $\{(D^k, A^k, c^k, b^k)\} \subset \Omega$ converging to (D, A, c, b),

$$\lim_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) = \alpha.$$

The next theorem characterizes the continuity of φ at a point $\omega = (D, A, c, b)$ where φ has the value $-\infty$.

Theorem 13.2. *Let $(D, A, c, b) \in \Omega$ and $\varphi(D, A, c, b) = -\infty$. Then, the optimal value function φ is continuous at (D, A, c, b) if and only if the system $Ax \geq b$ is regular.*

Proof. Suppose that $\varphi(D, A, c, d) = -\infty$ and φ is continuous at (D, A, c, b) but the system $Ax \geq b$ is irregular. By Remark 13.1, there exists a sequence $\{(A^k, b^k)\}$ in $R^{m\times n} \times R^m$ converging to (A, b) such that, for every k, the system $A^k x \geq b^k$ has no solutions. Since $\Delta(A^k, b^k) = \emptyset$, $\varphi(D, A^k, c, b^k) = +\infty$ for every k. Therefore, $\lim_{k\to\infty} \varphi(D, A^k, c, b^k) = +\infty$. On the other hand, since φ is continuous at (D, A, c, b) and since

$$(D, A^k, c, b^k) \longrightarrow (D, A, c, b) \quad \text{as } k \to \infty,$$

we obtain

$$+\infty = \lim_{k\to\infty} \varphi(D, A^k, c, b^k) = \varphi(D, A, c, b) = -\infty,$$

a contradiction. Thus $Ax \geq b$ must be a regular system.

Conversely, suppose that $\varphi(D, A, c, d) = -\infty$ and the system $Ax \geq b$ is regular. Let $\{(D^k, A^k, c^k, b^k)\} \subset \Omega$ be a sequence converging to (D, A, c, b). By the assumption, $\varphi(D, A, c, b) = -\infty$, hence there is a sequence $\{x^i\}$ in R^n such that $Ax^i \geq b$ and

$$f(x^i, c, D) \longrightarrow -\infty \quad \text{as} \quad i \to \infty. \tag{13.21}$$

By Lemma 13.1, for every i, there exists a sequence $\{y^{i_k}\}$ in R^n with the property that

$$A^k y^{i_k} \geq b^k, \tag{13.22}$$

$$\lim_{k\to\infty} y^{i_k} = x^i. \tag{13.23}$$

By (13.22),

$$\varphi(D^k, A^k, c^k, b^k) \leq \frac{1}{2}(y^{i_k})^T D^k y^{i_k} + (c^k)^T y^{i_k}. \tag{13.24}$$

From (13.23) and (13.24) it follows that

$$\limsup_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) \leq \frac{1}{2}(x^i)^T D x^i + c^T x^i. \tag{13.25}$$

Combining (13.25) with (13.21), we obtain

$$\limsup_{k\to\infty} \varphi(D^k, A^k, c_, b^k) = -\infty.$$

This implies that

$$\lim_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) = -\infty = \varphi(D, A, c, b).$$

Thus φ is continuous at (D, A, c, b). The proof is complete. $\square$

The following theorem characterizes the continuity of φ at a point $\omega = (D, A, c, b)$ where φ has the value $+\infty$.

Theorem 13.3. *Let $(D, A, c, b) \in \Omega$ and $\varphi(D, A, c, b) = +\infty$. Then, the optimal value function φ is continuous at (D, A, c, b) if and only if* $\mathrm{Sol}(D, A, 0, 0) = \{0\}$.

Proof. Suppose that $\varphi(D, A, c, b) = +\infty$ and that φ is continuous at (D, A, c, b) but $\mathrm{Sol}(D, A, 0, 0) \neq \{0\}$. Then there exists a nonzero vector $\bar{x} \in R^n$ such that

$$A\bar{x} \geq 0, \quad \bar{x}^T D \bar{x} \leq 0.$$

Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$. We define a matrix $M \in R^{m\times n}$ by setting $M = [m_{ij}]$, where

$$m_{ij} = \bar{x}_j \quad \text{for } 1 \leq i \leq m,\ 1 \leq j \leq n.$$

Let

$$D^k = D - \frac{1}{k}E, \quad A^k = A + \frac{1}{k}M,$$

where E is the unit matrix in $R^{n\times n}$. Consider the sequence

$$\{(D^k, A^k, c, b)\}.$$

A simple computation shows that

$$A^k\bar{x} > 0 \quad \text{for every } k.$$

By Lemma 13.2, for every k the system $A^kx \geq b$ is regular. Let z be a solution of the system $A^kx \geq b$. Since $A^k\bar{x} > 0$ and

$$\bar{x}^T D^k \bar{x} = \bar{x}^T D\bar{x} - \frac{\bar{x}^T\bar{x}}{k} < 0$$

for every k, we have

$$f(z + t\bar{x}, c, D^k) = \frac{1}{2}(z + t\bar{x})^T D^k(z + t\bar{x}) + c^T(z + t\bar{x}) \to -\infty$$

as $t \to \infty$. Since $z + t\bar{x} \in \Delta(A^k, b)$ for every k and for every $t > 0$, $\mathrm{Sol}(D^k, A^k, c, b) = \emptyset$. We have arrived at a contradiction, because φ is continuous at (D, A, c, b) and

$$-\infty = \lim_{k\to\infty} \varphi(D^k, A^k, c, b) = \varphi(D, A, c, b) = +\infty.$$

Conversely, assume that $\mathrm{Sol}(D, A, 0, 0) = \{0\}$ and

$$\{(D^k, A^k, c^k, b^k)\} \subset \Omega$$

is a sequence converging to (D, A, c, b). We shall show that

$$\liminf_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) = +\infty.$$

Suppose that $\liminf_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) < +\infty$. Without loss of generality we can assume that

$$\liminf_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) = \lim_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) < +\infty.$$

Then, there exist a positive integer k_1 and a constant $\gamma \geq 0$ such that

$$\varphi(D^k, A^k, c^k, b^k) \leq \gamma$$

for every $k \geq k_1$. As $\mathrm{Sol}(D, A, 0, 0) = \{0\}$, we can assume that there is an positive integer k_2 such that $\mathrm{Sol}(D^k, A^k, 0, 0) = \{0\}$ for every $k \geq k_2$. By Lemma 13.3 we can assume that

$$\mathrm{Sol}(D^k, A^k, c^k, b^k) \neq \emptyset$$

for every $k \geq k_2$. Hence there exists a sequence $\{x^k\}$ in R^n such that, for every $k \geq k_2$, we have

$$\varphi(D^k, A^k, c^k, b^k) = \frac{1}{2}(x^k)^T D^k x^k + (c^k)^T x^k \leq \gamma, \tag{13.26}$$

$$A^k x^k \geq b^k. \tag{13.27}$$

We now prove that $\{x^k\}$ is a bounded sequence. Suppose, contrary to our claim, that the sequence $\{x^k\}$ is unbounded. Without loss of generality we can assume that $\|x^k\| \neq 0$ for every k and that $\|x^k\| \to \infty$ as $k \to \infty$. Then the sequence $\{\|x^k\|^{-1}x^k\}$ has a convergent subsequence. We can assume that the sequence itself converges to a point $x^0 \in R^n$ with $\|x^0\| = 1$. By (13.27) we have

$$A^k \frac{x^k}{\|x^k\|} \geq \frac{b^k}{\|x^k\|},$$

hence

$$Ax^0 \geq 0. \tag{13.28}$$

By dividing both sides of the inequality in (13.26) by $\|x^k\|^2$ and taking the limits as $k \to \infty$, we get

$$(x^0)^T D x^0 \leq 0. \tag{13.29}$$

From (13.28) and (13.29) we deduce that $\mathrm{Sol}(D, A, 0, 0) \neq \{0\}$. This contradicts our assumption. Thus the sequence $\{x^k\}$ is bounded, and it has a convergent subsequence. Without loss of generality we can assume that $\{x^k\}$ converges to $\bar{x} \in R^n$. Letting $k \to \infty$, from (13.27) we obtain

$$A\bar{x} \geq b.$$

This means that $\Delta(A, b) \neq \emptyset$. We have arrived at a contradiction because $\varphi(D, A, c, b) = +\infty$. The proof is complete. $\square$

From Theorems 13.1–13.3 it follows that conditions (a), (b) in Theorem 13.1 are sufficient for the function $\varphi(\cdot)$ to be continuous at the given parameter value (D, A, c, b).

13.2 Semicontinuity of the Optimal Value Function

As it has been shown in the preceding section, continuity of the optimal value function holds under a special set of conditions. In some situations, only the upper semicontinuity or the lower semicontinuity of that function is required. So we wish to have simple sufficient conditions for the upper semicontinuity and the lower semicontinuity of φ at a given point. Such conditions are given in this section.

A sufficient condition for the upper semicontinuity of the function $\varphi(\cdot)$ at a given parameter value is given in the following theorem.

Theorem 13.4. *Let $(D, A, c, b) \in \Omega$. If the system $Ax \geq b$ is regular then $\varphi(\cdot)$ is upper semicontinuous at (D, A, c, b).*

Proof. As $Ax \geq b$ is regular, we have $\Delta(A, b) \neq \emptyset$. Hence

$$\varphi(D, A, c, b) < +\infty.$$

Let $\{(D^k, A^k, c^k, b^k)\} \subset \Omega$ be a sequence converging to (D, A, c, b). Since $\varphi(D, A, c, b) < +\infty$, there is a sequence $\{x^i\}$ in R^n such that $Ax^i \geq b$ and

$$f\left(x^i, c, D\right) = \frac{1}{2}\left(x^i\right)^T Dx^i + c^T x^i \longrightarrow \varphi(D, A, c, b) \quad \text{as } i \to \infty.$$

By Lemma 13.1 and by the regularity of the system $Ax \geq b$, for each i one can find a sequence $\{y^{i_k}\}$ in R^n such that $A^k y^{i_k} \geq b^k$ and

$$\lim_{k\to\infty} y^{i_k} = x^i.$$

Since $y^{i_k} \in \Delta(A^k, b^k)$,

$$\varphi(D^k, A^k, c^k, b^k) \leq f(y^i, c^k, D^k).$$

This implies that

$$\limsup_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) \leq f(x^i, c, D).$$

Taking limits in the last inequality as $i \to \infty$, we obtain

$$\limsup_{k\to\infty} \varphi(D^k, A^k, c^k, b^k) \leq \varphi(D, A, c, b).$$

We have proved that $\varphi(\cdot)$ is upper semicontinuous at (D, A, c, b). □

The next example shows that the regularity condition in Theorem 13.4 does not guarantee the lower semicontinuity of φ at (D, A, c, b).

Example 13.3. Consider the problem $QP(D,A,c,b)$ where $m = n = 1$, $D = [0]$, $A = [1]$, $c = (0), b = (0)$. It is clear that $Ax \geq 0$ is regular, $\mathrm{Sol}(D,A,c,b) = \Delta(A,b) = \{x : x \geq 0\}$, and $\varphi(D,A,c,b) = 0$. Consider the sequence $\{(D^k,A,c,b)\}$, where $D^k = D - \left[\frac{1}{k}\right]$. We have $\varphi(D^k,A,c,b) = -\infty$ for every k, so

$$\liminf_{k\to\infty} \varphi(D^k,A,c,b) < \varphi(D,A,c,b).$$

Thus φ is not lower semicontinuous at (D,A,c,b).

The following example is designed to show that the regularity condition in Theorem 13.4 is sufficient but not necessary for the upper semicontinuity of φ at (D,A,c,b).

Example 13.4. Choose a matrix $A \in R^{m\times n}$ and a vector $b \in R^m$ such that $\Delta(A,b) = \emptyset$ (then the system $Ax \geq b$ is irregular). Fix an arbitrary matrix $D \in R_S^{n\times n}$ and an arbitrary vector $c \in R^n$. Since $\varphi(D,A,c,b) = +\infty$, for any sequence $\{(D^k,A^k,c^k,b^k)\}$ converging to (D,A,c,b), we have

$$\limsup_{k\to\infty} \varphi(D^k,A^k,c^k,b^k) \leq \varphi(D,A,c,b).$$

Thus φ is upper semicontinuous at (D,A,c,b).

A sufficient condition for the lower semicontinuity of the function $\varphi(\cdot)$ is given in the following theorem.

Theorem 13.5. *Let* $(D,A,c,b) \in \Omega$. *If* $\mathrm{Sol}(D,A,0,0) = \{0\}$ *then* $\varphi(\cdot)$ *is lower semicontinuous at* (D,A,c,b).

Proof. Assume that $\mathrm{Sol}(D,A,0,0) = \{0\}$. Let

$$\{(D^k,A^k,c^k,b^k)\} \subset \Omega$$

be a sequence converging to (D,A,c,b). We claim that

$$\liminf_{k\to\infty} \varphi(D^k,A^k,c^k,b^k) \geq \varphi(D,A,c,b).$$

Indeed, suppose that

$$\liminf_{k\to\infty} \varphi(D^k,A^k,c^k,b^k) < \varphi(D,A,c,b).$$

Without loss of generality we can assume that

$$\liminf_{k\to\infty} \varphi(D^k,A^k,c^k,b^k) = \lim_{k\to\infty} \varphi(D^k,A^k,c^k,b^k).$$

Then there exist an index k_1 and a real number γ such that $\gamma < \varphi(D, A, c, b)$ and

$$\varphi(D^k, A^k, c^k, b^k) \le \gamma \quad \text{for every} \ \ k \ge k_1.$$

Since $\varphi(D^k, A^k, c^k, b^k) < +\infty$, we must have $\Delta(A^k, b^k) \ne \emptyset$ for every $k \ge k_1$. Since $\mathrm{Sol}(D, A, 0, 0) = \{0\}$, there exists an integer $k_2 \ge k_1$ such that

$$\mathrm{Sol}(D^k, A^k, 0, 0) = \{0\}$$

for every $k \ge k_2$. As $\Delta(A^k, b^k) \ne \emptyset$, by Lemma 13.3 we have $\mathrm{Sol}(D^k, A^k, c^k, b^k) \ne \emptyset$ for every $k \ge k_2$. Hence there exists a sequence $\{x^k\}$ such that we have $A^k x^k \ge b^k$ for every $k \ge k_2$, and

$$\frac{1}{2}(x^k)^T D^k x^k + (c^k)^T x^k = \varphi(D^k, A^k, c^k, b^k) \le \gamma.$$

The sequence $\{x^k\}$ must be bounded. Indeed, if $\{x^k\}$ is unbounded then, without loss of generality, we can assume that $\|x^k\| \ne 0$ for every k and $\|x^k\| \to \infty$ as $k \to \infty$. Then the sequence $\{\|x^k\|^{-1} x^k\}$ has a convergent subsequence. We can assume that this sequence itself converges to a vector $v \in R^n$ with $\|v\| = 1$. Since

$$A^k \frac{x^k}{\|x^k\|} \ge \frac{b^k}{\|x^k\|} \quad \text{for every } k \ge k_2,$$

we have $Av \ge 0$. On the other hand, since for each $k \ge k_2$ it holds

$$\frac{1}{2}\frac{(x^k)^T}{\|x^k\|} D^k \frac{x^k}{\|x^k\|} + (c^k)^T \frac{x^k}{\|x^k\|} \le \frac{\gamma}{\|x^k\|^2},$$

we deduce that

$$v^T D v \le 0.$$

Combining all the above we get $v \in \mathrm{Sol}(D, A, 0, 0) \setminus \{0\}$, a contradiction. We have thus proved that the sequence $\{x^k\}$ is bounded. Without loss of generality we can assume that $x^k \to \bar{x} \in R^n$. Since $A^k x^k \ge b^k$ for every k, we get $A\bar{x} \ge b$. Since

$$\frac{1}{2}(x^k)^T D^k x^k + (c^k)^T x^k \le \gamma,$$

we have

$$f(\bar{x}, c, D) = \frac{1}{2}\bar{x}^T D \bar{x} + c^T \bar{x} \le \gamma.$$

As $\gamma < \varphi(D, A, c, b)$, we see that $f(\bar{x}, c, D) < \varphi(D, A, c, b)$. This is an absurd because $\bar{x} \in \Delta(A, b)$. We have thus proved that $\varphi(\cdot)$ is lower semicontinuous at (D, A, c, b). $\square$

The next example shows that the condition $\mathrm{Sol}(D, A, 0, 0) = \{0\}$ in Theorem 13.5 does not guarantee the upper semicontinuity of φ at (D, A, c, b).

Example 13.5. Consider the problem $QP(D, A, c, b)$ where $m = n = 1$, $D = [1]$, $A = [0]$, $c = (0), b = (0)$. It is clear that $\mathrm{Sol}(D, A, 0, 0) = \{0\}$. Consider the sequence $\{(D, A, c, b^k)\}$, where $b^k = (\frac{1}{k})$. We have $\varphi(D, A, c, b) = 0$ and $\varphi(D, A, c, b^k) = +\infty$ for all k (because $\Delta(A, b^k) = \emptyset$ for all k). Therefore

$$\limsup_{k \to \infty} \varphi(D, A, c, b^k) = +\infty > 0 = \varphi(D, A, c, b).$$

Thus φ is not upper semicontinuous at (D, A, c, b).

The condition $\mathrm{Sol}(D, A, 0, 0) = \{0\}$ in Theorem 13.5 is sufficient but not necessary for the lower semicontinuity of φ at (D, A, c, b).

Example 13.6. Consider the problem $QP(D, A, c, b)$ where $m = n = 1$, $D = [-1]$, $A = [1]$, $c = (1), b = (0)$. It is clear that $\mathrm{Sol}(D, A, 0, 0) = \emptyset$. Since $\varphi(D, A, c, b) = -\infty$, for any sequence $\{(D^k, A^k, c^k, b^k)\}$ converging to (D, A, c, b), we have

$$\liminf_{k \to \infty} \varphi(D^k, A^k, c^k, b^k) \geq \varphi(D, A, c, b).$$

Thus φ is lower semicontinuous at (D, A, c, b).

13.3 Commentaries

The results presented in this chapter are due to Tam (2002).

Lemma 13.1 is a well-known fact (see, for example, Robinson (1975), Theorem 1, and Bank et al. (1982), Theorem 3.1.5).

In Best and Chakravarti (1990) and Best and Ding (1995) the authors have considered convex quadratic programming problems and obtained some results on the continuity and differentiability of the optimal value function of the problem as a function of a parameter specifying the magnitude of the perturbation. In Auslender and Coutat (1996), similar questions for the case of linear-quadratic programming problems were investigated. Continuity and Lipschitzian properties of the function $\varphi(D, A, \cdot, \cdot)$ (the matrices D and A are fixed) were studied in Bank et al. (1982), Bank and Hansel (1984), Klatte (1985), Rockafellar and Wets (1998).

We have considered indefinite QP problems and obtained several results on the continuity, the upper and lower semicontinuity of the optimal value function φ at a given point ω. In comparison with the preceding results of Best and Chakravarti (1990), Best and Ding (1995), the advantage here is that the quadratic objective function is allowed to be indefinite.

The obtained results can be used for analyzing algorithms for solving the indefinite QP problems.

Chapter 14

Directional Differentiability of the Optimal Value Function

In this chapter we establish an explicit formula for computing the directional derivative of the optimal value function in a general parametric QP programming problem. We will consider one illustrative example to see how the formula works for concrete QP problems.

In Section 14.1 we prove several lemmas. In Section 14.2 we introduce *condition* (G) and describe a general situation where (G) holds. Section 14.3 is devoted to proving the above-mentioned formula for computing the directional derivative of the optimal value function in indefinite QP problems. In the same section, the obtained result is compared with the corresponding results on differential stability in nonlinear programming of Auslender and Cominetti (1990), and Minchenko and Sakolchik (1996).

14.1 Lemmas

Consider the general quadratic programming problem (13.1) which is abbreviated to $QP(D, A, c, b)$. The problem depends on the parameter $\omega = (D, A, c, b) \in \Omega$, where

$$\Omega := R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m.$$

As in Chapter 13, the solution set of this problem will be denoted by $\mathrm{Sol}(D, A, c, b)$, and its optimal value function $\varphi : \Omega \longrightarrow \overline{R}$ is given by

$$\varphi(\omega) = \inf\{f(x,c,D) \,:\, x \in \Delta(A,b)\}.$$

The proofs of Theorem 14.1 and Theorem 14.2, the main results in this chapter, are based on some lemmas established in the present section.

Let $\omega = (D, A, c, b)$ and $\omega^0 = (D^0, A^0, c^0, b^0)$ be two elements of Ω. Denote

$$\begin{aligned}
\omega + t\omega^0 &= (D + tD^0, A + tA^0, c + tc^0, b + tb^0),\\
\varphi^+(\omega;\omega^0) &= \limsup_{t\downarrow 0} \frac{\varphi(\omega + t\omega^0) - \varphi(\omega)}{t},\\
\varphi^-(\omega;\omega^0) &= \liminf_{t\downarrow 0} \frac{\varphi(\omega + t\omega^0) - \varphi(\omega)}{t}.
\end{aligned}$$

If $\varphi^+(\omega;\omega^0) = \varphi^-(\omega;\omega^0)$ then the optimal value function $\varphi(\cdot)$ is directionally differentiable at ω in direction ω^0 (see Definition 1.8). The common value is denoted by $\varphi'(\omega;\omega^0)$ which is the directional derivative of φ at ω in direction ω^0. We have

$$\varphi'(\omega;\omega^0) = \lim_{t\downarrow 0} \frac{\varphi(\omega + t\omega^0) - \varphi(\omega)}{t}.$$

For every $\bar{x} \in \Delta(A,b)$, we set

$$I = \alpha(\bar{x}) = \{i \,:\, (A\bar{x})_i = b_i\},$$

and define

$$\begin{aligned}
F(\bar{x},\omega,\omega^0) = \{v \in R^n \,:\, &\exists\varepsilon > 0 \quad \text{such that}\\
&\bar{x} + tv \in \Delta(A + tA^0, b + tb^0) \quad \text{for every } t \in [0,\varepsilon]\},
\end{aligned}$$

$$R(\bar{x},\omega,\omega^0) = \begin{cases} R^n & \text{if } I = \emptyset\\ \{v \in R^n \,:\, A_I v + A_I^0\bar{x} - b_I^0 \geq 0\} & \text{if } I \neq \emptyset. \end{cases}$$

The following lemma is originated from Seeger (1988), Auslender and Cominetti (1990).

Lemma 14.1. *If the system $Ax \geq b$ is regular, then*

$$\emptyset \neq \operatorname{int} R(\bar{x},\omega,\omega^0) \subset F(\bar{x},\omega,\omega^0) \subset R(\bar{x},\omega,\omega^0) \tag{14.1}$$

for every $\bar{x} \in \Delta(A,b)$.

Proof. Let $\bar{x} \in \Delta(A, b)$, $I = \alpha(\bar{x}) = \{i : (A\bar{x})_i = b_i\}$. If $I = \emptyset$ then $A\bar{x} > b$. Thus, for every $v \in R^n$ there is an $\varepsilon = \varepsilon(v) > 0$ such that for each $t \in [0, \varepsilon]$ we have

$$A\bar{x} + t(Av + A^0\bar{x} - b^0 + tA^0v) \geq b.$$

The above inequality is equivalent to the following one

$$(A + tA^0)(\bar{x} + tv) \geq b + tb^0.$$

Hence $\bar{x} + tv \in \Delta(A + tA^0, b + tb^0)$ for each $t \in [0, \varepsilon]$. This implies that $F(\bar{x}, \omega, \omega^0) = R^n$. By definition, in this case we also have $R(\bar{x}, \omega, \omega^0) = R^n$. Therefore

$$F(\bar{x}, \omega, \omega^0) = R^n = R(\bar{x}, \omega, \omega^0),$$

and we have (14.1). Consider the case where $I \neq \emptyset$. We first show that

$$\mathrm{int}R(\bar{x}, \omega, \omega^0) \neq \emptyset.$$

Since $Ax \geq b$ is a regular system, there exists $x^0 \in R^n$ such that $Ax^0 > b$. Then we have

$$A_I x^0 > b_I.$$

As $A_I\bar{x} = b_I$ and $A_I x^0 > b_I$, we have

$$A_I(x^0 - \bar{x}) > 0.$$

Putting $\hat{v} = x^0 - \bar{x}$, we get

$$A_I\hat{v} > 0.$$

By Lemma 13.2, the inequality system (of the unknown v)

$$A_I v \geq b_I^0 - A_I^0\bar{x}$$

is regular, hence there exists $\bar{v} \in R^n$ such that

$$A_I\bar{v} > b_I^0 - A_I^0\bar{x}.$$

This proves that $\bar{v} \in \mathrm{int}R(\bar{x}, \omega, \omega^0)$, therefore $\mathrm{int}R(\bar{x}, \omega, \omega^0) \neq \emptyset$.

We now prove that

$$\mathrm{int}R(\bar{x}, \omega, \omega^0) \subset F(\bar{x}, \omega, \omega^0).$$

Suppose that $v \in \mathrm{int}R(\bar{x}, \omega, \omega^0)$. We have

$$A_I v + A_I^0 \bar{x} - b_I^0 > 0.$$

Hence there is $\varepsilon_1 > 0$ such that for each $t \in [0, \varepsilon_1]$ one has

$$A_I v + A_I^0 \bar{x} - b_I^0 + t A_I^0 v > 0.$$

Then, for each $t \in [0, \varepsilon_1]$,

$$t(A_I v + A_I^0 \bar{x} - b_I^0 + t A_I^0 v) \geq 0. \tag{14.2}$$

As $A_i \bar{x} > b_i$ for every $i \in \{1, \ldots, m\} \setminus I$, one can find $\varepsilon_2 > 0$ such that for each $t \in [0, \varepsilon_2]$ it holds

$$A_i \bar{x} + t(A_i v + A_i^0 \bar{x} - b_i^0 + t A_i^0 v) \geq b_i \tag{14.3}$$

for every $i \in \{1, \ldots, m\} \setminus I$. Let $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. It follows from (14.2) and (14.3) that

$$A\bar{x} + t(Av + A^0 \bar{x} - b^0 + t A^0 v) \geq b \tag{14.4}$$

for every $t \in [0, \varepsilon]$. This implies that

$$\bar{x} + tv \in \Delta(A + tA^0, b + tb^0)$$

for every $t \in [0, \varepsilon]$. Hence $v \in F(\bar{x}, \omega, \omega^0)$, and we have

$$\mathrm{int}R(\bar{x}, \omega, \omega^0) \subset F(\bar{x}, \omega, \omega^0).$$

Finally, we shall prove that

$$F(\bar{x}, \omega, \omega^0) \subset R(\bar{x}, \omega, \omega^0).$$

Take any $v \in F(\bar{x}, \omega, \omega^0)$. By definition, there is an $\varepsilon > 0$ such that for each $t \in [0, \varepsilon]$ we have

$$(A_I + tA_I^0)(\bar{x} + tv) \geq b + tb^0.$$

Consequently,

$$A_I \bar{x} + t(A_I v + A_I^0 \bar{x} - b_I^0 + t A_I^0 v) \geq b_I$$

for every $t \in [0, \varepsilon]$. As $A_I \bar{x} = b_I$, we have

$$t(A_I v + A_I^0 \bar{x} - b_I^0 + t A_I^0 v) \geq 0$$

for each $t \in [0, \varepsilon]$. Hence, for every $t \in (0, \varepsilon]$,

$$A_I v + A_I^0 \bar{x} - b_I^0 + t A_I^0 v \geq 0.$$

Letting $t \to 0$, we obtain

$$A_I v + A_I^0 \bar{x} - b_I^0 \geq 0.$$

This shows that $v \in R(\bar{x}, \omega, \omega^0)$, hence $F(\bar{x}, \omega, \omega^0) \subset R(\bar{x}, \omega, \omega^0)$. We have thus shown that the inclusions in (14.1) are valid. The proof is complete. $\square$

If $\bar{x} \in \mathrm{Sol}(D, A, c, b)$, then there exists a Lagrange multiplier $\lambda \in R^m$ such that

$$\begin{aligned} & D\bar{x} - A^T\lambda + c = 0, \\ & A\bar{x} \geq b, \quad \lambda \geq 0, \\ & \lambda^T(A\bar{x} - b) = 0 \end{aligned}$$

The set of all the Lagrange multipliers corresponding to $\bar{x}$ is denoted by $\Lambda(\bar{x}, \omega)$, where $\omega = (D, A, c, b)$.

The forthcoming result is well known in nonlinear programming (see, for instance, Gauvin (1977) and Dien (1985)). For the sake of completeness, we give a proof for the case of QP problems.

Lemma 14.2. *If the system $Ax \geq b$ is regular, then for every $\bar{x} \in \mathrm{Sol}(D, A, c, b)$ the set $\Lambda(\bar{x}, \omega)$ is compact.*

Proof. Let $\omega = (D, A, c, b)$. Suppose that there is $\bar{x} \in \mathrm{Sol}(D, A, c, b)$ such that $\Lambda(\bar{x}, \omega)$ is noncompact. Then there exists a sequence $\{\lambda^k\}$ in R^m such that $\|\lambda^k\| \neq 0$,

$$D\bar{x} - A^T\lambda^k + c = 0, \tag{14.5}$$

$$\lambda^k \geq 0, \tag{14.6}$$

$$(\lambda^k)^T(A\bar{x} - b) = 0, \tag{14.7}$$

for every k, and $\|\lambda^k\| \to \infty$ as $k \to \infty$. Without loss of generality we can assume that $\{\|\lambda^k\|^{-1}\lambda^k\}$ converges to $\bar{\lambda}$ with $\|\bar{\lambda}\| = 1$. Dividing each expression in (14.5)–(14.7) by $\|\lambda^k\|$ and taking the limits as $k \to \infty$, we get

$$A^T\bar{\lambda} = 0, \quad \bar{\lambda} \geq 0, \quad \bar{\lambda}^T(A\bar{x} - b) = 0. \tag{14.8}$$

Since $\bar{\lambda}^T A\bar{x} = \bar{x}^T(A^T\bar{\lambda}) = 0$, from (14.8) it follows that

$$A^T\bar{\lambda} = 0, \quad \bar{\lambda} \geq 0, \quad \bar{\lambda}^T b = 0.$$

For every $t > 0$, we set $b_t = b + t\bar{\lambda}$. Since $\bar{\lambda}^T\bar{\lambda} = \|\bar{\lambda}\|^2 = 1$,

$$\bar{\lambda}^T b_t = \bar{\lambda}^T b + t\bar{\lambda}^T\bar{\lambda} = \bar{\lambda}^T b + t = t.$$

Consequently, for every $t > 0$, $\bar{\lambda}$ is a solution of the following system

$$A^T\lambda = 0, \quad \lambda \geq 0, \quad \lambda^T b_t > 0.$$

Hence, for every $t > 0$, the system $Ax \geq b_t$ has no solutions (see Cottle et al. (1992), Theorem 2.7.8). Since $\Delta(A, b) \neq \emptyset$ and $\|b_t - b\| = t \to 0$ as $t \to 0$, the system $Ax \geq b$ is irregular (see Mangasarian (1980), Lemma 2.1), contrary to our assumption. The proof is complete. $\square$

Lemma 14.3 (cf. Auslender and Cominetti (1990), Lemma 2). *If the system $Ax \geq b$ is regular and $\bar{x} \in \mathrm{Sol}(D, A, c, b)$ then*

$$\inf_{v \in R(\bar{x},\omega,\omega^0)} (D\bar{x} + c)^T v = \max_{\lambda \in \Lambda(\bar{x},\omega)} (b^0 - A^0\bar{x})^T\lambda, \tag{14.9}$$

where $\Lambda(\bar{x}, \omega)$ stands for the Lagrange multiplier set corresponding to $\bar{x}$.

