# On a vector equality

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**Abstract**

In this paper, we recall a vector equality and give some of its applications.

Let us start by introducing the notations that will be used in this paper. By $T$ we will denote the triangle $A_1A_2A_3$. The inscribed circle of $T$ has center $I$ and is tanget to the side opposite to vertex $A_i$ at $B_i,\ i = 1, 2, 3$. Denote by $a_i$ and $h_i,\ i = 1, 2, 3$ the lengths of the side opposite to vertex $A_i$ and the length of the height from that same vertex, respectively. Let $K, p, R, r$ be the area, semiperimeter, circumradius, and inradius of $T$. Denote by $K_1, p_1, r_1$ the area, semiperimeter, and inradius of $B_1B_2B_3$.

**Theorem.** *$I$ is the center of mass of the system $B_1, B_2, B_3$ with masses $\frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}$ or*

$$\frac{\overrightarrow{IB_1}}{h_1} + \frac{\overrightarrow{IB_2}}{h_2} + \frac{\overrightarrow{IB_3}}{h_3} = 0. \tag{1}$$

*Proof.* We have

$$A_1B_2 = A_1B_3 = p - a_1,\ A_2B_3 = A_2B_1 = p - a_2,\ A_3B_1 = A_3B_2 = p - a_3.$$

It is not difficult to prove that for every point $M$ on the side $A_iA_j,\ i \neq j,\ i, j \in \{1, 2, 3\}$ we have

$$\overrightarrow{IM} = \frac{MA_j}{A_iA_j} \cdot \overrightarrow{IA_i} + \frac{MA_i}{A_iA_j} \cdot \overrightarrow{IA_j}.$$

When $M = B_1$ we get

$$\overrightarrow{IB_1} = \frac{B_1A_j}{A_iA_j} \cdot \overrightarrow{IA_i} + \frac{B_1A_i}{A_iA_j} \cdot \overrightarrow{IA_j}$$

or equivalently $a_1\overrightarrow{IB_1} = (p - a_3)\overrightarrow{IA_2} + (p - a_2)\overrightarrow{IA_3}$. Similarly, we obtain

$$a_2\overrightarrow{IB_2} = (p - a_1)\overrightarrow{IA_3} + (p - a_3)\overrightarrow{IA_1}$$

and

$$a_3\overrightarrow{IB_3} = (p - a_2)\overrightarrow{IA_1} + (p - a_1)\overrightarrow{IA_2}.$$

Adding the above equalities side by side, we obtain

$$a_1\overrightarrow{IB_1} + a_2\overrightarrow{IB_2} + a_3\overrightarrow{IB_3} = (2p - a_2 - a_3)\overrightarrow{IA_1} + (2p - a_3 - a_1)\overrightarrow{IA_2} + (2p - a_1 - a_2)\overrightarrow{IA_3}.$$

Note that $2p = a_1 + a_2 + a_3$ and recall that $a_1\overrightarrow{IA_1} + a_2\overrightarrow{IA_2} + a_3\overrightarrow{IA_3} = 0$, thus

$$a_1\overrightarrow{IB_1} + a_2\overrightarrow{IB_2} + a_3\overrightarrow{IB_3} = 0.$$

Using the formula $a_i = \frac{2K}{h_i}$ we conclude that

$$\frac{\overrightarrow{IB_1}}{h_1} + \frac{\overrightarrow{IB_2}}{h_2} + \frac{\overrightarrow{IB_3}}{h_3} = 0.$$

*Alternative proof.* For $i = 1, 2, 3$ we know that $\frac{\overrightarrow{IB_i}}{IB_i}$ is a unit vector perpendicular to the side opposite to vertex $A_i$, its direction being out of the triangle. By applying the "Porcupine Theorem" we obtain

$$a_1\frac{\overrightarrow{IB_1}}{IB_1} + a_2\frac{\overrightarrow{IB_2}}{IB_2} + a_3\frac{\overrightarrow{IB_3}}{IB_3} = 0.$$

Note that $IB_1 = IB_2 = IB_3 = r$ and by using the formula $a_i = \frac{2K}{h_i}$, $i = 1, 2, 3$ we deduce that

$$\frac{\overrightarrow{IB_1}}{h_1} + \frac{\overrightarrow{IB_2}}{h_2} + \frac{\overrightarrow{IB_3}}{h_3} = 0.$$

The proof is thus complete and we are ready to show some of the results due to this theorem.