Proof. Let $\bar{x} \in \mathrm{Sol}(D, A, c, b)$. If $I = \alpha(\bar{x}) = \{i : (A\bar{x})_i = b_i\}$ is empty then, by definition, $R(\bar{x}, \omega, \omega^0) = R^n$. As $\bar{x} \in \mathrm{Sol}(D, A, c, b)$ and $A\bar{x} > b$, Theorem 3.3 applied to $\bar{x}$ shows that $(D\bar{x} + c)^T v = 0$ for every $v \in R^n$. Then we have

$$\inf_{v \in R(\bar{x},\omega,\omega^0)} (D\bar{x} + c)^T v = 0.$$

Again, by the just cited first-order necessary optimality condition, for every $\bar{x}$ we have $\Lambda(\bar{x}, \omega) \neq \emptyset$. Since $A\bar{x} > b$, $\Lambda(\bar{x}, \omega) = \{0\}$. Therefore

$$\max_{\lambda \in \Lambda(\bar{x},\omega)} (b^0 - A^0\bar{x})^T\lambda = 0.$$

Thus, in the case $I = \emptyset$ the assertion of the lemma is valid. We now consider the case where $I = \alpha(\bar{x}) = \{i : (A\bar{x})_i = b_i\} \neq \emptyset$. We have

$$\inf_{v \in R(\bar{x},\omega,\omega^0)} (D\bar{x} + c)^T v = \inf\{(D\bar{x} + c)^T v : v \in R^n, A_I v \geq b_I^0 - A_I^0\bar{x}\}.$$

Consider a pair of dual linear programs

$$(P) \qquad \begin{cases} (D\bar{x} + c)^T v \longrightarrow \min \\ v \in R^n, \quad A_I v \geq b_I^0 - A_I^0\bar{x} \end{cases}$$

and

$$(P') \qquad \begin{cases} (b_I^0 - A_I^0\bar{x})^T\lambda_I \longrightarrow \max \\ \lambda_I \in R^{|I|}, \quad A_I^T\lambda_I = D\bar{x} + c, \quad \lambda_I \geq 0 \end{cases}$$

where $|I|$ is the number of the elements of I. From the definition of $\Lambda(\bar{x}, \omega)$ it follows that if λ_I is a feasible point of (P') then $(\lambda_I, 0_J) \in \Lambda(\bar{x}, \omega)$, where $J = \{1, \cdots, m\} \setminus I$. Conversely, if $\lambda = (\lambda_I, \lambda_J) \in \Lambda(\bar{x}, \omega)$ then $\lambda_J = 0_J$. The regularity of the system $Ax \geq b$ and Lemma 14.2 imply that $\Lambda(\bar{x}, \omega)$ is nonempty and compact. Therefore, by the above observation, the feasible domain of (P') is nonempty and compact. By the duality theorem in linear programming (see Theorem 1.10(iv)), the optimal values of (P) and (P') are both finite and equal to each other. Therefore

$$\begin{aligned}
&\inf\{(D\bar{x}+c)^T v \,:\, v \in R(\bar{v}, \omega, \omega^0)\} \\
&= \inf\{(D\bar{x}+c)^T v \,:\; v \in R^n, A_I v \geq b_I^0 - A_I^0\bar{x}\} \\
&= \max\{(b_I^0 - A_I^0\bar{x})^T\lambda_I \,:\; \lambda_I \in R^{|I|},\ \lambda_I \geq 0,\ A_I^T\lambda_I = D\bar{x} + c\} \\
&= \max\{(b^0 - A^0\bar{x})^T\lambda \,:\; \lambda \in R^m,\ \lambda = (\lambda_I, 0_J) \geq 0, \\
&\qquad A^T\lambda = D\bar{x} + c\} \\
&= \max\nolimits_{\lambda \in \Lambda(\bar{x}, \omega)} (b^0 - A_0\bar{x})^T\lambda.
\end{aligned}$$

Formula (14.9) is proved. □

Lemma 14.4. *Suppose that $\omega^k = (D^k, A^k, c^k, b^k)$, $k \in N$, is a sequence in Ω converging to $\omega = (D, A, c, b)$, $\{x^k\}$ is a sequence in R^n such that $x^k \in \mathrm{Sol}(D^k, A^k, c^k, b^k)$ for every k. If the system $Ax \geq b$ is regular and $\mathrm{Sol}(D, A, 0, 0) = \{0\}$ then there exists a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ such that $\{x^{k_i}\}$ converges to $\bar{x} \in \mathrm{Sol}(D, A, c, b)$ as $i \to \infty$.*

Proof. Suppose that $Ax \geq b$ is a regular system and $\mathrm{Sol}(D, A, 0, 0) = \{0\}$. We have

$$A^k x^k \geq b^k. \tag{14.10}$$

Take $x \in \Delta(A, b)$. Then there exists a sequence $\{y^k\}$ in R^n tending to x such that

$$A^k y^k \geq b^k \quad \text{for every} \quad k \tag{14.11}$$

(see Lemma 13.1). The inequality in (14.11) shows that $y^k \in \Delta(A^k, b^k)$. Since $x^k \in \mathrm{Sol}(D^k, A^k, c^k, b^k)$,

$$\frac{1}{2}(x^k)^T D^k x^k + (c^k)^T x^k \leq \frac{1}{2}(y^k)^T D^k y^k + (c^k)^T y^k. \tag{14.12}$$

We claim that the sequence $\{x^k\}$ is bounded. Suppose for a while that $\{x^k\}$ is unbounded. Then, without loss of generality, we may assume that $\|x^k\| \to \infty$ as $k \to \infty$ and $\|x^k\| \neq 0$ for every k. So the sequence $\{\|x^k\|^{-1}x^k\}$ has a convergent subsequence. We may assume that the sequence $\{\|x^k\|^{-1}x^k\}$ itself converges to $\hat{x} \in R^n$ with $\|\hat{x}\| = 1$. From (14.10) we have

$$A^k \frac{x^k}{\|x^k\|} \geq \frac{b^k}{\|x^k\|}.$$

Letting $k \to \infty$, we obtain

$$A\hat{x} \geq 0. \tag{14.12}$$

Dividing both sides of (14.12) by $\|x^k\|^2$ and taking limit as $k \to \infty$, we obtain

$$\hat{x}^T D\hat{x} \leq 0. \tag{14.14}$$

Combining (14.13) and (14.14), we have $\mathrm{Sol}(D, A, 0, 0) \neq \{0\}$, contrary to our assumptions. Thus the sequence $\{x^k\}$ is bounded and it has a convergent subsequence, say, $\{x^{k_i}\}$. Suppose that $\{x^{k_i}\}$ converges to $\bar{x}$. From (14.12) we have

$$\frac{1}{2}(x^{k_i})^T D^{k_i} x^{k_i} + (c^{k_i})^T x^{k_i} \leq \frac{1}{2}(y^{k_i})^T D^{k_i} y^{k_i} + (c^{k_i})^T y^{k_i}. \tag{14.15}$$

From (14.10) we have

$$A^{k_i} x^{k_i} \geq b^{k_i}. \tag{14.16}$$

Taking limits in (14.15) and (14.16) as $i \to \infty$, we obtain

$$\frac{1}{2}\bar{x}^T D\bar{x} + c^T\bar{x} \leq \frac{1}{2}x^T Dx + c^T x, \tag{14.17}$$

$$A\bar{x} \geq b. \tag{14.18}$$

As $x \in \Delta(A, b)$ is arbitrarily chosen, (14.17) and (14.18) yield $\bar{x} \in \mathrm{Sol}(D, A, c, b)$. The lemma is proved. $\square$

14.2 Condition (G)

Let $\omega = (D, A, c, b) \in \Omega$ be a given parameter value and $\omega^0 = (D^0, A^0, c^0, b^0) \in \Omega$ be a given direction. Consider the following condition which we call *condition* (G):

For every sequence $\{t^k\}$, $t^k \downarrow 0$, *for every sequence* $\{x^k\}$,

$$x^k \longrightarrow \bar{x} \in \mathrm{Sol}(D, A, c, b),$$

where $x^k \in \mathrm{Sol}(\omega + t^k\omega^0)$ *for each* k, *the following inequality is satisfied*

$$\liminf_{k\to\infty} \frac{(x^k - \bar{x})^T D(x^k - \bar{x})}{t^k} \geq 0.$$

Remark 14.1. If D is a positive semidefinite matrix, then condition (G) holds. Indeed, if D is positive semidefinite then $(x^k - \bar{x})^T D(x^k - \bar{x}) \geq 0$, hence the inequality in (G) is satisfied.

Remark 14.2. If the system $Ax \geq b$ is regular then (G) is weaker than the condition saying that the $(SOSC)_u$ property introduced in Auslender and Cominetti (1990) (applied to QP problems) holds at every $\bar{x} \in \mathrm{Sol}(D, A, c, b)$. It is interesting to note that if the system $Ax \geq b$ is regular then (G) is also weaker than the condition (H3) introduced by Minchenko and Sakolchik (1996) (applied to QP problems). There exist many QP problems where the conditions $(SOSC)_u$ and (H3) do not hold but condition (G) is satisfied. A detailed comparison of our results with the ones in Auslender and Cominetti (1990) and Minchenko and Sakolchik (1996) will be given in Section 14.3.

Now we describe a general situation where (G) is fulfilled.

Theorem 14.1. *If* $Ax \geq b$ *is a regular system and every solution* $\bar{x} \in \mathrm{Sol}(D, A, c, b)$ *is a locally unique solution of problem* (13.1), *then condition* (G) *is satisfied.*

Proof. From the statement of (G) it is obvious that the condition is satisfied if $\mathrm{Sol}(D, A, c, b) = \emptyset$. Consider the case where $\mathrm{Sol}(D, A, c, b) \neq \emptyset$. For any given $\bar{x} \in \mathrm{Sol}(D, A, c, b)$ we set $I = \alpha(\bar{x}) = \{i : (A\bar{x})_i = b_i\}$ and

$$F_{\bar{x}} = \{v \in R^n : (Av)_i \geq 0 \text{ for every } i \in I\}.$$

For every $\bar{x} \in \mathrm{Sol}(D, A, c, b)$, Theorem 3.7 shows that the following conditions are equivalent:

(a) $\bar{x}$ is a locally unique solution of problem (13.1),

(b) for every $v \in F_{\bar{x}} \setminus \{0\}$, if $(D\bar{x} + c)^T v = 0$ then $v^T Dv > 0$.

We shall use the above equivalence to prove our theorem. Suppose, contrary to our claim, that (G) does not hold. Then there exist a sequence $\{t^k\}$, $t^k \downarrow 0$, and a sequence $\{x^k\}$, $x^k \to \bar{x} \in$

$\mathrm{Sol}(D, A, c, b)$, $x^k \in \mathrm{Sol}(D + t^k D^0, A + t^k A^0, c + t^k c^0, b + t^k b^0)$ for every k, such that

$$\lim_{k\to\infty} \frac{(x^k - \bar{x})^T D(x^k - \bar{x})}{t^k} < 0. \tag{14.19}$$

By taking a subsequence if necessary, we can assume that

$$(x^k - \bar{x})^T D(x^k - \bar{x}) < 0, \quad \|x^k - \bar{x}\| \neq 0 \quad \text{for every } k, \tag{14.20}$$

and

$$\lim_{k\to\infty} \frac{\|x^k - \bar{x}\|}{t^k} = +\infty. \tag{14.21}$$

Then the sequence $\{\|x^k - \bar{x}\|^{-1}(x^k - \bar{x})\}$ has a convergent subsequence. Without loss of generality, we may assume that $\{\|x^k - \bar{x}\|^{-1}(x^k - \bar{x})\}$ converges to some $v \in R^n$ with $\|v\| = 1$. Dividing both sides of the inequality in (14.20) by $\|x^k - \bar{x}\|^2$ and letting $k \to \infty$, we get

$$v^T D v \leq 0. \tag{14.22}$$

Since $x^k \in \mathrm{Sol}(D + t^k D^0, A + t^k A^0, c + t^k c^0, b + t^k b^0)$, we have

$$(A_I + t^k A_I^0)x^k \geq b_I + t^k b_I^0,$$

where $I = \{i : (A\bar{x})_i = b_i\}$. Since $b_I = A_I\bar{x}$,

$$A_I(x^k - \bar{x}) \geq t^k(b_I^0 - A_I^0 x^k).$$

Dividing both sides of the inequality above by $\|x^k - \bar{x}\|$, taking account of (14.21) and letting $k \to \infty$, we obtain

$$A_I v \geq 0.$$

Then

$$v \in F_{\bar{x}} \setminus \{0\}. \tag{14.23}$$

Now we are going to show that $(D\bar{x} + c)^T v = 0$. We have

$$\begin{aligned}
&\varphi(\omega + t^k\omega^0) - \varphi(\omega)\\
&= \frac{1}{2}(x^k)^T(D + t^k D^0)x^k + (c + t^k c^0)^T x^k - \frac{1}{2}\bar{x}^T D\bar{x} - c^T\bar{x}\\
&= \frac{1}{2}(x^k - \bar{x})^T D(x^k - \bar{x}) + (D\bar{x} + c)^T(x^k - \bar{x})\\
&\quad + t^k\left(\frac{1}{2}(x^k)^T D^0 x^k + (c^0)^T x^k\right).
\end{aligned} \tag{14.24}$$

Since $Ax \geq b$ is a regular system, by Lemma 14.1 we have

$$F(\bar{x}, \omega, \omega^0) \neq \emptyset.$$

Take $\bar{v} \in F(\bar{x}, \omega, \omega^0)$. Then, for every small enough positive number t^k, we have

$$\bar{x} + t^k \bar{v} \in \Delta(A + t^k A^0, b + t^k b^0).$$

Hence, for small enough t^k, we have

$$\begin{aligned} &\varphi(\omega + t^k\omega^0) - \varphi(\omega) = \frac{1}{2}(x^k)^T(D + t^k D^0)x^k \\ &\quad +(c + t^k c^0)^T x^k + \left(-\frac{1}{2}\bar{x}^T D\bar{x} - c^T\bar{x}\right) \\ &\leq \frac{1}{2}(\bar{x} + t^k\bar{v})^T(D + t^k D^0)(\bar{x} + t^k\bar{v}) \\ &\quad +(c + t^k c^0)^T(\bar{x} + t^k\bar{v}) + \left(-\frac{1}{2}\bar{x}^T D\bar{x} - c^T\bar{x}\right). \end{aligned} \tag{14.25}$$

From (14.24) and (14.25), for k large enough, we have

$$\begin{aligned} &(D\bar{x} + c)^T(x^k - \bar{x}) + \frac{1}{2}(x^k - \bar{x})^T D(x^k - \bar{x}) \\ &\quad +t^k\left(\frac{1}{2}(x^k)^T D^0 x^k + (c^0)^T x^k\right) \\ &\leq t^k(c^0)^T(\bar{x} + t^k\bar{v}) + \frac{1}{2}t^k(\bar{x} + t^k\bar{v})^T D^0(\bar{x} + t^k\bar{v}) + \\ &\quad +t^k\left(c^T\bar{v} + \bar{v}^T D\bar{v} + \frac{1}{2}t^k\bar{v}^T D\bar{v}\right). \end{aligned} \tag{14.26}$$

Dividing both sides of (14.26) by $\|x^k - \bar{x}\|$, letting $k \to \infty$ and taking account of (14.21), we get

$$(D\bar{x} + c)^T v \leq 0. \tag{14.27}$$

As $\bar{x}$ is a solution of (13.1) and (14.23) is valid, we have $(D\bar{x}+c)^T v \geq 0$ (see Theorem 3.5). Combining this with (14.27), we conclude that

$$(D\bar{x} + c)^T v = 0. \tag{14.28}$$

Properties (14.22), (14.23) and (14.28) show that (b) does not hold. Thus $\bar{x}$ cannot be a locally unique solution of (13.1), a contrary to our assumptions. The proof is complete. $\square$

14.3 Directional Differentiability of $\varphi(\cdot)$

The following theorem describes a sufficient condition for $\varphi(\cdot)$ to be directionally differentiable and gives an explicit formula for computing the directional derivative of $\varphi(\cdot)$.

Theorem 14.2. *Let $\omega = (D, A, c, b) \in \Omega$ be a given point and $\omega^0 = (D^0, A^0, c^0, b^0) \in \Omega$ be a given direction. If* (G) *and the following two conditions*

(i) *the system $Ax \geq b$ is regular,*

(ii) $\mathrm{Sol}(D, A, 0, 0) = \{0\}$

are satisfied, then the optimal value function φ is directionally differentiable at $\omega = (D, A, c, b)$ in direction $\omega^0 = (D^0, A^0, c^0, b^0)$, and

$$\varphi'(\omega; \omega^0) = \inf_{\bar{x} \in \mathrm{Sol}(D,A,c,b)} \max_{\lambda \in \Lambda(\bar{x},\omega)} \left[\frac{1}{2}\bar{x}^T D^0 \bar{x} + (c^0)^T \bar{x} + (b^0 - A^0 \bar{x})^T \lambda\right], \tag{14.29}$$

where $\Lambda(\bar{x}, \omega)$ is the Lagrange multipliers set corresponding to the solution $\bar{x} \in \mathrm{Sol}(D, A, c, b)$.

Proof.

1) Suppose that the conditions (i) and (ii) are satisfied. According to Lemma 13.3, $\mathrm{Sol}(D, A, c, b)$ is a nonempty compact set. Take any $\bar{x} \in \mathrm{Sol}(D, A, c, b)$. By (i) and Lemma 14.1, $F(\bar{x}, \omega, \omega^0) \neq \emptyset$. Take any $v \in F(\bar{x}, \omega, \omega^0)$. For $t > 0$ small enough, we have

$$\bar{x} + tv \in \Delta(A + tA^0, b + tb^0),$$

hence

$$\begin{aligned}
\varphi(\omega + t\omega^0) - \varphi(\omega) &\leq \frac{1}{2}(\bar{x} + tv)^T (D + tD^0)(\bar{x} + tv) \\
&\quad + (c + tc^0)^T (\bar{x} + tv) - \left(\frac{1}{2}\bar{x}^T D \bar{x} + c^T \bar{x}\right) \\
&= t(D\bar{x} + c)^T v + t\left(\frac{1}{2}\bar{x}^T D^0 \bar{x} + (c^0)^T \bar{x}\right) \\
&\quad + \frac{1}{2}t^2 v^T D v + t^2 v^T D \bar{x} + \frac{1}{2}t^3 v^T D^0 v.
\end{aligned}$$

Multiplying the above double inequality by t^{-1} and taking lim sup as $t \to 0^+$, we obtain

$$\varphi^+(\omega; \omega^0) \leq \frac{1}{2}\bar{x}^T D^0 \bar{x} + (D\bar{x} + c)^T v + +(c^0)^T \bar{x}.$$

This inequality is valid for any $v \in F(\bar{x}, \omega, \omega^0)$ and any

$$\bar{x} \in \mathrm{Sol}(D, A, c, b).$$

Consequently,

$$\varphi^+(\omega; \omega^0) \leq \inf_{\bar{x} \in \mathrm{Sol}(D,A,c,b)} \inf_{v \in F(\bar{x},\omega,\omega^0)} \left[\frac{1}{2}\bar{x}D^0\bar{x} + (c^0)^T\bar{x} + (D\bar{x} + c)^T v\right].$$

By Lemmas 14.2 and 14.3,

$$\begin{aligned} \inf\nolimits_{v \in F(\bar{x},\omega,\omega^0)}(D\bar{x} + c)^T v &= \inf\nolimits_{v \in R(\bar{x},\omega,\omega^0)}(D\bar{x} + c)^T v \\ &= \max\nolimits_{\lambda \in \Lambda(\bar{x},\omega)}(b^0 - A^0\bar{x})^T\lambda. \end{aligned}$$

Hence

$$\varphi^+(\omega; \omega^0) \leq \inf_{\bar{x} \in \mathrm{Sol}(D,A,c,b)} \max_{\lambda \in \Lambda(\bar{x},\omega)} \left[\frac{1}{2}\bar{x}^T D^0\bar{x} + (c^0)^T\bar{x} + (b^0 - A^0\bar{x})^T\lambda\right]. \tag{14.30}$$

2) Let $\{t^k\}$ be a sequence of real numbers such that $t^k \downarrow 0$ and

$$\varphi^-(\omega; \omega^0) = \lim_{k \to \infty} \frac{\varphi(\omega + t^k\omega^0) - \varphi(\omega)}{t^k}.$$

Due to the assumptions (i) and (ii), taking account of Lemmas 13.1 and 13.3 and the openness of the set G defined in (13.3) we can assume that

$$\mathrm{Sol}(\omega + t^k\omega^0) \neq \emptyset \quad \text{for every } k.$$

Let $\{x^k\}$ be a sequence in R^n such that $x^k \in \mathrm{Sol}(\omega + t^k\omega^0)$ for every k. By Lemma 14.4, without loss of generality we can assume that $x^k \to \hat{x} \in \mathrm{Sol}(D, A, c, b)$ as $k \to \infty$. We have

$$\begin{aligned} \varphi(\omega + t^k\omega^0) - \varphi(\omega) = \frac{1}{2}(x^k)^T(D + t^k D^0)x^k \\ +(c + t^k c^0)^T x^k + \left(-\frac{1}{2}\hat{x}^T D\hat{x} - c^T\hat{x}\right). \end{aligned} \tag{14.31}$$

Take $\lambda \in \Lambda(\bar{x}, \omega)$. As

$$\lambda^T(A\hat{x} - b) = 0, \quad \lambda \geq 0,$$

and

$$(A + t^k A^0)x^k \geq b + t^k b^0,$$

from (14.31) we get

$$\begin{aligned}
&\varphi(\omega + t^k\omega^0) - \varphi(\omega) \geq \frac{1}{2}(x^k)^T(D + t^kD^0)x^k \\
&\quad +(c + t^kc^0)^Tx^k - \frac{1}{2}\hat{x}^TD\hat{x} - c^T\hat{x} \\
&\quad +\lambda^T(A\hat{x} - b) - [(A + t^kA_0)x^k - b - t^kb^0]^T\lambda \\
&= (D\hat{x} - A^T\lambda + c)^T(x^k - \hat{x}) + \frac{1}{2}(x^k - \hat{x})^TD(x^k - \hat{x}) \\
&\quad +t^k\left[\frac{1}{2}(x^k)^TD^0x^k + (c^0)^Tx^k + (b^0 - A^0x^k)^T\lambda\right].
\end{aligned}$$

Since $\lambda \in \Lambda(\hat{x}, \omega)$, $D\hat{x} - A^T\lambda + c = 0$. Then we have

$$\begin{aligned}
&\varphi(\omega + t^k\omega^0) - \varphi(\omega) \geq \frac{1}{2}(x^k - \hat{x})^TD(x^k - \hat{x}) \\
&\quad +t^k\left[\frac{1}{2}(x^k)^TD^0x^k + (c^0)^Tx^k + (b^0 - A^0x^k)^T\lambda\right].
\end{aligned}$$

Multiplying both sides of this inequality by $(t^k)^{-1}$, taking lim inf as $k \to \infty$ and using condition (G), we obtain

$$\varphi^-(\omega; \omega^0) \geq \left(\frac{1}{2}\hat{x}^TD^0\hat{x} + c^0\right)^T\hat{x} + (b^0 - A^0\hat{x})^T\lambda.$$

As $\lambda \in \Lambda(\hat{x}, \omega)$ can be chosen arbitrarily, we conclude that

$$\begin{aligned}
&\varphi^-(\omega; \omega^0) \geq \max\nolimits_{\lambda\in\Lambda(\hat{x},\omega)}\left[\frac{1}{2}\hat{x}^TD^0\hat{x} + (c^0)^T\hat{x} + (b^0 - A^0\hat{x})^T\lambda\right] \\
&\geq \inf\nolimits_{\bar{x}\in\mathrm{Sol}(D,A,c,b)}\max\nolimits_{\lambda\in\Lambda(\bar{x},\omega)}[\frac{1}{2}\bar{x}^TD^0\bar{x} + (c^0)^T\bar{x} + (b^0 - A^0\bar{x})^T\lambda].
\end{aligned}$$

Combining this with (14.30), we have

$$\varphi^-(\omega; \omega^0) = \varphi^+(\omega; \omega^0)$$

and, therefore,

$$\varphi'(\omega; \omega^0) = \inf_{\bar{x}\in\mathrm{Sol}(\omega)}\max_{\lambda\in\Lambda(\bar{x},\omega)}[\frac{1}{2}\bar{x}^TD^0\bar{x} + (c^0)^T\bar{x} + (b^0 - A^0\bar{x})^T\lambda].$$

The proof is complete. □

We now apply Theorem 14.2 to a concrete example.

Example 14.1. Let $n = 2$, $m = 3$,

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

$$b^T = (0,\ -1,\ 0), \quad c = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$D^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (A^0)^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$(b^0)^T = (0,\ -1,\ 0),\ c^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\omega = (D, A, c, b), \quad \omega^0 = (D^0, A^0, c^0, b^0).$$

It is easy to verify that $Ax \geq b$ is a regular system, $\mathrm{Sol}(D, A, 0, 0) = \{0\}$ and

$$\begin{aligned} \mathrm{Sol}(D, A, c, b) = \mathrm{Sol}(\omega) &= \{(x_1, x_2)^T \in R^2 : x_1 = x_2,\ 0 \leq x_1 \leq 1\} \\ \mathrm{Sol}(\omega + t\omega^0) &= \{(x_1, x_2)^T \in R^2 : x_1 = x_2, \quad 0 \leq x_1 \leq 1 + t\} \end{aligned}$$

for every $t \geq 0$. For $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathrm{Sol}(\omega)$, we have

$$\Lambda(\bar{x}, \omega) = \{(\lambda_1, \lambda_2, \lambda_3)^T \in R^3 : \lambda_1 = \bar{x}_1, \lambda_2 = \lambda_3 = 0\}.$$

Suppose that $x^k = (x_1^k, x_2^k) \in \mathrm{Sol}(\omega + t^k\omega^0)$ and the sequence $\{x^k\}$ converges to $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathrm{Sol}(\omega)$. We have $x_1^k = x_2^k$ and $\bar{x}_1 = \bar{x}_2$. Then

$$\frac{(x^k - \bar{x})^T D (x^k - \bar{x})}{t^k} = \frac{(x_1^k - \bar{x}_1)^2 - (x_2^k - \bar{x}_2)^2}{t^k} = 0,$$

hence condition (G) is satisfied. By Theorem 14.2,

$$\begin{aligned} \varphi'(\omega; \omega^0) &= \inf\nolimits_{\bar{x} \in \mathrm{Sol}(\omega)} \max_{\lambda \in \Lambda(\bar{x}, \omega)} \left(\left(\frac{1}{2} \bar{x}^T D^0 \bar{x} + (c^0)^T \bar{x} \right) + b^0 \right)^T \lambda \\ &= \inf\nolimits_{\bar{x} \in \mathrm{Sol}(\omega)} 0 = 0. \end{aligned}$$

Observe that, in Example 14.1, $x^T Dx$ is an indefinite quadratic form (the sign of the expression $x^T Dx$ depends on the choice of x) and the solutions of the QP problem are not locally unique, so the assumptions of Theorem 14.1 are not satisfied.

Consider problem (13.1) and assume that $\bar{x} \in \mathrm{Sol}(D, A, c, b)$ is one of its solutions. Let $u = \omega^0 = (D^0, A^0, c^0, b^0) \in \Omega$ be a given direction. Applied to the solution $\bar{x}$ of problem (13.1), condition $(SOSC)_u$ in Auslender and Cominetti (1990) is stated as follows:

$$(SOSC)_u \quad \begin{cases} \textit{For every vector } v \in F_{\bar{x}} \setminus \{0\}, & \textit{if } (D\bar{x} + c)^T v = 0 \\ \textit{then} \quad v^T Dv > 0, \end{cases}$$

where $F_{\bar{x}}$ is the cone of the feasible directions of $\Delta(A, b)$ at $\bar{x}$. That is

$$F_{\bar{x}} = \{v \in R^n : (Av)_i \geq 0 \text{ for every } i \text{ satisfying } (A\bar{x})_i = b_i\}.$$

Observe that, in the case of QP problems, condition $(SOSC)_u$ is equivalent to the requirement saying that $\bar{x}$ is a locally unique solution of (13.1) (see Theorem 3.7). This remark allows us to deduce from Theorem 1 in Auslender and Coutat (1990) the following result.

Proposition 14.1. *Let $\omega = (D, A, c, b) \in \Omega$ be a given point and $u = \omega^0 = (D^0, A^0, c^0, b^0) \in \Omega$ be a given direction. If all the solutions of problem* (13.1) *are locally unique and the two conditions*

(i) *the system $Ax \geq b$ is regular,*

(ii) $\mathrm{Sol}(D, A, 0, 0) = \{0\}$

are satisfied, then the optimal value function φ is directionally differentiable at $\omega = (D, A, c, b)$ in direction $u = \omega^0 = (D^0, A^0, c^0, b^0)$, and formula (14.29) *is valid.*

Proof. By Theorem 12.1, from the assumptions (i) and (ii) it follows that the map $\mathrm{Sol}(\cdot)$ is upper semicontinuous at (D, A, c, b). Besides, by Lemma 13.3, $\mathrm{Sol}(D, A, c, b)$ is a nonempty compact set. Then there exists a compact set $B \subset R^n$ and a constant $\varepsilon > 0$ such that

$$\emptyset \neq \mathrm{Sol}(\omega + t\omega^0) \subset B \quad \text{for every } t \in [0, \varepsilon].$$

Under the conditions of our proposition, all the assumptions of Theorem 1 in Auslender and Coutat (1990) are fulfilled. So the desired conclusion follows from applying Theorem 1 in Auslender and Coutat (1990). □

Observe that Proposition 14.1 is a direct corollary of our Theorems 14.1 and 14.2. It is worth noting that the result stated in Proposition 14.1 cannot be applied to the problem described in Example 14.1 (because condition $(SOSC)_u$, where $u := \omega^0$, does not hold at any solution $\bar{x} \in \mathrm{Sol}(\omega)$). That result cannot be applied also to convex QP problems whose solution sets have more than one element. This is because, for such a problem, the solution set is a convex set consisting of more than one element. Using Remark 14.1 we can conclude that Theorem 14.2 is applicable to convex QP problems.

Consider problem (13.1) and denote $\omega = (D, A, c, b)$. Suppose that $\omega^0 = (D^0, A^0, c^0, b^0) \in \Omega$ is a given direction. In this case, condition (H3) in Minchenko and Sakolchik (1996) is stated as follows:

(H3) *For every sequence* $\{t^k\}$, $t^k \downarrow 0$, *and every sequence* $\{x^k\}$, $x^k \to \bar{x} \in \mathrm{Sol}(D, A, c, b)$, $x^k \in \mathrm{Sol}(\omega + t^k\omega^0)$ *for each* k, *the following inequality is satisfied*

$$\limsup_{k\to\infty} \frac{\|x^k - \bar{x}\|^2}{t^k} < +\infty.$$

Applying Theorem 4.1 in Minchenko and Sakolchik (1996) to problem (13.1) we get the following result.

Proposition 14.2. *Let* $\omega = (D, A, c, b)$ *and* $\omega^0 = (D^0, A^0, c^0, b^0)$ *be given as in Proposition* 14.1. *If* (H3) *and the two conditions*

(i) *the system* $Ax \geq b$ *is regular,*

(ii) *there exist a compact set* $B \subset R^n$ *and a neighborhood* U *of* $(A, b) \in R^{m\times n} \times R^m$ *such that* $\Delta(A', b') \subset B$ *for every* $(A', b') \in U$

are satisfied, then the optimal value function φ *is directionally differentiable at* $\omega = (D, A, c, b)$ *in direction* $u = \omega^0 = (D^0, A^0, c^0, b^0)$, *and formula* (14.29) *is valid.*

Consider the problem described in Example 14.1. Choose $\bar{x} = (0, 0) \in \mathrm{Sol}(\omega)$, $t^k = k^{-1}$,

$$x^k = (k^{-\frac{1}{4}}, k^{-\frac{1}{4}}) \in \mathrm{Sol}(\omega + t^k\omega^0).$$

We have $x^k \to \bar{x}$ as $k \to \infty$ and

$$\limsup_{k\to\infty} \frac{\|x^k - \bar{x}\|^2}{t^k} = \limsup_{k\to\infty} \frac{k^{-\frac{1}{2}} + k^{-\frac{1}{2}}}{k^{-1}} = +\infty,$$

so (H3) does not hold and Proposition 14.2 cannot be applied to this QP problem.

We have shown that Theorem 14.2 can be applied even to some kinds of QP problems where the existing results on differential stability in nonlinear programming cannot be used.

Now we want to show that, for problem (13.1), if the system $Ax \geq b$ is regular then (H3) implies (G).