**Problem 1.** Prove that

$$\frac{B_1B_2{}^2}{h_1h_2} + \frac{B_2B_3{}^2}{h_2h_3} + \frac{B_3B_1{}^2}{h_3h_1} = 1. \tag{2}$$

*Solution.* By squaring both sides of the equation (1) and using the following identity

$$2\overrightarrow{IB_i}\overrightarrow{IB_j} = IB_i{}^2 + IB_j{}^2 - B_iB_j{}^2 = 2r^2 - B_iB_j{}^2$$

we obtain

$$\left[\left(\sum_{i=1}^{3}\frac{1}{h_i^2} + 2\cdot\sum_{i=1}^{3}\frac{1}{h_ih_j}\right)r^2 - \left(\frac{B_1B_2{}^2}{h_1h_2} + \frac{B_2B_3{}^2}{h_2h_3} + \frac{B_3B_1{}^2}{h_3h_1}\right)\right] = 0.$$

Using the fact that $\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{1}{r}$ it follows that

$$\frac{B_1B_2{}^2}{h_1h_2} + \frac{B_2B_3{}^2}{h_2h_3} + \frac{B_3B_1{}^2}{h_3h_1} = 1.$$

We will now present four corollaries following from **Problem 1.**

**Corollary 1.** The following inequality holds

$$a_1a_2B_1B_2 + a_2a_3B_2B_3 + a_3a_1B_3B_1 \leq \frac{4\sqrt{3}}{3}pK. \tag{3}$$

*Solution.* Knowing that $a_i = \frac{2K}{h_i}$, we can rewrite the inequality in the form

$$4K^2\left(\frac{B_1B_2}{h_1h_2} + \frac{B_2B_3}{h_2h_3} + \frac{B_3B_1}{h_3h_1}\right) \leq \frac{4\sqrt{3}}{3}pK.$$

Because $K = pr$, the above inequality becomes

$$\left(\frac{B_1B_2}{h_1h_2} + \frac{B_2B_3}{h_2h_3} + \frac{B_3B_1}{h_3h_1}\right) \leq \frac{\sqrt{3}}{3r}.$$

We will now prove that the last inequality is true. Indeed, by applying the Cauchy-Schwarz inequality we obtain

$$\left(\frac{B_1B_2}{h_1h_2} + \frac{B_2B_3}{h_2h_3} + \frac{B_3B_1}{h_3h_1}\right)^2 \leq \left(\frac{1}{h_1h_2} + \frac{1}{h_2h_3} + \frac{1}{h_3h_1}\right)\left(\frac{B_1B_2{}^2}{h_1h_2} + \frac{B_2B_3{}^2}{h_2h_3} + \frac{B_3B_1{}^2}{h_3h_1}\right).$$

By using the well-known inequality $ab + bc + ca \leq \frac{1}{3}(a+b+c)^2$ for all real number $a, b, c$ and the equality $\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{1}{r}$ it follows, from **Problem 1**, that

$$\left(\frac{B_1B_2}{h_1h_2} + \frac{B_2B_3}{h_2h_3} + \frac{B_3B_1}{h_3h_1}\right)^2 \leq \frac{1}{3r^2}$$

or equivalently

$$\frac{B_1B_2}{h_1h_2} + \frac{B_2B_3}{h_2h_3} + \frac{B_3B_1{}^2}{h_3h_1} \leq \frac{\sqrt{3}}{3r}.$$

The equality takes places if and only if $T$ is an equilateral triangle.

**Corollary 2.** If $T$ is an acute triangl, then

$$\max\{a_1, a_2, a_3\} \geq \sqrt{3}R \tag{4}$$

with equality if and only if $T$ is an equilateral triangle.

*Solution.* Draw three lines tanget to the circle circumscribed around $T$ from its vertices. Suppose that these three lines intersect at three points $C_1, C_2$, and $C_3$ where $C_i$ is the vertex opposite to the side passing through $A_i,\ i = 1, 2, 3$. The circle circumscribed to $T$ is the incercle of $C_1C_2C_3$. Thus $R$ is the length of the inradius of $C_1C_2C_3$. Denote by $l_1, l_2$, and $l_3$ the lengths of the altitudes in the triange $C_1C_2C_3$. Applying the result of **Problem 1** we obtain

$$\frac{A_1A_2{}^2}{l_1l_2} + \frac{A_2A_3{}^2}{l_2l_3} + \frac{A_3A_1{}^2}{l_3l_1} = 1$$

or

$$\frac{{a_3}^2}{l_1 l_2} + \frac{{a_1}^2}{l_2 l_3} + \frac{{a_2}^2}{l_3 l_1} = 1.$$

Hence

$$1 \le \max\{a_1^2, a_2^2, a_3^2\} \cdot \left(\frac{1}{l_1 l_2} + \frac{1}{l_2 l_3} + \frac{1}{l_3 l_1}\right) \le \frac{1}{3} \max\{a_1^2, a_2^2, a_3^2\} \left(\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3}\right)^2.$$

Note that $\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3} = \frac{1}{R}$, thus it follows that

$$\frac{1}{3R^2} \max\{a_1^2, a_2^2, a_3^2\} \ge 1$$

and we conclude that

$$\max\{a_1, a_2, a_3\} \ge \sqrt{3}R.$$

Equality occurs if and only if $l_1 = l_2 = l_3$ and thus $C_1C_2 = C_2C_3 = C_3C_1$. This imples that $C_1C_2C_3$ is equilateral which in turn means that $A_1A_2A_3$ is equilateral, and we are done.

**Corollary 3.**

$$p_1^2 \le \frac{pK}{2R}. \tag{5}$$

When does equality occur ?

*Solution.* We introduce the following inequality

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \ge \frac{(a+b+c)^2}{x+y+z}$$

for all positive reals $a, b$, and $c$ to prove the above inequality with equality if and only if $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$. Using the above inequality and the result of **Problem 1** we obtain

$$1 = \frac{{B_1B_2}^2}{h_1h_2} + \frac{{B_2B_3}^2}{h_2h_3} + \frac{{B_3B_1}^2}{h_3h_1} \ge \frac{(B_1B_2 + B_2B_3 + B_3B_1)^2}{h_1h_2 + h_2h_3 + h_3h_1}.$$

Observe that $B_1B_2 + B_2B_3 + B_3B_1 = 2p_1$ and

$$h_1h_2 + h_2h_3 + h_3h_1 = 4K^2\left(\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \frac{1}{a_3a_1}\right) = \frac{8K^2p}{a_1a_2a_3}.$$

By using the formulae $K = \frac{a_1a_2a_3}{4R}$ and $h_1h_2 + h_2h_3 + h_3h_1 = \frac{2Kp}{R}$ it follows that

$$p_1^2 \le \frac{pK}{2R}.$$

Equality occurs if and only if $\frac{B_1B_2}{h_1h_2} = \frac{B_2B_3}{h_2h_3} = \frac{B_3B_1}{h_3h_1}$ or

$$a_1a_2B_1B_2 = a_2a_3B_2B_3 = a_3a_1B_3B_1.$$

Because $a_1a_2B_1B_2 = a_2a_3B_2B_3$ then $a_1^2{B_1B_2}^2 = a_2{B_2B_3}^2$. By the Law of Cosines we obtain

$$ {B_1B_2}^2 = 2(p-a_3)^2(1-\cos A_3) = 2(p-a_3)^2\left(1-\frac{a_1^2+a_2^2-a_3^2}{2a_1a_2}\right). $$

Therefore

$$ {B_1B_2}^2 = \frac{4(p-a_3)^2(p-a_1)(p-a_2)}{a_1a_2}. $$

Similarly

$$ {B_2B_3}^2 = \frac{4(p-a_1)^2(p-a_2)(p-a_3)}{a_2a_3}. $$

Thus, the inequality $a_1^2{B_1B_2}^2 = a_2{B_2B_3}^2$ is equivalent to $a_1(p-a_3) = a_3(p-a_1)$ and so $a_1 = a_3$. Analogously we deduce that $a_2 = a_3$. Consequantly $a_1 = a_2 = a_3$ which proves that $T$ is equilateral and we are done.