Proposition 14.3. *Let* $\omega = (D, A, c, b)$ *and* $\omega^0 = (D^0, A^0, c^0, b^0)$ *be given as in Proposition* 14.1. *If the system* $Ax \geq b$ *is regular, then condition* (H3) *implies condition* (G).

Proof. Suppose that (H3) holds. Let $\{t^k\}$, $t^k \downarrow 0$, and $\{x^k\}$, where $x^k \in \mathrm{Sol}(\omega + t^k\omega^0)$ for each k, be arbitrary sequences. If

$$x^k \to \bar{x} \in \mathrm{Sol}(D, A, c, b)$$

then, by (H3), we have

$$\limsup_{k\to\infty} \frac{\|x^k - \bar{x}\|^2}{t^k} < +\infty. \tag{14.32}$$

We have to prove that the inequality written in condition (G) is satisfied. Let $\{(t^{k'})^{-1}(x^{k'} - \bar{x})^T D(x^{k'} - \bar{x})\}$ be a subsequence of $\{(t^k)^{-1}(x^k - \bar{x})^T D(x^k - \bar{x})\}$ satisfying

$$\begin{aligned} &\liminf_{k\to\infty}(t^k)^{-1}(x^k - \bar{x})^T D(x^k - \bar{x}) \\ &= \lim_{k'\to\infty}(t^{k'})^{-1}(x^{k'} - \bar{x})^T D(x^{k'} - \bar{x}). \end{aligned} \tag{14.33}$$

From (14.32) it follows that the sequence $\{(t^k)^{-1}\|x^k - \bar{x}\|^2\}$ is bounded. Then the sequence $\{(t^k)^{-1/2}\|x^k - \bar{x}\|\}$ is bounded. Without loss of generality, we may assume that

$$(t^k)^{-1/2}\|x^k - \bar{x}\| \to v \in R^n. \tag{14.34}$$

As $x^k \in \mathrm{Sol}(D + t^k D^0, A + t^k A^0, c + t^k c^0, b + t^k b^0)$, we have

$$(A_I + t^k A_I^0)x^k \geq b_I + t^k b_I^0,$$

where $I = \{i : (A\bar{x})_i = b_i\}$. Since $b_I = A_I\bar{x}$,

$$A_I(x^k - \bar{x}) \geq t^k(b_I^0 - A_I^0 x^k).$$

Multiplying both sides of this inequality by $(t^k)^{-1/2}$ and letting $k \to \infty$, due to (14.34) we can conclude that $A_I v \geq 0$. Hence $v \in F_{\bar{x}}$, where $F_{\bar{x}}$ is defined as in the formulation of condition $(SOSC)_u$. Furthermore, note that the expression (14.24) holds. As $Ax \geq b$ is a regular system, by Lemma 14.1 we have $F(\bar{x}, \omega, \omega^0) \neq \emptyset$. Take any $\bar{v} \in F(\bar{x}, \omega, \omega^0)$. Then, for k large enough,

$$\bar{x} + t^k\bar{v} \in \Delta(A + t^k A^0, b + t^k b^0).$$

Therefore, for k large enough, we have (14.25). From (14.24) and (14.25) we have (14.26). Multiplying both sides of (14.26) by

$$(t^k)^{-1/2},$$

letting $k \to \infty$ and taking account of (14.34), we get (14.27). As $\bar{x}$ is a solution of problem (13.1) and $v \in F_{\bar{x}}$, the situation $(D\bar{x}+c)^T v < 0$ cannot happen. Hence $(D\bar{x} + c)^T v = 0$. Since $\bar{x} \in \mathrm{Sol}(\omega)$, we must have $v^T Dv \geq 0$ (see Theorem 3.5). By (14.33) and (14.34),

$$\begin{aligned}
&\liminf_{k\to\infty}(t^k)^{-1}(x^k - \bar{x})^T D(x^k - \bar{x}) \\
&= \lim_{k'\to\infty} \left((t^{k'})^{-1/2}(x^{k'} - \bar{x})\right)^T D \left((t^{k'})^{-1/2}(x^{k'} - \bar{x})\right) \\
&= v^T Dv \geq 0.
\end{aligned}$$

Thus (G) is satisfied. □

14.4 Commentaries

The results presented in this chapter are due to Tam (2001b).

Best and Chakravarti (1990) considered parametric convex quadratic programming problems and obtained some results on the directional differentiability of the optimal value function. Auslender and Coutat (1996) investigated similar questions for the case of generalized linear-quadratic programs. A survey of some results on stability and sensitivity of nonlinear mathematical programming problems can be found in Bonnans and Shapiro (1998). A comprehensive theory on perturbation analysis of optimization problems was given by Bonnans and Shapiro (2000).

Chapter 15

Quadratic Programming under Linear Perturbations: I. Continuity of the Solution Maps

Continuity of the local solution map and the solution map of QP problems under linear perturbations is studied in this chapter.

Since it is impossible to give a satisfactory characterization for the usc property of the local solution map and since the usc property of the solution map can be derived from a result of Klatte (1985), Theorem 3, we will concentrate mainly on characterizing the lsc property of the local solution map and the solution map.

Consider the QP problem

$$\begin{cases} \text{Minimize } f(x) := \frac{1}{2}x^TDx + c^Tx \\ \text{subject to } x \in \Delta(A,b) := \{x \in R^n \,:\, Ax \geq b\} \end{cases} \tag{15.1}$$

depending on the parameter $w = (c,b) \in R^n \times R^m$, where the matrices $D \in R_S^{n\times n}$ and $A \in R^{m\times n}$ are not subject to change. The solution set, the local solution set and the KKT point set of this problem are denoted, respectively, by $\mathrm{Sol}(c,b)$, $\mathrm{loc}(c,b)$ and $S(D,A,c,b)$.

15.1 Lower Semicontinuity of the Local Solution Map

In this section we investigate the lsc property of the local solution map

$$\mathrm{loc}(\cdot) : R^n \times R^m \to 2^{R^m}, \quad (c', b') \in R^n \times R^m \mapsto \mathrm{loc}(c', b'). \quad (15.2)$$

Theorem 15.1. *The multifunction* (15.2) *is lower semicontinuous at $(c, b) \in R^n \times R^m$ if and only if the system $Ax \geq b$ is regular and the set* $\mathrm{loc}(c, b)$ *is nonempty and finite.*

Proof. *Necessity:* Since the multifunction (15.2) is lower semicontinuous at (c, b), $\mathrm{loc}(c, b)$ is nonempty and the regularity condition is satisfied. We now prove that $\mathrm{loc}(c, b)$ is finite. Define the sets Q_I ($I \subset \{1, \ldots, m\}$) and Q as in the proof of Theorem 11.3. Since Q is nowhere dense, there exists a sequence $\{(c^k, b^k)\}$ converging to (c, b) in $R^n \times R^m$ such that $(-c^k, b^k) \notin \mathrm{Q}$ for all $k \in N$. Fix a point $\bar{x} \in \mathrm{loc}(c, b)$. Since $\mathrm{loc}(\cdot)$ is lower semicontinuous at (c, b), there exist a subsequence $\{(c^{k_l}, b^{k_l})\}$ of $\{(c^k, b^k)\}$ and a sequence $\{x^{k_l}\}$ converging to $\bar{x}$ in R^n such that

$$x^{k_l} \in \mathrm{loc}(c^{k_l}, b^{k_l})$$

for all k_l. For any k_l, since $\mathrm{loc}(c^{k_l}, b^{k_l}) \subset S(D, A, c^{k_l}, b^{k_l})$, there exists $\lambda^{k_l} \in R^m$ such that (11.9) is satisfied. For every k_l, define

$$I_{k_l} = \{i \in \{1, \ldots, m\} : \lambda_i^{k_l} > 0\}.$$

By the same arguments as those used in the proof of Theorem 11.3, we obtain a subset $I \subset \{1, \ldots, m\}$ such that (11.10)–(11.13) hold. Next, let Z and X be defined as in the proof of Theorem 11.3. As before, X is a finite set and we have $\bar{x} \in X$. Since $\bar{x} \in \mathrm{loc}(c, b)$ can be chosen arbitrarily, we have $\mathrm{loc}(c, b) \subset X$. Hence $\mathrm{loc}(c, b)$ is a finite set.

Sufficiency: If the regularity condition is satisfied and the set $\mathrm{loc}(c, b)$ is finite, then from Theorem 5 in Phu and Yen (2001) it follows that the multifunction (15.2) is lower semicontinuous at $(c, b) \in R^n \times R^m$. □

Since $\mathrm{loc}(c, b) \subset S(D, A, c, b)$, from Theorem 15.1 we obtain the following corollary.

Corollary 15.1. *Let $(D, A, c, b) \in R_S^{n \times n} \times R^{m \times n} \times R^n \times R^m$. Suppose that the system $Ax \geq b$ is regular and the following conditions are satisfied:*

(i) *the set* $S(D, A, c, b)$ *is finite,*

(ii) *the set* $\mathrm{loc}(c, b)$ *is nonempty.*

Then, the multifunction (15.2) *is lower semicontinuous at* (c, b).

15.2 Lower Semicontinuity of the Solution Map

In this section, a complete characterization for the lower semicontinuity of the solution map

$$\mathrm{Sol}(\cdot) : R^n \times R^m \to 2^{R^n}, \quad (c', b') \mapsto \mathrm{Sol}(c', b') \tag{15.3}$$

of the QP problem (15.1) will be given. Before proving the result, we state some lemmas.

Let $(D, A, c, b) \in R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m$.

Lemma 15.1. *If the multifunction* (15.3) *is lower semicontinuous at* (c, b)*, then the system* $Ax \geq b$ *is regular and the set* $\mathrm{Sol}(c, b)$ *is nonempty and finite.*

Proof. (This proof is very similar to the first part of proof of Theorem 15.1.) It is clear that if the multifunction (15.3) is lower semicontinuous at (c, b) then regularity condition is satisfied and the set $\mathrm{Sol}(c, b)$ is nonempty. In order to prove the finiteness of $\mathrm{Sol}(c, b)$, we define Q_I ($I \subset \{1, \ldots, m\}$) and Q as in the proof of Theorem 13.3. Then, there exists a sequence $\{(c^k, b^k)\}$ converging to (c, b) such that $(-c^k, b^k) \notin \mathrm{Q}$ for all k. Fix any $\bar{x} \in \mathrm{Sol}(c, b)$. As $\mathrm{Sol}(\cdot)$ is lower semicontinuous at (c, b), there exist a subsequence $\{(c^{k_l}, b^{k_l})\}$ of $\{(c^k, b^k)\}$ and a sequence $\{x^{k_l}\}$ converging to $\bar{x}$ in R^n such that $x^{k_l} \in \mathrm{Sol}(c^{k_l}, b^{k_l})$ for all k_l. Since $\mathrm{Sol}(c^{k_l}, b^{k_l}) \subset S(D, A, c^{k_l}, b^{k_l})$, there exists $\lambda^{k_l} \in R^m$ such that (11.9) is satisfied. Constructing I and defining Z and X as in the proof of Theorem 11.3, we have that X is a finite set and $\bar{x} \in X$. Hence the solution set $\mathrm{Sol}(c, b)$ is finite. □

Lemma 15.2. *If the multifunction* (15.3) *is lower semicontinuous at* (c, b)*, then the set* $\mathrm{Sol}(c, b)$ *is a singleton.*

Proof. On the contrary, suppose that $\mathrm{Sol}(\cdot)$ is lower semicontinuous at (c, b) but there exist $\bar{x}$, $\bar{y} \in \mathrm{Sol}(c, b)$ such that $\bar{x} \neq \bar{y}$. Choose $c_0 \in R^n$ such that

$$c_0^T(\bar{y} - \bar{x}) = 1. \tag{15.4}$$

By Lemma 15.1, $\mathrm{Sol}(c,b)$ is a finite set. Combining this fact with (15.4) we see that there exists a open set $U \subset R^n$ containing $\bar{y}$ such that

$$\mathrm{Sol}(c,b) \cap U = \{\bar{y}\}$$

and

$$c_0^T(y - \bar{x}) > 0 \quad \text{for all } \ y \in U. \tag{15.5}$$

Let $\delta > 0$ be given arbitrarily. Choose $\varepsilon > 0$ so that

$$0 < \varepsilon < \frac{\delta}{\|c_0\|}.$$

Let $b' = b, \ c' = c + \varepsilon c_0$. We have

$$\|(c', b') - (c, b)\| = \varepsilon\|c_0\| < \delta.$$

We now show that

$$\mathrm{Sol}(c', b') \cap U = \emptyset.$$

For any $y \in \Delta(A, b') = \Delta(A, b)$, since $\bar{x}$, $\bar{y} \in \mathrm{Sol}(c, b)$, using (15.5) we have

$$\begin{aligned}
\frac{1}{2}y^T Dy + (c')^T y \quad &= \frac{1}{2}y^T Dy + (c + \varepsilon c_0)^T y \\
&= (\frac{1}{2}y^T Dy + c^T y) + \varepsilon c_0^T y \\
&\geq (\frac{1}{2}\bar{y}^T D\bar{y} + c^T \bar{y}) + \varepsilon c_0^T y \\
&> \left(\frac{1}{2}\bar{y}^T D\bar{y} + c^T \bar{y}\right) + \varepsilon c_0^T \bar{x} \\
&= (\frac{1}{2}\bar{x}^T D\bar{x} + c^T \bar{x}) + \varepsilon c_0^T \bar{x} \\
&= \frac{1}{2}\bar{x}^T D\bar{x} + (c + \varepsilon c_0)^T \bar{x}.
\end{aligned} \tag{15.6}$$

Since $\bar{x} \in \Delta(A, b')$, by (15.6) we have $y \notin \mathrm{Sol}(c', b')$. Consequently, for the chosen neighborhood U of $\bar{y} \in \mathrm{Sol}(c, b)$, for every $\delta > 0$ there exists $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta$ and $\mathrm{Sol}(c', b') \cap U = \emptyset$. This contradicts our assumption that $\mathrm{Sol}(\cdot)$ is lower semicontinuous at (c, b). We have shown that $\mathrm{Sol}(c, b)$ cannot have more than one element. Since $\mathrm{Sol}(\cdot)$ is lower semicontinuous at (c, b), we must have $\mathrm{Sol}(c, b) \neq \emptyset$. From what has been proved, we conclude that $\mathrm{Sol}(c, b)$ is a singleton. □

Lemma 15.3. *If $\Delta(A, b)$ is nonempty and if we have*

$$\inf\{v^T(Dx + c) \, : \, x \in R^n, \ Ax \geq b\} > 0 \tag{15.7}$$

for some $v \in R^n$, then there exists $\delta > 0$ such that

$$\inf\{v^T(Dx' + c') : x' \in R^n,\ Ax' \geq b'\} \geq 0 \tag{15.8}$$

for every $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta$. (By convention, $\inf \emptyset = +\infty$.)

Proof. By Corollary 7.2, for the given matrix A there exists a constant $\gamma(A) \geq 0$ such that

$$\Delta(A, b') \subset \Delta(A, b'') + \gamma(A)\|b'' - b'\|B_{R^n} \tag{15.9}$$

for all b', $b'' \in R^n$ satisfying $\Delta(A, b') \neq \emptyset$ and $\Delta(A, b'') \neq \emptyset$. Define

$$\mu = \inf\{v^T(Dx + c) : x \in R^n,\ Ax \geq b\}. \tag{15.10}$$

By (15.7), $\mu > 0$. Choose $\delta > 0$ such that

$$\gamma(A)\|v\|\|D\|\delta < \frac{\mu}{4}, \quad \|v\|\delta < \frac{\mu}{4}. \tag{15.11}$$

Here, as usual, $\|D\| = \max\{\|Dx\| : x \in R^n,\ \|x\| \leq 1\}$. Let $(c', b') \in R^n \times R^m$ be such that $\|(c', b') - (c, b)\| < \delta$. If $\Delta(A, b') = \emptyset$ inequality (15.8) is valid because the infimum in its left-hand-side equal $+\infty$. Now consider the case where $\Delta(A, b') \neq \emptyset$. By our assumption, $\Delta(A, b) \neq \emptyset$. Hence, for every $x' \in \Delta(A, b')$, from (15.9) it follows that there exists $x \in \Delta(A, b)$ such that

$$x' = x + \gamma(A)\|b' - b\|u \tag{15.12}$$

for some $u \in B_{R^n}$. By (15.11) and (15.12), we have

$$\begin{aligned}
&|v^T(Dx' + c') - v^T(Dx + c)| \\
&\leq |v^T D(x' - x)| + |v^T(c' - c)| \\
&\leq \|v\|\|D\|\|x' - x\| + \|v\|\|c' - c\| \\
&\leq \|v\|\|D\|\gamma(A)\|b' - b\|\|u\| + \|v\|\|c' - c\| \\
&\leq \|v\|\|D\|\gamma(A)\delta + \|v\|\|c' - c\| \\
&< \frac{\mu}{4} + \frac{\mu}{4} = \frac{\mu}{2}.
\end{aligned}$$

Combining this with (15.10), we obtain

$$\begin{aligned}
v^T(Dx' + c') &\geq v^T(Dx + c) - \frac{\mu}{2} \\
&\geq \mu - \frac{\mu}{2} \\
&= \frac{\mu}{2}.
\end{aligned}$$

From this we deduce that (15.8) holds for every $(c', b') \in R^n \times R^m$ with the property that $\|(c', b') - (c, b)\| < \delta$. $\square$

Lemma 15.4. *Let K denote the cone $\{v \in R^n : Av \geq 0,\ v^T Dv = 0\}$. Assume that the system $Ax \geq b$ is regular, the set* $\mathrm{Sol}(c, b)$ *is nonempty, and*

$$\inf\{v^T(Dx + c) : x \in R^n,\ Ax \geq b\} > 0$$

for every nonzero $v \in K$. Then there exists $\rho > 0$ such that $\mathrm{Sol}(c', b')$ *is nonempty for all $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \rho$.*

Proof. Since the regularity condition is satisfied, by Lemma 13.1 there exists $\rho_0 > 0$ such that for every $b' \in R^m$ satisfying $\|b' - b\| < \rho_0$ we have

$$\Delta(A, b') \neq \emptyset. \tag{15.13}$$

Since $\mathrm{Sol}(c, b) \neq \emptyset$, by Theorem 2.2 we have

$$v^T Dv \geq 0 \tag{15.14}$$

for all $v \in R^n$ satisfying $Av \geq 0$.

We now distinguish two cases.

Case 1. $K = \{v \in R^n : Av \geq 0,\ v^T Dv = 0\} = \{0\}$. In this case, we have

$$v^T(Dx + c') = 0 \tag{15.15}$$

for all $v \in K$, $x \in R^n$ and $c' \in R^n$. On account of Theorem 2.2 and the properties (15.13)–(15.15), we have $\mathrm{Sol}(c', b') \neq \emptyset$ for every $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \rho_0$.

Case 2. $K = \{v \in R^n : Av \geq 0,\ v^T Dv = 0\} \neq \{0\}$. In this case, from (15.14) it follows that K can be represented as the union of finitely many polyhedral convex cones (see Bank et al. (1982), Lemma 4.5.1). Suppose that

$$K = \bigcup_{j=1}^{s} K_j, \tag{15.16}$$

where K_j $(j = 1, 2, \ldots, s)$ are polyhedral convex cones. Fix an index $j \in \{1, 2, \ldots, s\}$. Let $v^1, v^2, \ldots, v^{k_j}$ be the generators (see Rockafellar (1970), p. 170) of the cone K_j. By our assumption, for every v^i, $1 \leq i \leq k_j$, we have

$$\inf\{(v^i)^T(Dx + c) : x \in R^n,\ Ax \geq b\} > 0.$$

By Lemma 15.3, there exists $\delta_i > 0$ such that

$$\inf\{(v^i)^T(Dx + c') : x \in R^n,\ Ax \geq b'\} \geq 0 \tag{15.17}$$

for all $(c', b') \in R^n \times R^m$ satisfying $\|(c', b') - (c, b)\| < \delta_i$. Define

$$\rho_j = \min\{\delta_i : i = 1, \cdots, k_j\}.$$

From (15.17) it follows that

$$(v^i)^T(Dx + c') \geq 0 \tag{15.18}$$

for all $i = 1, \cdots, k_j$ and for all $x \in R^n$ satisfying $Ax \geq b'$, provided that $\|(c', b') - (c, b)\| < \rho_j$. Define $\bar{\rho}_j = \min\{\rho_0,\ \rho_j\}$. Let $(c', b') \in R^n \times R^m$ be such that $\|(c', b') - (c, b)\| < \bar{\rho}_j$, $x \in \Delta(A, b')$ and $v \in K_j$. As $v^1, \cdots, v^{k_j}$ are the generators of K_j, there exist nonnegative real numbers $\alpha_1, \cdots, \alpha_{k_j}$ such that $v = \alpha_1 v^1 + \cdots + \alpha_{k_j} v^{k_j}$. By (15.18), we have

$$v^T(Dx + c') = \sum_{i=1}^{k_j} \alpha_i (v^i)^T(Dx + c') \geq 0. \tag{15.19}$$

Set $\rho = \min\{\bar{\rho}_j : j = 1, 2, \ldots, s\}$. Let $(c', b') \in R^n \times R^m$ be such that

$$\|(c', b') - (c, b)\| < \rho, \tag{15.20}$$

and let $x \in \Delta(A, b')$ and $v \in K$ be given arbitrarily. By (15.16), there exists $j \in \{1, 2, \ldots, s\}$ such that $v \in K_j$. Let $v = \alpha_1 v^1 + \cdots + \alpha_{k_j} v^{k_j}$, where $\alpha_i \geq 0$ for $i = 1, 2, \ldots, k_j$. By virtue of (15.19), we have $v^T(Dx + c') \geq 0$. Hence, taking account of (15.13), (15.14), and applying Theorem 2.2 we have $\mathrm{Sol}(c', b') \neq \emptyset$ for all $(c', b') \in R^n \times R^m$ satisfying (15.20). The lemma is proved. $\square$

We can now state the main result of this section.

Theorem 15.2. *The multifunction* (15.3) *is lower semicontinuous at* (c, b) *if and only if the system* $Ax \geq b$ *is regular and the following conditions are satisfied:*

(i) *for every nonzero vector*

$$v \in K := \{v \in R^n : Av \geq 0,\ v^T Dv = 0\}$$

it holds $\inf\{v^T(Dx + c) : x \in R^n,\ Ax \geq b\} > 0$,

(ii) *the set* $\mathrm{Sol}(c, b)$ *is a singleton.*

Proof. *Necessity:* If $\mathrm{Sol}(\cdot)$ is lower semicontinuous at (c,b) then by Lemmas 15.1 and 15.2, the system $Ax \geq b$ is regular and condition (ii) is satisfied. Suppose that the property (i) were false. Then we could find a nonzero vector $\bar{v} \in K = \{v \in R^n : Av \geq 0,\ v^TDv = 0\}$ such that

$$\inf\{\bar{v}^T(Dx+c) : x \in R^n,\ Ax \geq b\} \leq 0. \tag{15.21}$$

If the infimum in the left-hand-side of (15.21) is $-\infty$ then it is obvious that there exists $\bar{x} \in \Delta(A,b)$ satisfying $\bar{v}^T(D\bar{x}+c) < 0$. If that infimum is finite then, applying the Frank-Wolfe Theorem to the linear programming problem

$$\text{Minimize } \bar{v}^T(Dx+c) \quad \text{subject to } x \in R^n,\ Ax \geq b,$$

we find $\bar{x} \in \Delta(A,b)$ such that

$$\bar{v}^T(D\bar{x}+c) = \inf\{\bar{v}^T(Dx+c) : x \in R^n,\ Ax \geq b\} \leq 0.$$

So, in both cases, we can find $\bar{x} \in \Delta(A,b)$ such that

$$\bar{v}^T(D\bar{x}+c) \leq 0. \tag{15.22}$$

For every positive integer k, let $c^k := c - \frac{1}{k}\bar{v}$. By (15.22),

$$\bar{v}^T(D\bar{x}+c^k) = \bar{v}^T(D\bar{x}+c) - \frac{1}{k}\bar{v}^T\bar{v} < 0. \tag{15.23}$$

From (15.23) we see that condition (ii) in Theorem 2.2, where $(D,A,c,b) := (D,A,c^k,b)$, is violated. Hence, by Theorem 2.2 we have $\mathrm{Sol}(c^k,b) = \emptyset$ for all k. Since $c^k \to c$ as $k \to \infty$, the latter fact shows that the multifunction (15.3) is not lower semicontinuous at (c,b), a contradiction.

Sufficiency: Suppose that the system $Ax \geq b$ is regular and the conditions (i), (ii) are satisfied. By (ii), we can assume that $\mathrm{Sol}(c,b) = \{\bar{x}\}$ for some $\bar{x} \in R^n$. Let U be any open set containing $\bar{x}$. By the regularity assumption, by (i) and Lemma 15.4, there exists $\rho > 0$ such that $\mathrm{Sol}(c',b') \neq \emptyset$ for all $(c',b') \in R^n \times R^m$ satisfying $\|(c',b') - (c,b)\| < \rho$. By (ii) and Theorem 3 in Klatte (1985), there exists $\rho_1 > 0$ such that $\mathrm{Sol}(c',b') \subset U$ for all $(c',b') \in R^n \times R^m$ satisfying $\|(c',b') - (c,b)\| < \rho_1$. Hence, for $\rho_2 := \min\{\rho, \rho_1\}$, we have $\mathrm{Sol}(c',b') \cap U \neq \emptyset$ for all $(c',b') \in R^n \times R^m$ satisfying $\|(c',b') - (c,b)\| < \rho_2$. From what already been proved, it may be concluded that $\mathrm{Sol}(\cdot)$ is lower semicontinuous at (c,b). The proof is complete. □

Let us mention two direct corollaries of Theorem 15.2.

Corollary 15.2. *For $(D,A,c,b) \in R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m$, if $K := \{v \in R^n : Av \geq 0, v^TDv = 0\} = \{0\}$ then the multifunction (15.3) is lower semicontinuous at (c,b) if and only if the system $Ax \geq b$ is regular and the set $\mathrm{Sol}(c,b)$ is a singleton.*

Corollary 15.3. *Let $(D,A,c,b) \in R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m$. If D is a positive definite matrix then the multifunction (15.3) is lower semicontinuous at (c,b) if and only if condition the system $Ax \geq b$ is regular.*

In Theorem 15.2 we have established a complete characterization for the lower semicontinuity property of the solution map (15.3). Let us consider an example.

Example 15.1. Let $n=2$, $m=4$, and

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then we have the following QP problem

$$\begin{cases} \text{Minimize} & \frac{1}{2}(x_1^2 - x_2^2) + x_1 \\ \text{subject to} & x_1 - x_2 \geq 0,\ x_1 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0. \end{cases}$$

It is easily seen that the system $Ax \geq b$ is regular. For any $x = (x_1, x_2) \in \Delta(A,b)$, we have

$$\frac{1}{2}(x_1^2 - x_2^2) + x_1 \geq x_1 \geq 1.$$

The last two inequalities become equalities if and only if $x_1 = x_2 = 1$. These observations allow us to conclude that $\mathrm{Sol}(c,b) = \{(1,1)\}$. Clearly,

$$\mathrm{Sol}(0,0) = \{(x_1,x_2) \in R^2 : x_1 \geq 0,\ x_2 \geq 0,\ x_1 = x_2\}$$

and

$$\begin{aligned} K &= \{(v_1,v_2) \in R^2 : \ v_1 \geq 0,\ v_2 \geq 0, v_1^2 - v_2^2 = 0\} \\ &= \{(v_1,v_2) \in R^2 : \ v_1 \geq 0,\ v_2 \geq 0, v_1 = v_2\}. \end{aligned}$$

For any $v = (\mu, \mu) \in K \setminus \{0\}$ ($\mu > 0$) and for any $x = (x_1, x_2) \in \Delta(A, b)$, we have

$$\begin{aligned} v^T(Dx + c) &= (\mu, \mu)\binom{x_1+1}{-x_2} \\ &= \mu(x_1 + 1 - x_2) \geq \mu. \end{aligned}$$

So

$$\inf\{v^T(Dx + c) \,:\, x \in R^2,\ Ax \geq b\} \geq \mu > 0.$$

Since the system $Ax \geq b$ is regular and conditions (i), (ii) in Theorem 15.2 are satisfied, we conclude that the multifunction (15.3) is lower semicontinuous at (c, b). Meanwhile, since $\mathrm{Sol}(0, 0) \neq \{0\}$, from Theorem 12.3 it follows that the multifunction $(D', A', c', b') \mapsto \mathrm{Sol}(D', A', c', b')$, where $\mathrm{Sol}(D', A', c', b')$ denotes the solution set of the canonical QP problem

$$\begin{cases} \text{Minimize} & f(x) := \dfrac{1}{2}x^T D' x + (c')^T x \\ \text{subject to} & A'x \geq b', \quad x \geq 0, \end{cases}$$

is not lower semicontinuous at $(D, \widetilde{A}, c, \widetilde{b}) \in R_S^{2\times 2} \times R^{2\times 2} \times R^2 \times R^2$. Here

$$\widetilde{A} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad \widetilde{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

15.3 Commentaries

The results presented in this chapter are taken from Lee et al. (2002b, 2002c).

Theorem 15.2 is the main result of this chapter. Example 15.1 shows clearly the difference between the characterization given by Theorem 15.2 and the one provided by Theorem 12.3.

Chapter 16

Quadratic Programming under Linear Perturbations: II. Properties of the Optimal Value Function

In this chapter, we will consider the optimal value function $(c, b) \mapsto \varphi(c, b)$ of the parametric QP problem (15.1). It is proved that φ is directionally differentiable at any point $\bar{w} = (\bar{c}, \bar{b})$ in its effective domain $W := \{w = (c, b) \in R^n \times R^m : -\infty < \varphi(c, b) < +\infty\}$. Formulae for computing the directional derivative $\varphi'(\bar{w}; z)$ of φ at $\bar{w}$ in a direction $z = (u, v) \in R^n \times R^m$ are also obtained.

If D is positive semidefinite, then φ is piecewise linear-quadratic on the set W (which is a polyhedral convex cone). If D is not assumed to be positive semidefinite then W may be nonconvex, but it can be represented as the union of finitely many polyhedral convex cones. We present an example showing that, in general, φ is not piecewise linear-quadratic on W.

16.1 Auxiliary Results

Consider the standard QP problem (15.1) depending on the parameter $w = (c, b) \in R^n \times R^m$, where $D \in R_S^{n \times n}$ and $A \in R^{m \times n}$ are given matrices. Denote by $S(c, b)$, $\mathrm{Sol}(c, b)$, $\mathrm{loc}(c, b)$ and $\varphi(c, b)$, respectively, the set of the Karush-Kuhn-Tucker points, the set of the solutions, the set of the local solutions, and the optimal value of (15.1). Klatte (1985) established several fundamental facts on the

Lipschitzian continuity of the map $(c, b) \mapsto \mathrm{Sol}(c, b)$ and the function $(c, b) \mapsto \varphi(c, b)$. Among other results, he proved that $\varphi(\cdot, \cdot)$ is Lipschitzian on every bounded subset of its effective domain

$$W := \{(c, b) \in R^n \times R^m \, : \, -\infty < \varphi(c, b) < +\infty\}. \tag{16.1}$$

Following Klatte (1985), we consider the next auxiliary problem

$$\begin{cases} \text{Minimize} & \dfrac{1}{2}(c^T x + b^T \lambda) \\ \text{subject to} & (x, \lambda) \in P_{KKT}(c, b) \end{cases} \tag{16.2}$$

where

$$\begin{aligned} P_{KKT}(c, b) = \{(x, \lambda) \in R^n \times R^m \, : \quad & Dx - A^T\lambda + c = 0, \\ & Ax \geq b, \quad \lambda \geq 0, \\ & \lambda^T(Ax - b) = 0\}. \end{aligned} \tag{16.3}$$

Elements of $P_{KKT}(c, b)$ are the Karush-Kuhn-Tucker pairs of (15.1).

Let

$$\varphi_{KKT}(c, b) = \inf\left\{\frac{1}{2}(c^T x + b^T \lambda) \, : \, (x, \lambda) \in P_{KKT}(c, b)\right\} \tag{16.4}$$

be the optimal value of the auxiliary problem (16.2). By definition,

$$\varphi(c, b) = \inf\left\{\frac{1}{2}x^T Dx + c^T x \, : \, Ax \geq b, \; x \in R^n\right\}. \tag{16.5}$$

Denote by $\mathrm{Sol}_{KKT}(c, b)$ the solution set of (16.2).

Lemma 16.1. (See Klatte (1985), p. 820) *If* $\mathrm{Sol}(c, b)$ *is nonempty then* $\mathrm{Sol}_{KKT}(c, b)$ *is nonempty, and*

$$\mathrm{Sol}(c, b) = \pi_{R^n}(\mathrm{Sol}_{KKT}(c, b)), \tag{16.6}$$

$$\varphi(c, b) = \varphi_{KKT}(c, b), \tag{16.7}$$

where, by definition, $\pi_{R^n}(x, \lambda) = x$ *for every* $(x, \lambda) \in R^n \times R^m$.

Proof. Since $\mathrm{Sol}(c, b) \neq \emptyset$, we can select a point $\bar{x} \in \mathrm{Sol}(c, b)$. By Theorem 3.3, there exists $\bar{\lambda} \in R^m$ such that

$$D\bar{x} - A^T\bar{\lambda} + c = 0, \quad A\bar{x} \geq b, \quad \bar{\lambda} \geq 0, \quad \bar{\lambda}^T(A\bar{x} - b) = 0.$$

Let (x, λ) be a feasible point for (16.2), that is

$$Dx - A^T\lambda + c = 0, \; Ax \geq b, \quad \lambda \geq 0, \quad \lambda^T(Ax - b) = 0. \tag{16.8}$$

By (16.8), $x \in \Delta(A, b)$. Hence $f(x,c) \geq f(\bar{x}, c)$, where $f(x,c) := \frac{1}{2}x^T Dx + c^T x$. From (16.8) it follows that

$$\begin{aligned}\frac{1}{2}(c^T x + b^T \lambda) &= \frac{1}{2}(c^T x + x^T A^T \lambda) \\ &= \frac{1}{2}(c^T x + x^T Dx + x^T c) \\ &= f(x, c).\end{aligned}$$

Similarly, one has

$$\frac{1}{2}(c^T \bar{x} + b^T \bar{\lambda}) = f(\bar{x}, c). \tag{16.9}$$

Consequently,

$$\frac{1}{2}(c^T x + b^T \lambda) \geq \frac{1}{2}(c^T \bar{x} + b^T \bar{\lambda}).$$

Since this inequality holds for every $(x, \lambda) \in P_{KKT}(c, b)$, we conclude that $(\bar{x}, \bar{\lambda})$ is a solution of (16.2). Combining this with (16.4), (16.5) and (16.9), we obtain (16.7). In order to prove (16.6), we fix any $x \in \mathrm{Sol}(c, b)$. Let λ be a Lagrange multiplier corresponding to that x. The above arguments show that (x, λ) is a solution of (16.2). From this it follows that $x \in \pi_{R^n}(\mathrm{Sol}_{KKT}(c, b))$. Now, let (x, λ) be a solution of (16.2). Since (x, λ) satisfies the inequality system described in (16.8), we have

$$\frac{1}{2}(c^T x + b^T \lambda) = f(x, c).$$

Since $(\bar{x}, \bar{\lambda})$ and (x, λ) are from the solution set of (16.2), it holds

$$\frac{1}{2}(c^T x + b^T \lambda) = \frac{1}{2}(c^T \bar{x} + b^T \bar{\lambda}) = f(\bar{x}, c).$$

Consequently, $f(x, c) = f(\bar{x}, c)$. Since $x \in \Delta(A, b)$, from the last equality we deduce that $x \in \mathrm{Sol}(c, b)$. The equality (16.6) has been proved. $\square$

Note that the set W defined by (16.1) coincides with the effective domain of the multifunction $\mathrm{Sol}(\cdot, \cdot)$, that is

$$\begin{aligned} W = &\ \{(c, b) \in R^n \times R^m : -\infty < \varphi(c, b) < +\infty\} \\ = &\ \{(c, b) \in R^n \times R^m : \mathrm{Sol}(c, b) \neq \emptyset\}. \end{aligned} \tag{16.10}$$

Indeed, for any pair $(c, b) \in R^n \times R^m$, if $\mathrm{Sol}(c, b) \neq \emptyset$ then $-\infty < \varphi(c, b) < +\infty$. Conversely, if $-\infty < \varphi(c, b) < +\infty$ then $\Delta(A, b)$ is nonempty and the function $f(\cdot, c)$ is bounded below on $\Delta(A, b)$. By Theorem 2.1, $\mathrm{Sol}(c, b) \neq \emptyset$.

Taking account of (16.10), we can formulate the results from Klatte (1985) concerning the optimal value function $\varphi(c, b)$ as follows.

Lemma 16.2. (See Klatte (1985), Theorem 2) *The effective domain W of φ is the union of a finitely many polyhedral convex cones, i.e. there exists a finite number of polyhedral convex cones $W_i \subset R^n \times R^m$ $(i = 1, 2, \dots, s)$ such that*

$$W = \bigcup_{i=1}^{s} W_i. \tag{16.11}$$

Lemma 16.3. (See Klatte (1985), Theorem 3) *The function φ is Lipschitzian on every bounded subset $\Omega_0 \subset W$, i.e., for each bounded subset $\Omega_0 \subset W$ there exists a constant $k_{\Omega_0} > 0$ such that*

$$\|\varphi(c', b') - \varphi(c, b)\| \leq k_{\Omega_0}(\|c' - c\| + \|b' - b\|)$$

for any (c, b), $(c', b') \in \Omega_0$.

For each subset $I \subset \{1, 2, \dots, m\}$, we define

$$\begin{aligned} P^I_{KKT}(c, b) = \{(x, \lambda) \in R^n \times R^m :\ & Dx - A^T\lambda + c = 0, \\ & A_i x \geq b_i,\ ; \lambda_i = 0 \ \ (\forall i \in I), \\ & A_j x = b_j, \ \ \lambda_j \geq 0 \ \ (\forall j \notin I)\}, \end{aligned} \tag{16.12}$$

where A_i $(i \in \{1, \dots, m\})$ is the $i-$th row of the matrix A and b_i is the $i-$th component of b. It is clear that

$$P_{KKT}(c, b) = \bigcup_{I \subset \{1, \dots, m\}} P^I_{KKT}(c, b). \tag{16.13}$$

Note that $P^I_{KKT}(c, b)$ is the solution set of the following system of linear equalities and inequalities:

$$\begin{cases} Dx - A^T\lambda + c = 0, \\ A_I x \geq b_I, \quad \lambda_I = 0 \\ A_J x = b_J, \quad \lambda_J \geq 0, \\ x \in R^n, \quad \lambda \in R^m, \end{cases} \tag{16.14}$$

where $J = \{1, 2, \ldots, m\} \setminus I$ and, as usual, A_J denotes the matrix composed by the rows A_j $(j \in J)$ of A, and λ_I is the vector with the components λ_i $(i \in I)$.

Let

$$\varphi^I_{KKT}(c, b) = \inf \left\{ \frac{1}{2}(c^T x + b^T \lambda) \, : \, (x, \lambda) \in P^I_{KKT}(c, b) \right\}. \quad (16.15)$$

Thus $\varphi^I_{KKT}(c, b)$ is the optimal value of the *linear programming problem* whose objective function is $\frac{1}{2}(c^T x + b^T \lambda)$ and whose constraints are described by (16.14). Note that the pair (c, b) represents the right-hand-side perturbations of the linear system (16.14).

It turns out that, for any $I \subset \{1, 2, \ldots, m\}$, the effective domain of $\varphi^I_{KKT}(\cdot)$ is a polyhedral convex cone on which the function admits a linear-quadratic representation. Namely, using the concept of pseudo-matrix one can establish the following result.

Lemma 16.4. (See Bank et al. (1982), Theorem 5.5.2) *The effective domain*

$$\mathrm{dom}\varphi^I_{KKT} = \{(c, b) \in R^n \times R^m \, : \, -\infty < \varphi^I_{KKT}(c, b) < +\infty\}$$

is a polyhedral convex cone and there exist

$$M_I \in R^{(n+m)\times(n+m)} \quad \text{and} \quad q_I \in R^{n+m}$$

such that

$$\varphi^I_{KKT}(c, b) = \frac{1}{2}\begin{pmatrix} c \\ b \end{pmatrix}^T M_I \begin{pmatrix} c \\ b \end{pmatrix} + q_I^T \begin{pmatrix} c \\ b \end{pmatrix} \quad (16.16)$$

for every $(c, b) \in \mathrm{dom}\varphi^I_{KKT}$.

The following useful fact follows from Lemma 16.1.

Lemma 16.5. *For any* $(c, b) \in W$, *it holds*

$$\varphi(c, b) = \min\{\varphi^I_{KKT}(c, b) \, : \, I \subset \{1, 2, \ldots, m\}\}. \quad (16.17)$$

Proof. From (16.4), (16.13) and (16.15), we deduce that

$$\varphi_{KKT}(c, b) = \min\{\varphi^I_{KKT}(c, b) \, : \, I \subset \{1, 2, \ldots, m\}\}.$$

Combining this with (16.7) we obtain (16.17). □

Remark 16.1. From (16.17) it follows that, for for any $(c, b) \in W$ and for any $I \subset \{1, 2, \ldots, m\}$, we have $\varphi^I_{KKT}(c, b) > -\infty$.

Remark 16.2. It may happen that for some pairs $(c,b) \in W$ the function φ^I_{KKT} has the value $+\infty$. Note that $\varphi^I_{KKT}(c,b) = +\infty$ if and only if the solution set of (16.14) is empty. The example considered in Section 16.3 will illustrate this situation.

Remark 16.3. If D is a positive semidefinite matrix then (15.1) is a convex QP problem and the equality $\varphi^{I_1}_{KKT}(c,b) = \varphi^{I_2}_{KKT}(c,b)$ holds for any index sets I_1, $I_2 \subset \{1,2,\dots,m\}$ and for any point $(c,b) \in \mathrm{dom}\varphi^{I_1}_{KKT} \cap \mathrm{dom}\varphi^{I_2}_{KKT}$. The last equality is valid because any KKT point of a convex QP problem is a solution.

16.2 Directional Differentiability

In this section, we will prove that although φ is not a convex function but it enjoys the important property of convex functions of being directionally differentiable at any point in its effective domain. A formula for computing the directional derivative $\varphi'(\bar{w};z)$ of φ at any $\bar{w} = (\bar{c},\bar{b}) \in W$ in direction $z = (u,v) \in R^n \times R^m$ is also established.

Recall (Rockafellar (1970), p. 13) that a subset $K \subset R^p$ is called a *cone* if $tx \in K$ whenever $x \in K$ and $t > 0$. (The origin itself may or may not be included in K).

Proposition 16.1. *Let W be defined by* (16.1),

$$Z_1 = \{(c,b) \in R^n \times R^m \ : \ \varphi(c,b) = +\infty\},$$

$$Z_2 = \{(c,b) \in R^n \times R^m \ : \ \varphi(c,b) = -\infty\}, \ \textit{and}$$

$$L = \{b \in R^m \ : \ \Delta(A,b) \ \ \textit{is nonempty}\}.$$

Then Z_1 is an open cone, W is a closed cone, and Z_2 is a cone which is relatively open in the polyhedral convex cone $R^n \times L \subset R^n \times R^m$. Moreover, it holds

$$R^n \times L = W \cup Z_2, \ R^n \times R^m = W \cup Z_2 \cup Z_1, \ Z_1 = (R^n \times R^m) \setminus (R^n \times L). \tag{16.18}$$

The easy proof of this proposition is omitted.

Theorem 16.1. *The optimal value function φ defined in* (16.5) *is directionally differentiable on W, i.e., for any $\bar{w} = (\bar{c},\bar{b}) \in W$ and for any $z = (u,v) \in R^n \times R^m$ there exists the directional derivative*

$$\varphi'(\bar{w};z) := \lim_{t \downarrow 0} \frac{\varphi(\bar{w}+tz) - \varphi(\bar{w})}{t} \tag{16.19}$$

of φ at $\bar{w}$ in direction z.

Proof. Let $\bar{w} = (\bar{c}, \bar{b}) \in W$ and $z = (u, v) \in R^n \times R^m$ be given arbitrarily. If $z = 0$ then it is obvious that $\varphi'(\bar{w}; z) = 0$. Assume that $z \neq 0$. We first prove that one of the following three cases must occur:

(c1) There exists $\bar{t} > 0$ such that $\bar{w} + tz \in Z_1$ for every $t \in (0, \bar{t}]$,

(c2) There exists $\bar{t} > 0$ such that $\bar{w} + tz \in Z_2$ for every $t \in (0, \bar{t}]$,

(c3) There exists $\bar{t} > 0$ such that $\bar{w} + tz \in W$ for every $t \in (0, \bar{t}]$.

For this purpose, suppose that (c3) fails to hold. We have to show that, in this case, (c1) or (c2) must occur. Since (c3) is not valid, we can find a decreasing sequence $t_k \to 0+$ such that $\bar{w} + t_k z \notin W$ for every $k \in N$. By (16.18), for each $k \in N$, we must have $\bar{w} + t_k z \in Z_1$ or $\bar{w} + t_k z \in Z_2$. Hence, there exists a subsequence $\{t_{k_i}\}$ of $\{t_k\}$ such that

$$\bar{w} + t_{k_i} z \in Z_1 \quad (\forall i \in N), \tag{16.20}$$

or

$$\bar{w} + t_{k_i} z \in Z_2 \quad (\forall i \in N). \tag{16.21}$$

Consider the case where (16.20) is fulfilled. If there exists an $\hat{t} \in (0, t_{k_1})$ such that

$$\bar{w} + \hat{t} z \in R^n \times L$$

then, by the convexity of $R^n \times L$,

$$\{\bar{w} + tz : t \in [0, \hat{t}]\} \subset R^n \times L.$$

By virtue of the first equality in (16.18), this yields $\varphi(\bar{w} + tz) \neq +\infty$ for every $t \in [0, \hat{t}]$, contradicting (16.20). Thus (16.20) implies that $\bar{w} + tz \notin R^n \times L$ for every $t \in (0, t_{k_1})$. Then, the third equality in (16.18) shows that $\bar{w} + tz \in Z_1$ for every $t \in (0, t_{k_1})$. Putting $\bar{t} = t_{k_1}$, we see at once that (c1) holds.

Consider the case where (16.21) is fulfilled. Since $\bar{w} \in W \subset R^n \times L$ and

$$\bar{w} + t_{k_1} z \in Z_2 \subset R^n \times L,$$

it follows that

$$\{\bar{w} + tz \,:\, t \in [0, t_{k_1}]\} \subset R^n \times L.$$

Therefore, we can deduce from the first equality in (16.18) that, for every $t \in (0, t_{k_1})$, $\bar{w} + tz \in Z_2$ or $\bar{w} + tz \in W$. If there exists $i \in N$ such that

$$\bar{w} + tz \in Z_2 \quad (\forall t \in (0, t_{k_i})) \tag{16.22}$$

then (c2) is satisfied if we choose $\bar{t} = t_{k_i}$. If there is no $i \in N$ such that (16.22) is valid, then for every $i \in N$ there must exist some $t'_{k_i} \in (0, t_{k_i})$ such that $\bar{w} + t'_{k_i} z \in W$. By (16.11), there is an index $j(k_i) \in \{1, \ldots, s\}$ such that

$$\bar{w} + t'_{k_i} z \in W_{j(k_i)}. \tag{16.23}$$

Without loss of generality, we can assume that

$$0 < t'_{k_{i+1}} < t_{k_{i+1}} < t'_{k_i} < t_{k_i} \quad (\forall i \in N). \tag{16.24}$$

Since $j(k_i) \in \{1, \ldots, s\}$, there must exist a pair (i, j) such that $j > i$ and $j(k_j) = j(k_i)$. By (16.23) and by the convexity of $W_{j(k_i)}$, we have

$$\{\bar{w} + tz \,:\, t'_{k_j} \le t \le t'_{k_i}\} \subset W_{j(k_i)} \subset W. \tag{16.25}$$

From (16.21) and (16.24) we get $\varphi(\bar{w} + t_{k_{i+1}} z) = -\infty$ and $t'_{k_j} < t_{k_{i+1}} < t'_{k_i}$, a contradiction to (16.25). We have thus proved that if (16.21) is valid then (c2) must occur.

Summarizing all the above, we conclude that one of the three cases (c1)–(c3) must occur.

If (c1) occurs then, by (16.19), we have $\varphi'(\bar{w}; z) = +\infty$. Similarly, if (c2) happens then $\varphi'(\bar{w}; z) = -\infty$.

Now assume that (c3) takes place. Denote by F the collection of the index sets $I \subset \{1, 2, \ldots, m\}$ for which there exists $t_I \in (0, \bar{t})$, where $\bar{t} > 0$ is given by (c3), such that

$$\{\bar{w} + tz \,:\, t \in [0, t_I]\} \subset \mathrm{dom}\varphi^I_{KKT}. \tag{16.26}$$

Recall that $\mathrm{dom}\varphi^I_{KKT}$ is a closed convex set (see Lemma 16.4). If $F = \emptyset$ then for any $I \subset \{1, 2, \ldots, m\}$ and for any $t \in (0, \bar{t}]$ one has $\varphi^I_{KKT}(\bar{w} + tz) = +\infty$. By (c3), $\bar{w} + tz \in W$ for all $t \in (0, \bar{t}]$. Then, according to (16.18) we have

$$\varphi(\bar{w} + tz) = \min\{\varphi^I_{KKT}(\bar{w} + tz) \,:\, I \subset \{1, 2, \ldots, m\}\} = +\infty$$

for all $t \in (0, \bar{t}]$, which is impossible. We have shown that $F \neq \emptyset$. Define

$$\hat{t} = \min\{t_I \,:\, I \in F\} > 0.$$

By virtue of (c3) and of (16.18), one has

$$\varphi(\bar{w} + tz) = \min\{\varphi^I_{KKT}(\bar{w} + tz) \, : \, I \in F\} \quad (\forall t \in [0, \hat{t}]). \quad (16.27)$$

It follows from (16.26) that

$$\bar{w} + tz \in \mathrm{dom}\varphi^I_{KKT} \quad (\forall I \in F, \, \forall t \in [0, \hat{t}]).$$

For each $I \in F$, let $M_I \in R^{(n+m)\times(n+m)}$ and $q_I \in R^{n+m}$ be such that the representation (16.16) holds for all $(c, b) \in \mathrm{dom}\varphi^I_{KKT}$. Setting

$$\tilde{\varphi}^I_{KKT}(c, b) = \frac{1}{2}\begin{pmatrix} c \\ b \end{pmatrix}^T M_I \begin{pmatrix} c \\ b \end{pmatrix} + q_I^T \begin{pmatrix} c \\ b \end{pmatrix} \qquad (16.28)$$

for every $(c, b) \in R^n \times R^m$, we extend $\varphi^I_{KKT}(\cdot)$ from $\mathrm{dom}\varphi^I_{KKT}$ to the whole space $R^n \times R^m$. From (16.28) it follows that all the functions $\tilde{\varphi}^I_{KKT}(\cdot)$, $I \in F$, are smooth. According to Theorem 2.1 in Clarke (1975), the function

$$\tilde{\varphi}(c, b) = \min\{\tilde{\varphi}^I_{KKT}(c, b) \, : \, I \in F\}$$

is locally Lipschitz at $\bar{w} = (\bar{c}, \bar{b})$. Moreover, $\tilde{\varphi}$ is *Lipschitz regular* (see Definition 2.3.4 in Clarke (1983)) at $\bar{w}$, and

$$\tilde{\varphi}^0(\bar{w}; z) = \tilde{\varphi}'(\bar{w}; z) = \min\{(\tilde{\varphi}^I_{KKT})'(\bar{w}; z) \, : \, I \in F\}, \qquad (16.29)$$

where $\tilde{\varphi}^0(\bar{w}; z)$ (resp., $\tilde{\varphi}'(\bar{w}; z)$) denotes the Clarke generalized directional derivative (resp., the directional derivative) of $\tilde{\varphi}$ at $\bar{w}$ in direction z. Since

$$\tilde{\varphi}^I_{KKT}(c, b) = \varphi^I_{KKT}(c, b)$$

for all $(c, b) \in \mathrm{dom}\varphi^I_{KKT}$, from (16.27) and (16.29) it follows that the directional derivative $\varphi'(\bar{w}; z)$ exists, and we have

$$\varphi'(\bar{w}; z) = \min\{(\varphi^I_{KKT})'(\bar{w}; z) \, : \, I \in F\}. \qquad (16.30)$$

The proof is complete. $\square$

In the course of the above proof we have obtained some explicit formulae for computing the directional derivative of the function φ. Namely, we have proved the following result.

Theorem 16.2. *Let $\bar{w} \in W$ and $z = (u, v) \in R^n \times R^m$. The following assertions hold:*

(i) *If there exists* $\bar{t} > 0$ *such that*

$$\bar{w} + tz \in Z_1 = \{(c,b) \,:\, \Delta(A,b) = \emptyset\}$$

for all $t \in (0, \bar{t}])$, *then* $\varphi'(\bar{w}; z) = +\infty$.

(ii) *If there exists* $\bar{t} > 0$ *such that*

$$\bar{w} + tz \in Z_2 = \{(c,b) \,:\, \Delta(A,b) \neq \emptyset,\ \varphi(c,b) = -\infty\}$$

for all $t \in (0, \bar{t}])$, *then* $\varphi'(\bar{w}; z) = -\infty$.

(iii) *If there exists* $\bar{t} > 0$ *such that*

$$\bar{w} + tz \in W = \{(c,b) \,:\, \Delta(A,b) \neq \emptyset,\ \varphi(c,b) > -\infty\}$$

for all $t \in (0, \bar{t}])$, *then* $\varphi'(\bar{w}; z)$ *can be computed by formula* (16.30), *where* F *is the collection of all* $I \subset \{1, 2, \ldots, m\}$ *for which there exists some* $t_I \in (0, \bar{t})$ *satisfying condition* (16.26).

At the end of the next section we shall use Theorem 16.2 for computing directional derivative of the optimal value function in a concrete nonconvex QP problem.

16.3 Piecewise Linear-Quadratic Property

The notion of *piecewise linear-quadratic function* (plq function, for brevity) was introduced in Rockafellar (1988).

Definition 16.1. (See Rockafellar and Wets (1998), p. 440) A function $\psi : R^l \to \bar{R}$ is piecewise linear-quadratic (plq) if the set

$$\mathrm{dom}\psi = \{z \in R^l \,:\, -\infty < \psi(z) < +\infty\} \tag{16.31}$$

can be represented as the union of finitely many polyhedral convex sets, relative to each of which $\psi(z)$ is given by an expression of the form

$$\frac{1}{2} z^T Q z + d^T z + \alpha \tag{16.32}$$

for some $\alpha \in R$, $d \in R^l$, $Q \in R_S^{l \times l}$.

Note that in Rockafellar and Wets (1998) instead of (16.31) one has the following formula

$$\mathrm{dom}\psi = \{z \in R^l \,:\, \psi(z) < +\infty\}. \tag{16.33}$$

If there exists some $\bar{z} \in R^l$ with $\psi(\bar{z}) = -\infty$ then, since $\bar{z}$ belongs to the set defined in (16.33), one cannot represent the latter as the union of finitely many polyhedral convex sets, relative to each of which $\psi(z)$ is given by an expression of the form (16.32). Hence ψ cannot be a plq function. This is the reason why we prefer (16.31) to (16.33).

If D is a positive semidefinite matrix then, by using the Eaves Theorem we can prove that W is a polyhedral convex cone. Using Lemmas 16.4, 16.5, and Remark 16.3, it is not difficult to show that *the optimal value function $\varphi(c, b) = \varphi(c, b)$ of a convex QP problem is plq.*

Example 16.1. (See Rockafellar and Wets (1998)) Consider the function

$$\psi(z) = |z_1^2 + z_2^2 - 1|, \quad z = (z_1, z_2) \in R^2.$$

We have $R^2 = \Omega_1 \cup \Omega_2$, where

$$\Omega_1 = \{z \, : \, z_1^2 + z_2^2 \le 1\}, \quad \Omega_2 = \{z \, : \, z_1^2 + z_2^2 \ge 1\}.$$

The formulae

$$\psi(z) = -z_1^2 - z_2^2 + 1 \ (\forall z \in \Omega_1) \ \text{ and } \ \psi(z) = z_1^2 + z_2^2 - 1 \ (\forall z \in \Omega_2)$$

show that ψ admits a representation of the form (16.33) on each domain Ω_i $(i = 1, 2)$. Meanwhile, it can be proved that ψ is *not* a plq function.

Note that if the function $\varphi(c, b)$ defined by (16.5) is plq then, for any $\bar{b} \in R^m$, the function $\varphi(\cdot, \bar{b})$ is also plq on its effective domain. Indeed, assume that $\varphi(c, b)$ is plq, that is W admits a representation of the form $W = \bigcup_{i=1}^s W_i$, where every W_i is a polyhedral convex set and there exist $Q_i \in R_S^{(n+m)\times(n+m)}$, $d_i \in R^{n+m}$ and $\alpha_i \in R$ such that

$$\varphi(c, b) = \frac{1}{2}\begin{pmatrix} c \\ b \end{pmatrix}^T Q_i \begin{pmatrix} c \\ b \end{pmatrix} + d_i^T \begin{pmatrix} c \\ b \end{pmatrix} + \alpha_i \quad (\forall (c, b) \in W_i). \tag{16.34}$$

Let $\bar{b} \in R^m$ be given arbitrarily. Define

$$W' = \{c \in R^n \, : \, (c, \bar{b}) \in W\}, \quad W_i' = \{c \in R^n \, : \, (c, \bar{b}) \in W_i\}$$

for all $i = 1, \ldots, s$. It is obvious that $W' = \mathrm{dom}\varphi(\cdot, \bar{b})$ and $W' = \bigcup_{i=1}^s W_i'$. Moreover, for every $i \in \{1, \ldots, s\}$, from (16.34) it follows that

$$\varphi(c, \bar{b}) = \frac{1}{2}\begin{pmatrix} c \\ \bar{b} \end{pmatrix}^T Q_i \begin{pmatrix} c \\ \bar{b} \end{pmatrix} + d_i^T \begin{pmatrix} c \\ \bar{b} \end{pmatrix} + \alpha_i \quad (\forall c \in W_i').$$

Since the function in the right-hand-side of this formula is a linear-quadratic function of c and since each W_i' is a polyhedral convex set (maybe empty), we conclude that $\varphi(\cdot, \bar{b})$ is a plq function.

We are interested in solving the following question: *Whether the optimal value function in a general (indefinite) parametric quadratic programming problem is a plq function w.r.t. the linear parameters?*

It turns out that the plq property is not available in the general case.

Example 16.2. Consider the problem

$$\begin{cases} \text{Minimize } f(x,c) = \frac{1}{2}(x_1^2 + 2x_1x_2 - x_2^2) + c_1x_1 + c_2x_2 \\ \text{subject to} \quad x = (x_1, x_2) \in R^2, \quad \frac{1}{2}x_1 + x_2 \geq 0, \\ \qquad\qquad\quad x_2 - x_1 \geq 0, \quad -x_2 \geq -2, \end{cases} \tag{16.35}$$

and denote by $\varphi(c)$, $c = (c_1, c_2) \in R^2$, the optimal value of this nonconvex QP problem.

In the remainder of this section we will compute the values $\varphi(c)$, $c \in R^2$. In the next section it will be shown that the function $\varphi(c)$ is not plq. Then we can conclude that the optimal value function $\varphi(c, b)$, $c = (c_1, c_2) \in R^2$ and $b = (b_1, b_2, b_3) \in R^3$, of the following parametric QP problem is not plq:

$$\begin{cases} \text{Minimize } f(x,c) = \frac{1}{2}(x_1^2 + 2x_1x_2 - x_2^2) + c_1x_1 + c_2x_2 \\ \text{subject to} \quad x = (x_1, x_2) \in R^2, \quad \frac{1}{2}x_1 + x_2 \geq b_1, \\ \qquad\qquad\quad x_2 - x_1 \geq b_2, \quad -x_2 \geq b_3. \end{cases} \tag{16.36}$$

Indeed, if $\varphi(c, b)$ is plq then the arguments given after Example 16.1 show that $\varphi(c) = \varphi(c, \bar{b})$, where $\bar{b} = (0, 0, -2)$, is a plq function, which is impossible.

In order to write (16.35) in the form (15.1), we put

$$D = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad b = \bar{b} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Note that the feasible domain $\Delta(A, \bar{b})$ of (16.36) is a triangle with the vertexes $(0, 0)$, $(2, 2)$ and $(-4, 2)$. Since $\Delta(A, \bar{b})$ is compact, $\varphi(c, \bar{b}) \in R$ for every $c \in R^2$. In other words, $\mathrm{dom}\varphi(\cdot, \bar{b}) = R^2$.

In agreement with (16.2) and (16.3), the auxiliary problem corresponding to (16.35) is the following one

$$\begin{cases} \text{Minimize } \frac{1}{2}(c^Tx + b^T\lambda) = \frac{1}{2}(c_1x_1 + c_2x_2) - \lambda_3 \\ \text{subject to } \quad (x,\lambda) = (x_1, x_2, \lambda_1, \lambda_2, \lambda_3) \in R^2 \times R^3, \\ x_1 + x_2 - \frac{1}{2}\lambda_1 + \lambda_2 + c_1 = 0, \\ x_1 - x_2 - \lambda_1 - \lambda_2 + \lambda_3 + c_2 = 0, \\ \frac{1}{2}x_1 + x_2 \geq 0, \quad \lambda_1 \geq 0, \quad \lambda_1(\frac{1}{2}x_1 + x_2) = 0, \\ x_2 - x_1 \geq 0, \quad \lambda_2 \geq 0, \quad \lambda_2(x_2 - x_1) = 0, \\ x_2 \leq 2, \quad \lambda_3 \geq 0, \quad \lambda_3(2 - x_2) = 0. \end{cases} \tag{16.37}$$

We shall apply formula (16.17) to compute the values $\varphi(c,b)$, $c \in R^2$, $b = \bar{b}$. To do so, we have to compute the optimal value $\varphi^I_{KKT}(c,b)$ defined by (16.15), where $I \subset \{1,2,3\}$ is an arbitrary subset. Since there are 8 possibilities to choose such index set I, we have to consider 8 linear subproblems of the problem (16.37).

Case 1. $I = I_1 = \{1,2,3\}$. In the corresponding subproblem we must have $\lambda_I = (\lambda_1, \lambda_2, \lambda_3) = (0,0,0)$. Taking account of (16.37), we can write that subproblem as follows

$$\begin{cases} \frac{1}{2}(c_1x_1 + c_2x_2) \to \min \\ x_1 + x_2 + c_1 = 0, \quad x_1 - x_2 + c_2 = 0, \\ \frac{1}{2}x_1 + x_2 \geq 0, \quad x_2 - x_1 \geq 0, \quad x_2 \leq 2. \end{cases} \tag{I_1}$$

In accordance with (16.15), we denote the optimal value of (I_1) by $\varphi^{I_1}_{KKT}(c,\bar{b})$. An elementary investigation on (I_1) gives us the following result:

$$\begin{cases} \varphi^{I_1}_{KKT}(c,\bar{b}) = \frac{1}{4}(-c_1^2 - 2c_1c_2 + c_2^2), \\ \mathrm{dom}\varphi^{I_1}_{KKT}(\cdot,\bar{b}) = \{c = (c_1,c_2) \,:\, -3c_1 + c_2 \geq 0, \\ \qquad\qquad\qquad\qquad c_2 \geq 0,\ -c_1 + c_2 \leq 4\}. \end{cases} \tag{16.38}$$

The exact meaning of (16.38) is the following: We have

$$\varphi^{I_1}_{KKT}(c,\bar{b}) = \frac{1}{4}(-c_1^2 - 2c_1c_2 + c_2^2)$$

for every $c \in \mathrm{dom}\varphi^{I_1}_{KKT}(\cdot,\bar{b})$ and

$$\varphi^{I_1}_{KKT}(c,\bar{b}) = +\infty$$

for every $c \notin \mathrm{dom}\varphi^{I_1}_{KKT}(\cdot,\bar b)$. A similar interpretation applies to the results of the forthcoming 7 cases.