**Corollary 4.** Let $O$ be the circumcenter of $T$. Then

$$ \frac{OB_1}{h_1}+\frac{OB_2}{h_2}+\frac{OB_3}{h_3} \le \frac{R}{r}-1. \tag{6} $$

*Solution.* By using (1) and the equality $\frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3}=\frac{1}{r}$ we obtain

$$ \overrightarrow{OI} = r\left(\frac{\overrightarrow{OB_1}}{h_1}+\frac{\overrightarrow{OB_2}}{h_2}+\frac{\overrightarrow{OB_3}}{h_3}\right). $$

On squaring both sides and using the identity $2\overrightarrow{OB_i}\overrightarrow{OB_j} = {OB_i}^2+{OB_j}^2-{B_iB_j}^2,\ i,j = 1,2,3$ we get

$$ OI^2 = r^2\left(\sum_{i=1}^{3}\frac{{OB_i}^2}{h_i^2}+\sum_{1\le i<j\le 3}\frac{{OB_i}^2+{OB_j}^2}{h_ih_j}+\sum_{1\le i<j\le 3}\frac{{B_iB_j}^2}{h_ih_j}\right). $$

By using the result from **Problem 1** and Euler's theorem $OI^2 = R^2-2Rr$ it follows that

$$ \frac{{OB_1}^2}{h_1^2}+\frac{{OB_2}^2}{h_2^2}+\frac{{OB_3}^2}{h_3^2}+\frac{OB_1+{OB_2}^2}{h_1h_2}+\frac{OB_2+{OB_3}^2}{h_2h_3}+\frac{OB_3+{OB_1}^2}{h_3h_1} = \left(\frac{R}{r}-1\right)^2. $$

Because ${OB_i}^2+{OB_j}^2 \ge 2OB_iOB_j,\ i,j = 1,2,3$ and the identity $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$ we obtain

$$ \left(\frac{OB_1}{h_1}+\frac{OB_2}{h_2}+\frac{OB_3}{h_3}\right) \le \left(\frac{R}{r}-1\right)^2. $$

Since $R \geq 2r > r$ we obtain the desired result. Equality occurs if and only if $T$ is an equilateral triangle.

**Problem 2.** Prove that

$$\frac{MB_1}{h_1} + \frac{MB_2}{h_2} + \frac{MB_3}{h_3} \geq 1 \tag{7}$$

for all points $M$.

*Solution.* We will use the follwing two facts $|\overrightarrow{u}| \cdot |\overrightarrow{v}| \geq \overrightarrow{u} \cdot \overrightarrow{v}$ and $\overrightarrow{MB_i} = \overrightarrow{MI} + \overrightarrow{IB_i},\ i = 1, 2, 3$ to obtain

$$\sum_{i=1}^{3} \frac{MB_i}{h_i} \geq \frac{1}{r} \sum_{i=1}^{3} \frac{\overrightarrow{MB_i}\overrightarrow{MB_i}}{h_i} = \frac{1}{r} \left( \sum_{i=1}^{3} \frac{IB_i}{h_i} \right) \overrightarrow{MI} + r \sum_{i=1}^{3} \frac{1}{h_i}.$$

The desired inequality follows if we keep in mind that $\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{1}{r}$. Equality occurs if and only if $M = I$.

**Problem 3.** Prove that

$$\frac{{MB_1}^2}{h_1} + \frac{{MB_2}^2}{h_2} + \frac{{MB_3}^2}{h_3} = \frac{MI^2}{r} + r \tag{8}$$

for all points $M$ in the plane.

*Solution.* We have

$$\sum_{i=1}^{3} \frac{{MB_i}^2}{h_i} = \sum_{i=1}^{3} \frac{\left(\overrightarrow{MI} + \overrightarrow{IB_i}\right)}{h_i} = MI^2 \sum_{i=1}^{3} \frac{1}{h_i} + 2 \left( \sum_{i=1}^{3} \frac{\overrightarrow{IB_i}}{h_i} \right) \overrightarrow{MI} + \sum_{i=1}^{3} \frac{{IB_i}^2}{h_i}.$$

Thus,

$$\sum_{i=1}^{3} \frac{{MB_i}^2}{h_i} = \frac{MI^2}{r} + r$$

and we are done

## References


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