Case 2. $I = I_2 = \{1,2\}$. We have $\lambda_I = (\lambda_1,\lambda_2) = (0,0)$, $\lambda_3 \geq 0$. The corresponding subproblem is

$$\begin{cases} \frac{1}{2}(c_1x_1 + c_2x_2) - \lambda_3 \to \min \\ x_1 + x_2 + c_1 = 0, \quad x_1 - x_2 + \lambda_3 + c_2 = 0, \\ \frac{1}{2}x_1 + x_2 \geq 0, \quad x_2 - x_1 \geq 0, \quad x_2 = 2, \quad \lambda_3 \geq 0. \end{cases} \tag{I_2}$$

Then

$$\begin{cases} \varphi^{I_2}_{KKT}(c,\bar b) = -\frac{1}{2}c_1^2 - 2c_1 + 2c_2 - 4, \\ \mathrm{dom}\varphi^{I_2}_{KKT}(\cdot,\bar b) = \{c = (c_1,c_2) \,:\, 2 - c_1 \geq 0, \\ \qquad\qquad 4 + c_1 \geq 0,\ 4 + c_1 - c_2 \geq 0\}. \end{cases} \tag{16.39}$$

Case 3. $I = I_3 = \{2,3\}$. We have $\lambda_I = (\lambda_2,\lambda_3) = (0,0)$, $\lambda_1 \geq 0$. The corresponding subproblem is

$$\begin{cases} \frac{1}{2}(c_1x_1 + c_2x_2) \to \min \\ x_1 + x_2 - \frac{1}{2}\lambda_1 + c_1 = 0, \quad x_1 - x_2 - \lambda_1 + c_2 = 0, \\ \frac{1}{2}x_1 + x_2 = 0, \quad \lambda_1 \geq 0, \quad x_2 - x_1 \geq 0, \quad x_2 \leq 2. \end{cases} \tag{I_3}$$

Then

$$\begin{cases} \varphi^{I_3}_{KKT}(c,\bar b) = 2c_1^2 + \frac{1}{2}c_2^2 - 2c_1c_2, \\ \mathrm{dom}\varphi^{I_3}_{KKT}(\cdot,\bar b) = \{c = (c_1,c_2) \,:\, c_2 \leq 3c_1, \\ \qquad\qquad c_2 - 2c_1 \geq 0, \quad c_2 - 2c_1 \leq 2\}. \end{cases} \tag{16.40}$$

Case 4. $I = I_4 = \{1,3\}$. We have $\lambda_I = (\lambda_1,\lambda_3) = (0,0)$, $\lambda_2 \geq 0$. The corresponding subproblem is

$$\begin{cases} \frac{1}{2}(c_1x_1 + c_2x_2) \to \min \\ x_1 + x_2 + \lambda_2 + c_1 = 0, \quad x_1 - x_2 - \lambda_2 + c_2 = 0, \\ \frac{1}{2}x_1 + x_2 \geq 0, \quad x_2 - x_1 = 0, \quad \lambda_2 \geq 0, \quad x_2 \leq 2. \end{cases} \tag{I_4}$$

Then

$$\begin{cases} \varphi^{I_4}_{KKT}(c,\bar b) = -\frac{1}{4}(c_1 + c_2)^2, \\ \mathrm{dom}\varphi^{I_4}_{KKT}(\cdot,\bar b) = \{c = (c_1,c_2) \,:\, c_1 + c_2 \leq 0, \\ \qquad\qquad c_1 + c_2 \geq -4, \quad c_2 \geq 0\}. \end{cases} \tag{16.41}$$

Case 5. $I = I_5 = \{1\}$. We have $\lambda_1 = 0,\ \lambda_2 \geq 0,\ \lambda_3 \geq 0$. The corresponding subproblem is

$$\begin{cases} \dfrac{1}{2}(c_1x_1 + c_2x_2) - \lambda_3 \to \min \\ x_1 + x_2 + \lambda_2 + c_1 = 0, \quad x_1 - x_2 - \lambda_2 + \lambda_3 + c_2 = 0, \\ \dfrac{1}{2}x_1 + x_2 \geq 0, \quad x_2 - x_1 = 0, \quad \lambda_2 \geq 0, \quad x_2 = 2, \quad \lambda_3 \geq 0. \end{cases} \tag{I_5}$$

Then

$$\begin{cases} \varphi_{KKT}^{I_5}(c, \bar{b}) = 2c_1 + 2c_2 + 4, \\ \mathrm{dom}\varphi_{KKT}^{I_4}(\cdot, \bar{b}) = \{c = (c_1, c_2)\ :\ c_1 + 4 \leq 0, \quad c_1 + c_2 + 4 \leq 0\}. \end{cases} \tag{16.42}$$

Case 6. $I = I_6 = \{2\}$. We have $\lambda_2 = 0,\ \lambda_1 \geq 0,\ \lambda_3 \geq 0$. The corresponding subproblem is

$$\begin{cases} \dfrac{1}{2}(c_1x_1 + c_2x_2) - \lambda_3 \to \min \\ x_1 + x_2 - \dfrac{1}{2}\lambda_1 + c_1 = 0, \quad x_1 - x_2 - \lambda_1 + \lambda_3 + c_2 = 0, \\ \dfrac{1}{2}x_1 + x_2 = 0, \quad \lambda_1 \geq 0, \quad x_2 - x_1 \geq 0, \quad x_2 = 2, \quad \lambda_3 \geq 0. \end{cases} \tag{I_6}$$

Then

$$\begin{cases} \varphi_{KKT}^{I_6}(c, \bar{b}) = -4c_1 + 2c_2 - 2, \\ \mathrm{dom}\varphi_{KKT}^{I_6}(\cdot, \bar{b}) = \{c = (c_1, c_2)\ :\ c_1 - 2 \geq 0, \\ \qquad\qquad\qquad\qquad\qquad 2 + 2c_1 - c_2 \geq 0\}. \end{cases} \tag{16.43}$$

Case 7. $I = I_7 = \{3\}$. We have $\lambda_3 = 0,\ \lambda_1 \geq 0,\ \lambda_2 \geq 0$. The corresponding subproblem is

$$\begin{cases} \dfrac{1}{2}(c_1x_1 + c_2x_2) \to \min \\ x_1 + x_2 - \dfrac{1}{2}\lambda_1 + \lambda_2 + c_1 = 0, \quad x_1 - x_2 - \lambda_1 - \lambda_2 + c_2 = 0, \\ \dfrac{1}{2}x_1 + x_2 = 0, \quad \lambda_1 \geq 0, \quad x_2 - x_1 = 0, \quad \lambda_2 \geq 0, \quad x_2 \leq 2. \end{cases} \tag{I_7}$$

Then

$$\begin{cases} \varphi_{KKT}^{I_7}(c, \bar{b}) = 0, \\ \mathrm{dom}\varphi_{KKT}^{I_7}(\cdot, \bar{b}) = \{c = (c_1, c_2)\ :\ c_1 + c_2 \geq 0, \quad c_2 - 2c_1 \geq 0\}. \end{cases} \tag{16.44}$$

Case 8. $I = I_8 = \emptyset$. We have $\lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \lambda_3 \geq 0$. The corresponding subproblem is

$$\begin{cases} \frac{1}{2}(c_1x_1 + c_2x_2) - \lambda_3 \to \min \\ x_1 + x_2 - \frac{1}{2}\lambda_1 + \lambda_2 + c_1 = 0,\ x_1 - x_2 - \lambda_1 - \lambda_2 + \lambda_3 + c_2 = 0, \\ \frac{1}{2}x_1 + x_2 = 0,\ \lambda_1 \geq 0, \quad x_2 - x_1 = 0,\ \lambda_2 \geq 0,\ x_2 = 2,\ \lambda_3 \geq 0. \end{cases} \tag{I_8}$$

Then

$$\begin{cases} \varphi^{I_8}_{KKT}(c, \bar{b}) = +\infty \quad \text{for every } c \in R^2, \\ \text{dom}\varphi^{I_8}_{KKT}(\cdot, \bar{b}) = \emptyset. \end{cases} \tag{16.45}$$

Consider the following polyhedral convex subsets of R^2:

$$\begin{aligned}
\Omega_1 = &\ \{c = (c_1, c_2) : c_2 \leq -c_1 - 4, \quad c_1 \leq -4\}, \\
\Omega_2 = &\ \{c = (c_1, c_2) : c_2 \geq -c_1 - 4, \quad c_2 \leq -c_1, \quad c_2 \geq c_1 + 4\}, \\
\Omega_3 = &\ \{c = (c_1, c_2) : c_2 \geq -c_1, \quad c_2 \geq c_1 + 4, \quad c_2 \geq 2c_1 + 2\}, \\
\Omega_4 = &\ \{c = (c_1, c_2) : c_2 \leq 2c_1 + 2, \quad c_2 \geq 2c_1 + 1, \quad c_1 \geq 2\}, \\
\Omega_5 = &\ \{c = (c_1, c_2) : c_2 \leq 2c_1 + 1, \quad c_1 \geq 2\}, \\
\Omega_6 = &\ \{c = (c_1, c_2) : c_1 \leq 2, \quad c_2 \leq 2c_1, \quad c_2 \geq 0\}, \\
\Omega_7 = &\ \{c = (c_1, c_2) : c_2 \leq 0, \quad c_1 \geq -4, \quad c_1 \leq 2\}, \\
\Omega_8 = &\ \{c = (c_1, c_2) : c_2 \geq 0, \quad c_2 \leq -c_1, \\
&\qquad c_2 \leq (\sqrt{2} - 1)(c_1 + 4)\}, \\
\Omega_9 = &\ \{c = (c_1, c_2) : c_2 \geq (\sqrt{2} - 1)(c_1 + 4), \\
&\qquad c_2 \leq -c_1, \quad c_2 \leq c_1 + 4\}, \\
\Omega_{10} = &\ \{c = (c_1, c_2) : c_2 \geq -c_1,\ c_2 \leq c_1 + 4,\ c_1 \leq 2,\ c_2 \geq 2c_1\}.
\end{aligned}$$

Using formulae (16.17) and (16.38)–(16.45), one can show that

$$\begin{aligned}
\varphi(c, \bar{b}) = &\ \varphi^{I_5}_{KKT}(c, \bar{b}) = 2c_1 + 2c_2 + 4 \quad \text{for every } c \in \Omega_1, \\
\varphi(c, \bar{b}) = &\ \varphi^{I_4}_{KKT}(c, \bar{b}) = -\frac{1}{4}(c_1 + c_2)^2 \quad \text{for every } c \in \Omega_2 \cup \Omega_9, \\
\varphi(c, \bar{b}) = &\ \varphi^{I_7}_{KKT}(c, \bar{b}) = 0 \quad \text{for every } c \in \Omega_3 \cup \Omega_4, \\
\varphi(c, \bar{b}) = &\ \varphi^{I_6}_{KKT}(c, \bar{b}) = -4c_1 + 2c_2 - 2 \quad \text{for every } c \in \Omega_5,
\end{aligned}$$

and

$$\varphi(c, \bar{b}) = \varphi^{I_2}_{KKT}(c, \bar{b}) = -\frac{1}{2}c_1^2 - 2c_1 + 2c_2 - 4$$

for every $c \in \Omega_6 \cup \Omega_7 \cup \Omega_8$.

We will pay a special attention to the behavior of $\varphi(\cdot, \bar{b})$ on the region Ω_{10}. In order to compute $\varphi(c, \bar{b})$ for $c \in \Omega_{10}$, we divide Ω_{10}

into two subsets:

$$\begin{aligned}
\Omega'_{10} = & \ \{c = (c_1, c_2) \in \Omega_{10} \, : \, c_2 \geq 3c_1\} \\
= & \ \{c = (c_1, c_2) \, : \, c_2 \geq -c_1, \quad c_2 \geq 3c_1, \quad c_2 \leq c_1 + 4\} \\
\Omega''_{10} = & \ \{c = (c_1, c_2) \in \Omega_{10} \, : \, c_2 \leq 3c_1\} \\
= & \ \{c = (c_1, c_2) \, : \, c_1 \leq 2, \quad c_2 \geq 2c_1, \quad c_2 \leq 3c_1\}.
\end{aligned}$$

For $c \in \Omega'_{10}$, by (16.17) and (16.38)–(16.45) we have

$$\varphi(c, \bar{b}) = \min\{\varphi^{I_1}_{KKT}(c, \bar{b}), \ \varphi^{I_2}_{KKT}(c, \bar{b}), \ \varphi^{I_7}_{KKT}(c, \bar{b})\}.$$

Since

$$\begin{aligned}
&\varphi^{I_1}_{KKT}(c, \bar{b}) - \varphi^{I_2}_{KKT}(c, \bar{b}) \\
&\quad = \frac{1}{4}(-c_1^2 - 2c_1c_2 + c_2^2) + \frac{1}{2}c_1^2 + 2c_1 - 2c_2 + 4 \\
&\quad = \frac{1}{4}(c_2 - (c_1 + 4))^2 \geq 0
\end{aligned}$$

for every $c \in \Omega'_{10}$, we have

$$\varphi(c, \bar{b}) = \min\{\varphi^{I_2}_{KKT}(c, \bar{b}), \ \varphi^{I_7}_{KKT}(c, \bar{b})\} \quad (\forall c \in \Omega'_{10}). \tag{16.46}$$

For $c \in \Omega''_{10}$, by (16.17) and (16.38)–(16.45) we have

$$\varphi(c, \bar{b}) = \min\{\varphi^{I_2}_{KKT}(c, \bar{b}), \ \varphi^{I_3}_{KKT}(c, \bar{b}), \ \varphi^{I_7}_{KKT}(c, \bar{b})\}.$$

Since

$$\begin{aligned}
&\varphi^{I_3}_{KKT}(c, \bar{b}) - \varphi^{I_7}_{KKT}(c, \bar{b}) \\
&\quad = 2c_1^2 + \frac{1}{2}c_2^2 - 2c_1c_2 \\
&\quad = \frac{1}{2}(2c_1 - c_2)^2 \geq 0
\end{aligned}$$

for every $c \in \Omega''_{10}$, we have

$$\varphi(c, \bar{b}) = \min\{\varphi^{I_2}_{KKT}(c, \bar{b}), \ \varphi^{I_7}_{KKT}(c, \bar{b})\} \quad (\forall c \in \Omega''_{10}). \tag{16.47}$$

From (16.46) and (16.47) it follows that

$$\begin{aligned}
\varphi(c, \bar{b}) = & \ \min\{\varphi^{I_2}_{KKT}(c, \bar{b}), \ \varphi^{I_7}_{KKT}(c, \bar{b})\} \\
= & \ \min\{-\frac{1}{2}c_1^2 - 2c_1 + 2c_2 - 4, 0\}
\end{aligned} \tag{16.48}$$

for all $c \in \Omega_{10} = \Omega'_{10} \cup \Omega''_{10}$. Consider the parabola

$$\Gamma = \{(c_1, c_2) \in R^2 \, : \, c_2 = \frac{1}{4}c_1^2 + c_1 + 2\}. \tag{16.49}$$

By (16.48), for each $c \in \Omega_{10}$ we have

$$\varphi(c, \bar{b}) = \begin{cases} 0 & \text{if } c_2 \geq \frac{1}{4}c_1^2 + c_1 + 2 \\ -\frac{1}{2}c_1^2 - 2c_1 + 2c_2 - 4 & \text{if } c_2 \leq \frac{1}{4}c_1^2 + c_1 + 2. \end{cases} \tag{16.50}$$

This amounts to saying that $\varphi(c, \bar{b}) = \varphi^{I_7}_{KKT}(c, \bar{b})$ for all the points $c \in \Omega_{10}$ lying above the parabola Γ, and $\varphi(c, \bar{b}) = \varphi^{I_2}_{KKT}(c, \bar{b})$ for all the points $c \in \Omega_{10}$ lying below the curve Γ.

We have thus computed the values $\varphi(c, \bar{b})$ for all $c \in R^2$. To have a better knowledge of the behavior of the function $\varphi(\cdot, \bar{b})$, the reader can draw a plane R^2 with the regions $\Omega_1, \ldots, \Omega_{10}$ and the parabola Γ.

Proposition 16.2. *The obtained optimal value function $\varphi(c, \bar{b})$ $(c \in R^2)$ cannot be a piecewise linear-quadratic function.*

A detailed proof of this proposition will be given in the next section.

By virtue of Proposition 16.2 and the observation stated just after Example 16.1, we can conclude that the optimal value function $(c, b) \mapsto \varphi(c, b)$ of problem (16.36) cannot be a plq function. Thus, *if D is not assumed to be a positive semidefinite matrix then the optimal value function $\varphi(\cdot, \cdot)$ of* (15.1) *can fail to be piecewise linear-quadratic.*

We now apply formula (16.30) to compute directional derivative of the function $\varphi(\cdot, \bar{b})$ studied in this section.

Let $\bar{c} = \bar{c}(\mu) = (0, \mu)$, $\mu \in R$. Let $\varphi_1(c) := \varphi(c, \bar{b})$. For $\bar{w}(\mu) = (\bar{c}(\mu), \bar{b})$ and $\bar{z} = (\bar{u}, \bar{v})$, where $\bar{u} = (1, 0) \in R^2$ and $\bar{v} = (0, 0, 0) \in R^3$, we have $\varphi'(\bar{w}(\mu); \bar{z}) = \varphi_1'(\bar{c}(\mu); \bar{u})$. Using formulae (16.30) and (16.38)–(16.45), we obtain

$$\begin{aligned} &\varphi'(\bar{w}(\mu); \bar{z}) \\ &= \varphi_1'(\bar{c}(\mu); \bar{u}) \\ &= \begin{cases} (\varphi^{I_7}_{KKT})'(\bar{c}(\mu); \bar{u}) & \text{for } \mu > 2, \\ \min\{(\varphi^{I_7}_{KKT})'(\bar{c}(\mu); \bar{u}),\ (\varphi^{I_2}_{KKT})'(\bar{c}(\mu); \bar{u})\} & \text{for } \mu = 2, \\ (\varphi^{I_2}_{KKT})'(\bar{c}(\mu); \bar{u}) & \text{for } \mu < 2. \end{cases} \end{aligned}$$

Therefore

$$\varphi'(\bar{w}(\mu); \bar{z}) = \varphi_1'(\bar{c}(\mu); \bar{u}) = \begin{cases} 0 & \text{for } \mu > 2, \\ -2 & \text{for } \mu \leq 2. \end{cases}$$

By Lemma 16.3, the function $\varphi_1(\cdot) = \varphi(\cdot, \bar{b})$ is locally Lipschitz on R^2. From Theorems 16.1 and 16.2 it follows that $\varphi_1(\cdot)$ is directionally differentiable at every $c \in R^2$ and, for every $u \in R^2$, the directional derivative $\varphi_1(c; u)$ is finite. One can expect that $\varphi_1(\cdot)$ is regular in the sense of Clarke (1983), i.e. for every $c \in R^2$ it holds $\varphi_1^0(c; u) = \varphi_1'(c; u)$, where

$$\varphi_1^0(c; u) := \limsup_{c' \to c,\ t \downarrow 0} \frac{\varphi_1(c' + tu) - \varphi_1(c')}{t}$$

denotes the generalized directional derivative of φ_1 at c in direction u. Unfortunately, *the function* $\varphi_1(\cdot)$ *is not Lipschitz regular.* Indeed, for $\bar{c} = (0, 2)$ and $\bar{u} = (0, 1)$, using (16.50) it is not difficult to show that

$$0 = \varphi_1^0(\bar{c}; \bar{u}) > \varphi_1'(\bar{c}; \bar{u}) = -2.$$

16.4 Proof of Proposition 16.2

Suppose, contrary to our claim, that the function $\varphi(\cdot, \bar{b})$ is plq. Then the set $\mathrm{dom}\varphi(\cdot, \bar{b}) = R^2$ can be represented in the form

$$R^2 = \bigcup_{j \in J} \Delta_j, \tag{16.51}$$

where J is a finite index set and Δ_j $(j \in J)$ are polyhedral convex sets. Moreover, for every $j \in J$, one has

$$\varphi(c, \bar{b}) = \frac{1}{2} c^T Q_j c + d_j^T c + \alpha_j \tag{16.52}$$

for all $c \in \Delta_j$, where $\alpha_j \in R,\ d_j \in R^2,\ Q_j \in R_S^{2\times 2}$. Let

$$\Delta_j' = \Delta_j \cap \Omega_{10} \quad (j \in J).$$

Note that some of the sets Δ_j' can be empty. From (16.51) we deduce that

$$\Omega_{10} = \bigcup_{j \in J} \Delta_j'. \tag{16.53}$$

Note also that on each set Δ_j' $(j \in J)$ the function $\varphi(\cdot, \bar{b})$ has the linear-quadratic representation (16.52). Define

$$\Omega_{10}^I = \{c = (c_1, c_2) \in \Omega_{10} : c_2 \geq \frac{1}{4} c_1^2 + c_1 + 2\},$$

$$\Omega_{10}^{II} = \{c = (c_1, c_2) \in \Omega_{10} \, : \, c_2 \le \frac{1}{4}c_1^2 + c_1 + 2\}.$$

It is evident that Ω_{10}^I is a convex set. Note that Ω_{10}^I and Ω_{10}^{II} are compact sets which admit the curve $\Gamma \cap \Omega_{10}$, where Γ is the parabola defined by (16.49), as the common boundary. The set Ω_{10}^I (resp., Ω_{10}^{II}) has nonempty interior. Indeed, let $\hat{c} := (0,3)$ and $\tilde{c} := (0,1)$. Substituting the coordinates of these vectors into the inequalities defining Ω_{10}^I and Ω_{10}^{II}, one see at once that $\hat{c} \in \mathrm{int}\Omega_{10}^I$ and $\tilde{c} \in \mathrm{int}\Omega_{10}^{II}$.

Fix any index $j \in J$ for which $\Delta_j' \neq \emptyset$.

We first consider the case $\mathrm{int}\Delta_j' \neq \emptyset$. If

$$\mathrm{int}\Delta_j' \cap \mathrm{int}\Omega_{10}^I \neq \emptyset \tag{16.54}$$

then we must have $\Delta_j' \subset \Omega_{10}^I$. Indeed, by (16.54) there must exist a ball $B \subset R^2$ of positive radius such that

$$B \subset \Delta_j' \cap \Omega_{10}^I.$$

By (16.50), $\varphi(c, \bar{b}) = 0$ for every $c \in \Omega_{10}^I$. Then, it follows from (16.52) that

$$\varphi(c, \bar{b}) = \frac{1}{2}c^T Q_j c + d_j^T c + \alpha_j = 0$$

for every $c \in B$. This implies that $Q_j = 0$, $d_j = 0$ and $\alpha_j = 0$. Consequently,

$$\varphi(c, \bar{b}) = 0 \quad (\forall c \in \Delta_j'). \tag{16.55}$$

We observe from (16.50) that $\varphi(c, \bar{b}) < 0$ for every $c \in \Omega_{10} \setminus \Omega_{10}^I$. Hence (16.55) clearly forces $\Delta_j' \subset \Omega_{10}^I$. If

$$\mathrm{int}\Delta_j' \cap \mathrm{int}\Omega_{10}^I = \emptyset$$

then we must have $\mathrm{int}\Delta_j' \subset \Omega_{10}^{II}$. Since Ω_{10}^{II} is closed, we conclude that $\Delta_j' \subset \Omega_{10}^{II}$. Therefore, if $\Delta_j' \cap \Omega_{10}^I \neq \emptyset$ then $\Delta_j' \cap \Omega_{10}^I = \Delta_j' \cap \Gamma$. In this case, it is easy to show that $\Delta_j' \cap \Gamma$ is a singleton.

We now consider the case $\mathrm{int}\Delta_j' = \emptyset$. Since Δ_j' is a compact polyhedral convex set in R^2, there are only two possibilities:

(i) Δ_j' is a singleton,

(ii) Δ_j' is a line segment.

In both situations, if $\Delta_j' \cap \Omega_{10}^I$ is nonempty then it is a compact polyhedral convex set (a point or a line segment).

From (16.53) and from the above discussion, we can conclude that Ω_{10}^I is the union of the following finite collection of polyhedral convex sets:

$$\begin{array}{ll} \Delta'_j & (j \in J \text{ is such that } \operatorname{int}\Delta'_j \cap \operatorname{int}\Omega^I_{10} \neq \emptyset), \\ \Delta'_j \cap \Gamma & (j \in J \text{ is such that } \operatorname{int}\Delta'_j \neq \emptyset,\ \operatorname{int}\Delta'_j \cap \operatorname{int}\Omega^I_{10} = \emptyset), \\ \Delta'_j \cap \Omega^I_{10} & (j \in J \text{ is such that } \operatorname{int}\Delta'_j = \emptyset,\ \Delta'_j \cap \Omega^I_{10} \neq \emptyset). \end{array}$$

As Ω^I_{10} is convex, it coincides with the convex hull of the above-named compact polyhedral convex sets. According to Theorem 19.1 in Rockafellar (1970), this convex hull is a compact polyhedral convex set. So it has only a finite number of extreme points (see Rockafellar (1970), p. 162). Meanwhile, it is a simple matter to show that *every point from the infinite set* $\Gamma \cap \Omega_{10}$ *is an extreme point of* Ω^I_{10}. We have arrived at a contradiction. The proof is complete. $\square$

16.5 Commentaries

The results presented in this chapter are taken from Lee et al. (2002a).

In this chapter we have studied a class of optimal value functions in parametric (nonconvex) quadratic programming. It has been shown that these functions are directionally differentiable at any point from their effective domains but, in general, they are not piecewise linear-quadratic and they may be not Lipschitz regular at some interior points in their effective domains.

The class of plq functions has been investigated systematically in Rockafellar and Wets (1998). In particular, the topics like subdifferential calculation, dualization, and optimization involving plq functions, are studied in the book.

The reader is referred to Gauvin and Tolle (1977), Gauvin and Dubeau (1982), Rockafellar (1982), Fiacco (1983), Clarke (1983), Janin (1984), Minchenko and Sakolchik (1996), Bonnans and Shapiro (1998, 2000), Ward and Lee (2001), and references therein, for different approaches in the study of differential properties of the optimal value functions in nonlinear optimization problems.

It would be desirable to find out what additional conditions one has to impose on the pair of matrices $(D, A) \in R^{n\times n}_S \times R^{m\times n}$, where D need not be a positive semidefinite matrix, so that the optimal value function

$$(c, b) \mapsto \varphi(c, b)$$

of the parametric problem (15.1) is piecewise linear-quadratic on $R^n \times R^m$.

Both referees of the paper Lee et al. (2001a) informed us that D. Klatte had constructed an example of an optimal value function in a linearly perturbed QP problem which is not plq. Being unaware of that (unpublished) example, we have constructed Example 16.2. One referee gave us some hints in detail on the example of Klatte. Namely, letting two components of the data perturbation of a QP problem considered by Klatte (1985) be fixed, one has the problem

$$\begin{cases} \text{Minimize } x_1 x_2 \\ \text{subject to } x = (x_1, x_2) \in R^2, \ -1 \le x_1 \le b_1, \ b_2 \le x_2 \le 1, \end{cases}$$

where $b = (b_1, b_2) \in R^2$, $b_1 \ge 0$ and $b_2 \le 0$, represents the perturbation of the feasible region. Denote by $\varphi(b_1, b_2)$ the optimal value function of this problem. It is easy to verify that

$$\varphi(b_1, b_2) = \begin{cases} -1 & \text{if } -1 \le b_1 b_2 \\ b_1 b_2 & \text{if } -1 > b_1 b_2. \end{cases}$$

If $b_1 < 0$ or $b_2 > 0$, then we put $\varphi(b_1, b_2) = +\infty$. Arguments similar to those of the proof of Proposition 16.2 show that $\varphi(b_1, b_2)$ is not a plq function. The main difference between this example and Example 16.2 is that here the feasible region is perturbed, while in Example 16.2 the objective function is perturbed.

Chapter 17

Quadratic Programming under Linear Perturbations: III. The Convex Case

The problem of finding the nearest point in a polyhedral convex set to a given point is a convex QP problem. That nearest point is called the *metric projection* of the given point onto the polyhedral convex set.

In this chapter we will see that the metric projection from a given point onto a moving polyhedral convex set is Lipschitz continuous with respect to the perturbations on the right-hand-sides of the linear inequalities defining the set. The property leads to a simple sufficient condition for Lipschitz continuity of a locally unique solution of parametric variational inequalities with a moving polyhedral constraint set. Applications of these results to traffic network equilibrium problems will be discussed in detail.

17.1 Preliminaries

We will study sensitivity of solutions to a parametric variational inequality (PVI, for brevity) with a parametric polyhedral constraint. Let

$$K(\lambda) = \{x \in R^n : Ax \geq \lambda, \ x \geq 0\}, \tag{17.1}$$

$$\Lambda = \{\lambda \in R^r : K(\lambda) \neq \emptyset\}, \tag{17.2}$$

where $A \in R^{r \times n}$ is a given matrix. Let $M \subset R^m$ be any subset and $f : R^n \times M \to R^n$ be a given function. Consider the following PVI depending on a pair of parameters $(\mu, \lambda) \in M \times \Lambda$:

$$\begin{cases} \text{Find} \quad x \in K(\lambda) \quad \text{such that} \\ \langle f(x,\mu), y-x\rangle \geq 0 \quad \text{for all} \quad y \in K(\lambda). \end{cases} \tag{17.3}$$

Assume that $\bar{x}$ is a solution of the following problem

$$\begin{cases} \text{Find} \quad x \in K(\bar{\lambda}) \quad \text{such that} \\ \langle f(x,\bar{\mu}), y-x\rangle \geq 0 \quad \text{for all} \quad y \in K(\bar{\lambda}), \end{cases} \tag{17.4}$$

where $(\bar{\mu}, \bar{\lambda}) \in M \times \Lambda$ are given parameters.

Our aim is to prove that under some appropriate conditions on f in a neighborhood of $(\bar{x}, \bar{\mu})$ and *no* conditions on the matrix A, there exist $k > 0$ and neighborhoods X, U, V of $\bar{x}, \bar{\mu}$, and $\bar{\lambda}$, respectively, such that

(i) *For every* $(\mu, \lambda) \in (M \cap U) \times (\Lambda \cap V)$ *there is a unique solution* $x = x(\mu, \lambda)$ *of* (17.3) *in* X;

(ii) *For every* $(\mu, \lambda), (\mu', \lambda') \in (M \cap U) \times (\Lambda \cap V)$,

$$\|x(\mu', \lambda') - x(\mu, \lambda)\| \leq k(\|\mu' - \mu\| + \|\lambda' - \lambda\|).$$

To this aim, in Section 17.2 we obtain a property of the metric projection onto a moving polyhedral convex set which can be stated simply, as follows: For a given a matrix $A \in R^{r \times n}$ there exists a constant $k_1 > 0$ such that for all $y \in R^n$ and $\lambda, \lambda' \in \Lambda$, we have

$$\|P_{K(\lambda')}y - P_{K(\lambda)}y\| \leq k_1\|\lambda' - \lambda\|, \tag{17.5}$$

where $K(\lambda)$ and Λ are defined by (17.1) and (17.2), $P_{K(\lambda)}y$ is the unique point in $K(\lambda)$ with the minimal distance to y. (The map $P_{K(\lambda)}(.)$ is said to be the *metric projection* onto $K(\lambda)$.)

Property (17.5) is established by using a result on linear complementarity problems in Mangasarian and Shiau (1987). Then the scheme for proving Lemma 2.4 in Dafermos (1988) enables us to get, in Section 17.3, the desired sensitivity result for PVI. The latter can be interpreted as a condition for Lipschitz continuity of the equilibrium flow in a traffic network with changing costs and demands. This fact is considered in Section 17.4.

17.2 Projection onto a Moving Polyhedral Convex Set

To establish property (17.5) we will consider $P_{K(\lambda)}y$ as the unique solution of a quadratic program with parameters (y, λ). By the standard procedure (see Murty (1976)) we reduce this program to an equivalent linear complementarity problem. Although the assumption on uniqueness of solutions of Theorem 3.2 in Mangasarian and Shiau (1987) is violated in our LCP problem, we will show that the partition method for obtaining that theorem is well adequate for our purpose.

So, let $y \in R^n$ and $\lambda \in \Lambda$ (see (17.2)) be given. From the definition it follows that $\xi := P_{K(\lambda)}y$ is the unique solution of the problem

$$\text{Minimize} \quad \|x - y\|^2 \quad \text{subject to} \quad Ax \geq \lambda, \ x \geq 0,$$

which is equivalent to the following one

$$\text{Minimize} \quad (-2y^T x + x^T x) \quad \text{subject to} \quad Ax \geq \lambda, \ x \geq 0. \tag{17.6}$$

It is clear that (17.6) is a particular case of the following convex QP problem

$$\text{Minimize} \quad \frac{1}{2}x^T Dx + c^T x \quad \text{subject to} \quad Ax \geq \lambda, \ x \geq 0, \tag{17.7}$$

where $c \in R^n$, D is a symmetric positive semidefinite matrix. Indeed, (17.7) becomes (17.6) if one takes $c = -2y$ and $D = 2E$, where E denotes the unit matrix of order n.

The next lemma follows easily from Corollary 3.1 and the convexity of problem (17.7).

Lemma 17.1. *Vector $\xi \in R^n$ is a solution of* (17.7) *if and only if there exists $\eta \in R^r$ such that*

$$z := \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

is a solution to the following LCP problem:

$$Mz + q \geq 0, \quad z \geq 0, \quad (Mz + q)^T z = 0, \tag{17.8}$$

where

$$M := \begin{pmatrix} D & -A^T \\ A & 0 \end{pmatrix} \quad \text{and} \quad q := \begin{pmatrix} c \\ -\lambda \end{pmatrix}. \tag{17.9}$$

We set $s = n + r$. For any subset $J \subset \{1, \cdots, s\}$, observe (see Mangasarian and Shiau (1987), p. 591) that every solution of the following system of $2s$ linear equalities and inequalities

$$\begin{cases} M_j z + q_j \geq 0, & z_j = 0, \quad j \in J, \\ M_j z + q_j = 0, & z_j \geq 0, \quad j \notin J, \end{cases} \tag{17.10}$$

is a solution of (17.8). For every $J \subset \{1, \cdots, s\}$, symbol $Q(J)$ denotes the set of all vectors q such that (17.10) has a solution. Note that $Q(J)$ is a closed convex cone which is called a *complementary cone* of (M, q) (see Murty (1976), Mangasarian and Shiau (1987)). The union $\cup\{Q(J) : J \subset \{1, \cdots, s\}\}$ is the set of all q such that (17.8) is solvable.

For each subset $J \subset \{1, \cdots, s\}$, according to Corollary 7.3, we can find a constant $\theta = \theta_J > 0$ such that if z^1 is a solution of (17.10) at $q = q^1$ and the solution set of (17.10) at $q = q^2$ is nonempty, then there exists a solution z^2 of (17.10) at $q = q^2$ such that

$$\|z^2 - z^1\| \leq \theta_J \|q^2 - q^1\|.$$

Let us set

$$k_0 = \max\{\theta_J : J \subset \{1, \cdots, s\}\}. \tag{17.11}$$

The next technical lemma is crucial for applying Corollary 7.3 to linear complementarity problems.

Lemma 17.2. (See Mangasarian and Shiau (1987), p. 591) *Let $q^1, q^2 \in R^s$ be two distinct vectors. Assume that for every $t \in [0, 1]$ system (17.8) is solvable for $q = q(t) := (1-t)q^1 + tq^2$. Then there is a partition $0 = t_0 < t_1 < \cdots < t_\ell = 1$ such that for every $i \in \{1, \cdots, \ell\}$,*

$$q(t_{i-1}) \in Q(J_i), \quad q(t_i) \in Q(J_i) \quad \text{for some} \quad J_i \subset \{1, \cdots, s\}. \tag{17.12}$$

The proof of this lemma is based on the observation that the intersection of each complementary cone of (17.8) with the segment $[q^1, q^2]$ is a closed interval (which may reduce to a single point or to the empty set). Since (17.8) is solvable for every $q \in [q^1, q^2]$, this segment is contained in the union of such intervals. Excluding some redundant intervals in that union and let t_i be 0, 1, or a point in the intersection of two neighbouring intervals, we get the desired partition.

The following theorem will be useful for obtaining the results in Section 17.3.

Theorem 17.1. *Given a matrix $A \in R^{r\times n}$, define the sets $K(\lambda)$ and Λ by* (17.1) *and* (17.2). *Then there exists a constant $k_1 > 0$ such that*

$$\|P_{K(\lambda')}y - P_{K(\lambda)}y\| \leq k_1\|\lambda' - \lambda\|, \tag{17.13}$$

for all $y \in R^n$ and $\lambda, \lambda' \in \Lambda$, where $P_{K(\lambda)}y$ is the metric projection of y onto $K(\lambda)$.

From the discusion at the beginning of this section we see that Theorem 17.1 is a direct consequence of the next result.

Theorem 17.2. (See Cottle et al. (1992), p. 696) *Let $A \in R^{r\times n}$, $K(\lambda)$ and Λ be defined as in* (17.1) *and* (17.2). *Let $D \in R_S^{n\times n}$ be a positive definite matrix. Define M and q by* (17.9), *k_0 by* (17.11). *Then for every $\lambda, \lambda' \in \Lambda$ and $c, c' \in R^n$ we have*

$$\|x(c', \lambda') - x(c, \lambda)\| \leq k_0\left(\|c' - c\| + \|\lambda' - \lambda\|\right), \tag{17.14}$$

where $x(c, \lambda)$ and $x(c', \lambda')$ are the unique solution of (17.7) *at the parameters (c, λ) and (c', λ'), respectively.*

Proof. We will follow the arguments for proving Theorem 3.2 in Mangasarian and Shiau (1987). Let there be given vectors $\lambda, \lambda' \in \Lambda$ and $c, c' \in R^n$. We set

$$q^1 = \begin{pmatrix} c \\ -\lambda \end{pmatrix}, \quad q^2 = \begin{pmatrix} c' \\ -\lambda' \end{pmatrix},$$

$$\begin{aligned} c(t) &= (1-t)c + t\,c', \ \lambda(t) = (1-t)\lambda + t\lambda', \\ q(t) &= (1-t)q^1 + tq^2 \quad \text{for every} \quad t \in [0,1]. \end{aligned}$$

If $\lambda = \lambda'$ and $c = c'$, then (17.14) holds. Consider the other case where at least one of these equalities does not hold. Then we have $q^1 \neq q^2$. From the definition we see that Λ is a closed convex cone. Thus $\lambda(t) \in \Lambda$ for every $t \in [0,1]$. This means that $K(\lambda(t)) \neq \emptyset$ for every $t \in [0,1]$. Since D is assumed to be a symmetric positive definite matrix, for each $t \in [0,1]$ program (17.7), where $(c(t), \lambda(t))$ are in the place of (c, λ), must have a unique solution, denoted by $\xi(t)$. Using Lemma 17.1 we find a vector $\eta(t) \in R^r$ such that

$$z(t) := \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix}$$

is a solution of (17.8), where

$$q = \begin{pmatrix} c(t) \\ -\lambda(t) \end{pmatrix} = q(t).$$

(Note that vector $\eta(t)$ may not be uniquely defined.) Hence, according to Lemma 17.2 there is a partition $0 = t_0 < t_1 < \cdots < t_\ell = 1$ such that for every $1 \le i \le \ell$ condition (17.12) holds.

Consequently, for every $1 \le i \le \ell$ vectors $q(t_{i-1})$ and $q(t_i)$ belong to the cone $Q(J_i)$ for a subset $J_i \subset \{1, \cdots, s\}$. Hence the systems of linear equalities and inequalities

$$\begin{cases} M_j z + (q(t_i))_j \ge 0, & z_j = 0, \quad j \in J_i \\ M_j z + (q(t_i))_j = 0, & z_j \ge 0, \quad j \notin J_i \end{cases} \tag{17.15}$$

and

$$\begin{cases} M_j z + (q(t_{i-1}))_j \ge 0, & z_j = 0, \quad j \in J_i \\ M_j z + (q(t_{i-1}))_j = 0, & z_j \ge 0, \quad j \notin J_i \end{cases} \tag{17.16}$$

are solvable. Let

$$\bar{z}(t_i) = \begin{pmatrix} \bar{\xi}(t_i) \\ \bar{\eta}(t_i) \end{pmatrix}$$

be a solution of (17.15). According to Corollary 7.3 there exists a solution

$$\bar{z}(t_{i-1}) = \begin{pmatrix} \bar{\xi}(t_{i-1}) \\ \bar{\eta}(t_{i-1}) \end{pmatrix}$$

of (17.16) satisfying

$$\begin{aligned} \|\bar{z}(t_i) - \bar{z}(t_{i-1})\| &\le \theta_J \|q(t_i) - q(t_{i-1})\| \\ &= \theta_J (t_i - t_{i-1}) \|q^1 - q^2\|. \end{aligned}$$

This implies

$$\|\bar{\xi}(t_i) - \bar{\xi}(t_{i-1})\| \le k_0 \, (t_i - t_{i-1}) \|q^1 - q^2\|. \tag{17.17}$$

Since $\bar{z}(t_i)$ solves (17.15) it also solves (17.8) at $q = q(t_i)$. By virtue of Lemma 17.1, $\bar{\xi}(t_i)$ is a solution of (17.7), where $(c, \lambda) = (c(t_i), \lambda(t_i))$. As the latter problem has a unique solution, we have $\bar{\xi}(t_i) = \xi(t_i)$. Similarly, since $\bar{z}(t_{i-1})$ solves (17.16) it also solves (17.8) at $q = q(t_{i-1})$. Hence, $\bar{\xi}(t_{i-1}) = \xi(t_{i-1})$. These facts and (17.17) imply

$$\|\xi(t_i) - \xi(t_{i-1})\| \le k_0 \, (t_i - t_{i-1}) \|q^1 - q^2\|.$$

Consequently,

$$\|\xi(t_\ell) - \xi(t_0)\| \le \sum_{i=1}^{\ell} \|\xi(t_i) - \xi(t_{i-1})\| \le k_0 \, \|q^1 - q^2\|.$$

Since $\xi(t_\ell) = \xi(1) = x(c', \lambda')$ and $\xi(t_0) = \xi(0) = x(c, \lambda)$, we obtain

$$\|x(c', \lambda') - x(c, \lambda)\| \le k_0 \, (\|c' - c\| + \|\lambda' - \lambda\|).$$

The proof is complete. $\square$

17.3 Application to Variational Inequalities

Consider problem (17.3) and suppose that for a pair $(\bar{\mu}, \bar{\lambda}) \in M \times \Lambda$ vector $\bar{x}$ is a solution of (17.4). Following Dafermos (1988) we assume that there exist neighborhoods X of $\bar{x}$, U of $\bar{\mu}$, and two constants $\alpha > 0, l > 0$, such that

$$\|f(x', \mu') - f(x, \mu)\| \leq l(\|x' - x\| + \|\mu' - \mu\|) \tag{17.18}$$

for all μ, μ' in $M \cap U$, x, x' in X, and

$$\langle f(x', \mu) - f(x, \mu), x' - x \rangle \geq \alpha \|x' - x\|^2 \tag{17.19}$$

for all $\mu \in M \cap U$, x and x' in X. Without loss of generality we can assume that X is a polyhedral convex set and $\alpha < l$. Condition (17.18) means that f is locally Lipschitz at $(\bar{x}, \bar{\mu})$. Condition (17.19) means that $f(., \mu)$ is locally strongly monotone around $\bar{x}$ with a common coefficient for all $\mu \in M \cap U$. Using the notation of Dafermos (1988) we put

$$G(x, \mu, \lambda) = P_{K(\lambda) \cap X}[x - \rho f(x, \mu)] \quad \text{for all } (x, \mu, \lambda) \in R^n \times M \times \Lambda, \tag{17.20}$$

where $\rho > 0$ is a fixed number and $P_{K(\lambda) \cap X} y$ is the metric projection of y onto $K(\lambda) \cap X$. Let us consider a number ρ satisfying

$$0 < \rho \leq \frac{\alpha}{l^2}. \tag{17.21}$$

For every $\lambda \in \Lambda$ such that $K(\lambda) \cap X \neq \emptyset$, Lemma 2.2 from Dafermos (1988) shows that

$$\|G(x', \mu, \lambda) - G(x, \mu, \lambda)\| \leq \beta \|x' - x\| \tag{17.22}$$

for all x and x' in X, $\mu \in M \cap U$, where

$$\beta := (1 - \rho\alpha)^{1/2} < 1. \tag{17.23}$$

According to the Banach contractive mapping principle, there is a unique vector $x = x(\mu, \lambda) \in X$ satisfying

$$x(\mu, \lambda) = G(x(\mu, \lambda), \mu, \lambda). \tag{17.24}$$

For the map $K(\lambda)$ defined by (17.1) we apply Corollary 7.3 to find an $\theta > 0$ such that if $\lambda, \lambda' \in \Lambda$ and $x \in K(\lambda)$, then there exists $x' \in K(\lambda')$ satisfying

$$\|x' - x\| \le \theta \|\lambda' - \lambda\|. \tag{17.25}$$

Since $\bar{x} \in K(\bar{\lambda})$, from (17.25) it follows that there is a neighborhood V_1 of $\bar{\lambda}$ such that

$$K(\lambda) \cap X \neq \emptyset \quad \text{for every} \quad \lambda \in \Lambda \cap V_1. \tag{17.26}$$

Since X is a polyhedral convex set we can find a matrix C of order $r_1 \times n$ and a vector $b \in R^{r_1}$ such that $X = \{x \in R^n : Cx \ge b\}$. Therefore

$$K(\lambda) \cap X = \{x \in R^n : Ax \ge \lambda,\ Cx \ge b,\ x \ge 0\}. \tag{17.27}$$

So, taking (17.26) into account we can apply Theorem 17.1 for system (17.27) to choose a constant $k_1 > 0$ such that

$$\|P_{K(\lambda')\cap X} y - P_{K(\lambda)\cap X} y\| \le k_1 \|\lambda' - \lambda\| \tag{17.28}$$

for all $y \in R^n$, λ and λ' in $\Lambda \cap V_1$. (Note that k_1 depends not only on A but also on C, that is, on the neighborhood X.)

Lemma 17.3. *Let* (17.18) *and* (17.19) *be fulfilled. Assume that* $k_1 > 0$ *is a constant satisfying* (17.28). *Then for any* $\rho > 0$ *satisfying* (17.21) *there exist neighborhoods* $\bar{U}$ *and* $\bar{V}$ *of* $\bar{\mu}$ *and* $\bar{\lambda}$, *respectively, such that:*

(i) *For every* $(\mu, \lambda) \in (M \cap \bar{U}) \times (\Lambda \cap \bar{V})$ *vector* $x(\mu, \lambda) \in X$ *defined by* (17.24) *is the unique solution of* (17.3) *in* X;

(ii) *For all* $\mu,\ \mu' \in M \cap \bar{U}$ *and* $\lambda,\ \lambda' \in \Lambda \cap \bar{V}$,

$$\|x(\mu', \lambda') - x(\mu, \lambda)\| \le \frac{1}{1-\beta}(\rho l \|\mu' - \mu\| + k_1 \|\lambda' - \lambda\|),$$

where β *is defined in* (17.23).

This lemma can be proved similarly as Lemma 2.1 in Yen (1995a). Note that the scheme given on p. 424 in Dafermos (1988) is our key argument.

Proof. Fixing any ρ satisfying (17.21), for every $(\mu, \lambda) \in (M \cap U) \times (\Lambda \cap V_1)$ we denote by $x(\mu, \lambda)$ the unique vector in X satisfying

(17.24). Let $(\mu,\lambda),(\mu',\lambda') \in (M\cap U)\times(\Lambda\cap V_1)$. Using (17.22) we have

$$\begin{aligned}
&\|x(\mu',\lambda') - x(\mu,\lambda)\| \\
&\quad = \|G(x(\mu',\lambda'),\mu',\lambda') - G(x(\mu,\lambda),\mu,\lambda)\| \\
&\quad \le \|G(x(\mu',\lambda'),\mu',\lambda') - G(x(\mu,\lambda),\mu',\lambda')\| \\
&\qquad + \|G(x(\mu,\lambda),\mu',\lambda') - G(x(\mu,\lambda),\mu,\lambda)\| \\
&\quad \le \beta\|x(\mu',\lambda') - x(\mu,\lambda)\| \\
&\qquad + \|G(x(\mu,\lambda),\mu',\lambda') - G(x(\mu,\lambda),\mu,\lambda)\|.
\end{aligned} \tag{17.29}$$

Formula (17.20) and the fact that the metric projection onto a fixed closed convex set is a nonexpansive mapping yield

$$\begin{aligned}
&\|G(x(\mu,\lambda),\mu',\lambda') - G(x(\mu,\lambda),\mu,\lambda)\| \\
&\quad = \|P_{K(\lambda')\cap X}[x(\mu,\lambda) - \rho f(x(\mu,\lambda),\mu')] \\
&\qquad - P_{K(\lambda)\cap X}[x(\mu,\lambda) - \rho f(x(\mu,\lambda),\mu)]\| \\
&\quad \le \|P_{K(\lambda')\cap X}[x(\mu,\lambda) - \rho f(x(\mu,\lambda),\mu')] \\
&\qquad - P_{K(\lambda')\cap X}[x(\mu,\lambda) - \rho f(x(\mu,\lambda),\mu)]\| \\
&\qquad + \|P_{K(\lambda')\cap X}[x(\mu,\lambda) - \rho f(x(\mu,\lambda),\mu)] \\
&\qquad - P_{K(\lambda)\cap X}[x(\mu,\lambda) - \rho f(x(\mu,\lambda),\mu)]\| \\
&\quad \le \rho\|f(x(\mu,\lambda),\mu') - f(x(\mu,\lambda),\mu)\| \\
&\qquad + \|P_{K(\lambda')\cap X}y(\mu,\lambda) - P_{K(\lambda)\cap X}y(\mu,\lambda)\|,
\end{aligned} \tag{17.30}$$

where

$$y(\mu,\lambda) := x(\mu,\lambda) - \rho f(x(\mu,\lambda),\mu).$$

From (17.18), (17.29) and (17.30) it follows that

$$\begin{aligned}
&\|x(\mu',\lambda') - x(\mu,\lambda)\| \\
&\quad \le \frac{1}{1-\beta}(\rho l\|\mu'-\mu\| + \|P_{K(\lambda')\cap X}y(\mu,\lambda) - P_{K(\lambda)\cap X}y(\mu,\lambda)\|).
\end{aligned} \tag{17.31}$$

Now we can find neighborhoods $\bar{U}$ and $\bar{V}$ of $\bar{\mu}$ and $\bar{\lambda}$ such that (i) and (ii) are fulfilled. Indeed, since $\bar{x}$ is a solution of (17.4), it is easy to show that

$$\bar{x} = P_{K(\bar{\lambda})}[\bar{x} - \rho f(\bar{x},\bar{\mu})].$$

Therefore $\bar{x}$ is the unique fixed point in X of the map $G(.,\bar{\mu},\bar{\lambda})$ defined by (17.20). Hence $\bar{x} = x(\bar{\mu},\bar{\lambda})$. Using this and putting $\bar{y} = \bar{x} - \rho f(\bar{x},\bar{\mu})$, we substitute $(\mu,\lambda) = (\bar{\mu},\bar{\lambda})$ into (17.31) to obtain

$$\|x(\mu',\lambda') - \bar{x}\| \le \frac{1}{1-\beta}(\rho l\|\mu'-\bar{\mu}\| + \|P_{K(\lambda')\cap X}\bar{y} - P_{K(\bar{\lambda})\cap X}\bar{y}\|).$$

Taking account of (17.28) we have

$$\|x(\mu', \lambda') - \bar{x}\| \le \frac{1}{1-\beta}(\rho l \|\mu' - \bar{\mu}\| + k_1 \|\lambda' - \bar{\lambda}\|, \tag{17.32}$$

for all $(\mu', \lambda') \in (M \cap U) \times (\Lambda \cap V_1)$. Due to (17.32), there exist neighborhoods $\bar{U} \subset U$ of $\bar{\mu}$ and $\bar{V} \subset V_1$ of $\bar{\lambda}$ such that $x(\mu, \lambda)$ belongs to the interior of X for every $(\mu, \lambda) \in (M \cap \bar{U}) \times (\Lambda \cap \bar{V})$. For such a pair (μ, λ), since the vector $x(\mu, \lambda)$ satisfies (17.24) and belongs to the interior of X, Lemma 2.1 in Dafermos (1988) shows that $x(\mu, \lambda)$ is the unique solution of (17.3) in X. We have thus established the first assertion of the lemma. The second assertion follows easily from (17.31) and (17.28). □

Now we can formulate the main result of this section.

Theorem 17.3. *Let $\bar{x}$ be a solution of* (17.4). *If conditions* (17.18) *and* (17.19) *are satisfied, then there exist constants $k_{\bar{\mu}} > 0$ and $k_{\bar{\lambda}} > 0$, neighborhoods $\bar{U}$ of $\bar{\mu}$ and $\bar{V}$ of $\bar{\lambda}$ such that:*

(i) *For every $(\mu, \lambda) \in (M \cap \bar{U}) \times (\Lambda \cap \bar{V})$ there exists a unique solution of* (17.3) *in X, denoted by $x(\mu, \lambda)$;*

(ii) *For all $(\mu', \lambda'), (\mu, \lambda) \in (M \cap \bar{U}) \times (\Lambda \cap \bar{V})$,*

$$\|x(\mu', \lambda') - x(\mu, \lambda)\| \le k_{\bar{\mu}} \|\mu' - \mu\| + k_{\bar{\lambda}} \|\lambda' - \lambda\|.$$

Proof. It suffices to apply Lemma 17.3 with any ρ satisfying (17.21), and put

$$k_{\bar{\mu}} = \frac{1}{1-\beta}\rho l, \quad k_{\bar{\lambda}} = \frac{1}{1-\beta} k_1. \quad \square$$

17.4 Application to a Network Equilibrium Problem

Let us consider problem (17.3) with $K(\lambda)$ defined in the following way:

$$K(\lambda) = \{x \in R^n : x = Zh,\ \Gamma h = \lambda,\ h \ge 0\}, \tag{17.33}$$

where $h \in R^p$, $\lambda \in R^r$, Γ is an $r \times p$-matrix, Z is an $n \times p$-matrix. This is the variational inequality model for the traffic equilibrium problem (see Smith (1979), Dafermos (1980), De Luca and Maugeri

(1989), Qiu and Magnanti (1989)) which we have studied in Chapter 9. The matrices, the vectors, and the function $f(x, \mu)$ in the model have the following interpretations (see Qiu and Magnanti (1989), and Chapter 9 of this book):

> x = vector of flows on arcs, h = vector of flows on paths, Γ = the incidence matrix of the relation "paths - OD (origin-destination) pairs", Z = the incidence matrix of the relation "arcs - paths", λ = vector of demands for the OD pairs, $f(x, \mu)$ = vector of the costs on arcs when the network is loaded with flow x, μ = parameter of the perturbation of the costs on arcs.

For a given pair (μ, λ), solutions of (17.3) are interpreted as the equilibrium flows on the traffic network, corresponding to vector λ of demands an function $f(., \mu)$ of the costs on arcs.

Since $K(\lambda)$ is defined by (17.33) rather by (17.1), Theorem 17.1 cannot be applied directly. However, a property like the one in (17.13) is valid.

Lemma 17.4. *Assume that $K(\lambda)$ is given by (17.33), $H(\lambda) := \{h \in R^p : \Gamma h = \lambda,\ h \geq 0\}$, $\Lambda := \{\lambda \in R^r : H(\lambda) \neq \emptyset\}$. Then there exists a constant $k > 0$ such that*

$$\|P_{K(\lambda')}y - P_{K(\lambda)}y\| \leq k\,\|\lambda' - \lambda\|, \tag{17.34}$$

for every $y \in R^n$ and $\lambda, \lambda' \in \Lambda$.

Proof. Since $K(\lambda) = Z\,(H(\lambda)) := \{Zh : h \in H(\lambda)\}$, then $\{\lambda \in R^r : K(\lambda) \neq \emptyset\} = \{\lambda \in R^r : H(\lambda) \neq \emptyset\} = \Lambda$. For each $\lambda \in \Lambda$ and $y \in R^n$ we consider two quadratic programming problems:

$$\text{Minimize} \quad \|y - x\|^2 \quad \text{subject to} \quad x - Zh = 0,\ \Gamma h = \lambda,\ h \geq 0; \tag{17.35}$$

and

$$\text{Minimize} \quad \|y - Zh\|^2 \quad \text{subject to} \quad \Gamma h = \lambda,\ h \geq 0. \tag{17.36}$$

Observe that if h is a solution of (17.36) then $x := Zh$ is a solution of (17.35) and, hence, $x = P_{K(\lambda)}y$. Moreover, since (17.35) has a unique solution, for arbitrary two solutions h^1, h^2 of (17.36) we have $Zh^1 = Zh^2$, and $x := Zh^1 = Zh^2$ is the unique solution of (17.35). Also, note that (17.36) is solvable, because (17.35) is solvable.

Since $\|y-Zh\|^2 = \|y\|^2-2y^TZh+h^TZ^TZh$, (17.36) is equivalent to the following problem

$$\text{Minimize} \quad \frac{1}{2}h^TDh + c^Th \quad \text{subject to} \quad Ah \geq \hat{\lambda},\ h \geq 0, \tag{17.37}$$

where $c := -2Z^Ty,\ D := 2Z^TZ$,

$$A := \begin{pmatrix} \Gamma \\ -\Gamma \end{pmatrix}, \quad \text{and} \quad \hat{\lambda} := \begin{pmatrix} \lambda \\ -\lambda \end{pmatrix}.$$

It is clear that D is a symmetric positive semidefinite matrix. Hence the scheme for reducing a convex quadratic programming problem to an equivalent LCP problem, recalled in Section 17.2, is applicable to (17.37). In particular, Lemma 17.1 asserts that $h \in R^p$ is a solution of (17.37) if and only if there exists $\eta \in R^{2r}$ such that

$$z := \begin{pmatrix} h \\ \eta \end{pmatrix}$$

is a solution of the LCP problem (17.8), where

$$M := \begin{pmatrix} D & -A^T \\ A & 0 \end{pmatrix} \quad \text{and} \quad q := \begin{pmatrix} c \\ -\hat{\lambda} \end{pmatrix}.$$

Let $s := p + 2r$ and k_0 be the constant defined by (17.11). We are going to prove that $k := \sqrt{2}k_0\|Z\|$ is a constant satisfying (17.34). Indeed, let $y \in R^n$ and $\lambda, \lambda' \in \Lambda$ be given arbitrarily. For each $t \in [0,1]$ we set

$$q(t) = (1-t)q^1 + tq^2,\ \lambda(t) = (1-t)\lambda + t\lambda',\ \hat{\lambda}(t) = (1-t)\hat{\lambda} + t\hat{\lambda}', \tag{17.38}$$

where

$$q^1 := \begin{pmatrix} c \\ -\hat{\lambda} \end{pmatrix}, \quad q^2 := \begin{pmatrix} c \\ -\hat{\lambda}' \end{pmatrix}, \quad c := -2Z^Ty, \quad \hat{\lambda} := \begin{pmatrix} \lambda \\ -\lambda \end{pmatrix},$$

$$\hat{\lambda}' := \begin{pmatrix} \lambda' \\ -\lambda' \end{pmatrix}.$$

Since (17.36) has a solution for every $\lambda \in \Lambda$, (17.37) is solvable for each

$$\hat{\lambda} = \begin{pmatrix} \lambda \\ -\lambda \end{pmatrix},$$

where $\lambda \in \Lambda$. Consequently, for every $t \in [0,1]$ problem (17.8) is solvable for $q = q(t)$, where $q(t)$ is defined in (17.38). Applying Lemma 17.2 one can find a partition $0 = t_0 < t_1 < \cdots < t_\ell = 1$ such that for every $1 \le i \le \ell$ condition (17.12) holds. Therefore, for each $1 \le i \le \ell$ there is a subset $J_i \subset \{1, \cdots, s\}$ such that the vectors $q(t_{i-1})$ and $q(t_i)$ belong to the cone $Q(J_i)$. This implies that (17.15) and (17.16) are solvable. Let

$$z(t_i) = \begin{pmatrix} h(t_i) \\ \eta(t_i) \end{pmatrix}$$

be a solution of (17.15). According to Corollary 7.3, there is a solution

$$z(t_{i-1}) = \begin{pmatrix} h(t_{i-1}) \\ \eta(t_{i-1}) \end{pmatrix}$$

of (17.16) satisfying

$$\|z(t_i) - z(t_{i-1})\| \le \theta_J \|q(t_i) - q(t_{i-1})\| = \theta_J (t_i - t_{i-1}) \|q^1 - q^2\|. \tag{17.39}$$

Since $z(t_i)$ solves (17.15) it also solves (17.8) at $q = q(t_i)$. Hence $h(t_i)$ is a solution of (17.37) at $\hat{\lambda} = \hat{\lambda}(t_i)$ and of (17.36) at $\lambda = \lambda(t_i)$. Thus, as remarked before, $x := Zh(t_i)$ is the unique solution of (17.35) at $\lambda = \lambda(t_i)$. In our notation,

$$Zh(t_i) = P_{K(\lambda(t_i))} y. \tag{17.40}$$

Arguing similarly with the solution $z(t_{i-1})$ of (17.16) we conclude that

$$Zh(t_{i-1}) = P_{K(\lambda(t_{i-1}))} y. \tag{17.41}$$

As (17.39) implies

$$\|h(t_i) - h(t_{i-1})\| \le \theta_J (t_i - t_{i-1}) \|q^1 - q^2\|,$$

(17.40) and (17.41) yield

$$\begin{aligned} \|P_{K(\lambda(t_{i-1}))} y - P_{K(\lambda(t_i))} y\| &= \|Zh(t_{i-1}) - Zh(t_i)\| \\ &\le \theta_J (t_i - t_{i-1}) \|q^1 - q^2\| \|Z\|. \end{aligned}$$

Therefore

$$\begin{aligned} &\|P_{K(\lambda(t_0))} y - P_{K(\lambda(t_1))} y\| \\ &\le \textstyle\sum_{i=1}^{\ell} \|P_{K(\lambda(t_{i-1}))} y - P_{K(\lambda(t_i))} y\| \\ &\le k_0 \|q^1 - q^2\| \|Z\| \\ &= k_0 \|\hat{\lambda} - \hat{\lambda}'\| \|Z\| \\ &= \sqrt{2} k_0 \|\lambda - \lambda'\| \|Z\|. \end{aligned}$$

Since $\lambda(t_0) = \lambda(0) = \lambda, \lambda(t_\ell) = \lambda(1) = \lambda'$ and $k = \sqrt{2}k_0\|Z\|$, the above estimation implies (17.34). □

Now, let $K(\lambda)$ be defined by (17.33) and $\bar{x}$ be a solution of (17.4). Let the function $f(x,\mu)$ satisfy conditions (17.18) and (17.19). Again, we assume that X is a polyhedral convex set and $\alpha < l$. Let $\bar{h} \in H(\bar{\lambda})$ be a vector such that $\bar{x} = Z\bar{h}$. Then $(\bar{x}, \bar{h})$ is a solution at parameter $\lambda = \bar{\lambda}$ of the following system of linear inequalities and equalities

$$x - Zh = 0, \quad \Gamma h = \lambda, \quad h \geq 0. \tag{17.42}$$

Applying Corollary 7.3 we can find $\theta > 0$ such that for every $\lambda \in \Lambda$ there exists a solution (x, h) of (17.42) satisfying

$$\|(x,h) - (\bar{x}, \bar{h})\| \leq \theta\|\lambda - \bar{\lambda}\|.$$

This implies that $x \in K(\lambda)$ and $\|x - \bar{x}\| \leq \theta\|\lambda - \bar{\lambda}\|$. Consequently, there is a neighborhood V_1 of $\bar{\lambda}$ such that

$$K(\lambda) \cap X \neq \emptyset \quad \text{for every} \quad \lambda \in \Lambda \cap V_1. \tag{17.43}$$

Now, let C be a matrix of order $r_1 \times n$ and b be a vector from R^{r_1}, such that $X = \{x \in R^n : Cx \geq b\}$. We have

$$\begin{aligned} K(\lambda) \cap X \ &= \{x \in R^n : x = Zh,\ Cx \geq b,\ \Gamma h = \lambda,\ h \geq 0\} \\ &= \{x \in R^n : x = Zh,\ CZh \geq b,\ \Gamma h = \lambda,\ h \geq 0\}. \end{aligned}$$

Since this set has the same structure as the one in (17.33), taking account of (17.43) we can apply Lemma 17.4 (see also the arguments for proving it) to find a constant $k > 0$ such that

$$\|P_{K(\lambda')\cap X}y - P_{K(\lambda)\cap X}y\| \leq k\|\lambda' - \lambda\| \tag{17.44}$$

for all $y \in R^n$ and $\lambda, \lambda' \in \Lambda \cap V_1$.

Using property (17.44) instead of (17.28) one can see that Lemma 17.3 (with k_1 being replaced by k) holds for the case where $K(\lambda)$ is given by (17.33). This fact gives us the following result.

Theorem 17.4. *Let $K(\lambda)$ be defined by (17.33) and $\bar{x}$ be a solution of (17.4), where $\bar{\mu} \in M \times \Lambda$ is a given pair of parameters. If conditions (17.18) and (17.19) are satisfied, then there exist constants $k_{\bar{\mu}} > 0$ and $k_{\bar{\lambda}} > 0$, neighborhoods $\bar{U}$ of $\bar{\mu}$ and $\bar{V}$ of $\bar{\lambda}$ such that:*

(i) *For every $(\mu, \lambda) \in (M \cap \bar{U}) \times (\Lambda \cap \bar{V})$ there exists a unique solution of (17.3) in X, denoted by $x(\mu, \lambda)$;*

(ii) *For all* $(\mu', \lambda'), (\mu, \lambda) \in (M \cap \bar{U}) \times (\Lambda \cap \bar{V})$,

$$\|x(\mu', \lambda') - x(\mu, \lambda)\| \leq k_{\bar{\mu}} \|\mu' - \mu\| + k_{\bar{\lambda}} \|\lambda' - \lambda\|.$$

Theorem 17.4 can be interpreted by saying that: *"In a traffic network with locally Lipschitz, locally strongly monotone function of costs on arcs, the equilibrium arcs flow is locally unique and is a locally Lipschitz function of the perturbations of costs on arcs and of the vector of demands."*

17.5 Commentaries

The material of this chapter is taken from Yen (1995b).

Stability and sensitivity analysis is a central topic in the optimization theory (see Robinson (1979), Fiacco (1983), Bank et al. (1983), Malanowski (1987), Levitin (1994), Bonnans and Shapiro (2000), and references therein). Recently, much attention has been devoted to stability and sensitivity analysis of variational inequalities. Although the methods here resemble those used in nonlinear parametric mathematical programming, specific features of variational inequalities pose new problems. The case of PVI with a fixed constraint set is studied, for example, in Kyparisis (1988). The case of PVI whose constraint set depends on a parameter is considered, for example, in Tobin (1986), Dafermos (1988), Harker and Pang (1990), Kyparisis (1990), Yen (1995b), Domokos (1999), Kien (2001).

Chapter 18

Continuity of the Solution Map in Affine Variational Inequalities

This chapter presents a systematic study of the usc and lsc properties of the solution map in parametric AVI problems. We will follow some ideas of Chapters 10 and 11, where the usc and lsc properties of the Karush-Kuhn-Tucker point set map in parametric QP problems were investigated.

In Section 18.1 we obtain a necessary condition for the usc property of the solution map at a given point. We will show that the obtained necessary condition is not a sufficient one. Then, in the same section, we derive some sufficient conditions for the usc property of the solution map and consider several useful illustrative examples. The lsc property of the solution map is studied in Section 18.2.

18.1 USC Property of the Solution Map

Consider the following AVI problem

$$\begin{cases} \text{Find} \quad x \in \Delta(A,b) \quad \text{such that} \\ \langle Mx + q, y - x \rangle \geq 0 \quad \text{for all} \quad y \in \Delta(A,b), \end{cases} \tag{18.1}$$

which depends on the parameter $(M, A, q, b) \in R^{n\times n} \times R^{m\times n} \times R^n \times R^m$. Here $\Delta(A,b) := \{x \in R^n : Ax \geq b\}$. We will abbreviate (18.1) to $\mathrm{AVI}(M,A,q,b)$, and denote its solution set by $\mathrm{Sol}(M,A,q,b)$.

The multifunction $(M,A,q,b) \mapsto \mathrm{Sol}(M,A,q,b)$ is called the *solution map* of (18.1) and is abbreviated to $\mathrm{Sol}(\cdot)$. For a fixed pair $(q,b) \in R^n \times R^m$, the symbol $\mathrm{Sol}(\cdot,\cdot,q,b)$ stands for the multifunction $(M,A) \mapsto \mathrm{Sol}(M,A,q,b)$. Similarly, for a fixed pair $(M,A) \in R^{n\times n} \times R^{m\times n}$, the symbol $\mathrm{Sol}(M,A,\cdot,\cdot)$ stands for the multifunction $(q,b) \mapsto \mathrm{Sol}(M,A,q,b)$.

According to Theorem 5.3, solutions of an AVI problem can be characterized via Lagrange multipliers. Namely, $x \in R^n$ is a solution of $\mathrm{AVI}(M,A,q,b)$ if and only if there exists $\lambda \in R^m$ such that

$$Mx - A^T\lambda + q = 0, \quad Ax \geq b, \quad \lambda \geq 0, \quad \langle \lambda, Ax - b\rangle = 0. \tag{18.2}$$

Vector $\lambda \in R^m$ satisfying (18.2) is called a *Lagrange multiplier* corresponding to x.

The following theorem gives a necessary condition for the usc property of the multifunction $\mathrm{Sol}(\cdot,\cdot,q,b)$ and the solution map $\mathrm{Sol}(\cdot)$.

Theorem 18.1. *Let* $(M,A,q,b) \in R^{n\times n}\times R^{m\times n}\times R^n\times R^m$. *Suppose that the solution set* $\mathrm{Sol}(M,A,q,b)$ *is bounded. Then, the following statements are valid:*

(i) *If the multifunction* $\mathrm{Sol}(\cdot,\cdot,q,b)$ *is upper semicontinuous at* (M,A), *then*
$$\mathrm{Sol}(M,A,0,0) = \{0\}. \tag{18.3}$$

(ii) *If the solution map* $\mathrm{Sol}(\cdot)$ *is upper semicontinuous at* (M,A,q,b), *then* (18.3) *is valid.*

Proof. (We shall follow the proof scheme of Theorem 10.1). Clearly, if $\mathrm{Sol}(\cdot)$ is upper semicontinuous at (M,A,q,b) then the multifunction $\mathrm{Sol}(\cdot,\cdot,q,b)$ is upper semicontinuous at (M,A). Hence (i) implies (ii). It remains to prove (i).

To obtain a contradiction, suppose that $\mathrm{Sol}(M,A,q,b)$ is bounded, the multifunction $\mathrm{Sol}(\cdot,\cdot,q,b)$ is upper semicontinuous at (M,A), and
$$\mathrm{Sol}(M,A,0,0) \neq \{0\}. \tag{18.4}$$

Since $0 \in \mathrm{Sol}(M,A,0,0)$, (18.4) implies that there exists a nonzero vector $\bar{x} \in R^n$ such that $\bar{x} \in \mathrm{Sol}(M,A,0,0)$. Hence there exists $\bar{\lambda} \in R^m$ such that

$$M\bar{x} - A^T\bar{\lambda} = 0, \quad A\bar{x} \geq 0, \quad \bar{\lambda} \geq 0, \quad \langle \bar{\lambda}, A\bar{x}\rangle = 0. \tag{18.5}$$

For every $t \in (0,1)$, we define

$$x_t = t^{-1}\bar{x}, \quad \lambda_t = t^{-1}\bar{\lambda}. \tag{18.6}$$

We shall show that for every $t \in (0,1)$ there exist $M_t \in R^{n\times n}$ and $A_t \in R^{m\times n}$ such that

$$M_t x_t - A_t^T \lambda_t + q = 0, \tag{18.7}$$

$$A_t x_t \geq b, \quad \lambda_t \geq 0, \tag{18.8}$$

$$\langle \lambda_t, A_t x_t - b\rangle = 0, \tag{18.9}$$

and $\|M_t - M\| \to 0$, $\|A_t - A\| \to 0$ as $t \to 0$.

We will find M_t and A_t in the following forms:

$$M_t = M + tM_0, \quad A_t = A + tA_0, \tag{18.10}$$

where the matrices $M_0 \in R^{n\times n}$ and $A_0 \in R^{m\times n}$ will be chosen so that they do not depend on t. If M_t and A_t are of the forms described in (18.10), then we have

$$\begin{aligned} M_t x_t - A_t^T \lambda_t + q &= t^{-1}(M + tM_0)\bar{x} - t^{-1}(A + tA_0)^T \bar{\lambda} + q \\ &= t^{-1}(M\bar{x} - A^T\bar{\lambda}) + (M_0\bar{x} - A_0^T\bar{\lambda} + q), \end{aligned}$$

$$A_t x_t = t^{-1}A\bar{x} + A_0\bar{x}, \quad \lambda_t = t^{-1}\bar{\lambda},$$

and

$$\langle \lambda_t, A_t x_t - b\rangle = \langle t^{-1}\bar{\lambda}, t^{-1}A\bar{x} + A_0\bar{x} - b\rangle.$$

On account of (18.5), we have

$$M_t x_t - A_t^T \lambda_t + q = M_0\bar{x} - A_0^T\bar{\lambda} + q,$$

$$A_t x_t - b = t^{-1}A\bar{x} + A_0\bar{x} - b \geq A_0\bar{x} - b, \quad \lambda_t \geq 0,$$

$$\begin{aligned} \langle \bar{\lambda}_t, A_t x_t - b\rangle &= \langle t^{-1}\bar{\lambda}, t^{-1}A\bar{x} + A_0\bar{x} - b\rangle \\ &= \langle t^{-1}\bar{\lambda}, A_0\bar{x} - b\rangle. \end{aligned}$$

It is clear that conditions (18.7)–(18.9) will be satisfied if we choose M_0 and A_0 so that

$$M_0\bar{x} - A_0^T\bar{\lambda} + q = 0, \tag{18.11}$$

$$A_0\bar{x} - b = 0. \tag{18.12}$$

Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ and let $I = \{i : \bar{x}_i \neq 0\}$. Since $\bar{x} \neq 0$, I is nonempty. Let i_0 be any element in I. We define

$$A_0 = [c_1 \ldots c_n] \in R^{m\times n},$$

where each c_i $(1 \le i \le n)$ is a column with m components given by the following formula

$$c_i = \begin{cases} (\bar{x}_i)^{-1} b & \text{for } i = i_0 \\ 0 & \text{for } i \ne i_0. \end{cases}$$

We check at once that this A_0 satisfies (18.12). Similarly, we define

$$M_0 = [d_1 \dots d_n] \in R^{n \times n},$$

where each d_i $(1 \le i \le n)$ is a column with n components given by

$$d_i = \begin{cases} (\bar{x}_i)^{-1}(A_0^T \bar{\lambda} - q) & \text{for } i = i_0 \\ 0 & \text{for } i \ne i_0. \end{cases}$$

This M_0 satisfies (18.11). So, for these matrices M_0 and A_0, conditions (18.7)–(18.9) are satisfied. According to Theorem 5.3, we have

$$x_t \in \mathrm{Sol}(M_t, A_t, q, b)$$

for every $t \in (0, 1)$. Since $\mathrm{Sol}(M, A, q, b)$ is bounded, there exists a bounded open set $V \subset R^n$ such that $\mathrm{Sol}(M, A, q, b) \subset V$. Since $\mathrm{Sol}(\cdot, \cdot, q, b)$ is upper semicontinuous at (M, A), there exists $\delta > 0$ such that

$$\mathrm{Sol}(M', A', q, b) \subset V$$

for all $(M', A') \in R^{n \times n} \times R^{m \times n}$ satisfying $\|(M', A') - (M, A)\| < \delta$. As $\|M_t - M\| < 2^{-1/2}\delta$ and $\|A_t - A\| < 2^{-1/2}\delta$ for all $t > 0$ small enough, we have $\mathrm{Sol}(M_t, A_t, q, b) \subset V$ for all $t > 0$ small enough. Hence $x_t \in V$ for every $t > 0$ sufficiently small. This is impossible, because V is bounded and $\|x_t\| = t^{-1}\|\bar{x}\| \to +\infty$ as $t \to 0$. The proof is complete. □

It is easy to verify that the solution set $\mathrm{Sol}(M, A, 0, 0)$ of the homogeneous AVI problem $\mathrm{AVI}(M, A, 0, 0)$ is a closed cone. Condition (18.3) requires that this cone consists of only one element 0.

We can characterize condition (18.3) as follows.

Proposition 18.1 (cf. Proposition 3 in Gowda and Pang (1994a)). *Condition* (18.3) *holds if and only if for every* $(q, b) \in R^n \times R^m$ *the set* $\mathrm{Sol}(M, A, q, b)$ *is bounded.*

Proof. Suppose that (18.3) holds. If there is a pair $(\bar{q}, \bar{b}) \in R^n \times R^m$ such that $\mathrm{Sol}(M, A, \bar{q}, \bar{b})$ is unbounded, there exists an unbounded sequence $\{x^k\}$ such that $x^k \in \mathrm{Sol}(M, A, \bar{q}, \bar{b})$ for all k. Without loss of generality we can assume that $\|x^k\| \ne 0$ for all k, $\|x^k\| \to \infty$ and

$\|x^k\|^{-1}x^k \to \bar{x}$ for some $\bar{x} \in R^n$ with $\|\bar{x}\| = 1$ as $k \to \infty$. Since $x^k \in \mathrm{Sol}(M, A, \bar{q}, \bar{b})$, for any $y \in \Delta(A, \bar{b})$ we have

$$\langle Mx^k + \bar{q}, y - x^k\rangle \geq 0, \quad Ax^k \geq \bar{b}, \tag{18.13}$$

for all k. Dividing the first inequality in (18.13) by $\|x^k\|^2$, the second inequality by $\|x^k\|$, and letting $k \to \infty$ we get

$$\langle M\bar{x}, -\bar{x}\rangle \geq 0, \qquad A\bar{x} \geq 0. \tag{18.14}$$

Since $A(x^k + \bar{x}) = Ax^k + A\bar{x} \geq \bar{b}$, we have $x^k + \bar{x} \in \Delta(A, \bar{b})$. Substituting $x^k + \bar{x}$ for y in the first inequality in (18.13), we have $\langle Mx^k + \bar{q}, \bar{x}\rangle \geq 0$. Dividing this inequality by $\|x^k\|$ and letting $k \to \infty$ we get

$$\langle M\bar{x}, \bar{x}\rangle \geq 0. \tag{18.15}$$

From (18.14) and (18.15) it follows that

$$\langle M\bar{x}, \bar{x}\rangle = 0. \tag{18.16}$$

Let z be any point in $\Delta(A, 0)$. Clearly, $x^k + z \in \Delta(A, \bar{b})$. Substituting $x^k + z$ for y in the first inequality in (18.13), we have $\langle Mx^k + \bar{q}, z\rangle \geq 0$. Dividing this inequality by $\|x^k\|$ and letting $k \to \infty$ we get $\langle M\bar{x}, z\rangle \geq 0$. From this and (18.16) we deduce that $\langle M\bar{x}, z - \bar{x}\rangle \geq 0$. Since $z \in \Delta(A, 0)$ is arbitrary, we conclude that $\bar{x} \in \mathrm{Sol}(M, A, 0, 0) \setminus \{0\}$. This contradicts our assumption that $\mathrm{Sol}(M, A, 0, 0) = \{0\}$.

We now suppose that $\mathrm{Sol}(M, A, q, b)$ is bounded for every $(q, b) \in R^n \times R^m$. Since the solution set $\mathrm{Sol}(M, A, 0, 0)$ is a cone, from its boundedness we see that (18.3) is valid. □

The following example shows that condition (18.3) and the boundedness of $\mathrm{Sol}(M, A, q, b)$ are not sufficient for the upper semicontinuity of $\mathrm{Sol}(\cdot)$ at (M, A, q, b).

Example 18.1. Consider problem (18.1) with

$$M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad q = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

For each $t \in (0, 1)$, we set

$$A_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -t & -1 \end{bmatrix}.$$

Using Theorem 5.3, we find that

$$\mathrm{Sol}(M, A, q, b) = \{(0,0), (0,1)\}, \quad \mathrm{Sol}(M, A, 0, 0) = \{(0,0)\},$$

and

$$\mathrm{Sol}(M, A_t, q, b) = \left\{(0,0), (0,1), \left(\frac{1}{t}, 0\right), \left(\frac{t}{t^2+1}, \frac{1}{t^2+1}\right)\right\}.$$

Since $(t^{-1}, 0) \in \mathrm{Sol}(M, A_t, q, b)$ for all $t \in (0,1)$, for any bounded open subset $V \subset R^2$ containing $\mathrm{Sol}(M, A, q, b)$ there exists $\delta_V > 0$ such that

$$\mathrm{Sol}(M, A_t, q, b) \setminus V \neq \emptyset$$

for every $t \in (0, \delta_V)$. Since $\|A_t - A\| \to 0$ as $t \to 0$, we conclude that $\mathrm{Sol}(\cdot)$ is not upper semicontinuous at (M, A, q, b).

Our next goal is to find some sets of conditions which guarantee that the solution map $\mathrm{Sol}(\cdot)$ is upper semicontinuous at a given point $(M, A, q, b) \in R^{n\times n} \times R^{m\times n} \times R^n \times R^m$.

In order to obtain some sufficient conditions for the usc property of $\mathrm{Sol}(\cdot)$ to hold, we will pay attention to the behavior of the quadratic form $\langle Mv, v\rangle$ on the cone $\Delta(A, 0) = \{v \in R^n : Av \geq 0\}$ and to the regularity of the inequality system $Ax \geq b$.

The next proposition shows that, for a given pair $(M, A) \in R^{n\times n} \times R^{m\times n}$, for almost all $(q, b) \in R^n \times R^m$ the set $\mathrm{Sol}(M, A, q, b)$ is bounded (may be empty).

Proposition 18.2 (cf. Lemma 1 in Oettli and Yen (1995)). *Let $(M, A) \in R^{n\times n} \times R^{m\times n}$. The set*

$$W = \{(q, b) \in R^n \times R^m \ : \ \mathrm{Sol}(M, A, q, b) \ \text{is bounded}\} \tag{18.17}$$

is of full Lebesgue measure in $R^n \times R^m$.

Proof. The set $\mathrm{Sol}(M, A, q, b)$ is nonempty if and only if system (18.2) has a solution $(x, \lambda) \in R^n \times R^m$. We check at once that (18.2) has a solution if and only if there exists a subset $\alpha \subset I$, where $I := \{1, 2, \ldots, m\}$, such that the system

$$\begin{cases} Mx - A_\alpha^T \lambda_\alpha + q = 0, \\ A_\alpha x = b_\alpha, \quad \lambda_\alpha \geq 0, \\ A_{\bar\alpha} x \geq b_{\bar\alpha}, \quad \lambda_{\bar\alpha} = 0. \end{cases} \tag{18.18}$$

has a solution $(x, \lambda_\alpha, \lambda_{\bar\alpha})$, where $\bar\alpha = I \setminus \alpha$. If $\alpha = \emptyset$ (resp., $\bar\alpha = \emptyset$) then the terms indexed by α (resp., by $\bar\alpha$) are absent in (18.18). Hence

$$\mathrm{Sol}(M, A, q, b) = \bigcup \{\mathrm{Sol}(M, A, q, b)_\alpha \ : \ \alpha \subset I\},$$

where

$$\begin{aligned}\mathrm{Sol}(M,A,q,b)_\alpha \;:=&\; \{x \in R^n \,:\, \text{there exists } \lambda \in R^m \text{ such that} \\ &\qquad (x, \lambda_\alpha, \lambda_{\bar\alpha}) \text{ is a solution of } (18.18)\}.\end{aligned}$$

Note that the set $\mathrm{Sol}(M,A,q,b)$ is unbounded if and only if there exists $\alpha \subset I$ such that $\mathrm{Sol}(M,A,q,b)_\alpha$ is unbounded. We denote by $S(M,A,q,b)_\alpha$ the set of all $(x,\lambda_\alpha) \in R^n \times R^{|\alpha|}$ satisfying the system

$$Mx - A_\alpha^T\lambda_\alpha + q = 0, \quad A_\alpha x = b_\alpha,$$

where $|\alpha|$ is the number of elements of α. Let

$$\Omega_\alpha = \{(q,b) \in R^n \times R^m \;:\; S(M,A,q,b)_\alpha \text{ is unbounded}\}.$$

Obviously, if $\mathrm{Sol}(M,A,q,b)_\alpha$ is unbounded then $S(M,A,q,b)_\alpha$ is unbounded. So, on account of (18.19), we have

$$\begin{aligned}&\{(q,b) \in R^n \times R^m \;:\; \mathrm{Sol}(M,A,q,b) \text{ is unbounded}\} \\ &\subset \bigcup \{\Omega_\alpha : \alpha \subset I\}.\end{aligned} \tag{18.20}$$

Clearly,

$$\begin{aligned}\Omega_\alpha \;=&\; \Big\{(q,b) \in R^n \times R^m \;:\; \det \widetilde{M}_\alpha = 0 \text{ and there exists} \\ &\qquad (x,\lambda) \in R^n \times R^m \text{ such that } \widetilde{M}_\alpha\tbinom{x}{\lambda_\alpha} = \tbinom{q}{b_\alpha}\Big\},\end{aligned}$$

where

$$\widetilde{M}_\alpha = \begin{bmatrix} -M & A_\alpha^T \\ A_\alpha & 0 \end{bmatrix}.$$

If $\det \widetilde{M}_\alpha = 0$, then the image of the linear operator

$$\widetilde{M}_\alpha : R^n \times R^{|\alpha|} \longrightarrow R^n \times R^{|\alpha|}$$

corresponding to the matrix $\widetilde{M}_\alpha$ is a proper linear subspace of $R^n \times R^{|\alpha|}$. Hence Ω_α is a proper linear subspace of $R^n \times R^m$. So the set Ω_α is of Lebesgue measure 0 in $R^n \times R^m$. Therefore, from (18.20) we deduce that the set

$$\Omega := \{(q,b) \in R^n \times R^m \;:\; \mathrm{Sol}(M,A,q,b) \text{ is unbounded}\}$$

is of Lebesgue measure 0 in $R^n \times R^m$. Since $W = (R^n \times R^m) \setminus \Omega$ by (18.17), the desired conclusion follows. $\square$

In Example 18.1, the system $Ax \geq 0$ is irregular and $\mathrm{Sol}(\cdot)$ is not upper semicontinuous (M, A, q, b). The following theorem shows that if the system $Ax \geq 0$ is regular, then condition (18.3) is necessary and sufficient for the usc property of $\mathrm{Sol}(\cdot)$.

Theorem 18.2. *Let $(M, A, q, b) \in R^{n\times n} \times R^{m\times n} \times R^n \times R^m$. Suppose that the system $Ax \geq 0$ is regular. Then,* (18.3) *holds if and only if for every $(q, b) \in R^n \times R^m$ the solution map* $\mathrm{Sol}(\cdot)$ *is upper semicontinuous at (M, A, q, b).*

Proof. Suppose that (18.3) holds. We have to prove that for every $(q, b) \in R^n \times R^m$ the solution map $\mathrm{Sol}(\cdot)$ is upper semicontinuous at (M, A, q, b). To obtain a contradiction, suppose that there is a pair $(\bar{q}, \bar{b}) \in R^n \times R^m$ such that $\mathrm{Sol}(\cdot)$ is not upper semicontinuous at $(M, A, \bar{q}, \bar{b})$. Then there exist an open set $V \subset R^n$ containing $\mathrm{Sol}(M, A, \bar{q}, \bar{b})$, a sequence $\{(M^k, A^k, q^k, b^k)\}$ converging to $(M, A, \bar{q}, \bar{b})$ in $R^{n\times n} \times R^{m\times n} \times R^n \times R^m$ and a sequence $\{x^k\}$ in R^n such that

$$x^k \in \mathrm{Sol}(M^k, A^k, q^k, b^k) \setminus V. \tag{18.21}$$

Let y be any point in $\Delta(A, \bar{b})$. Since the system $Ax \geq 0$ is regular, the system $Ax \geq \bar{b}$ is regular (see Lemma 13.2). By Lemma 13.1, there exists a subsequence $\{k'\}$ of $\{k\}$ and a sequence $\{y^{k'}\}$ in R^n converging to y such that

$$A^{k'}y^{k'} \geq b^{k'} \quad \text{for all} \quad k'. \tag{18.22}$$

From (18.21) and (18.22) it follows that

$$A^{k'}x^{k'} \geq b^{k'}, \quad \langle M^{k'}x^{k'} + q^{k'}, y^{k'} - x^{k'}\rangle \geq 0. \tag{18.23}$$

We claim that $\{x^{k'}\}$ is bounded. If $\{x^{k'}\}$ is unbounded, then without loss of generality we can assume that $\|x^{k'}\| \neq 0$ for all k', $\|x^{k'}\| \to \infty$ and $\|x^{k'}\|^{-1}x^{k'} \to \bar{x}$ for some $\bar{x} \in R^n$ with $\|\bar{x}\| = 1$, as $k' \to \infty$.

Dividing the first inequality in (18.23) by $\|x^{k'}\|$, the second inequality in (18.23) by $\|x^{k'}\|^2$, and letting $k' \to \infty$, we get

$$A\bar{x} \geq 0, \quad \langle M\bar{x}, -\bar{x}\rangle \geq 0. \tag{18.24}$$

Let v be any point in $\Delta(A, 0)$. As $Ax \geq 0$ is regular, by Lemma 13.1 there exists a subsequence $\{k''\}$ of $\{k'\}$ and a sequence $\{v^{k''}\}$ converging to v such that

$$A^{k''}v^{k''} \geq 0 \quad \text{for all} \quad k''. \tag{18.25}$$

By (18.23) and (18.25) we have

$$A^{k''}(x^{k''}+v^{k''}) \geq b^{k''} \quad \text{for all } k''.$$

From (18.21) it follows that

$$\langle M^{k''}x^{k''}+q^{k''}, y-x^{k''}\rangle \geq 0 \quad \text{for all} \quad y \in \Delta(A^{k''}, b^{k''}). \tag{18.26}$$

Substituting $x^{k''}+v^{k''}$ for y in (18.26), we obtain

$$\langle M^{k''}x^{k''}+q^{k''}, v^{k''}\rangle \geq 0. \tag{18.27}$$

Dividing (18.27) by $\|x^{k''}\|$ and letting $k'' \to \infty$ we get $\langle M\bar{x}, v\rangle \geq 0$. From the last inequality and (18.26) it follows that

$$\langle M\bar{x}, v-\bar{x}\rangle \geq 0, \quad A\bar{x} \geq 0. \tag{18.28}$$

Since v is arbitrary in $\Delta(A,0)$, from (18.28) we deduce that $\bar{x} \in \mathrm{Sol}(M,A,0,0) \setminus \{0\}$, a contradiction. Thus the sequence $\{x^{k'}\}$ is bounded. There is no loss of generality in assuming that $x^{k'} \to \hat{x}$ as $k' \to \infty$. By (18.21),

$$\hat{x} \in \mathrm{Sol}(M,A,\bar{q},\bar{b}) \subset V.$$

Since $x^{k'} \in R^n \setminus V$ and V is open, we have $\hat{x} \in R^n \setminus V$, which is impossible. We have thus proved that for every $(q,b) \in R^n \times R^m$ the map $\mathrm{Sol}(\cdot)$ is upper semicontinuous at (M,A,q,b).

Conversely, suppose that for every $(q,b) \in R^n \times R^m$ the solution map $\mathrm{Sol}(\cdot)$ is upper semicontinuous at (M,A,q,b). By Proposition 18.1, there exists $(\bar{q},\bar{b}) \in R^n \times R^m$ such that $\mathrm{Sol}(M,A,\bar{q},\bar{b})$ is bounded. Therefore, according to Theorem 18.1, condition (18.3) is satisfied. The proof is complete. □

Theorem 18.3. *Let $(M,A,b) \in R^{n\times n} \times R^{m\times n} \times R^m$. Let*

$$K^- := \{v \in R^n : \langle Mv, v\rangle \leq 0, \quad Av \geq 0\}.$$

If $K^- = \{0\}$ and the system $Ax \geq b$ is regular then, for any $q \in R^n$, the solution map $\mathrm{Sol}(\cdot)$ is upper semicontinuous at (M,A,q,b).

Proof. Suppose that $K^- = \{0\}$ and the system $Ax \geq b$ is regular. Suppose that the assertion of the theorem is false. Then there exists $q \in R^n$ such that $\mathrm{Sol}(\cdot)$ is not upper semicontinuous at (M,A,q,b). Thus there exist an open set $V \subset R^n$ containing $\mathrm{Sol}(M,A,q,b)$, a sequence $\{(M^k, A^k, q^k, b^k)\}$ converging to (M, A, q, b) in $R^{n\times n} \times R^{m\times n} \times R^n \times R^m$, and a sequence $\{x^k\}$ in R^n such that

$$x^k \in \mathrm{Sol}(M^k, A^k, q^k, b^k) \setminus V \quad \text{for all} \;\; k. \tag{18.29}$$

Let $y \in \Delta(A, b)$. Since $Ax \geq b$ is regular, by Lemma 13.1 there exist a subsequence $\{k'\}$ of $\{k\}$ and a sequence $\{y^{k'}\}$ converging to y in R^n such that

$$A^{k'} y^{k'} \geq b^{k'} \;\; \text{for all} \;\; k'. \tag{18.30}$$

From (18.29) and (18.30) it follows that

$$\langle M^{k'} x^{k'} + q^{k'}, y^{k'} - x^{k'} \rangle \geq 0 \;\; \text{for all} \;\; k'. \tag{18.31}$$

We claim that the sequence $\{x^{k'}\}$ is bounded. Indeed, if $\{x^{k'}\}$ is unbounded then without loss of generality we can assume that $\|x^{k'}\| \neq 0$ for all k', $\|x^{k'}\| \to \infty$ and $\|x^{k'}\|^{-1} x^{k'} \to \bar{v}$ as $k' \to \infty$ for some $\bar{v} \in R^n$ with $\|\bar{v}\| = 1$. Dividing the inequalities in (18.31) by $\|x^{k'}\|^2$ and letting $k' \to \infty$, we get

$$\langle M\bar{v}, -\bar{v} \rangle \geq 0. \tag{18.32}$$

Since $Ax^{k'} \geq b^{k'}$, we have $A\bar{v} \geq 0$. Since $\bar{v} \neq 0$, from the last inequality and (18.32) we deduce that $K^- \neq \{0\}$, which is impossible. Thus the sequence $\{x^{k'}\}$ is bounded; so it has a convergent subsequence. Without loss of generality we can assume that $\{x^{k'}\}$ itself converges to some $\bar{x}$ in R^n. From (18.31) it follows that

$$\langle M\bar{x} + q, y - \bar{x} \rangle \geq 0. \tag{18.33}$$

Since y is arbitrary in $\Delta(A, b)$ and $\bar{x} \in \Delta(A, b)$, from (18.33) we deduce that $\bar{x} \in \mathrm{Sol}(M, A, q, b) \subset V$. On the other hand, since $x^k \in \mathrm{Sol}(M^k, A^k, q^k, b^k) \setminus V$ for every k and V is open, it follows that $\bar{x} \notin V$. We have thus arrived at a contradiction. The proof is complete. $\square$

Remark 18.1. It is a simple matter to show that if $K^- = \{0\}$ then (18.3) holds.

Corollary 18.1. *Let $(A, b) \in R^{m\times n} \times R^m$. Suppose that the system $Ax \geq b$ is regular and the set $\Delta(A, b)$ is bounded. Then, for any $(M, q) \in R^{n\times n} \times R^n$, the solution map $\mathrm{Sol}(\cdot)$ is upper semicontinuous at (M, A, q, b).*

Proof. Since $Ax \geq b$ is regular, $\Delta(A, b) \neq \emptyset$. Since $\Delta(A, b)$ is bounded, we deduce that $\Delta(A, 0) = \{0\}$. Let there be given any $(M, q) \in R^{n\times n} \times R^n$. Since $K^- \subset \Delta(A, 0)$ and $\Delta(A, 0) = \{0\}$, we have $K^- = \{0\}$. By Theorem 18.2, Sol$(\cdot)$ is upper semicontinuous at (M, A, q, b). $\square$

Corollary 18.2. *Let $(M, A, b) \in R^{n\times n} \times R^{m\times n} \times R^m$. Suppose that the matrix M is positive definite and the system $Ax \geq b$ is regular. Then, for every $q \in R^n$, the solution map* Sol$(\cdot)$ *is upper semicontinuous at (M, A, q, b).*

Proof. Since M is positive definite, we have $v^T Mv > 0$ for every nonzero $v \in R^n$. Hence $K^- = \{0\}$. Applying Theorem 18.2 we see that, for every $q \in R^n$, Sol$(\cdot)$ is upper semicontinuous at (M, A, q, b). $\square$

The next examples show that without the regularity condition imposed on the system $Ax \geq b$, the assertion of Theorem 18.2 may be false or may be true, as well.

Example 18.2. Consider problem (18.1), where $m = n = 1$, $M = [1]$, $A = [0]$, $q = 0$, and $b = 0$. It is easily seen that $Ax \geq b$ is irregular. For every $t \in (0, 1)$, let $A_t = [t^2]$ and $b_t = t$. We have

$$K^- = \{v \in R \,:\, \langle Mv,\ v\rangle \leq 0,\ \ Av \geq 0\} = \{v \in R : v^2 \leq 0\} = \{0\},$$

$$\mathrm{Sol}(M, A, q, b) = \{0\}, \quad \mathrm{Sol}(M, A_t, q, b_t) = \{t^{-1}\}.$$

Fix any bounded open set V satisfying $\mathrm{Sol}(M, A, q, b) = \{0\} \subset V$. Since $t^{-1} \in \mathrm{Sol}(M, A_t, q, b_t)$ for every $t \in (0, 1)$, the inclusion $\mathrm{Sol}(M, A_t, q, b_t) \subset V$ does not hold for $t > 0$ small enough. Since $A_t \to A$, $b_t \to b$ as $t \to 0$, the multifunction Sol$(\cdot)$ cannot be upper semicontinuous at (M, A, q, b).

Example 18.3. Consider problem (18.1), where $n = 1$, $m = 2$,

$$M = [-1], \quad A = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad q = (0), \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In this case we have

$$K^- = \{v \in R \,:\, -v^2 \leq 0,\ \ -v \geq 0,\ \ v \geq 0\} = \{0\},$$

$$\Delta(A, b) = \{x \in R \,:\, -x \geq 1,\ \ x \geq 0\} = \emptyset.$$

Obviously, the system $Ax \geq b$ is irregular. Since $\mathrm{Sol}(M, A, q, b) \subset \Delta(A, b)$, we conclude that $\mathrm{Sol}(M, A, q, b) = \emptyset$. Moreover, we can find $\delta > 0$ such that for any $(A', b') \in R^{m\times n} \times R^m$ with $\|(A', b') - (A, b)\| \le \delta$, $\Delta(A', b') = \emptyset$. Hence the multifunction $\mathrm{Sol}(\cdot)$ is upper semicontinuous at (M, A, q, b).

Lemma 18.1 (cf. Lemma 3 in Robinson (1977) and Lemma 10.2 in this book). *Let $(A, b) \in R^{m\times n} \times R^m$. Let*

$$\Lambda_0[A] = \{\lambda \in R^m \ : \ A^T\lambda = 0, \ \ \lambda \ge 0\}.$$

Then, the system $Ax \ge b$ is regular if and only if $\langle \lambda, b\rangle < 0$ for every $\lambda \in \Lambda_0[A] \setminus \{0\}$.

Proof. Suppose that $Ax \ge b$ is regular, that is there exists $x_0 \in R^n$ such that $Ax_0 > b$. Let $y_0 = Ax_0 - b > 0$. Let $\lambda \in \Lambda_0[A] \setminus \{0\}$, that is $A^T\lambda = 0$, $\lambda \ge 0$ and $\lambda \ne 0$. We have

$$\langle \lambda, b\rangle = \langle \lambda, Ax_0\rangle - \langle \lambda, y_0\rangle = \langle A^T\lambda, x_0\rangle - \langle \lambda, y_0\rangle = -\langle \lambda, y_0\rangle < 0.$$

Conversely, suppose that $\langle \lambda, b\rangle < 0$ for every $\lambda \in \Lambda_0[A] \setminus \{0\}$. If $Ax \ge b$ is irregular then there exists a sequence $\{b^k\}$ converging to b in R^m such that the system $Ax \ge b^k$ has no solutions for all k. According to Theorem 22.1 from Rockafellar (1970), there exists a sequence $\{\lambda^k\}$ in R^m such that

$$A^T\lambda^k = 0, \quad \lambda^k \ge 0, \quad \langle \lambda^k, b^k\rangle > 0. \tag{18.34}$$

Since $\lambda^k \ne 0$, by the homogeneity of the inequalities in (18.34) we can assume that $\|\lambda^k\| = 1$ for every k. Thus the sequence $\{\lambda^k\}$ has a convergent subsequence. We can suppose that the sequence $\{\lambda^k\}$ itself converges to some $\bar\lambda$ with $\|\bar\lambda\| = 1$. Taking the limits in (18.34) as $k \to \infty$ we get

$$A^T\bar\lambda = 0, \quad \bar\lambda \ge 0, \quad \langle \bar\lambda, b\rangle \ge 0.$$

Hence $\bar\lambda \in \Lambda_0[A] \setminus \{0\}$ and $\langle \bar\lambda, b\rangle \ge 0$. We have arrived at contradiction. □

Theorem 18.4. *Let $(M, A, b) \in R^{n\times n} \times R^{m\times n} \times R^m$. Let*

$$K^+ := \{v \in R^n \ : \ \langle Mv, v\rangle \ge 0, \ \ Av \ge 0\}.$$

If $K^+ = \{0\}$ and the system $Ax \ge -b$ is regular then, for any $q \in R^n$, the solution map $\mathrm{Sol}(\cdot)$ is upper semicontinuous at (M, A, q, b).

Proof. Suppose that $K^+ = \{0\}$ and the system $Ax \ge -b$ is regular. Suppose that the assertion of the theorem is false. Then there exists

$q \in R^n$ such that Sol$(\cdot)$ is not upper semicontinuous at (M, A, q, b). Thus there exist an open set $V \subset R^n$ containing Sol(M, A, q, b), a sequence $\{(M^k, A^k, q^k, b^k)\}$ converging to (M, A, q, b) in $R^{n \times n} \times R^{m \times n} \times R^n \times R^m$, and a sequence $\{x^k\}$ in R^n such that

$$x^k \in \mathrm{Sol}(M^k, A^k, q^k, b^k) \setminus V \quad \text{for all} \quad k. \tag{18.35}$$

Since $x^k \in \mathrm{Sol}(M^k, A^k, q^k, b^k)$, there exists λ^k such that

$$M^k x^k - (A^k)^T \lambda^k + q^k = 0, \tag{18.36}$$

$$A^k x^k \geq b^k, \quad \lambda^k \geq 0, \quad \langle \lambda^k, A^k x^k - b^k \rangle = 0. \tag{18.37}$$

We claim that the sequence $\{(x^k, \lambda^k)\}$ is unbounded. Indeed, if $\{(x^k, \lambda^k)\}$ is bounded then $\{x^k\}$ and $\{\lambda^k\}$ are bounded sequences, so each of them has a convergent subsequence. Without loss of generality we can assume that $x^k \to \bar{x}$, $\lambda^k \to \bar{\lambda}$ as $k \to \infty$, where $\bar{x} \in R^n$ and $\bar{\lambda} \in R^m$. From (18.36) and (18.37) we deduce that

$$M\bar{x} - A^T \bar{\lambda} + q = 0,$$

$$A\bar{x} \geq b, \quad \bar{\lambda} \geq 0, \quad \langle \bar{\lambda}, A\bar{x} - b \rangle = 0.$$

Hence $\bar{x} \in \mathrm{Sol}(M, A, q, b) \subset V$. This is impossible because V is open and $x^k \in \mathrm{Sol}(M^k, A^k, q^k, b^k) \setminus V$ for all k. Thus $\{(x^k, \lambda^k)\}$ is unbounded.

There is no loss of generality in assuming that $\|(x^k, \lambda^k)\| \neq 0$ for all k, $\|(x^k, \lambda^k)\| \to \infty$,

$$\|(x^k, \lambda^k)\|^{-1}(x^k, \lambda^k) \to (\bar{v}, \bar{\lambda}) \in R^n \times R^m, \tag{18.38}$$

where $\|(\bar{v}, \bar{\lambda})\| = 1$. Dividing the equality in (18.36) and the first two inequalities in (18.37) by $\|(x^k, \lambda^k)\|$, the equality in (18.37) by $\|(x^k, \lambda^k)\|^2$ and letting $k \to \infty$ we get

$$M\bar{v} - A^T \bar{\lambda} = 0, \quad A\bar{v} \geq 0, \quad \bar{\lambda} \geq 0, \quad \langle \bar{\lambda}, A\bar{v} \rangle = 0. \tag{18.39}$$

From (18.39) it follows that $\langle M\bar{v}, \bar{v} \rangle = 0$ and $A\bar{v} \geq 0$. Thus $\bar{v} \in K^+$. As $K^+ = \{0\}$, we have $\bar{v} = 0$. By (18.38), $\|\bar{\lambda}\| = 1$. Since $\bar{v} = 0$, from (18.39) we deduce that $\bar{\lambda} \in \Lambda_0[A]$. Since $Ax \geq -b$ is regular, by Lemma 18.1 we have $\langle \bar{\lambda}, -b \rangle < 0$. From (18.36) and (18.37) it follows that

$$\langle M^k x^k + q^k, x^k \rangle = \langle \lambda^k, b^k \rangle. \tag{18.40}$$

If there exists an integer k_0 such that $\langle \lambda^k, b^k \rangle \le 0$ for all $k \ge k_0$ then without loss of generality we can suppose that $\langle \lambda^k, b^k \rangle \le 0$ for all k. Dividing the last inequality by $\|(x^k, \lambda^k)\|$ and letting $k \to \infty$ we have

$$\langle \bar{\lambda}, b \rangle \le 0,$$

which contradicts the fact that $\langle \bar{\lambda}, -b \rangle < 0$. So there exists a subsequence $\{k'\}$ of $\{k\}$ such that $\langle \lambda^{k'}, b^{k'} \rangle > 0$. From this and (18.40) we deduce that

$$\langle M^{k'} x^{k'} + q^{k'}, x^{k'} \rangle = \langle \lambda^{k'}, b^{k'} \rangle, \quad \langle M^{k'} x^{k'} + q^{k'}, x^{k'} \rangle > 0 \quad (18.41)$$

for all k'. If $\{x^{k'}\}$ is bounded then, dividing the equality in (18.41) by $\|(x^{k'}, \lambda^{k'})\|$ and letting $k' \to \infty$ we obtain $\langle \bar{\lambda}, b \rangle = 0$, which contradicts the fact that $\langle \bar{\lambda}, -b \rangle < 0$. Thus $\{x^{k'}\}$ is unbounded. Without loss of generality we can assume that $\|x^{k'}\| \ne 0$ for all k' and the sequence $\|x^{k'}\|^{-1} x^{k'}$ itself converges to some $\hat{v} \in R^n$ with $\|\hat{v}\| = 1$. Dividing the inequality in (18.41) by $\|x^{k'}\|^2$ and letting $k' \to \infty$ we get $\langle M\hat{v}, \hat{v} \rangle \ge 0$. By (18.37), we have $A^{k'} x^{k'} \ge b^{k'}$ for all k'. Dividing the last inequality by $\|x^{k'}\|$ and letting $k' \to \infty$ we get $A\hat{v} \ge 0$. From this and the inequality $\langle M\hat{v}, \hat{v} \rangle \ge 0$ we see that $\hat{v} \in K^+ \setminus \{0\}$, contrary to the assumption $K^+ = \{0\}$. The proof is complete. □

Remark 18.2. It is easily verified that if $K^+ = \{0\}$ then (18.3) holds.

Corollary 18.3. *Let $(A, b) \in R^{m \times n} \times R^m$. Suppose that the system $Ax \ge -b$ is regular and $\Delta(A, b)$ is nonempty and bounded. Then, for any $(M, q) \in R^{n \times n} \times R^n$ the solution map* $\mathrm{Sol}(\cdot)$ *is upper semicontinuous at (M, A, q, b).*

Proof. Let $(M, q) \in R^{n \times n} \times R^n$ be given arbitrarily. Since $\Delta(A, b)$ is nonempty and bounded, $\Delta(A, 0) = \{0\}$. As $K^+ \subset \Delta(A, 0)$, we have $K^+ = \{0\}$. Applying Theorem 18.4 we conclude that the solution map $\mathrm{Sol}(\cdot)$ is upper semicontinuous at (M, A, q, b). □

Corollary 18.4. *Let $(M, A, b) \in R^{n \times n} \times R^{m \times n} \times R^m$. If the matrix M is negative definite and the system $Ax \ge -b$ is regular then, for any $q \in R^n$, the solution map* $\mathrm{Sol}(\cdot)$ *is upper semicontinuous at (M, A, q, b).*

Proof. Since M is negative definite, we have $v^T M v < 0$ for any nonzero vector $v \in R^n$. Since $K^+ \subset \{v \in R^n : \langle Mv, v \rangle \ge 0\}$, we deduce that $K^+ = \{0\}$. Applying Theorem 18.4 we see that, for every $q \in R^n$, $\mathrm{Sol}(\cdot)$ is upper semicontinuous at (M, A, q, b). □

Corollary 18.5. *Let $(A, b) \in R^{m\times n} \times R^m$. Suppose that $\Delta(A, 0) = \{0\}$ and $\langle \lambda, b\rangle \neq 0$ for every nonzero λ satisfying $A^T\lambda = 0, \lambda \geq 0$. Then, for any $(M, q) \in R^{n\times n} \times R^n$,* Sol$(\cdot)$ *is upper semicontinuous at (M, A, q, b).*

Proof. Let $(M, q) \in R^{n\times n} \times R^n$ be given arbitrarily. Since $\Delta(A, 0) = \{0\}$, we have $K^- = K^+ = \{0\}$. Since $\Lambda[A] = \{\lambda \in R^m : A^T\lambda = 0,\ \lambda \geq 0\}$, we see that $\Lambda[A]$ is a pointed convex cone. From the assumption that $\langle \lambda, b\rangle \neq 0$ for every nonzero $\lambda \in \Lambda[A]$ we deduce that one and only one of the following two cases occurs:

(i) $\langle \lambda, b\rangle > 0$ for every $\lambda \in \Lambda[A] \setminus \{0\}$;

(ii) $\langle \lambda, b\rangle < 0$ for every $\lambda \in \Lambda[A] \setminus \{0\}$.

In case (i), the system $Ax \geq b$ is regular by Lemma 18.1. Since $K^- = \{0\}$, the desired conclusion follows from Theorem 18.2. In case (ii), $Ax \geq -b$ is regular by Lemma 18.1. Since $K^+ = \{0\}$, the assertion follows from Theorem 18.4. □

The following examples show that without the regularity of the system $Ax \geq -b$, the assertion in Theorem 18.4 may be true or may be false, as well.

Example 18.4. Consider problem (18.1), where $n = 1$, $m = 2$,

$$M = [1], \quad A = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad q = (0), \quad b = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

We can check at once that $K^- = K^+ = \{0\}$, $\Delta(A, b) = \{x \in R : 0 \leq x \leq 1\}$, and $\Delta(A, -b) = \emptyset$. Note that the system $Ax \geq b$ is regular and the system $Ax \geq -b$ is irregular. The usc property of Sol$(\cdot)$ at (M, A, q, b) follows from Theorem 18.2.

Example 18.5. Consider problem (18.1), where $n = 2$, $m = 3$ and (M, A, q, b) is defined as in Example 18.1. For this problem we find that $K^+ = \{0\}$ and $\Delta(A, -b) = \emptyset$. In particular, the system $Ax \geq -b$ is irregular. As it has been shown in Example 18.1, the solution map Sol$(\cdot)$ is not upper semicontinuous at (M, A, q, b).

18.2 LSC Property of the Solution Map

In this section we will find necessary and sufficient conditions for the lower semicontinuity of the solution map in parametric AVI problems.

Theorem 18.5. *Let* $(M,A,c,b) \in R^{n\times n}\times R^{m\times n}\times R^n\times R^m$. *If the multifunction* $\mathrm{Sol}(M,A,\cdot,\cdot)$ *is lower semicontinuous at* (q,b) *then*

(a) *the set* $\mathrm{Sol}(M,A,q,b)$ *is finite, and*

(b) *the system* $Ax \ge b$ *is regular.*

Proof. We will omit the easy proof of (b). To prove (a), for every index set $I \subset \{1,\cdots,m\}$ we define a matrix $S_I \in R^{(n+|I|)\times(n+|I|)}$, where $|I|$ is the number of elements of I, by setting

$$S_I := \begin{bmatrix} M & -A_I^T \\ A_I & O \end{bmatrix}$$

(If $I = \emptyset$ then we set $S_I = M$). Let

$$\mathcal{P}_I = \Big\{(u,v) \in R^n \times R^m : \tbinom{u}{v_I} = S_I \tbinom{x}{\lambda_I} \\ \text{for some } (x,\lambda) \in R^n \times R^m\Big\},$$

and

$$\mathcal{P} = \bigcup \{\mathcal{P}_I : I \subset \{1,\cdots,m\},\ \ \det S_I = 0\}.$$

If $\det S_I = 0$ then $\mathcal{P}_I$ is a proper linear subspace of $R^n \times R^m$. Hence, by the Baire Lemma (see Brezis (1987)), $\mathcal{P}$ is nowhere dense in $R^n \times R^m$. Then there exists a sequence $\{(q^k,b^k)\}$ converging to (q,b) in $R^n \times R^m$ such that $(-q^k,b^k) \notin \mathcal{P}$ for all k.

Fix any $\bar{x} \in \mathrm{Sol}(M,A,q,b)$. As the multifunction $\mathrm{Sol}(M,A,\cdot,\cdot)$ is lower semicontinuous at (q,b), there exist a subsequence $\{(q^{k_l},b^{k_l})\}$ of $\{(q^k,b^k)\}$ and a sequence $\{x^{k_l}\}$ converging in R^n to $\bar{x}$ such that $x^{k_l} \in \mathrm{Sol}(M,A,q^{k_l},b^{k_l})$ for all k_l. Since $x^{k_l} \in \mathrm{Sol}(M,A,q^{k_l},b^{k_l})$, there exists $\lambda^{k_l} \in R^m$ such that

$$\begin{cases} Mx^{k_l} - A^T\lambda^{k_l} + q^{k_l} = 0, \\ Ax^{k_l} \ge b^{k_l}, \quad \lambda^{k_l} \ge 0, \\ (\lambda^{k_l})^T(Ax^{k_l} - b^{k_l}) = 0. \end{cases} \tag{18.42}$$

For every k_l, let $I_{k_l} = \{i \in \{1\ldots,m\} : \lambda_i^{k_l} > 0\}$. It is clear that there must exists an index set $I \subset \{1,\cdots,m\}$ such that $I_{k_l} = I$ for infinitely many k_l. Without loss of generality, we may assume that $I_{k_l} = I$ for all k_l. By (18.42), we have

$$Mx^{k_l} - A_I^T\lambda_I^{k_l} + q^{k_l} = 0, \quad A_I x^{k_l} = b_I^{k_l},$$

or, equivalently,

$$S_I \begin{pmatrix} x^{k_l} \\ \lambda_I^{k_l} \end{pmatrix} = \begin{pmatrix} -q^{k_l} \\ b_I^{k_l} \end{pmatrix}. \tag{18.43}$$

We claim that S_I is nonsingular. Indeed, if $\det S_I = 0$ then, by (18.43) and by the definitions of $\mathcal{P}_I$ and $\mathcal{P}$, we have

$$(-q^{k_l}, b^{k_l}) \in \mathcal{P}_I \subset \mathcal{P},$$

which is impossible because $(-q^k, b^k) \notin \mathcal{P}$ for all k. So S_I is nonsingular. From (18.43) it follows that

$$\begin{pmatrix} x^{k_l} \\ \lambda_I^{k_l} \end{pmatrix} = S_I^{-1} \begin{pmatrix} -q^{k_l} \\ b_I^{k_l} \end{pmatrix}.$$

Letting $l \to \infty$, we get

$$\lim_{l\to\infty} \begin{pmatrix} x^{k_l} \\ \lambda_I^{k_l} \end{pmatrix} = S_I^{-1} \begin{pmatrix} -q \\ b_I \end{pmatrix}. \tag{18.44}$$

If $I = \emptyset$ then (18.44) has the following form

$$\lim_{l\to\infty} x^{k_l} = M^{-1}(-q).$$

By (18.44), the sequence $\{\lambda_I^{k_l}\}$ must converge to some $\lambda_I \geq 0$ in $R^{|I|}$. As $x^{k_l} \to \bar{x}$, from (18.44) we obtain

$$\begin{pmatrix} \bar{x} \\ \lambda_I \end{pmatrix} = S_I^{-1} \begin{pmatrix} -q \\ b_I \end{pmatrix}. \tag{18.45}$$

Let

$$Z = \{(x, \lambda) \in R^n \times R^m : \begin{array}{l} \text{there exists such } J \subset \{1, \cdots, m\} \\ \text{that } \det S_J \neq 0 \text{ and } \binom{x}{\lambda_J} = S_J^{-1}\binom{-q}{b_J}\} \end{array}$$

and

$$X = \{x \in R^n \ : \text{ there exists } \lambda \in R^m \text{ such that } (x, \lambda) \in Z\}.$$

Similarly as in the proof of Theorem 11.3, X is a finite set. From (18.45) we conclude that $\bar{x} \in X$. We have thus proved that

$$\mathrm{Sol}(M, A, q, b) \subset X.$$

In particular, $\mathrm{Sol}(M, A, q, b)$ is a finite set. The proof is complete. □

The following example shows that, in general, the above conditions (a) and (b) are not sufficient for the lsc property of $\mathrm{Sol}(M, A, \cdot, \cdot)$ at (q, b).

Example 18.6. Consider the AVI problem (18.1) in which $n = 2$, $m = 3$, and

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad q = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}.$$

For every $\varepsilon > 0$, we set $q(\varepsilon) = (-1, -\varepsilon)$. We can perform some computations to show that

$$\mathrm{Sol}(M, A, q, b) = \{(0, 2),\ (1, 0)\}$$

and $\mathrm{Sol}(M, A, q(\varepsilon), b) = \{(0, 2)\}$ for every $\varepsilon > 0$. It is clear that the system $Ax \geq b$ is regular. Let

$$V = \left\{(x_1, x_2) \in R^2 : \frac{1}{2} < x_1 < \frac{3}{2},\ -1 < x_2 < 1\right\}.$$

Since $\mathrm{Sol}(M, A, q, b) \cap V = \{(1, 0)\}$ and $\mathrm{Sol}(M, A, q(\varepsilon), b) \cap V = \emptyset$ for every $\varepsilon > 0$, we conclude the multifunction $\mathrm{Sol}(M, A, \cdot, \cdot)$ is not lower semicontinuous at (q, b).

Let $(M, A, q, b) \in R^{n\times n} \times R^{m\times n} \times R^n \times R^m$. Let $x \in \mathrm{Sol}(M, A, q, b)$ and let $\lambda \in R^m$ be a Lagrange multiplier corresponding to x. Let $I = \{1, 2, \ldots, m\}$, and let K and J be defined, respectively, by (11.14) and (11.15). We set $I_0 = K \cup J$.

The following theorem gives a sufficient condition for the lsc property of the multifunction $\mathrm{Sol}(M, A, \cdot, \cdot)$ at a given point. By definition (see (Cottle et al. 1992)), a square matrix is called a *P-matrix* if the determinant of each of its principal submatrices is positive.

Theorem 18.7. *Let $(M, A, q, b) \in R^{n\times n} \times R^{m\times n} \times R^n \times R^m$. Suppose that*

(i) *the set $\mathrm{Sol}(M, A, q, b)$ is finite, nonempty,*

(ii) *the system $Ax \geq b$ is regular,*

and suppose that for every $x \in \mathrm{Sol}(M, A, q, b)$ there exists a Lagrange multiplier λ corresponding to x such that at least one of the following conditions holds:

(c1) $v^T M v \geq 0$ *for every* $v \in R^n$ *with* $A_{I_0} v \geq 0$ *and* $(Mx+q)^T v = 0$,

(c2) $J = K = \emptyset$,

(c3) $J = \emptyset$, $K \neq \emptyset$, *and the system* $\{A_i : i \in K\}$ *is linearly independent,*

(c4) $J \neq \emptyset$, $K = \emptyset$, M *is nonsingular and* $A_J M^{-1} A_J^T$ *is a P-matrix,*

where K *and* J *are defined via* (x, λ) *by* (11.14) *and* (11.15). *Then, the multifunction* $\mathrm{Sol}(M, A, \cdot, \cdot)$ *is lower semicontinuous at* (q, b).

Proof. Since $\mathrm{Sol}(M, A, q, b)$ is nonempty, in order to prove that $\mathrm{Sol}(M, A, \cdot, \cdot)$ is lower semicontinuous at (q, b) we only need to show that, for any $x \in \mathrm{Sol}(M, A, q, b)$ and for any open neighborhood V_x of x, there exists $\delta > 0$ such that

$$\mathrm{Sol}(M, A, q', b') \cap V_x \neq \emptyset \tag{18.46}$$

for every $(q', b') \in R^n \times R^m$ satisfying $\|(q', b') - (q, b)\| < \delta$.

Let $x \in \mathrm{Sol}(M, A, q, b)$ and let V_x be an open neighborhood of x. By our assumptions, there exists a Lagrange multiplier λ corresponding to x such that at least one of the four conditions (c1)-(c4) holds.

Consider the case where (c1) holds. Since $\mathrm{Sol}(M, A, q, b)$ is finite by (i), x is an isolated solution of (18.1). By Corollary 10 in Gowda and Pang (1994b), from our assumptions it follows that there exists $\delta > 0$ such that (18.46) is valid for every (q', b') satisfying $\|(q', b') - (q, b)\| < \delta$.

Analysis similar to that in the proof of Theorem 11.4 shows that if one of the conditions (c2)–(c4) holds then we can find $\delta > 0$ such that (18.46) is valid for every (q', b') satisfying $\|(q', b') - (q, b)\| < \delta$.

So we can conclude that the multifunction $\mathrm{Sol}(M, A, \cdot, \cdot)$ is lower semicontinuous at (q, b). $\square$

It is interesting to see how the conditions (c1)–(c4) in Theorem 18.7 can be verified for concrete AVI problems.

Writing the necessary optimality conditions for the QP problems in Examples 11.3–11.5 as AVI problems we obtain the following examples.

Example 18.7. Consider problem (18.1) with $n = 2$, $m = 2$,

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}, \quad q = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We can show that $\mathrm{Sol}(M, A, q, b) = \{\bar{x}, \hat{x}, \widetilde{x}\}$, where $\bar{x}$, $\hat{x}$, $\widetilde{x}$ are the same as in Example 11.3. Note that $\widetilde{\lambda} := (0, 0)$ is a Lagrange multiplier corresponding to $\tilde{x}$. We observe that conditions (i) and (ii) in Theorem 18.7 are satisfied and, for each point $x \in \mathrm{Sol}(M, A, q, b)$, either (c1) or (c2) is satisfied. More precisely, if $x = \bar{x}$ or $x = \hat{x}$ then (c1) is satisfied; if $x = \widetilde{x}$ then (c2) is satisfied. By Theorem 18.7, the multifunction $\mathrm{Sol}(M, A, \cdot, \cdot)$ is lower semicontinuous at (q, b).

Example 18.8. Consider problem (18.1) with $n = 2$, $m = 3$,

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad q = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It is easy to verify that $\mathrm{Sol}(M, A, q, b) = \{\bar{x}, \hat{x}, \widetilde{x}\}$, where $\bar{x}$, $\hat{x}$, $\widetilde{x}$ are the same as in the preceding example. Note that $\widetilde{\lambda} := (0, 0, 0)$ is a Lagrange multiplier corresponding to $\widetilde{x}$. For $x = \bar{x}$ and $x = \hat{x}$, assumption (c1) is satisfied. For the pair $(\widetilde{x}, \widetilde{\lambda})$, we have $K = \emptyset$, $J = \{3\}$. Since $A_J = (1\ 0)$, $M^{-1} = M$, we get $A_J M^{-1} A_J^T = 1$. Thus (c4) is satisfied. By Theorem 18.7, $\mathrm{Sol}(M, A, \cdot, \cdot)$ is lower semicontinuous at (q, b).

Example 18.9. Consider problem (18.1) with $n = 2$, $m = 3$,

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad q = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

We can show that $\mathrm{Sol}(M, A, q, b) = \{\bar{x}, \hat{x}, \widetilde{x}\}$, where $\bar{x} = (2, -1)$, $\hat{x} = (2, 1)$ and $\widetilde{x} = (2, 0)$. Note that $\widetilde{\lambda} := (0, 0, 1)$ is a Lagrange multiplier corresponding to $\widetilde{x}$. For $x = \bar{x}$ and $x = \hat{x}$, condition (c1) is satisfied. For the pair $(\widetilde{x}, \widetilde{\lambda})$, we have $K = \{3\}$, $J = \emptyset$. Since

$$\{A_i : i \in K\} = \{A_3\} = \{(1\ 0)\},$$

assumption (c3) is satisfied. According to Theorem 18.7, $S(D, A, \cdot, \cdot)$ is lower semicontinuous at (q, b).

Let $(M, A, q, b) \in R^{n \times n} \times R^{m \times n} \times R^n \times R^m$. Let $x \in \mathrm{Sol}(M, A, q, b)$ and let $\lambda \in R^m$ be a Lagrange multiplier corresponding to x. We define K and J by (11.14) and (11.15), respectively. Consider the case where both the sets K and J are nonempty. If the matrix

$$Q_K = \begin{bmatrix} M & -A_K^T \\ A_K & 0 \end{bmatrix} \in R^{(n+|K|) \times (n+|K|)}$$

is nonsingular, then we denote by S_J the Schur complement of Q_K in the following matrix

$$\begin{bmatrix} M & -A_K^T & -A_J^T \\ A_K & 0 & 0 \\ A_J & 0 & 0 \end{bmatrix} \in R^{(n+|K|+|J|)\times(n+|K|+|J|)}.$$

This means that

$$S_J = [A_J\ 0]Q_K^{-1}[A_J\ 0]^T.$$

Since M is not assumed to be symmetric, S_J is not necessarily a symmetric matrix. Consider the following condition:

(c5) *$J \neq \emptyset$, $K \neq \emptyset$, the system $\{A_i : i \in K\}$ is linearly independent, $v^T Mv \neq 0$ for every nonzero vector v satisfying $A_K v = 0$, and S_J is a $P-$matrix.*

It can be shown that if $J \neq \emptyset$, $K \neq \emptyset$, the system $\{A_i : i \in K\}$ is linearly independent, and $v^T Mv \neq 0$ for every nonzero vector v satisfying $A_K v = 0$, then Q_K is nonsingular.

It can be proved that the assertion of Theorem 18.7 remains valid if instead of (c1)–(c4) we use (c1)–(c3) and (c5).

18.3 Commentaries

The material of this chapter is taken from Lee et al. (2002b-d).

As it has been noted in Chapter 5, the affine variational inequality problem is a natural and important extension of the linear complementarity problem. Both of the problems are closely related to the Karush-Kuhn-Tucker conditions in quadratic programming.

Various continuity properties of the solution maps in parametric affine variational inequality problems and parametric linear complementarity problems have been investigated (see Robinson (1979, 1981), Bank et al. (1982), Jansen and Tijs (1987), Cottle et al. (1992), Gowda (1992), Gowda and Pang (1994a), Oettli and Yen (1995), Gowda and Sznajder (1996), and the references therein).

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